ON THE COMMUTATOR SUBGROUP OF THE FUNDAMENTAL GROUP OF THE COMPLEMENT TO A PLANE CURVE

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ABSTRACT. In this article the following theorem is proven. The commutator subgroup of the fundamental group of the complement to an irreducible curve in $\mathbb{P}^2$ is finitely presented.

0. Let $\overline{D} \subset \mathbb{P}^2$ be a projective algebraic curve. Denote by $\overline{G} = \pi_1(\mathbb{P}^2 \setminus \overline{D})$ the fundamental group of the complement of $\overline{D}$ in $\mathbb{P}^2$.

As a real subvariety, $\bar{D}$ is of real codimension 2 in $\mathbb{P}^2$. This situation is similar to one in the knot theory: a knot k is of real codimension 2 in the three-dimensional sphere S^3. It is well known that the set of knots is divided into two parts according to the properties of their groups, that is, the fundamental groups of their complements in S^3. Denote by $G = \pi_1(S^3 \setminus k)$ the group of a knot k and by $N = [G, G]$ its commutator subgroup. By theorem of Stallings [S], N is a finitely presented group if and only if k is a fibred knot, that is, $S^3 \setminus k$ admits a structure of fibration over S^1 with Seifert surfaces as fibres.

The aim of this short note is to prove the following theorem.

Theorem 1. *If $\overline{D} \subset \mathbb{P}^2$ is an irreducible curve, then the commutator subgroup $\overline{N} = [\overline{G}, \overline{G}]$ of $\overline{G} = \pi_1(\mathbb{P}^2 \setminus \overline{D})$ is finitely presented.*

Theorem 1 is a simple consequence from the following analog of this theorem in the affine case.

Theorem 2. *If $D \subset \mathbb{C}^2$ is an affine irreducible curve such that its projective closure $\overline{D} \subset \mathbb{P}^2$ and the line at infinity $L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2$ meet transversally, then the commutator subgroup $N = [G, G]$ of $G = \pi_1(\mathbb{C}^2 \setminus D)$ is finitely presented.*

In [K] it was proven that the commutator subgroup $N = [G, G]$ of $G = \pi_1(\mathbb{C}^2 \setminus D)$ is finitely generated for any irreducible affine curve. To prove Theorems 1 and 2 we essentually base on the ideas and results from [K].

We shall consider more general situation when $\overline{D} = \overline{D}_1 + \cdots + \overline{D}_n$ is a reducible reduced curve.

Let $L_\infty \subset \mathbb{P}^2$ be a straight line and define $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$, $D_i = \overline{D}_i \cap \mathbb{C}^2$. By $f_i(x,y) = 0$ denote an equation of D_i, where $f_i(x,y) \in \mathbb{C}[x,y]$ is an irreducible polynomial.

By

$$F : X = \mathbb{C}^2 \setminus D \to \mathbb{C}^* = \mathbb{C} \setminus \{0\} \tag{1}$$

denote the morphism defined by equation

$$z = \prod_{i=1}^{n} f_i(x,y)$$

We shall assume that the following condition is satisfied:

(*) *A general fiber* $F^{-1}(z) = Y_z$ *is connected.*

If D is connected in $\mathbb{C}^2$, then F satisfies the condition (*).

Theorem 2′. *If* $\overline{D} = \overline{D}_1 + \cdots + \overline{D}_n \subset \mathbb{P}^2$ *and* L_∞ *meet transversally and* D *satisfies the condition (*), then the kernel* N *of the induced homomorphism* $F_* : \pi_1(\mathbb{C}^2 \setminus D) \to \pi_1(\mathbb{C}^*)$ *is a finitely presented group.*

Theorem 2 is a corollary of Theorem 2′, since if D is irreducible, then $\ker F_*$ coincides with the commutator subgroup of $\pi_1(\mathbb{C}^2 \setminus \bar{D})$.

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1. Theorem 2 implies Theorem 1. Indeed, consider an irreducible projective curve $\overline{D} \subset \mathbb{P}^2$ and choose a line at infinity $L_\infty \subset \mathbb{P}^2$ such that $\overline{D}$ and L_∞ meet transversally. We have a natural homomorphism

$$i_* : G = \pi_1(\mathbb{C}^2 \setminus D) \to \overline{G} = \pi_1(\mathbb{P}^2 \setminus \overline{D})$$

induced by inclusion $i : \mathbb{C}^2 \setminus D \to \mathbb{P}^2 \setminus \overline{D}$. Obviously, i_* is an epimorphism. By [N], since $\overline{D}$ and L_∞ meet transversally, the kernel of i_* is an infinite cyclic group generated by a simple circuit around the line at infinity. Denote this generator of $\ker i_*$ by γ_∞. Since i_* is an epimorphism, the restriction $j : N \to \overline{N} = [\overline{G}, \overline{G}]$ of i_* to N is also epimorphism.

Let $f(x,y) = 0$ be an equation of D in $\mathbb{C}^2$, where $f(x,y)$ is an irreducible polynomial. The polynomial $f(x,y)$ determines a morphism $F : \mathbb{C}^2 \to \mathbb{C}^1$ defined by equation $f(x,y) = z$ such that $D = F^{-1}(0)$ is a fibre over zero. Consider the restriction $\varphi : \mathbb{C}^2 \setminus D \to \mathbb{C}^1 \setminus \{0\} = C^*$ of F to $\mathbb{C}^2 \setminus D$. The induced homomorphism $\varphi_* : G \to \pi_1(\mathbb{C}^*) \simeq \mathbb{Z}$ is an epimorphism, since a general fibre of φ is connected.

On the other hand, it is well known that $\pi_1(\mathbb{C}^2 \setminus D)$ is generated by the following geometric generators. By definition, a geometric generator γ is a loop consisting of a path l, a small circuit s around D and a path l^{-1}, where l connects the base point of the fundamental group with a point x close to D, s is a curcle (with positive orientation) lying in a real plane passing frough x and meeting transversally D at a point $y \in D$ which is the center of s. If D is irreducible, then all geometric generators are conjugated to each other. Therefore, the natural epimorphism α : $G \to G/N \simeq H_1(\mathbb{C}^2 \setminus D, \mathbb{Z}) \simeq \mathbb{Z}$, $N = [G, G]$, sends all geometric generators of G to a generator of $\mathbb{Z}$. It is easy to see that φ_* also sends all geometric generators of G to a generator of $\mathbb{Z}$. Hence, φ_* and α coincide. Moreover, the homorphism α allows us to consider G as a semidirect product $G \simeq N \ltimes \mathbb{Z}$. We fix one of the geometric generators, say γ, as a generator of the second factor $\mathbb{Z}$. Then γ_∞ can be represented as a product: $\gamma_\infty = \nu\gamma^d$, where $d = \deg f(x, y)$ is the degree of the curve D and ν is some element of N. Since the intersection $\ker i_* \cap N$ is trivial, the homomorphism $j : N \to \overline{N}$ is an isomorphism.

2.1.. *Proof* of Theorem 2′. Consider the map F defined by (1) and denote by $X = \mathbb{C}^2 \setminus D$ the complement of D. It is well known that there exists a finite subset

$$\{z_1, ..., z_n\} \subset \mathbb{C}^*$$

such that

$$F : X \setminus F^{-1}(\{z_1, ..., z_n\}) \to \mathbb{C}^* \setminus \{z_1, ...z_n\}$$

is a locally trivial C^∞-bundle. As in [K], let B_i be a disk of center z_i and radius $r_i \ll 1$, and let ∂B_i be its boundary. Choose two distinct points $z_{i,1}$, $z_{i,2}$ belonging to ∂B_i. The points $z_{i,1}$, $z_{i,2}$ divide ∂B_i into two arcs $\gamma_{i,1}$ and $\gamma_{i,2}$. Choose non-intersecting paths γ_i connecting the points $z_{i,1}$ and $z_{i+1,2}$ ($z_{n+1,2} = z_{1,2}$), and let $\gamma_{i,1}$ be the arc of ∂B_i such that $l_{in} = (\cup\gamma_{i,1}) \cup (\cup\gamma_i)$ is the boundary of a restricted set V containing the origin $o \in \mathbb{C}^1$, and such that $z_i \notin V$ for all i, $1 \le i \le n$ (see Figure 1 in [K]). Let l_{ex} be the boundary of the set $V \cup (\cup B_i)$. Put $T = (\cup B_i) \cup (\cup \gamma_i)$. The set $Z = F^{-1}(T)$ is called a *necklace* of D .

Since T is a retract of C^* and the fibration $F : X \setminus Z \to \mathbb{C}^* \setminus T$ is a locally trivial, we have the following

Proposition 1. [K] *If D satisfies the condition (*), then $X = \mathbb{C}^2 \setminus D$ and the necklace Z of D are homotopy equivalent.*

Thus $\pi_1(\mathbb{C}^2 \setminus D) \simeq \pi_1(Z)$, moreover, we have the following commutative diagram

$$\begin{array}{ccccc} \pi_1(\mathbb{C}^2 \setminus D) & \xleftarrow[\sim]{} & \pi_1(Z) & & \\ {\scriptstyle F_*}\downarrow & & \downarrow{\scriptstyle F_*} & & \\ \pi_1(\mathbb{C}^*) & \xleftarrow[\sim]{} & \pi_1(T) & \xrightarrow[\sim]{} & \mathbb{Z}. \end{array}$$

If D satisfies the condition (*), then F_* is an epimorphism.

Let $z_0 \in \gamma_n$ be a point and let $Y = F^{-1}(z_0)$ be the fiber over z_0. The embedding $Y \subset Z$ induces the homomorphism $\psi : \pi_1(Y) \to \pi_1(Z)$. Obviously, $Im\,\psi \subset \ker F_*$. In [K], it was shown that the following theorem is true.

Theorem 1. *If $D \subset \mathbb{C}^2$ satisfies the condition (*), then the following sequence*

$$\pi_1(Y) \xrightarrow{\ \psi\ } \pi_1(\mathbb{C}^2 \setminus D) \xrightarrow{\ F_*\ } \mathbb{Z} \longrightarrow 0$$

is exact.

Corollary 1. *If $D \subset \mathbb{C}^2$ satisfies the condition (*), then $N = \ker F_*$ is a finitely generated group.*

Denote by $Z_{ex} = F^{-1}(l_{ex})$ the preimage of l_{ex}. The inclusion $Y \subset Z_{ex} \subset Z$ and the morphism F give rise to the following commutative diagram

$$(2)\qquad \begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(Y) & \xrightarrow{\ \alpha_{ex}\ } & \pi_1(Z_{ex}) & \xrightarrow{\ F_*\ } & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \beta_{ex}} & & \downarrow{\scriptstyle \simeq} & & \\
1 & \longrightarrow & N & \xrightarrow{\ \alpha\ } & \pi_1(Z) & \xrightarrow{\ F_*\ } & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

The map $F : Z_{ex} \to l_{ex}$ is a locally trivial fibration. Thus all rows in this diagram are exact.

Denote by h_{ex} the diffeomorphism of Y determined by the circuit along l_{ex}.

2.2. We fix a point $y_0 \in Y_0 = Y$. Let $l_i \subset l_{ex}$ be a path joining z_0 with $z_{i,2}$ (we use notations from 2.1) and consisting of the part of γ_n up to the point $z_{1,1}$, the path from $z_{1,1}$ to $z_{1,2}$ along $\gamma_{1,2}$, the path γ_1, the path from $z_{2,1}$ to $z_{2,2}$ along $\gamma_{2,2}$, and so on up to the point $z_{i,1}$. If we fix local trivializations of the bundle $F : Z_{ex} \to l_{ex}$ over some covering of l_{ex}, then the paths l_i lift uniquely to paths $\bar{l}_i \subset Z_{ex}$ starting at the point y_0. We denote by y_i the end of the path $\bar{l}_i$.

Let $\overline{B}_i = F^{-1}(B_i)$. The above paths $\bar{l}_i$ define homomorphisms $\rho_i : \pi_1(\overline{B}_i, y_i) \to \pi_1(Z, y_o)$. We denote by $\psi_i : \pi_1(Y_i, y_i) \to \pi_1(\overline{B}_i, y_i)$ the homomorphism induced by inclusion, where $Y_i = F^{-1}(z_{i,1})$. By Lemma 2 in [K], the homomorphisms ψ_i are epimorphisms.

Since $F : Z_{ex} \to l_{ex}$ is a locally trivial bundle, the above paths $\bar{l}_i$ define isomorphisms $\alpha_i : \pi_1(Y_i, y_i) \to \pi_1(Y_0, y_0)$. Hence in what follows we shall identify the groups $\pi_1(Y_i, y_i)$ with the group $\pi_1(Y_0, y_0)$. Thus we obtain epimorphisms $\psi_i : \pi_1(Y_0, y_0) \to \pi_1(\overline{B}_i, y_i)$.

2.3. We identify $\pi_1(l_{ex})$ with $\pi_1(T)$ by means of isomorphism induced by inclusion $l_{ex} \subset T$. The exact sequence

$$1 \longrightarrow N \longrightarrow \pi_1(Z, y_0) \xrightarrow{\ F_*\ } \pi_1(T, z_0) \longrightarrow 1$$

defines an infinite cyclic covering $\tilde{g} : \widetilde{Z} \to Z$ fitting into a commutative diagram

$$\begin{array}{ccc}
\widetilde{Z} & \xrightarrow{\ \widetilde{F}\ } & \widetilde{T} \\
{\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\
Z & \xrightarrow[\ F\ ]{} & T
\end{array}$$

in which $g : \widetilde{T} \to T$ is the universal covering. Pick a point $\tilde{y}_0 \in \tilde{g}^{-1}(y_0)$, and let $\tilde{z}_0 = \widetilde{F}(\tilde{y}_0)$. Then $\pi_1(\widetilde{Z}, \tilde{y}_0) = N$.

The space $\widetilde{T}$ is a disjoint union of countably many discs $B_{i,j}$, $j \in \mathbb{Z}$, such that $B_{i,j} \subset g^{-1}(B_i)$. These discs are joined by intervals (see Figure 2 in [K]) and form a chain. We number the discs $B_{i,j}$ in the order induced by the order in the chain as is shown on Figure 2. In each interval joining neighboring discs we pick a point $\tilde{z}_i$ (in the interval joining the discs $B_{n,-1}$ and $B_{1,1}$ we take the above point $\tilde{z}_0$) and number them in order induced by the order in the chain (the point $\tilde{z}_0$ has the number 0).

We denote by $\widetilde{T}_{kn,mn}$ the "part" of the space $\widetilde{T}$ lying between the points $\tilde{z}_{kn}$ and $\tilde{z}_{mn}$, $m > k$, where n is the number of discs B_i belonging to T. Let $\widetilde{Z}_{kn,mn} = \widetilde{F}^{-1}(\widetilde{T}_{kn,mn})$.

Lemma 1. *$\pi_1(\widetilde{Z}_{kn,mn})$ is a finitely presented group.*

Proof. Let $\tilde{z}_{0,i,j}$ be the center of the disc $B_{i,j}$. Consider a space

$$\widetilde{Z}^0_{kn,mn} = \widetilde{F}^{-1}(\widetilde{T}_{kn,mn} \setminus \bigcup_{i=1}^{n} \bigcup_{j=k}^{m} \tilde{z}_{0,i,j}).$$

Since fibrations $\widetilde{F} : \widetilde{F}^{-1}(B_{i,j} \setminus \{\tilde{z}_{0,i,j}\}) \to B_{i,j} \setminus \{\tilde{z}_{0,i,j}\}$ are locally trivial C^∞-bundles over punctured discs with punctured Riemann surfaces as fibres, the fundamental groups $\pi_1(\widetilde{F}^{-1}(B_{i,j} \setminus \{\tilde{z}_{0,i,j}\}))$ are finitely presented. Applying Seifert - van Kampen theorem, we obtain that $\pi_1(\widetilde{Z}^0_{kn,mn})$ is a finitely presented group. The preimage $\widetilde{F}^{-1}(\bigcup_{i=1}^{n} \bigcup_{j=k}^{m} \tilde{z}_{0,i,j})$ is the union of a finite number of Riemann surfaces and the kernel of the natural epimorphism $\pi_1(\widetilde{Z}^0_{kn,mn}) \to \pi_1(\widetilde{Z}_{kn,mn})$ is generated by geometric generators which are circuits around these surfaces. Since for each irreducible Riemann surface any two circuits around it are conjugated, we obtain that $\pi_1(\widetilde{Z}_{kn,mn})$ is a finitely presented group.

2.4.

Lemma 2. *If $f(x,y)$ is reduced and $\overline{D}$ and L_∞ meet transversally, then $h^d_{ex} = id$, where $d = \deg f(x,y)$*

Proof. The morphism F defines a rational map

$$F : \mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty \to \mathbb{P}^1 = \mathbb{C}^1 \cup \{\infty\}.$$

Let $\sigma : \overline{\mathbb{P}}^2 \to \mathbb{P}^2$ be a composition of σ-processes such that $\overline{F} = F \cdot \sigma : \overline{\mathbb{P}}^2 \to \mathbb{P}^1$ is a morphism.

The equation $z^d = f(x,y)$ defines a normal projective surface $\widetilde{X}_d \subset \mathbb{P}^3$ and a morphism $\tilde{\phi}_d : \widetilde{X}_d \to \mathbb{P}^2$. The preimage $\tilde{\phi}_d^{-1}(L_\infty) = \overline{Y}_\infty$ is a non-singular curve. Let $\overline{X}_d$ be the normalization of $\overline{\mathbb{P}}^2$ in the field $\mathbb{C}(x,y,z)$ and $\phi_d : \overline{X}_d \to \overline{\mathbb{P}}^2$ the corresponding morphism.

Choose a neighborhood U of the point $\infty \in \mathbb{P}^1$ which is isomorphic to the disc $\Delta = \{ u \in \mathbb{C} \mid | u | \leq 1 \}$ (the origin $u = 0$ corresponds to the point $\infty \in U$) and such that the map $\overline{F} : \overline{F}^{-1}(U \setminus \infty) \to U \setminus \infty$ is a smooth proper morphism. Put $\overline{U} = \overline{F}^{-1}(U)$ and $\widetilde{U} = \phi_{-1}^{d}(\overline{U})$. We obtain the following commutative diagram:

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\phi_d} & \overline{U} \\ \downarrow{\scriptstyle \overline{F}_d} & & \downarrow{\scriptstyle \overline{F}} \\ \Delta & \xrightarrow{\psi_d} & U, \end{array} \tag{3}$$

where ψ_d is defined by equation $u = v^d$ (v is a coordinate in Δ). Since preimage $\overline{F}_d^{-1}(0) = \phi_d^{-1}(L_\infty)$ is a non-degenerate fibre of $\overline{F}_d$, the monodromy h_d, acting on a general fibre and defined by circuit around the boundary of Δ, is trivial. On the other hand, it follows from commutative diagram (3) that $h_d = h_{ex}^d$. Lemma 2 is proven.

2.5. The following Lemma completes the proof of Theorem 2′.

Lemma 3. *If $f(x,y)$ and $\overline{D}$ are as in lemma 2, then $\pi_1(\widetilde{Z}_{kdn,mdn})$ are isomorphic to $\pi_1(\widetilde{Z}_{0,dn})$ for all k and m.*

Proof. Let $\tilde{l}_{ex} = g^{-1}(l_{ex})$ and $\widetilde{Z}_{ex} = \tilde{g}^{-1}(Z_{ex})$. Then $\widetilde{F} : \widetilde{Z}_{ex} \to \tilde{l}_{ex}$ is a trivial C^∞-bundle. Consider fibres $Y_s = \widetilde{F}^{-1}(\tilde{z}_s)$ of this bundle. If we choose a trivialization, then we can identify all these fibres, in other words, the choosed trivialization induces diffeomorphisms $\alpha_{i,j} : Y_i \to Y_j$. If $kdn \le s \le mdn$, then $Y_s \subset \widetilde{Z}_{kdn,mdn}$ and this inclusion induces an epimorphism $\psi_{s,kdn,mdn} : \pi_1(Y_s) \to \pi_1(\widetilde{Z}_{kdn,mdn})$ such that if $kdn \le r \le mdn$, then $\psi_{s,kdn,mdn}\alpha_{r,s*} = \psi_{r,kdn,mdn}$.

All spaces $\widetilde{Z}_{kdn,(k+1)dn}$, $k \in \mathbb{Z}$, are naturally diffeomorphic to each other, since these spaces are the preimages of d circuits along the necklace T starting at the point z_0. These diffeomorphisms allow us to identify the fundamental groups $\pi_1(\widetilde{Z}_{kdn,(k+1)dn}) \simeq \pi$ for all k. This identification is compatible with the above identification of the fibres Y_r, that is, the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(Y_r) & \xrightarrow{\alpha_{r,s*}} & \pi_1(Y_s) \\ {\scriptstyle \psi_{r,kdn,(k+1)dn}}\downarrow & & \downarrow{\scriptstyle \psi_{s,mdn,(m+1)dn}} \\ \pi_1(\widetilde{Z}_{kdn,(k+1)dn}) & \xrightarrow{\simeq} & \pi_1(\widetilde{Z}_{mdn,(m+1)dn}) \end{array}$$

To obtain $\widetilde{Z}_{0,2dn}$ from $\widetilde{Z}_{0,dn}$ and $\widetilde{Z}_{dn,2dn}$ (similarly, for $\widetilde{Z}_{-2dn,0}$), we must paste these two subspaces of $\widetilde{Z}_{0,2dn}$ along the diffeomorphic fibres $Y_{dn} \subset \widetilde{Z}_{0,dn}$ and $Y_{dn} \subset \widetilde{Z}_{dn,2dn}$. The rule of pasting is defined by monodromy h_{ex}^d. In our case, by Lemma 2, $h_{ex}^d = id$. Applying Seifert - van Kampen theorem, we obtain that there exists an isomorphism $\psi_{0,2} : \pi_1(\widetilde{Z}_{0,dn}) \to \pi_1(\widetilde{Z}_{0,2dn})$ compatible with the epimorphisms α_*. The obvious induction completes the proof of this Lemma.

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