Topological Coupling of Calabi-Yau Orbifold

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Introduction

The purpose of this paper is to describe a general method of computing the second Chern class of a CY (Calabi-Yau) orbifold and the cubic form by cup product on its second integral cohomology, which will also be called the toplogical coupling in the paper. The CY spaces we are concerned with are mainly the CY hypersurfaces of weighted projective 4–spaces and the mirror pairs constructed from them in [3, 7]. By the theory of Wall [8], the diffeomorphic classes of such CY spaces V are determined by $H^3(V,\mathbb{Z})$, cubic form on $H^2(V,\mathbb{Z})$ and the linear form on $H^2(V,\mathbb{Z})$ given by the second Chern class of the manifold. The third cohomology has been known and is determined by the Vafa's formula [6, 7]. As a consequence, the result of this note will give an effective means to determine the diffeomorphic type of the CY spaces we are dealing with. The cubic form on $H^2(V,\mathbb{Z})$ have been a main ingredient for the study of rational curves in a general CY space [9]. The method given here can give an explicit expression of cubic forms even though the existence of rational curves on such CY spaces is obvious in these cases. In fact for the rational curve problem, one tends to reduce to a similar situation for a general CY manifold through the behavior of the cubic form. On the other hand, a problem in string theory raised by Aspinwall and Lütken [1] concerns that the possibility of "flip" between CY spaces with different topologies implies the ambiguity of the "large radius limit" of a given conformal field model. We shall describe a large class of examples of CY spaces with such phenomena. A natural question which arises here is how to exploit the significance of this difference for "large radius limit" in the context of conformal field theory. Work along this line is under consideration.

The organization of this paper is as follows. In Sect. 1, we consider the case when the CY space is obtained by resolving the space with only "curve-singularities" occurred, and describe the method of computing its cubic form from the normal data of singularities in the original space. In Sect. 2, the same problem is considered for CY resolution of spaces with only "point-singularities". We shall illustrate the difference of the topological couplings for different resolutions through some example. In Sect. 3, the more general situation is considered where both "curve-singularities" and "point-singularities" appear in the construction of CY spaces, and the method is applied to the mirror of Fermat quintic. In Sect. 4, we describe the method of obtaining the expression of the second Chern class of CY resolution through toric geometry. For technique reasons and for the purpose of illustration, most of the discussion in this paper is followed by some specific calculational examples.

I am most pleased to acknowledge many fruitful discussions with Professor B. Greene during the preparation of this note. And I also wish to thank Professor

F. Hirzebruch for the opportunity of visiting Max-Planck-Instutut für Mathematik while this work was done.

Section 1

In this paper, we shall use the convention of writing $O(D)$ as the line bundle over a complex manifold having a section with zero being the divisor D.

First we shall derive an easy lemma on the local structure near singular sets in the examples which we shall work with later on.

Lemma 1. Let X be a quasi-smooth hypersurface in $WP^{N-1}_{(n_i)}$ defined by a quasi-homogeneous polynomial

$$f(Z) = f(Z_1, \ldots, Z_N) = 0,$$

here we assume gcd($n_i|i \neq j$)=1 for all j. Suppose for some $m < N$, X intersects with $Z_{m+1} = \ldots = Z_N = 0$ transversely, (i.e., for $a \in \mathbf{C}^N - \{0\}$, $f(a) = Z_{m+1}(a) = \ldots = Z_N(a) = 0$ implies $\frac{\partial f}{\partial Z_i}(a) \neq 0$ for some $i \leq m$.) Denote

$$Y = X \cap \{Z_{m+1} = \ldots = Z_N = 0\},$$
$$d = \gcd(n_1, \ldots, n_m),$$
$$\mathsf{H}^k = \text{ the line bundle over } Y \text{ corresponding to the restriction of } \mathcal{O}_{\mathsf{WP}^{N-1}}(k).$$

If $y = [y_i]$ is an element of Y with $\gcd(n_i|y_i \neq 0) = d$, then the following spaces are isomorphic as germs of analytic spaces:

$$(X, Y, y) \simeq \left(\left(\oplus_{j=m+1}^N \mathsf{H}^j\right)/\mathbf{Z}_d\,, Y, y\right),$$

here the generator of $\mathbf{Z}_d$ acts on $\oplus_{j=m+1}^N \mathsf{H}^j$ by $(h_j)_{j=m+1}^N \rightsquigarrow \left(e^{n_j \frac{2\pi i}{d}} h_j\right)_{j=m+1}^N$, and the space Y on the right hand side is identified with the zero section.

Proof. Denote

$$C(X) = \{Z \in \mathbf{C}^5 - \{0\} | f(Z) = 0\},$$
$$C(Y) = C(X) \cap \{Z_{m+1} = \ldots = Z_N = 0\},$$
$$\varphi : C(X) \to \mathbf{C}^{N-m}, \quad (Z_i)_{i=1}^N \rightsquigarrow (Z_i)_{i=m+1}^N.$$

Then the map φ is $\mathbf{C}^*$– equivariant with the $\mathbf{C}^*$—actions defined by $\lambda \cdot (Z_i) := (\lambda^{n_i} Z_i)$. The transversal condition of X with $Z_j = 0$, $j > m$, implies

$$(C(X), C(Y)) \simeq \left(C(Y) \times \mathbf{C}^{N-m}, C(Y) \times 0\right).$$

Therefore for y satisfying the condition of this lemma, we have

$$(X,Y,y) \simeq \left(Y \times \left(\mathbf{C}^{N-m}/\mathbf{Z}_d\right), Y \times [0], y \times [0]\right)$$

here the $\mathbf{Z}_d$—action on $\mathbf{C}^{N-m}$ is given by $\left([k], (\zeta_j)_{j=m+1}^{N}\right) \rightsquigarrow \left(e^{kn_j\frac{2\pi i}{d}}\zeta_j\right)_{j=m+1}^{N}$. Then the result follows from the definition of H^{ν}. q.e.d.

The following theorem will be used for the computation of couplings when only curve-singularity appears in the construction of CY resolution.

Theorem 1. Let L_i $(i = 1,2)$ be line bundles over a complex manifold M, and G be the group of dth roots of unity in $\mathbf{C}^*$. Consider the action of G on $\mathsf{L}_1 \oplus \mathsf{L}_2$,

$$g \cdot (\ell_1, \ell_2) = \left(g\ell_1, g^{-1}\ell_2\right) \qquad g \in G,\ \ell_i \in \mathsf{L}_i\,.$$

Denote

$$\begin{aligned} \mathcal{X} &= (\mathsf{L}_1 \oplus \mathsf{L}_2)/G\,, \\ \sigma : \hat{\mathcal{X}} &\to \mathcal{X} \text{ the minimal resolution}, \\ D_0 &= \text{the proper transform of } (0 \times \mathsf{L}_2)/G, \\ D_d &= \text{the proper transform of } (\mathsf{L}_1 \times 0)/G, \\ \pi : \hat{\mathcal{X}} &\to M \text{ the fiber bundle induced by} \\ &\qquad \text{the projection of } \mathsf{L}_1 \oplus \mathsf{L}_2 \text{ to } M. \end{aligned}$$

Then

(i) $\sigma^{-1}(\mathrm{Sing}(\mathcal{X}))$ is the union of D_0, D_d with the exceptional divisors D_j, $1 \leq j \leq d-1$. Only intersection among D_i $(0 \leq i \leq d)$ are

$$M_k := D_k \cap D_{k-1} \overset{\pi_{rest}}{\simeq} M \quad \text{for} \quad 1 \leq k \leq d.$$

(The D_j, M_j are shown in Figure 1.)

(ii) The following relations hold:

$$\begin{aligned} \bigotimes_{j=0}^{d} \mathcal{O}(D_j) &= \pi^*(\mathsf{L}_1 \otimes \mathsf{L}_2) \quad \text{over } \hat{\mathcal{X}}\,, \\ \mathcal{O}(D_{k-1})_{|M_k} &\simeq \mathsf{L}_1^{k} \otimes \mathsf{L}_2^{k-d}\quad, \\ \mathcal{O}(D_k)_{|M_k} &\simeq \mathsf{L}_1^{1-k} \otimes \mathsf{L}_2^{d-k+1} \quad \text{over } M_k \end{aligned}$$

for $1 \leq k \leq d$.

Proof. (i) follows from the construction of the minimal resolution $\hat{\mathcal{X}}$. We are going to show (ii) in the following two steps.

Step (I). We shall describe the local structure along the fiber of $\pi : \hat{\mathcal{X}} \to M$. We have the isomorphism

$$\pi^{-1}(m) \simeq \mathbf{C}^2/G \qquad \text{for } m \in M, \qquad (1)$$

here G acts on $\mathbf{C}^2$ by

$$g \cdot (z_1, z_2) = \left(g z_1, g^{-1} z_2\right), \; g \in G \quad z_i \in \mathbf{C}.$$

The local coordinate system of the minimal resolution $\hat{\mathbf{C}^2/G}$ can be described by toric data as a compactification of $\mathbf{C}^2/G$. We shall denote (z_1, z_2) the coordinates of $\mathbf{C}^2$. Let

$$\mathfrak{n} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2 | \mathrm{dia}\left[e^{2\pi i x_1}, e^{2\pi i x_2}\right] \in G \right\},$$

$$\triangle = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^2 | \sum_{i=1}^{2} x_i = 1, \, x_i \geq 0 \text{ for all } i \right\}.$$

Then $\mathfrak{n} \cap \triangle = \left\{\rho^i\right\}_{i=0}^{d}$ with $\rho^i := \begin{pmatrix} \frac{d-i}{d} \\ \frac{i}{d} \end{pmatrix}$. For each ρ^i, there associates a divisor D_{ρ^i} in $\hat{\mathbf{C}^2/G}$. D_{ρ^0}, D_{ρ^d} are the proper transform for $(0\times\mathbf{C})/G$, $(\mathbf{C}\times 0)/G$, and $D_{\rho^i}, 1 \leq i \leq d-1$, are the exceptional divisors. Let $\{e^1, e^2\}$ be the standard base of $\mathbf{R}^2$, and $\{e_1, e_2\}$ its dual. We have

$$\left(\rho^{k-1}, \rho^k\right) = (e^1, e^2) \begin{pmatrix} \frac{d-k+1}{d} & \frac{d-k}{d} \\ \frac{k-1}{d} & \frac{k}{d} \end{pmatrix},$$

$$\begin{pmatrix} \rho_*^{k-1} \\ \rho_*^k \end{pmatrix} = \begin{pmatrix} k & k-d \\ 1-k & d-k+1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

here $\{\rho_*^{k-1}, \rho_*^k\}$ is the dual base of $\{\rho^{k-1}, \rho^k\}$. Let (s_k, t_k) be the local coordinate system in $\hat{\mathbf{C}^2/G}$ corresponding to $\{\rho^{k-1}, \rho^k\}$. The relations

$$\begin{cases} s_k = z_1^k z_2^{k-d} \\ t_k = z_1^{1-k} z_2^{d-k+1} \end{cases} \qquad (2)$$

hold for $1 \leq k \leq d$. The local defining equations for $D_{\rho^{k-1}}, D_{\rho^k}$ are given by

$$\begin{aligned} D_{\rho^{k-1}} : \quad & s_k = z_1^k z_2^{k-d} = 0 \\ D_{\rho^k} : \quad & t_k = z_1^{1-k} z_2^{d-k+1} = 0 \end{aligned} \qquad , \qquad (3)$$

and by the relation $z_1 z_2 = s_k t_k$, the defining equation for $\sum_{j=0}^{d} D_{\rho^j}$ is

$$\sum_{j=0}^{d} D_{\rho^j} : \quad z_1 z_2 = 0 \qquad . \qquad (4)$$

Step (II). We now apply the analyses of Step (I) to the study of the divisors D_j of $\hat{\mathcal{X}}$. It is known that for $m \in M$, $D_j \cap \pi^{-1}(m)$ corresponds to D_{ρ^j} in the isomorphism (1). Over an open set of M, let ℓ_i be coordinates of L_i $(i = 1, 2)$. For $1 \leq k \leq d$, $\ell_1^k \ell_2^{k-d}$, $\ell_1^{1-k}\ell_2^{d-k+1}$ are considered as local functions of $\hat{\mathcal{X}}$ by (2), (3) and the local generators of the ideals of D_j's are given by:

$$\mathcal{I}_{D_{k-1}} = < \ell_1^k \ell_2^{k-d} >,$$
$$\mathcal{I}_{D_k} = < \ell_1^{1-k}\ell_2^{d-k+1} > .$$

Then it follows:

$$\mathsf{O}(D_{k-1})_{|M_k} \simeq \mathsf{L}_1^k \otimes \mathsf{L}_2^{k-d} ,$$
$$\mathsf{O}(D_k)_{|M_k} \simeq \mathsf{L}_1^{1-k} \otimes \mathsf{L}_2^{d-k+1}$$

over M_k for $1 \leq k \leq d$. Since $\ell_1 \otimes \ell_2$ is invariant under the action of G, it defines a holomorphic section of the line bundle $\pi^*(\mathsf{L}_1 \otimes \mathsf{L}_2)$ over $\hat{\mathcal{X}}$. By (4), the zeros of this section is equal to $\sum_{j=0}^{d} D_j$, therefore

$$\bigotimes_{j=0}^{d} \mathsf{O}(D_j) = \pi^*(\mathsf{L}_1 \otimes \mathsf{L}_2) \quad \text{over } \hat{\mathcal{X}} .$$

q.e.d.

Example 1. Let X be Fermat hypersurface in $\mathsf{WP}^4_{(2,2,2,1,1)}$

$$Z_1^4 + Z_2^4 + Z_3^4 + Z_4^8 + Z_5^8 = 0 \quad .$$

The singularity of X is given by

$$\mathrm{Sing}(X) = X \cap \{Z_4 = Z_5 = 0\}$$

which is a Riemann surface of genus 3. The CY resolution $\hat{X}$ of X has only one exceptional divisor D. By Lemma 1, the structure of X near $\mathrm{Sing}(X)$ is described as in the assumption of Theorem 1 with

$$M = \mathrm{Sing}(X),$$
$$\mathsf{L}_1 = \mathsf{L}_2 = \mathsf{H} \ (:= \text{the restriction of } \mathsf{O}_{\mathsf{WP}^4}(1)),$$
$$G = \mathbb{Z}/2\mathbb{Z}$$

We have $D = D_1$ and

$$\begin{aligned} D^3 &= D^2(\pi^*\mathsf{H}^2 - D_0 - D_2) \\ &= D(D \cdot \pi^*\mathsf{H}^2) - c_1\Big(\mathsf{O}(D_1)_{|D_1 \cap D_0}\Big) - c_1\Big(\mathsf{O}(D_1)_{|D_2 \cap D_1}\Big) \\ &= 4(-2) - c_1(\mathsf{H}^2) - c_1(\mathsf{H}^2) = -16 \quad . \end{aligned}$$

Denote h the element in $H^2(\hat{X}, \mathbf{Z})$ which represents the pull-back of $O_X(1)$. Then the coupling μ for $H^2(\hat{X}, \mathbf{Z})$ has the expression:

$$\mu(t \cdot h + s \cdot c_1(D)) = 2t^3 - 16s^3 .$$

q.e.d.

Example 2. Let X be the quotient of Fermat quintic

$$Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 + Z_5^5 = 0 \quad \text{in} \quad \mathbf{P}^4$$

by the order 5 group generated by

$$[Z_1, Z_2, Z_3, Z_4, Z_5] \rightsquigarrow [Z_1, Z_2, Z_3, \omega Z_4, \omega^4 Z_5]$$

with $\omega^5 = 1$. Then

$$\mathrm{Sing}(X) = X \cap \{Z_4 = Z_5 = 0\}$$

which is a Riemann surface of genus 6. The CY resolution $\hat{X}$ contains 4 exceptional divisors D_j, $1 \leq j \leq 4$, each of which is a $\mathbf{P}^1-$ bundle over $\mathrm{Sing}(X)$. Denote D_0, D_5 the divisors in $\hat{X}$ obtained by the proper transform of $Z_4 = 0$, $Z_5 = 0$ respectively. The classes $c_1(D_j), 1 \leq j \leq 4$, together with h (:= the class of pull back of $\mathcal{O}_X(5)$) form a base of $H^2(\hat{X}, \mathbf{Z})$. The coupling for $H^2(\hat{X}, \mathbf{Z})$ is the expression:

$$\mu\left(t \cdot h + \sum_{i=1}^{4} t_i \cdot c_1(D_i)\right) =$$

$$125t^3 + \sum_{i=1}^{4} D_i^3 t_i^3 + \sum_{i=1}^{3} \{(D_i^2 D_{i+1}) t_i^2 t_{i+1} + (D_i D_{i+1}^2) t_i t_{i+1}^2\}.$$

By Lemma 1, we can apply Theorem 1 on the local structure of X near $\mathrm{Sing}(X)$ by setting

$$M = \mathrm{Sing}(X),$$
$$\mathsf{L}_1 = \mathsf{L}_2 = \mathsf{H} \text{ (:= the restriction of hyperplane bundle)},$$
$$G = \mathbf{Z}/5\mathbf{Z}.$$

Then

$$D_1^2 D_2 = c_1([D_1]_{D_2 \cap D_1}) = c_1(\mathsf{H}^{-1}) = -5,$$

$$
\begin{aligned}
D_1 D_2^2 &= c_1\left([D_2]_{D_2\cap D_1}\right) = c_1(\mathsf{H}^3) = 15,\\
D_2^2 D_3 &= c_1\left([D_2]_{D_3\cap D_2}\right) = c_1(\mathsf{H}) = 5,\\
D_2 D_3^2 &= c_1\left([D_3]_{D_3\cap D_2}\right) = c_1(\mathsf{H}) = 5,\\
D_3^2 D_4 &= c_1\left([D_3]_{D_4\cap D_3}\right) = c_1(\mathsf{H}^3) = 15,\\
D_3 D_4^2 &= c_1\left([D_4]_{D_4\cap D_3}\right) = c_1(\mathsf{H}^{-1}) = -5.
\end{aligned}
$$

Also for $1 \le k \le 4$, we have

$$
[D_k] = \pi^*(\mathsf{H}^2) - \sum_{\substack{0\le j\le 5\\ j\ne k}} D_j \,,
$$

$$
\begin{aligned}
D_k^2\mathsf{H} &= 5D_k(\text{a } \mathsf{P}^1\text{–fiber in } D_k \text{ under } \pi)\\
&= -5\sum_{\substack{0\le j\le 5\\ j\ne k}} D_j(\text{a } \mathsf{P}^1\text{–fiber in } D_k \text{ under } \pi)\\
&= -10\,,\\
D_k^3 &= 2(D_k^2\mathsf{H}) - \sum_{\substack{0\le j\le 5\\ j\ne k}} D_k^2 D_j\\
&= -20 - c_1\left([D_k]_{D_k\cap D_{k-1}}\right) - c_1\left([D_k]_{D_k\cap D_{k+1}}\right)\\
&= -20 - 4c_1(\mathsf{H}) = -40\,.
\end{aligned}
$$

Hence the coupling for $\hat{X}$ is given by

$$
\mu\left(t\cdot h + \sum_{i=1}^{4} t_i\cdot c_1(D_i)\right) =
$$

$$
125t^3 - 40\sum_{i=1}^{4} t_i^3 - 5t_1^2t_2 + 15t_1t_2^2 + 5t_2^2t_3 + 5t_2t_3^2 + 15t_3^2t_4 - 5t_3t_4^2.
$$

q.e.d.

Section 2

In this section we compute the couplings of exceptional divisors in CY spaces obtained from the point-singularities. We shall use the toric data of the resolution to describe to the results.

Let G be a finite diagonal subgroup of $SL_3(\mathsf{C})$, $V = \mathsf{C}^3/G$, and $\hat{V}$ a CY resolution of V

$$
\sigma : \hat{V} \to V\,.
$$

We shall denote (z_1, z_2, z_3) the coordinate of $\mathbf{C}^3$. Let

$$\begin{aligned}
\mathfrak{n} &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3 | \mathrm{dia}[e^{2\pi i x_1}, e^{2\pi i x_2}, e^{2\pi i x_3}] \in G \right\}, \\
\triangle &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3 | \sum_{i=1}^{3} x_i = 1, \ x_i \geq 0 \ \text{ for all } \ i \right\}, \\
\Gamma &= \mathfrak{n} \cap \triangle \ .
\end{aligned} \tag{5}$$

Γ is finite subset of the lattice $\mathfrak{n}$, and contains the standard base $\{e^i\}_{i=1}^3$ of $\mathbf{R}^3$. There associates a divisor D_γ in $\hat{V}$ for each $\gamma \in \Gamma$. The dual configuration for intersections among D_γ's is given by the simplicial decomposition $\mathcal{S}$ of $\triangle$ with the property

$$\Gamma = \{\text{vertex in } \mathcal{S}\}.$$

It is known that

$$\begin{aligned}
D_{e^i} &= \text{the proper transform of } (z_i = 0)/G, \\
\left\{ D_\gamma | \gamma \in \Gamma - \{e^i\}_{i=1}^3 \right\} &= \left\{ \text{exceptional divisors in } \hat{V} \right\}.
\end{aligned}$$

The σ-image of an exceptional divisor D_γ is a point or a curve. We have

$$\sigma(D_\gamma) = \text{a point} \quad \Leftrightarrow \quad \gamma \in \Gamma \cap \text{interior}(\triangle),$$

in which case, $\sigma(D_\gamma)$ is the singular point of V corresponding to 0 of $\mathbf{C}^3$. Then the vertices of a 2-simplex $\{\alpha, \beta, \gamma\}$ of $\mathcal{S}$ form an integral base of $\mathfrak{n}$. Hence there corresponds a local coordinate system of $\hat{V}$, denoted by (w_1, w_2, w_3). From $\{\alpha, \beta, \gamma\} \subset \triangle$, the relation

$$w_1 w_2 w_3 = z_1 z_2 z_3$$

holds as functions of $\hat{V}$. As $z_1 z_2 z_3$ defines a global function of $\hat{V}$, we have

$$\mathbf{O}\left(\sum_{\gamma \in \Gamma} D_\gamma \right) = \text{the trivial bundle of } \hat{V}. \tag{6}$$

Theorem 2. (i) For 3 distinct elements $\alpha, \beta, \gamma \in \Gamma$,

$$D_\alpha D_\beta D_\gamma \neq 0 \ \Leftrightarrow \ \{\alpha, \beta, \gamma\} \text{ is a } 2-\text{simplex in } \mathcal{S} \ ,$$

in which case, we have $D_\alpha D_\beta D_\gamma = 1$.

(ii) For distinct $\alpha, \beta \in \Gamma \cap \text{interior}(\Delta)$, $D_\alpha^2 D_\beta = 0$ unless $\{\alpha, \beta\}$ is a 1–simplex of $\mathcal{S}$. When $\{\alpha, \beta\}$= 1–simplex of $\mathcal{S}$, there exist exactly 2 elements δ_1, δ_2 in Γ such that $\{\alpha, \beta, \delta_i\}$ are 2–simplexes of $\mathcal{S}$, and the following relations holds as vectors in $\mathbf{R}^3$:

$$\delta_1 + \delta_2 + (D_\alpha^2 D_\beta)\alpha + (D_\alpha D_\beta^2)\beta = 0.$$

(iii) For $\gamma \in \Gamma \cap \text{interior}(\Delta)$, let $\{\delta_i\}_{i=1}^L$ be the set of all the elements in Γ which can be connected to γ by 1–simplexes of $\mathcal{S}$. By the suitable indices, we assume $\{\gamma, \delta_i, \delta_{i+1}\}$ is a 2–simplex of $\mathcal{S}$ for $1 \leq i \leq L$, $(D_{L+1} := D_1)$, Define the integer n_i $(1 \leq i \leq L)$ by the equation

$$\delta_{i-1} + \delta_{i+1} + n_i \gamma + n_i' \delta_i = 0$$

for some n_i'. Then we have

$$D_\gamma^3 = -\sum_{i=1}^{L} n_i \ .$$

Proof. (i) is obvious.

(ii) Let α, β be elements in $\Gamma \cap \text{interior}(\Delta)$ such that $\{\alpha, \beta\}$= a 1–simplex of $\mathcal{S}$. It is easy to see that there are exactly 2 elements δ_1, δ_2 in Γ such that $\{\alpha, \beta, \delta_i\}$ are 2–simplexes of $\mathcal{S}$. Since both $\{\alpha, \beta, \delta_i\}$ are bases for n, we have the relation

$$(\delta_1, \alpha, \beta) = (\delta_2, \alpha, \beta) \begin{pmatrix} -1 & 0 & 0 \\ m & 1 & 0 \\ n & 0 & 1 \end{pmatrix}$$

for some integers m, n. Denote $(x_i)_{i=1}^3$, $(y_i)_{i=1}^3$ the local coordinate systems of $\hat{V}$ corresponding to $\{\delta_1, \alpha, \beta\}$, $\{\delta_2, \alpha, \beta\}$ respectively. One has

$$\left\{ \begin{array}{l} y_1 = x_1^{-1} \\ y_2 = x_1^m x_2 \\ y_3 = x_1^n x_3 \quad . \end{array} \right. \qquad (7)$$

The local defining equations for D_α, D_β are given by

$$\begin{aligned} D_\alpha : \ & x_2 = 0 \ , \ y_2 = 0, \\ D_\beta : \ & x_3 = 0 \ , \ y_3 = 0, \\ D_\alpha \cap D_\beta = \mathbf{P}^1 \ & \text{with affine coordinates } x_1, y_1. \end{aligned}$$

By (7), it follows that $D_\alpha^2 D_\beta = -m \, , D_\alpha D_\beta^2 = -n$, hence we obtain (ii).

(3) Let γ, δ_i, n_i, n'_i be the same as in the condition (iii). By (6),

$$D_\gamma^3 = -D_\gamma^2 \sum_{\substack{\alpha\in\Gamma \\ \alpha\neq\gamma}} D_\alpha = -\sum_{i=1}^{L} D_\gamma^2 D_{\delta_i} .$$

As the relation

$$\delta_{i-1} + \delta_{i+1} + n_i\gamma + n'_i\delta_i = 0$$

holds, the same argument as (ii) gives $D_\gamma^2 D_{\delta_i} = n_i$, hence $D_\gamma^3 = -\sum_{i=1}^{L} n_i$. q.e.d.

Example 3. Let X be the quotient of Fermat quintic in $\mathbf{P}^4$ by the group generated by

$$[Z_1, Z_2, Z_3, Z_4, Z_5] \rightsquigarrow [Z_1, \omega Z_2, \omega^2 Z_3, \omega^3 Z_4, \omega^4 Z_5]$$
$$[Z_1, Z_2, Z_3, Z_4, Z_5] \rightsquigarrow [Z_1, \omega Z_2, Z_3, \omega^2 Z_4, \omega^2 Z_5]$$

with $\omega^5 = 1$. (Example in [1]). Then X has only isolated singularities and

$$\mathrm{Sing}(X) = \{p_{ij}\ ,\ 1 \le i < j \le 5\}$$

here p_{ij} is the element of X with the coordinate $Z_k = 0$ for $k \neq i, j$. The structure near a singular point $p = p_{ij}$ is given by

$$(X, p) \simeq (\mathbf{C}^3/G, 0)$$

here G is the group generated by $\mathrm{dia}[\omega, \omega^2, \omega^2]$. The unique CY resolution of $\mathbf{C}^3/G$ is described by the simplicial decomposition of Δ as shown in Figure 2. In this case, $\Gamma = \{e^1, e^2, e^3, \alpha, \beta\}$ with $\alpha = \frac{1}{5}e^1 + \frac{1}{5}e^2 + \frac{3}{5}e^3$, $\beta = \frac{2}{5}e^1 + \frac{2}{5}e^2 + \frac{1}{5}e^3$. We have

$$e^1 + e^2 + 3e^3 - 5\alpha = 0,\ e^1 + e^2 + \alpha - 3\beta = 0,\ e^3 - 2\alpha + \beta = 0.$$

By Theorem 2,

$$D_\alpha^2 D_\beta = 1,\ \ D_\alpha D_\beta^2 = -3,\ \ D_\alpha^3 = 8,\ \ D_\beta^3 = 9.$$

$\hat{X}$ is obtained by resolving the singular points p_{ij} of X. Let A_{ij}, B_{ij} be the exceptional divisors over the singular point p_{ij} which correspond to the D_α, D_β in the above construction. Then the coupling for $\mathrm{H}^2(X, \mathbf{Z})$ has the expression:

$$\mu\left(t \cdot h + \sum_{1\le i<j\le 5} \{u_{ij}c_1(A_{ij}) + v_{ij}c_1(B_{ij})\}\right) =$$
$$25t^3 + \sum_{1\le i<j\le 5} (8u_{ij}^3 + 9v_{ij}^3) + \sum_{1\le i<j\le 5} (u_{ij}^2 v_{ij} - 3u_{ij}v_{ij}^2)$$

here h = class of pull − back of $\mathbf{O}_X(5)$. q.e.d.

Example 4. Let X be the quotient of

$$Z_1^4 Z_2 + Z_2^4 Z_3 + Z_3^4 Z_4 + Z_4^4 Z_5 + Z_5^4 Z_1 = 0 \quad \text{in } \mathbf{P}^4$$

by the order 41 group generated by

$$[Z_1, Z_2, Z_3, Z_4, Z_5] \rightsquigarrow \left[\omega Z_1, \omega^{37} Z_2, \omega^{16} Z_3, \omega^{18} Z_4, \omega^{10} Z_5\right]$$

with $\omega^{41} = 1$. (Example in [4]). Then

$$\text{Sing}(X) = \{p_i \,,\, 1 \le i \le 5\}$$

here p_i is the element of X with the coordinate $Z_k = 0$ for $k \neq i$. The structure near a singular point p_i is isomorphic to the quotient of $\mathbf{C}^3$ by an order 41 element of $SL_3(\mathbf{C})$. It contributes 20 exceptional divisors of the CY resolution $\hat{X}$. Hence we can obtain the cubic form of $\mathrm{H}^2\left(\hat{X}, \mathbf{Z}\right)$ using the method of Theorem 2 by the simplicial data attached to singular points. However this coupling depends on the triangulations of the simplicial data, which have several different ways in this example. We are going to illustrate their difference by comparing two triangulation information associated to the resolutions. We shall only work on the local situation at one singular point as it already reveal the nature of the topological couplings be effected by different resolutions for a CY orbifold. Consider the local structure near the singular point p_1. We have

$$(X, p_1) \simeq \left(\mathbf{C}^3/\text{dia}\left[\omega^{15}, \omega^{17}, \omega^9\right], 0\right).$$

The set Γ now consists of standard base elements together 20 points lying in the interior of Δ, in particular it contains the following 4 elements:

$$\alpha = \frac{1}{41}\begin{pmatrix} 2 \\ 5 \\ 34 \end{pmatrix}, \; \beta = \frac{1}{41}\begin{pmatrix} 9 \\ 2 \\ 30 \end{pmatrix}, \; \gamma = \frac{1}{41}\begin{pmatrix} 4 \\ 10 \\ 27 \end{pmatrix}, \; \delta = \frac{1}{41}\begin{pmatrix} 11 \\ 7 \\ 23 \end{pmatrix}.$$

One has

$$\alpha + \delta = \beta + \gamma \;,$$

and both $(\alpha, \delta, \beta), (\alpha, \delta, \gamma), (\alpha, \beta, \gamma), (\delta, \beta, \gamma)$ are integral bases of the lattice $\mathbf{n}$. Consider triangulations $\mathcal{S}_1, \mathcal{S}_2$ of Δ such that they differ only on the convex set spanned by the 4 elements $\alpha, \beta, \gamma, \delta$, while on this convex part $\mathcal{S}_1$ contains the 2–simplexes $(\alpha, \delta, \beta), (\alpha, \delta, \gamma)$ while $\mathcal{S}_2$ contains $(\alpha, \beta, \gamma), (\delta, \beta, \gamma)$. (See Figure 3). Let $\hat{X}_i$ be the CY resolution corresponding to the simplicial decomposition $\mathcal{S}_i$ for $i = 1, 2$. The classes $c_1(D_\gamma)$ $\left(\gamma \in \Gamma - \left\{e^i\right\}_{i=1}^3\right)$ form the base of $\mathrm{H}^2\left(\hat{X}_i, \mathbf{Z}\right)$.

By Theorem 2, the couplings μ_i for $\mathrm{H}^2\left(\hat{X}_i, \mathbf{Z}\right)$ are the same except $D_\lambda^2 D_{\lambda'}$, D_λ^3 for $\lambda, \lambda' \in \{\alpha, \beta, \gamma, \delta\}$, and we have

$$\mu_2\left(\sum_{\lambda\in\Gamma-\{e^i\}} t_\lambda c_1(D_\lambda)\right) - \mu_1\left(\sum_{\lambda\in\Gamma-\{e^i\}} t_\lambda c_1(D_\lambda)\right) = t_\alpha^3 + t_\delta^3 - t_\beta^3 - t_\gamma^3 - t_\beta^2 t_\gamma$$
$$-t_\beta t_\gamma^2 + t_\alpha^2 t_\delta + t_\alpha t_\delta^2 - t_\alpha^2 t_\beta + t_\alpha t_\beta^2 - t_\alpha^2 t_\gamma + t_\alpha t_\gamma^2 + t_\beta^2 t_\delta - t_\beta t_\delta^2 - t_\delta^2 t_\gamma + t_\delta^2 t_\gamma.$$

q.e.d.

Section 3

For the CY manifolds from "orbifold construction", the singular space we started with in general possesses curve-singularities together point-singularities on them. The couplings of exceptional divisors can be determined by the method of Sect. 2 except those couplings of divisors all contracting to curves of the singular space. In the latter situation, the computations are more complicated than the cases we described in the previous 2 sections. We shall give a general method for the computation of those remaining parts. For this purpose we shall work only couplings with divisors contracting to the same curve in the singularity, and formulate the problem in the local version near the curve-singularity.

Let M be a compact Riemann surface, and G' a finite abelian group acting on M. Denote

$$G = \{g \in G' \mid g \text{ acts trivially on } M\},$$

and in this section we shall always assume the order of G to be positive

$$d := |G| > 0.$$

Denote

$$\rho : M \to M/G' \quad \text{the projection},$$
$$\{p_1, \ldots, p_N\} = \text{the branched locus of } \rho \text{ in } M/G',$$
$$\tilde{p_j} = \text{the divisor} \sum_{m\in\rho^{-1}(p_j)} m \text{ in } M,$$
$$I_j = \text{the } G' - \text{isotropy subgroup at } p \in \rho^{-1}(p_j),$$
$$d_j = \text{the integer } \frac{|I_j|}{d}.$$

Suppose $\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}$ are line G'-bundles over M such that the following conditions hold:

$$(a)\ \text{There is a section } s \in \Gamma(M, \mathsf{L}) \text{ with } (s = 0) = \sum_{j=1}^{N} \tilde{p_j}$$

(b) The quotient space $\mathcal{X} := (\mathsf{L}_1 \oplus \mathsf{L}_2)/G'$ has trivial canonical sheaf, (8)

(c) The G' − action on $\mathsf{L}_1 \otimes \mathsf{L}_2 \otimes \mathsf{L}$ induces a line bundle E on M/G'

$$\begin{array}{ccc} \mathsf{L}_1 \otimes \mathsf{L}_2 \otimes \mathsf{L} & \rightarrow & \mathsf{E} := (\mathsf{L}_1 \otimes \mathsf{L}_2 \otimes \mathsf{L})/G' \\ \downarrow & & \downarrow \\ M & \rightarrow & M/G' \end{array} .$$

Consider a CY resolution

$$\sigma : \hat{\mathcal{X}} \rightarrow \mathcal{X}$$

and define π by the following diagram:

$$\begin{array}{ccc} \hat{\mathcal{X}} & \xrightarrow{\sigma} & \mathcal{X} \\ & \searrow_{\pi} & \downarrow \\ & & M/G' \end{array} .$$

Lemma 2. (i) G is the group generated by

$$\begin{array}{c} \mathsf{L}_1 \oplus \mathsf{L}_2 \rightarrow \mathsf{L}_1 \oplus \mathsf{L}_2 \\ (\ell_1, \ell_2) \rightsquigarrow (\omega \ell_1, \omega^{-1} \ell_2) \end{array}$$

with $\omega = e^{\frac{2\pi i}{d}}$.

(ii) Let D_0, D_d be the divisors in $\hat{\mathcal{X}}$ defined by

$$\begin{array}{l} D_0 = \text{the proper transform of } (0 \times \mathsf{L}_2)/G', \\ D_d = \text{the proper transform of } (\mathsf{L}_1 \times 0)/G'. \end{array}$$

Then there are exactly $d-1$ exceptional divisors $D_1, \ldots, D_{d-1}$ lying generically over M/G' through the map π, and only intersections among D_j's $(0 \leq j \leq d)$ are

$$\overline{M_k} := D_k \cap D_{k-1} \overset{\pi_{rest}}{\simeq} M/G' \quad \text{for } 1 \leq k \leq d.$$

Proof. Since G acts as scalar multiplications on line bundles L_i, the conclusion of (i) follows from the assumption on the trivial canonical sheaf of $\mathcal{X}$. (ii) follows from the structure of $\hat{\mathcal{X}}$. q.e.d.

We now describe the structure of $\hat{\mathcal{X}}$ near $\pi^{-1}(p_j)$ for a given j. For convenience of notations, we shall identify M with the zero section of $\mathsf{L}_1 \oplus \mathsf{L}_2$, and consider M as a G'—submanifold of $\mathsf{L}_1 \oplus \mathsf{L}_2$. We know that

$$(\mathcal{X}, p_j) \simeq ((\mathsf{L}_1 \oplus \mathsf{L}_2)/I_j, q_j) \simeq (\mathbf{C}^3/I_j, 0) ,$$

here q_j is an element in $\rho^{-1}(p_j)$, the action of I_j on $\mathbf{C}^3$ on the right hand side is considered as a diagonal subgroup of $SL_3(\mathbf{C})$. The coordinate (z_1, z_2, z_3) of $\mathbf{C}^3$

can be regarded as a local coordinate system of $L_1 \oplus L_2$ near q_j such that $\{z_3 = 0\}$ corresponds to the fiber $(L_1 \oplus L_2)_{q_j}$ and $0 \times L_2 \leftrightsquigarrow \{z_1 = 0\}$, $L_1 \times 0 \leftrightsquigarrow \{z_2 = 0\}$. Then we have

$$\left(\hat{\mathcal{X}}\ , \pi^{-1}(p_j)\right) \simeq \left(\hat{\mathbf{C}^3/I_j}, \hat{0}\right) \ , \qquad (9)$$

here $\hat{0}$= the union of exceptional divisors contracting to 0. The combinatorial data for the toric variety $\hat{\mathbf{C}^3/I_j}$ is now given by a simplicial decomposition $\mathcal{S}$ of

$$\Delta = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3 \mid \sum_{i=1}^{3} x_i = 1, \ \ x_i \geq 0 \right\}$$

having

$$\Gamma = \Delta \cap \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3 \mid \mathrm{dia}\left[e^{2\pi i x_1}, e^{2\pi i x_2}, e^{2\pi i x_3}\right] \in I_j \right\}$$

as the set of all its vertices [5]. Since G is a subgroup of I_j, it follows

$$\Gamma \cap \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbf{R}^3 \mid x_3 = 0 \right\} = \{\gamma_j\}_{j=0}^{d} \ , \ \ \gamma_j := \begin{pmatrix} \frac{d-j}{d} \\ \frac{j}{d} \\ 0 \end{pmatrix} .$$

Under the isomorphism (9), the divisor D_j in Lemma 2 corresponds to the toric divisor D_{γ_j} of $\hat{\mathbf{C}^3/I_j}$ associated to γ_j for $0 \leq j \leq d$. For $1 \leq k \leq d$, there is an unique element δ_k in Γ such that $\{\gamma_{k-1}, \gamma_k, \delta_k\}$= a 2-simplex in $\mathcal{S}$ (See Figure 4). One can write

$$\delta_k = \begin{pmatrix} r_{jk} \\ s_{jk} \\ \frac{1}{d_j} \end{pmatrix} \quad \text{with} \quad r_{jk}, s_{jk} \in \frac{1}{dd_j}\mathbf{Z}, \ \ d_j r_{jk} + d_j s_{jk} = d_j - 1.$$

Define

$$m_{jk} = -k(d_j - 1) + dd_j s_{jk}, \quad m'_{jk} = (k-1)(d_j - 1) - dd_j s_{jk}$$

for $1 \leq j \leq N$, $1 \leq k \leq d$. Note that $m_{jk} + m'_{jk} + d_j = 1$. Let (t_1, t_2, t_3) be the local coordinate system of $\hat{\mathbf{C}^3/I_j}$ attached to $\{\gamma_{k-1}, \gamma_k, \delta_k\}$. Its relation with the coordinate (z_1, z_2, z_3) of $\mathbf{C}^3$ are obtained from the toric data as follows. From the relation

$$(\gamma_{k-1}, \gamma_k, \delta_k) = (e^1, e^2, e^3) \begin{pmatrix} \frac{d-k+1}{d} & \frac{d-k}{d} & r_{jk} \\ \frac{k-1}{d} & \frac{k}{d} & s_{jk} \\ 0 & 0 & \frac{1}{d_j} \end{pmatrix},$$

and their duals

$$\begin{pmatrix} \gamma_{k-1\bullet} \\ \gamma_{k\bullet} \\ \delta_{k\bullet} \end{pmatrix} = \begin{pmatrix} k & k-d & m_{jk} \\ 1-k & d-k+1 & m'_{jk} \\ 0 & 0 & d_j \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},$$

we have

$$\left\{ \begin{array}{l} t_1 = z_1^k z_2^{k-d} z_3^{m_{jk}} \\ t_2 = z_1^{1-k} z_2^{d-k+1} z_3^{m'_{jk}} \\ t_3 = z_3^{d_j} \end{array} \right. ,$$

hence

$$t_1 t_2 t_3 = z_1 z_2 z_3 \ .$$

The local equations for the divisors $D_{\gamma_{k-1}}$, D_{γ_k}, $\left(D_{\gamma_{k-1}} + D_{\gamma_k} + D_{\delta_k}\right)$ are given by

$$\begin{array}{ll} D_{\gamma_{k-1}}: & t_1 = z_1^k z_2^{k-d} z_3^{m_{jk}} = 0 \ , \\ D_{\gamma_k}: & t_2 = z_1^{1-k} z_2^{d-k+1} z_3^{m'_{jk}} = 0, \\ \left(D_{\gamma_{k-1}} + D_{\gamma_k} + D_{\delta_k}\right): & t_1 t_2 t_3 = z_1 z_2 z_3 = 0 \ . \end{array} \tag{10}$$

Theorem 3. The following relations hold for $1 \leq k \leq d$,

$$|G'|\left(D_{k-1}^2 D_k\right) = kc_1(\mathsf{L}_1) + (k-d)c_1(\mathsf{L}_2) + \sum_{j=1}^{N} m_{jk}\left|\rho^{-1}(p_j)\right|,$$

$$|G'|\left(D_{k-1} D_k^2\right) = (1-k)c_1(\mathsf{L}_1) + (d-k+1)c_1(\mathsf{L}_2) + \sum_{j=1}^{N} m'_{jk}\left|\rho^{-1}(p_j)\right|,$$

$$\pi^*(\mathsf{E}) = \mathsf{O}\left(\sum_{i=0}^{d} D_i\right) \otimes \mathsf{O}\left(\sum_{\pi(D)=p_j} D\right) \quad \text{near} \ \bigcup_{k=1}^{d-1} D_k \ .$$

Proof. In order to show the first two relations, it suffices to show the following equalities hold for line bundles over M,

$$\rho^*\left(\mathsf{O}(D_{k-1})_{|\overline{M_t}}\right) = \mathsf{L}_1^k \otimes \mathsf{L}_2^{k-d} \otimes \mathsf{O}\left(\sum_{j=1}^{N} m_{jk}\tilde{p_j}\right),$$

$$\rho^*\left(\mathsf{O}(D_k)_{|\overline{M_t}}\right) = \mathsf{L}_1^{1-k} \otimes \mathsf{L}_2^{d-k+1} \otimes \mathsf{O}\left(\sum_{j=1}^{N} m'_{jk}\tilde{p_j}\right). \tag{11}$$

Consider the local trivializations of $\mathsf{L}_1, \mathsf{L}_2, \mathsf{L}$ over some G'-invariant open neighborhood U of $\rho^{-1}(p_j)$:

$$\begin{array}{ccccccc} \mathsf{L}_U \simeq & U \times \mathsf{C} & \qquad & \ell & \leftrightarrow & (u, \ \zeta_U(\ell)) \\ \downarrow & \downarrow & & \downarrow & & \downarrow \\ U = & U & & u & = & u \end{array} ,$$

$$\begin{array}{ccc} (\mathsf{L}_i)_U \simeq & U\times \mathsf{C} & \qquad \ell_i \leftrightsquigarrow \left(u,\ \zeta_{U,i}(\ell_i)\right) \\ \downarrow & \downarrow & \qquad \downarrow \qquad \downarrow \\ U = & U & \qquad u \ = \ u \end{array} .$$

The section $s \in \Gamma(M, \mathsf{L})$ in (8) on the open U corresponds to a function $s_U : U \to \mathsf{C}$ via the above trivilization of L. We denote $\tilde{s_U} : (\mathsf{L}_1 \oplus \mathsf{L}_2)_U \to \mathsf{C}$ the composition of s_U with the bundle projection map. The function s_U defines a local coordinate of M near the element $q_j \in \rho^{-1}(p_j)$. Therefore the map

$$\begin{array}{ccc} (\mathsf{L}_1 \oplus \mathsf{L}_2)_U & \stackrel{(\zeta_{U,1},\zeta_{U,2},\tilde{s_U})}{\rightarrow} & \mathsf{C}^3 \\ (\ell_1, \ell_2) & \rightsquigarrow & \left(\zeta_{U,1}(\ell_1), \zeta_{U,2}(\ell_2), \tilde{s_U}(\ell_1,\ell_2)\right), \end{array} \quad (12)$$

defines a local coordinate of the 3–fold $\mathsf{L}_1 \oplus \mathsf{L}_2$ near the point q_j. From (10), we have the expression of local generators of the following ideal sheaves near $\overline{M_k} \cap \pi^{-1}(p_j)$:

$$\begin{array}{rl} \mathcal{I}_{D_{\gamma_{k-1}}} & = < \zeta_{U,1}^{k}\zeta_{U,2}^{k-d}\tilde{s_U}^{m_{jk}} >, \\ \mathcal{I}_{D_{\gamma_k}} & =< \zeta_{U,1}^{1-k}\zeta_{U,2}^{d-k+1}\tilde{s_U}^{m'_{jk}} >, \end{array} \quad (13)$$

$$\mathcal{I}_{\left(D_{\gamma_{k-1}}+D_{\gamma_{k-1}}+D_{\delta_k}\right)} =< \zeta_{U,1}\zeta_{U,2}\tilde{s_U} > . \quad (14)$$

For $q \in M - \bigcup_{j=1}^{N} \rho^{-1}(p_j)$ and some neighborhood U in M, the map

$$\begin{array}{ccc} (\mathsf{L}_1 \oplus \mathsf{L}_2)_U & \stackrel{(\zeta_{U,1},\zeta_{U,2},\mathrm{proj.})}{\rightarrow} & \mathsf{C}^2 \times U \\ (\ell_1, \ell_2) & \rightsquigarrow & \left(\zeta_{U,1}(\ell_1), \zeta_{U,2}(\ell_2), u\right), \end{array} \quad (15)$$

(here$(\ell_1, \ell_2) \in (\mathsf{L}_1 \oplus \mathsf{L}_2)_u$) gives a local coordinate system for $\mathsf{L}_1 \oplus \mathsf{L}_2$ near the point q. Through the above map, the structure of $\mathcal{X}$ near the point $\rho(q)$ is given the isomorphism

$$(\mathcal{X},\ \rho(q)) \simeq \left((\mathsf{C}^2/G) \times U,\ [0] \times q\right),$$

here G acts on C^2 as the diagonal subgroup of $SL_2(\mathsf{C})$ generated by the element of order d. From the discussion of Sect. 1, the local generators of the following ideal sheaves near $\overline{M_k}$ are given by

$$\begin{array}{rl} \mathcal{I}_{D_{\gamma_{k-1}}} & =< \zeta_{U,1}^{k}\zeta_{U,2}^{k-d} >, \\ \mathcal{I}_{D_{\gamma_k}} & =< \zeta_{U,1}^{1-k}\zeta_{U,2}^{d-k+1} >, \end{array} \quad (16)$$

$$\mathcal{I}_{\left(D_{\gamma_{k-1}}+D_{\gamma_{k-1}}\right)} =< \zeta_{U,1}\zeta_{U,2} > . \quad (17)$$

Since $\rho^*\left(\mathrm{O}(-D_{k-1})_{|\overline{M_k}}\right)$, $\rho^*\left(\mathrm{O}(-D_k)_{|\overline{M_k}}\right)$ are the line bundles corresponding to $\rho^*\left(\mathcal{I}_{D_{k-1}} \underset{\mathcal{O}_{\hat{X}}}{\otimes} \mathcal{O}_{\overline{M_k}}\right)$, $\rho^*\left(\mathcal{I}_{D_k} \underset{\mathcal{O}_{\hat{X}}}{\otimes} \mathcal{O}_{\overline{M_k}}\right)$ respectively, one obtain (11) by computing transition functions of the line bundles from the relations (13) and (16). Similarly the third relation of this theorem follows from (14) and (17).

Example 5. (Mirror of Fermat quintic). Let X be the quotient of Fermat quintic in P^4 by the group , denoted by SD, generated by

$$\begin{aligned}
[Z_1, Z_2, Z_3, Z_4, Z_5] &\rightsquigarrow [\omega Z_1, \omega^4 Z_2, Z_3, Z_4, Z_5] \\
[Z_1, Z_2, Z_3, Z_4, Z_5] &\rightsquigarrow [Z_1, \omega Z_2, \omega^4 Z_3, Z_4, Z_5] \\
[Z_1, Z_2, Z_3, Z_4, Z_5] &\rightsquigarrow [Z_1, Z_2, \omega Z_3, \omega^4 Z_4, Z_5]
\end{aligned}$$

with $\omega^5 = 1$. Then

$$\mathrm{Sing}(X) = \bigcup_{i<j} (Z_i = Z_j = 0)/SD ,$$

and each $(Z_i = Z_j = 0)/SD$ is a rational curve, which intersects the others on 3 points. Then the CY resolution $\hat{X}$ of X is the mirror of Fermat quintic with the following properties [4, 7]:

$$\mathrm{H}^{1,1}\left(\hat{X}\right) \simeq \mathrm{H}^{2,1}(\text{quintic}),\ \mathrm{H}^{2,1}\left(\hat{X}\right) \simeq \mathrm{H}^{1,1}(\text{quintic}).$$

The exceptional divisors, together with the pull-back of $\mathrm{O}_X(5)$, give a base of $\mathrm{H}^2\left(\hat{X}, \mathrm{Z}\right)$. One can obtain the couplings on $\mathrm{H}^2\left(\hat{X}, \mathrm{Z}\right)$ using the method in Sect. 2 except those with all the divisors lying generically over the same curve $(Z_i = Z_j = 0)/SD$ for some i, j. For convenience of notations, we shall work only the case for $(i,j) = (1,2)$. Apply Theorem 3 on this case and set

$$\begin{aligned}
M &= \{[X_3, X_4, X_5] \in \mathrm{P}^2 \mid X_3^5 + X_4^5 + X_5^5 = 0\}, \\
G' &= \left\{[X_3, X_4, X_5] \rightsquigarrow \left[\omega^i X_3, \omega^j X_4, \omega^k X_5\right],\ i,j,k \in \mathrm{Z}\right\}, \\
\mathrm{L}_1 &= \mathrm{L}_2 = \mathrm{H} \text{ the restriction of hyperplane bundle,} \\
\mathrm{L} &= \mathrm{H}^3 .
\end{aligned}$$

The section $s \in \Gamma(M, \mathrm{L})$ in (8) is equal to $Z_3 Z_4 Z_5$. So $G' \simeq (\mathrm{Z}/5\mathrm{Z})^2$, and we can identify M/G' with

$$\left\{[W_3, W_4, W_5] \in \mathrm{P}^2 \mid W_3 + W_4 + W_5 = 0\right\}.$$

The projection $\rho : M \to M/G'$ is now given by $W_j = Z_j^5$ with the branched locus

$$\{p_1, p_2, p_3\} = \{[W_3, W_4, W_5] = [0, 1, -1], [1, 0, -1], [1, -1, 0]\}.$$

Then the line E in (8) is the hyperplane bundle $\mathcal{O}_{M/G'}(1)$ of the $[W_3, W_4, W_5]$ line. There are 4 exceptional divisors in this case, i.e., $d = 5$, $d_j = 5$ for all j. As the case in Sect. 2, the couplings depend on how the singularities resolved near p_1, p_2, p_3. We shall work only two cases to illustrate the method of computation using Theorem 3. The same procedure can be applied to the more general cases. Assume now the simplicial data in the CY resolution associated to p_1, p_2, p_3 are all the same, and equal to $\mathcal{S}_1$ or $\mathcal{S}_2$ as indicated in Figure 5.

For the case of $\mathcal{S}_1$, we have

$$m_{jk} = k - 5, \ \ m'_{jk} = 1 - k \quad \text{for } 1 \leq k \leq 5.$$

By Theorem 3, we obtain the following couplings:

$$D_{k-1}^2 D_k = k - 4, \ \ D_{k-1} D_k^2 = 2 - k \ ,$$
$$D_k^3 = D_k^2(\pi^* \mathcal{O}(1) - D_{k-1} - D_{k+1}) = -2 + 1 = -1$$

for $1 \leq k \leq 5$.

For the case of $\mathcal{S}_2$,

$$m_{j1} = m'_{j5} = 1, \ m_{j2} = m'_{j4} = 2, \ m_{j3} = m'_{j3} = -2,$$
$$m_{j4} = m'_{j2} = -6, \ m_{j5} = m'_{j1} = -5.$$

By Theorem 3, we obtain the following couplings:

$$D_0^2 D_1 = D_4 D_5^2 = 0, \ \ D_1^2 D_2 = D_3 D_4^2 = 1 \ , \ D_2^2 D_3 = D_2 D_3^2 = -1 \ ,$$
$$D_3^2 D_4 = D_1 D_2^2 = -3 \ , \ D_4^2 D_5 = D_0 D_1^2 = -2,$$
$$D_1^3 = -1, \quad D_2^3 = 2, \ \ D_3^3 = 2, \ \ D_4^3 = -1.$$

q.e.d.

Section 4

In this section we shall compute the second Chern class of the CY orbifolds.

Let $d_j \in \mathbf{Z}_{>1}, 1 \leq j \leq 5$, and $d := \operatorname{lcm}(d_1, \ldots, d_5)$, $n_j := \frac{d}{d_j}$, $q_j := \frac{1}{d_j}$. Denote

$$\{e^i\}_{i=1}^5 = \text{the standard base of } \mathbf{R}^5,$$
$$T = \text{the algebraic torus } (\mathbf{C}^*)^5,$$
$$q = \sum_{i=1}^5 q_i e^i \ ,$$
$$\mathcal{C} = \left\{ \sum_{i=1}^5 x_i e^i \in \mathbf{R}^5 \mid x_i \geq 0 \right\},$$

and define

$$\exp_q : \mathbf{R}^5 \to T \ , \ \exp_q(x) = \begin{bmatrix} e^{2\pi i q_1 x_1} \\ \vdots \\ e^{2\pi i q_5 x_5} \end{bmatrix},$$

$$\mathrm{tr}_q \colon \mathbf{R}^5 \to \mathbf{R} \ , \ \mathrm{tr}_q(x) = \sum_{i=1}^{5} q_i x_i \quad \text{for } x = \sum_{i=1}^{5} x_i e^i \ ,$$

$$SD_q = \left\{ \begin{bmatrix} t_1 \\ \vdots \\ t_5 \end{bmatrix} \in T \mid \prod_{i=1}^{5} t_i = 1, \ t_i^{d_i} = 1 \text{ for all } i \right\},$$

$$Q = \text{the group generated by } \exp_q\left(\sum_{i=1}^{5} e^i \right).$$

In this section G shall always be a group with the property

$$Q \subset G \subset SD_q \, . \qquad (18)$$

Let N_G (M_G) be the group of 1–parameter subgroups (characters) of the algebraic torus T/G:

$$N_G = \mathrm{Hom}_{\mathrm{alg.\,group}}(\mathbf{C}^*, T/G),$$
$$M_G = \mathrm{Hom}_{\mathrm{alg.\,group}}(T/G, \mathbf{C}^*).$$

We shall identify N_G, M_G with the following lattices:

$$N_G = \exp_q^{-1}(G),$$
$$M_G = \left\{ \sum_{i=1}^{5} k_i e^i \mid \prod_{i=1}^{5} Z_i^{k_i} \text{ is } G - \text{invaraint} \right\}.$$

The above lattices are connected to the structure of CY mirror pairs obtained from Fermat hypersurfaces in weighted 4–spaces [7]. But they are also naturally associated to the birational geometry of $\mathbf{WP}^4_{(n_i)}/G$ which we are now going to discuss. Denote

$$\mathbf{V} = \text{the vector space } \mathbf{R}^5/\mathbf{R}q,$$
$$\overline{N_G} = \text{the lattice } N_G/\mathbf{R}q \text{ in } \mathbf{V},$$
$$\overline{T/G} = \text{the algebraic } 4-\text{torus which is the quotient of } T/G \text{ by the } 1-\text{parameter subgroup } dq \in N_G.$$

The lattice structure of N_G in $\mathbf{R}^5$ induces $\mathbf{Z}$—structure of $\mathbf{V}$ with $\overline{N_G}$ as the lattice. Any rational cone decomposition $\{C_\alpha\}_{\alpha \in \mathrm{A}}$ of the boundary $\partial \mathcal{C}$ of the first quadrant cone C gives a rational simplicial cone decomposition $\{\overline{C_\alpha}\}_{\alpha\in A}$ of V here $\overline{C_\alpha} := C_\alpha + \mathbf{R}q \subset \mathrm{V}$. Hence the data $\{\overline{C_\alpha}\}_{\alpha\in A}$ induces a compactification of T/G which will be denoted by $\mathrm{P}_{\{C_\alpha\}_{\alpha\in A}}$. For the case when $\{C_\alpha\}_{\alpha\in A} = \{\text{coordinate faces of } C\}$, the corresponding compactification of T/G is simply the quotient $W\mathrm{P}^4_{(n_i)}/G$. In the case where each 4–dimensional cone in $\{C_\alpha\}_{\alpha\in A}$ is generated by part of Z—base of N_G, $\mathrm{P}_{\{C_\alpha\}_{\alpha\in A}}$ is a smooth projective resolution of $W\mathrm{P}^4_{(n_i)}/G$,

$$\Phi : \mathrm{P}_{\{C_\alpha\}_{\alpha\in A}} \to W\mathrm{P}^4_{(n_i)}/G \ . \qquad (19)$$

Let $\{Y_\ell\}_{\ell=1}^{L}$ be the collection of all toric divisors in $\mathrm{P}_{\{C_\alpha\}_{\alpha\in A}}$, i.e., $\bigcup_{\ell=1}^{L} Y_\ell = \Phi^{-1}\left(\bigcup_{j=1}^{5} (Z_j = 0)/G\right)$. Then we have the following expression of Chern classes of $\mathrm{P}_{\{C_\alpha\}}$:

Lemma 3. The total Chern class of the smooth compactification $\mathrm{P}_{\{C_\alpha\}}$ of $\overline{T_G}$ is given by

$$c(\mathrm{P}_{\{C_\alpha\}}) = \prod_{\ell=1}^{L} (1 + c_1(Y_\ell)) \, .$$

Proof. There is an exact sequence of sheaves over $\mathrm{P}_{\{C_\alpha\}}$,

$$0 \to \mathcal{O}_{\mathrm{P}_{\{C_\alpha\}}}\left(\mathrm{T}\left(\log \sum_{\ell=1}^{L} Y_\ell\right)\right) \to \mathcal{O}_{\mathrm{P}_{\{C_\alpha\}}}(\mathrm{T}) \to \bigoplus_{\ell=1}^{L} \mathcal{O}_{Y_\ell}(Y_\ell) \to 0 \, ,$$

here T is the tangent bundle of $\mathrm{P}_{\{C_\alpha\}}$, and $\mathcal{O}_{\mathrm{P}_{\{C_\alpha\}}}\left(\mathrm{T}\left(\log \sum_{\ell=1}^{L} Y_\ell\right)\right) =$ dual of $\Omega^1_{\mathrm{P}_{\{C_\alpha\}}}\left(\log \sum_{\ell=1}^{L} Y_\ell\right)$. Since $\Omega^1_{\mathrm{P}_{\{C_\alpha\}}}\left(\log \sum_{\ell=1}^{L} Y_\ell\right)$ is a free $\mathcal{O}_{\mathrm{P}_{\{C_\alpha\}}}$—module, we have

$$\begin{aligned} c(\mathrm{P}_{\{C_\alpha\}}) &= c\left(\mathrm{T}\left(\log \sum_{\ell} Y_\ell\right)\right) \prod_{\ell=1}^{L} c(\mathcal{O}_{Y_\ell}(Y_\ell)) \\ &= \prod_{\ell=1}^{L} (1 + c_1(Y_\ell)) \, . \end{aligned}$$

q.e.d.

We now compute the Chern classes of CY orbifolds using the above results of $\mathrm{P}_{\{C_\alpha\}}$, Consider a degree d quasi-smooth hypersurface in $W\mathrm{P}^4_{(n_i)}$ defined by

$$f(Z) = Z_1^{d_1} + Z_2^{d_2} + Z_3^{d_3} + Z_4^{d_4} + Z_5^{d_5} + \lambda Z_1 Z_2 Z_3 Z_4 Z_5 = 0$$

with $\lambda \in \mathbf{C}$. Assume d, n_j satisfy the condition

$$d = \sum_{j=1}^{5} n_j \quad ,$$

and G the same as (18). As G preserves the polynomial $f(Z)$ and the form $dZ_1 \wedge \ldots \wedge dZ_5$, the quotient space

$$X := \Big([Z] \in \mathbf{WP}^4_{(n_i)} \,|\, f(Z) = 0\Big)/G$$

has the trivial canonical sheaf with the singularity

$$\mathrm{Sing}(X) = \bigcup \{X_I \,|\, I \subset \{1, \ldots, 5\},\ c_I > 1\}$$

here $X_I = X \cap \bigcap_{i \in I} (Z_i = 0)$, $c_I = |\{g \in G | g(Z) = Z \text{ for } Z_i = 0,\ i \in I\}|$. Note that $X_I = \emptyset$ for $|I| \geq 4$. The exceptional divisors of the CY resolution

$$\sigma : \hat{X} \to X$$

are described in a certain part of the lattice N_G [7]. In fact the combinatorial data of exceptional divisors over X_I ($c_I > 1$) is a simplicial cone decomposition of the Ith face of $\mathcal{C}$ $\left(:=\left\{x = \sum_{i=1}^{5} x_i e^i \in \mathcal{C} \,|\, x_j = 0 \text{ for } j \notin I\right\}\right)$ having

$$\{\mathbf{R}_{\geq 0} v | v \in (I\text{th face of } \mathcal{C}) \cap \{x \text{ with } \mathrm{tr}_q(x) = 1\}\}$$

as the set of all 1–dimensional cones. With these given data on the simplicial cone decomposition of $\bigcup_{|I| \leq 3}$ (Ith faces of $\mathcal{C}$), one can extend it to a simplicial cone decomposition of the whole $\mathcal{C}$ in such a way that every 4–dimensional cone C_α is generated by part of $\mathbf{Z}$—base of N_G. Then the corresponding space $\mathbf{P}_{\{C_\alpha\}_{\alpha \in A}}$ is a projective resolution of $\mathbf{WP}^4_{(n_i)}/G$. By the construction of the toroidal resolutions, $\hat{X}$ is a smooth hypersurface of $\mathbf{P}_{\{C_\alpha\}_{\alpha \in A}}$, and in fact it is the proper transform of X of the birational morphism Φ in (19). Note that $\hat{X}$ is disjoint with the exceptional divisors of $\mathbf{P}_{\{C_\alpha\}_{\alpha \in A}}$ lying over points of $WP^4_{(n_i)}/G$ with vanishing coordinates except one.

Theorem 4. Let E_i ($1 \leq i \leq e$) be all the divisors in $\hat{X}$ contained in $\bigcup_{j=1}^{5} \sigma^{-1}(Z_j = 0)$. Denote $E_I = \bigcap_{i \in I} E_i$ for $I \subset \{1, \ldots, e\}$, and $[E_I]$ the Poincare dual of E_I. Then the second Chern class and Euler number of $\hat{X}$ are given by

$$c_2\Big(\hat{X}\Big) = \sum_{|I|=2} [E_I],$$

$$\chi\Big(\hat{X}\Big) = -2 \sum_{|I|=3} |E_I| + \sum_{|I|=2} \chi(E_I)\,.$$

Proof. Let $\mathsf{P}_{\{C_\alpha\}_{\alpha\in A}}$ be a resolution of $WP^4_{(n_i)}/G$ we have just described above. There is an exact sequence of vector bundles over $\hat{X}$,

$$0 \to \mathsf{T}\left(\hat{X}\right) \to \mathsf{T}(\mathsf{P}_{\{C_\alpha\}})_{|\hat{X}} \to \mathsf{N}_{\hat{X},\mathsf{P}_{\{C_\alpha\}}} \to 0,$$

here $\mathsf{T}\left(\hat{X}\right)$, $\mathsf{T}(\mathsf{P}_{\{C_\alpha\}})$ are the tangent bundles of $\hat{X}$, $\mathsf{P}_{\{C_\alpha\}_{\alpha\in A}}$ respectively and $\mathsf{N}_{\hat{X},\mathsf{P}_{\{C_\alpha\}}}$ is the normal of $\hat{X}$ in $\mathsf{P}_{\{C_\alpha\}}$. Then by Lemma 3,

$$c\left(\hat{X}\right)(1 + c_1(\mathsf{N}_{\hat{X}})) = c\left(\mathsf{T}(\mathsf{P}_{\{C_\alpha\}})_{|\hat{X}}\right),$$

$$c_1\left(\hat{X}\right) + c_1(\mathsf{N}_{\hat{X}}) = \sum_{i=1}^{e} [E_i],$$

$$c_2\left(\hat{X}\right) + c_1\left(\hat{X}\right)c_1(\mathsf{N}_{\hat{X}}) = \sum_{|I|=2} [E_I],$$

$$c_3\left(\hat{X}\right) + c_2\left(\hat{X}\right)c_1(\mathsf{N}_{\hat{X}}) = \sum_{|I|=3} [E_I] \ .$$

As $c_1\left(\hat{X}\right) = 0$, we obtain the first relation of this theorem. Also we have

$$\chi\left(\hat{X}\right) = -\sum_{i=1}^{e}\sum_{|I|=2} [E_i][E_I] + \sum_{|I|=3} |E_I|$$

$$= -3\sum_{|I|=3} |E_I| - \sum_{|I|=2}\left(\sum_{i\in I}[E_i]\right)[E_I] + \sum_{|I|=3} |E_I|.$$

For $|I| = 2$, $\bigoplus_{i\in I} \mathsf{O}(E_i)$ is isomorphic to $\mathsf{N}_{E_I,\hat{X}}$ (:= the normal of E_I in $\hat{X}$), hence

$$\left(\sum_{i\in I}[E_i]\right)[E_I] = \int_{E_I} c_1\left(\bigwedge^2 \mathsf{N}_{E_I,\hat{X}}\right) = -\int_{E_I} c_1(\mathsf{T}(E_I)) = -\chi(E_I).$$

Therefore

$$\chi\left(\hat{X}\right) = -2\sum_{|I|=3} |E_I| + \sum_{|I|=2} \chi(E_I).$$

q.e.d.

Example 6. Let X be the Fermat hypersurface in $WP^4_{(2,2,2,1,1)}$, $\hat{X}$ its CY resolution with the exceptional divisor D and the class h in $\mathrm{H}^2\left(\hat{X},\mathbf{Z}\right)$ as in Example 1. We have 6 divisors E_i in Theorem 4 for this case, which are defined by

$$E_i = \text{the proper transform of } (Z_i = 0) \text{ for } 1 \le i \le 5,$$
$$E_6 = \text{the exceptional divisor } D.$$

Then the curves E_I are connected Riemann surfaces of genus 9 for

$$I = \{1,2\}, \{1,3\}, \{2,3\};$$

of genus 3 for

$$I = \{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,6\}, \{5,6\};$$

E_I are the union of 4 disjoint $\mathbf{P}^1$ for

$$I = \{1,6\}, \{2,6\}, \{3,6\};$$

and $E_{\{4,5\}} = \emptyset$. For $|I| = 3$, we have

$$|E_I| = \begin{cases} 8, & I = \{1,2,3\}, \\ 0, & I=\{1,2,6\},\{2,3,6\},\{1,3,6\},\{1,4,5\},\{2,4,5\},\{3,4,5\},\{4,5,6\}, \\ 4, & I=\{1,2,4\},\{1,2,5\},\{2,3,4\},\{2,3,5\},\{1,3,4\},\{1,3,5\},\{1,4,6\},\{1,5,6\},\{2,4,6\},\{2,5,6\},\{3,4,6\},\{3,5,6\}. \end{cases}$$

Hence by Theorem 4, we have

$$\chi\left(\hat{X}\right) = -2(8+48) + 3(-16) + 8(-4) + 3 \times 8 = -168,$$
$$[E_6]c_2\left(\hat{X}\right) = \sum_{\substack{|I|=3 \\ 6 \in I}} |E_I| + \sum_{\substack{|I|=2 \\ 6 \in I}} [E_6][E_I] = 24 + \sum_{\substack{|I|=2 \\ 6 \in I}} [E_6][E_I].$$

Using Theorem 1, with the same computation as in Example 1, we have

$$[E_6][E_I] = \begin{cases} 4 & \text{for } I = \{4,6\}, \{5,6\}. \\ -8 & \text{for } I = \{1,6\}, \{2,6\}, \{3,6\}. \end{cases}$$
$$[E_6]c_2\left(\hat{X}\right) = 24 - 16 = 8.$$

One can also have the following results:

$$h[E_I] = \begin{cases} 4 & \text{for } I=\{1,2\},\{1,3\},\{2,3\},\{1,4\},\{2,4\},\{3,4\},\{1,5\},\{2,5\},\{3,5\}. \\ 2 & \text{for } I = \{4,6\}, \{5,6\}. \\ 0 & \text{for } I = \{1,6\}, \{2,6\}, \{3,6\}, \{4,5\}. \end{cases}$$

This implies

$$hc_2\left(\hat{X}\right) = 40.$$

The linear form on $\mathrm{H}^2\left(\hat{X}, \mathbf{Z}\right)$ given by the second Chern class is now expressed by

$$t \cdot h + s \cdot c_1(D) \rightsquigarrow 40t + 8s.$$

q.e.d.

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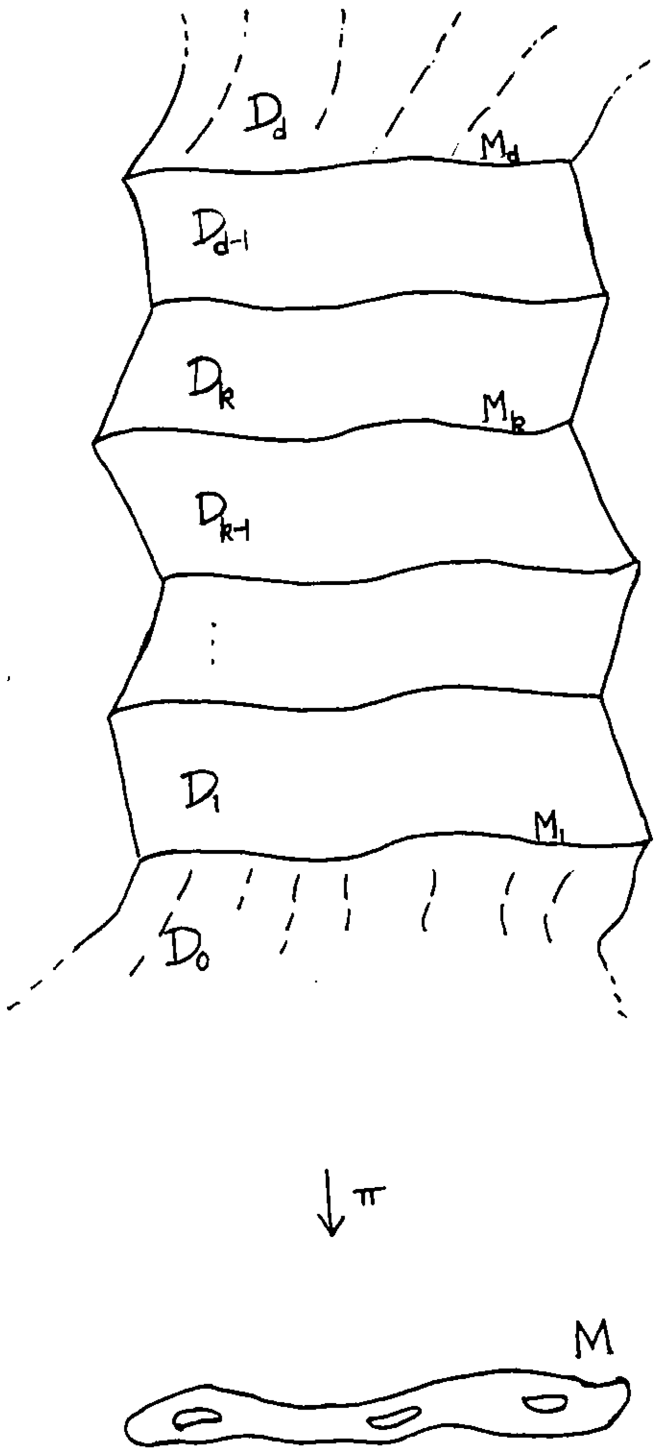


Figure 1

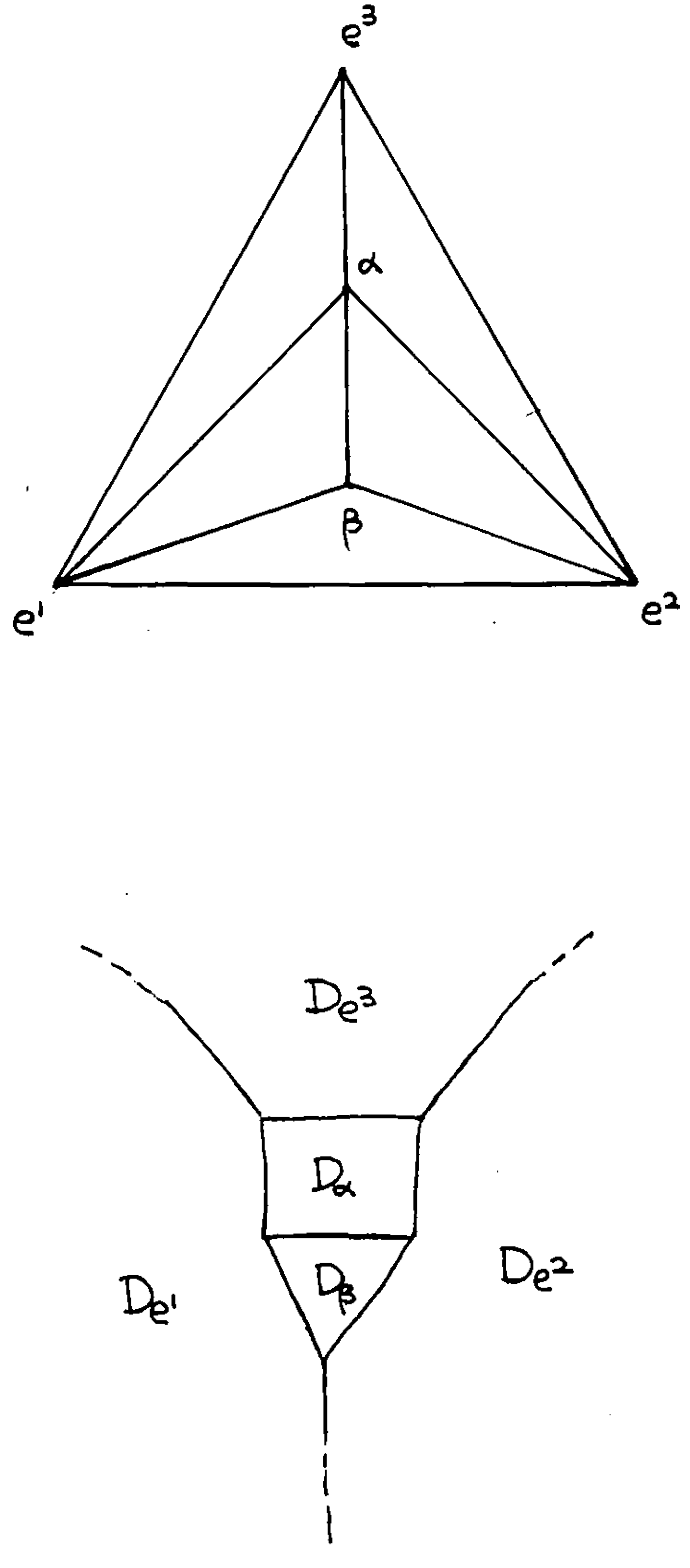
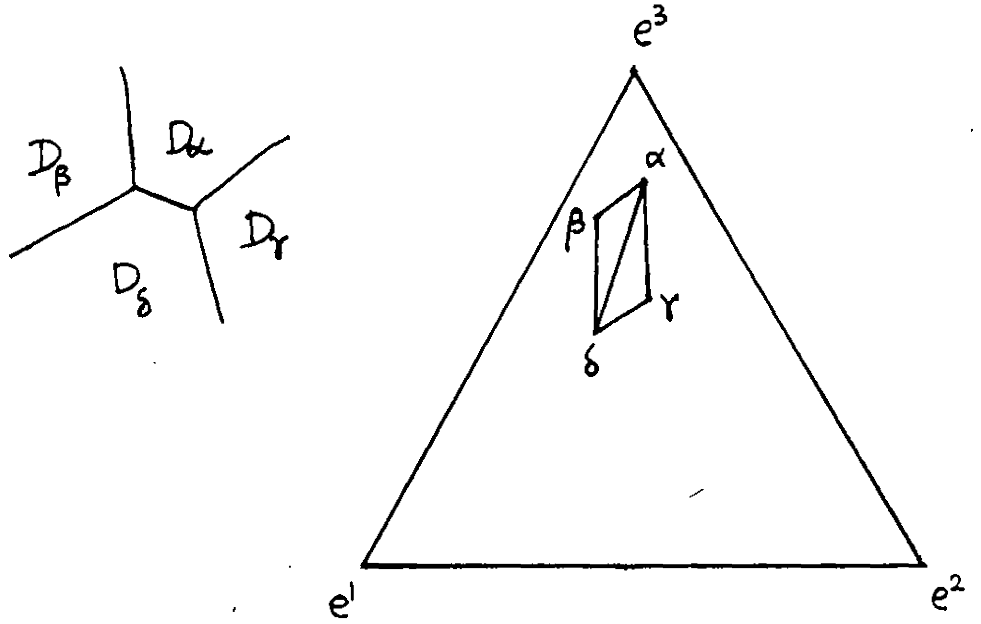
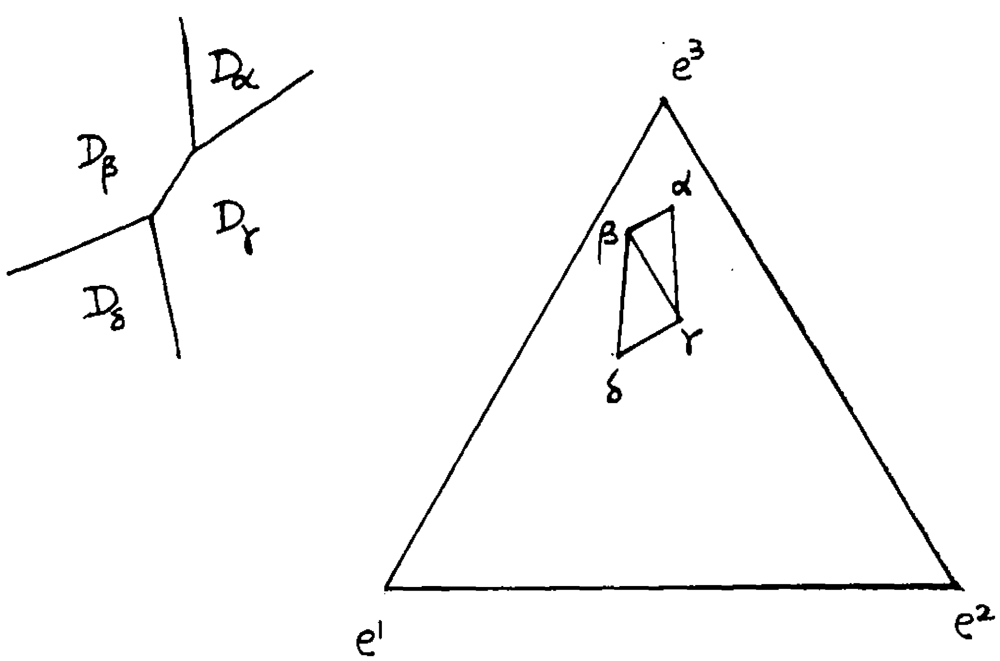


Figure 2

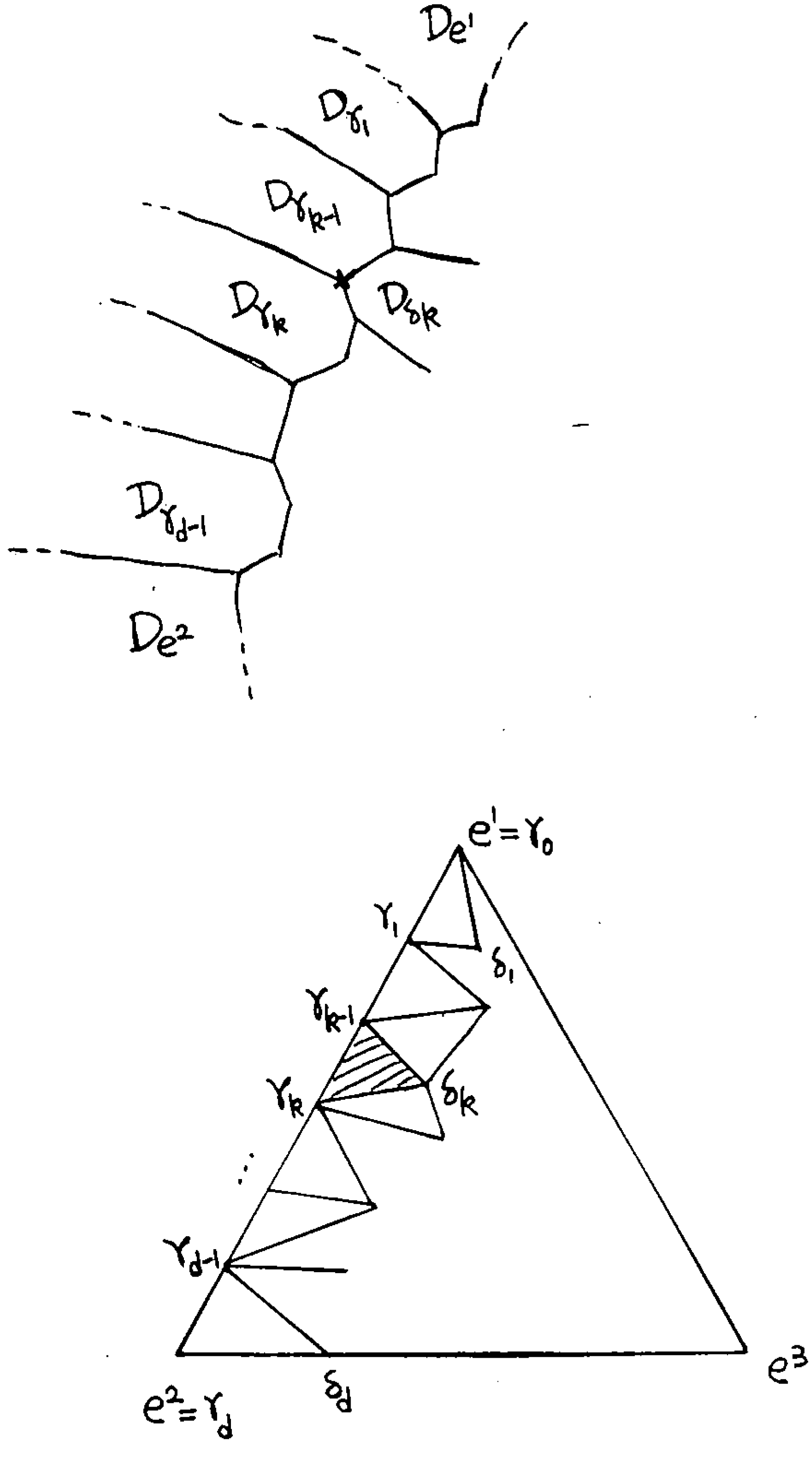


S_1



S_2

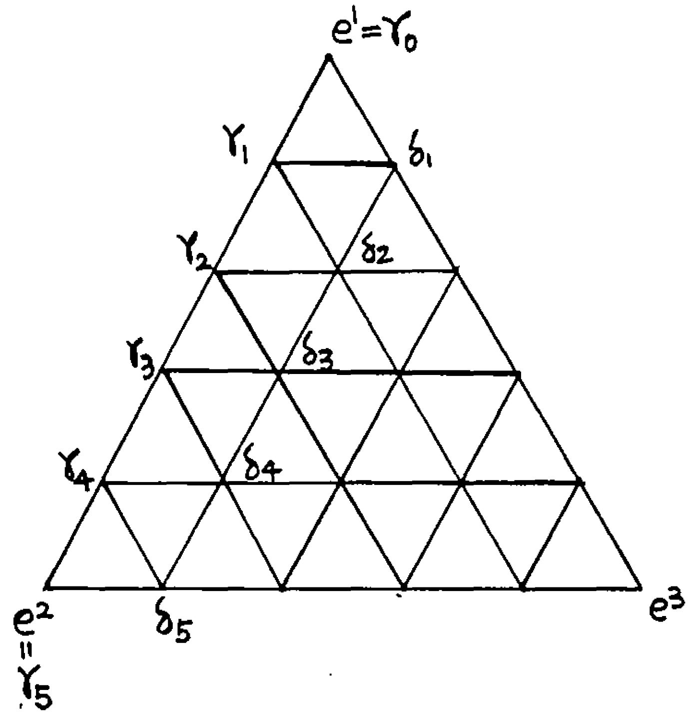
Figure 3



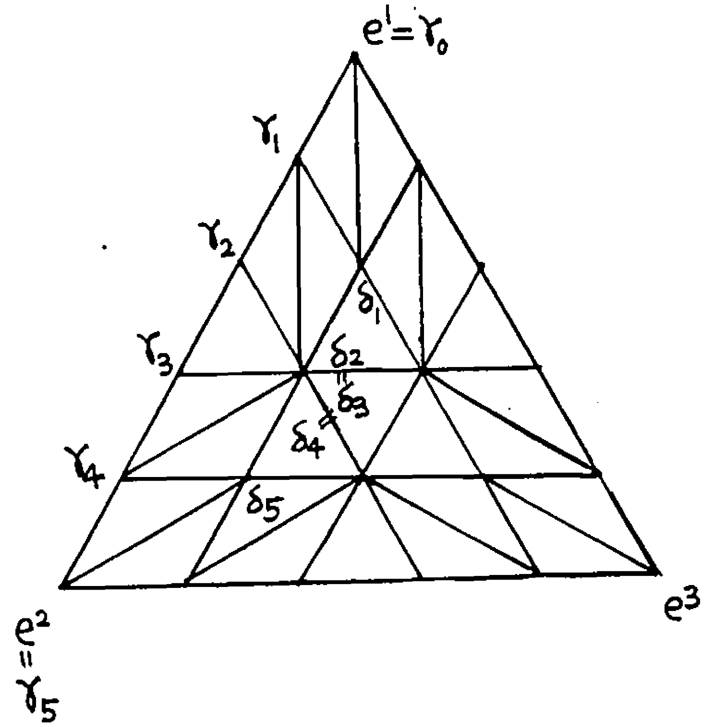
$$\gamma_k = \frac{d-k}{d} e^1 + \frac{k}{d} e^2$$

$$\delta_k = r_{jk} e^1 + s_{jk} e^2 + \frac{1}{d_j} e^3$$

Figure 4



S_1



S_2

Figure 5