TOWARDS MULTIPLE ELLIPTIC POLYLOGARITHMS

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Contents

1. Introduction

1.1. History and motivation.

1.1.1. *Classical polylogarithms and* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. For the purposes of their proof of the weak version of Zagier's conjecture, Beilinson and Deligne [3] gave a geometric interpretation of the classical polylogarithm functions from the XVIIIth century

$$\mathrm{Li}_k(z) := \sum_{n>0} \frac{z^n}{n^k}, \quad k \geqslant 1 \tag{1}$$

as periods of certain unipotent variations of mixed Hodge structures on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. On the differential side, the main feature is that the generating series

$$L(z) := \sum_{n \geqslant 0} \mathrm{Li}_n(z) t^n$$

satisfies the following differential equation

$$dL(z) \;=\; E_0 \mathrm{dlog(z)} + \mathrm{E_1 dlog(1-z)},$$

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where E_0, E_1 are explicit topologically nilpotent linear operators on $\mathbb{C}[\![t]\!]$.

1.1.2. *Generalisations.* One may consider the following generalisation with several integer indices:

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) := \sum_{n_1>n_2>\cdots>n_r>0} \frac{z^{n_1}}{n_1^{s_1}\cdots n_r^{s_r}}, \quad r>0, s_i>0 \tag{2}$$

The noncommutative generating series

$$1+\sum_{s_1,\ldots,s_r>0} \mathrm{Li}_{s_1,\ldots,s_r}(z) e_0^{s_1-1}e_1\cdots e_0^{s_r-1}e_1$$

is a horizontal section of the trivial vector bundle $\mathcal{P}$ on $\mathbb{P}^1\setminus\{0,1,\infty\}$ with fiber $\mathbb{C}\langle\!\langle e_0,e_1\rangle\!\rangle$, the completed free associative algebra with generators e_0, e_1, and connection

$$\nabla = d - e_0\mathrm{dlog(z)} - \mathrm{e_1 dlog(1-z)}, \tag{3}$$

where e_0, e_1 act by left multiplication within $\mathbb{C}\langle\!\langle e_0,e_1\rangle\!\rangle$.

As a vector bundle with connection, $\mathcal{P}$ is universal in the sense that any nilpotent vector bundle with regular singular connection on $\mathbb{P}^1\setminus\{0,1,\infty\}$ is isomorphic to the trivial bundle with fiber V and connection given by (3), for some $E_0, E_1 \in \mathsf{GL}_{\mathbb{C}}(V)$. This is best expressed in the language of tannakian categories. In the terminology of Deligne [8], $\mathcal{P}$ is the De Rham version of a *fundamental torsor* of motivic paths on $\mathbb{P}^1\setminus\{0,1,\infty\}$ (loc. cit., §12). This fundamental torsor provides the geometric interpretation of these generalized polylogarithms.

One can push the definition further and consider the *multiple polylogarithm*

$$\mathrm{Li}_{s_1,\ldots,s_r}(z_1,\ldots,z_r) := \sum_{n_1>n_2>\cdots>n_r>0} \frac{z^{n_1}\cdots z^{n_r}}{n_1^{s_1}\cdots n_r^{s_r}} \tag{4}$$

which can be interpretated in terms of fundamental groups and torsors of the moduli spaces $\mathscr{M}_{0,n}$ of genus 0 curves with n marked points [13].

1.1.3. Special values of these functions, and especially the *multiple zeta values* (MZVs):

$$\zeta(s_1\ldots,s_r) := \sum_{n_1>\cdots>n_r>0} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}} \tag{5}$$

are of prime importance in the theory of mixed Tate motives. There is a very precise conjectural picture [10, 12] of the algebraic rational relations that occur among them. In particular, the pro-unipotent Grothendieck-Teichmüller program in genus 0 relates them to a conjectural description of nilpotent quotients of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_\infty))$ [14]. It may be worthwile to notice that one can extract from explicit formulas like (5) a lot of information in a combinatorial way [19, 11, 6], whose interpretation in terms of differential equations is rather non-trivial [13].

1.1.4. *Elliptic polylog.* Another direction of generalization was taken by Beĭlinson and the first author [4]. The polylogarithmic sheaf can be characterized canonically by its formal properties with respect to the inclusion of $\mathbb{P}^1\setminus\{0,1,\infty\}$ into $\mathbb{G}_{\mathrm{m}}$ [3]. This abstract description has an elliptic parallel that allows to define elliptic polygarithmic mixed sheaf on a punctured elliptic curve. There is a relative version for families of elliptic curves, and a continuity property to the ordinary polylogarithmic sheaf at cusps.

This approach was further investigated by Wildeshaus [23] who generalized it to the complement of a mixed Shimura variety into another.

1.1.5. *Our goal: multiple elliptic polylogarithms.* In this article, we start to study multiple elliptic polylogarithms in terms of De Rham fundamental torsors approach of [8].

It seems rather natural that the sheaf theoretic elliptic analog of (2) should come from the whole fundamental torsor of paths on a punctured elliptic curve, whereas the full multiple elliptic polylogarithms should be related to fundamental torsors of the moduli stacks $\mathscr{M}_{1,n}$.

At this point, one should mention that in [8], Deligne used systems of realisations (among them the De Rham realisation) as a substitute for the conjectural category of mixed Tate motives. Such a category is now well defined, and the motivic fundamental groups and torsor have received direct motivic definitions [10]. However, as of this writing, mixed elliptic motives seem to be still far out of reach.

1.2. **What we do.** We provide here an explicit description of the De Rham fundamental torsor of an elliptic curve X minus its origin and extend this to families of those, *i.e.*, to $\mathscr{M}_{1,2}$. In equivalent and more down-to-earth words, we give a complete and explicit classification of vector bundles with (relatively) nilpotent connection on a (family of) elliptic curves.

Explicit formulas in the genus 0 case, such as (3), arise from the fact that any nilpotent connection on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is the restriction of a connection with simple poles at $0,1,\infty$ on a *trivial* bundle on $\mathbb{P}^1$. In general, the relevant category to consider for the De Rham fundamental group(oid) of a variety given as complement of a normal crossing divisor D in a proper variety X is the category $\mathrm{NConn}(X;D)$ of nilpotent vector bundles *on the whole of* X with connection having simple poles along D [7, 5, 8]:

$$\nabla\colon \mathcal{V} \longrightarrow \mathcal{V}\otimes\Omega^1_X(D) \tag{6}$$

Such nilpotent bundles are trivial if one assumes that $H^1(X,\mathcal{O}_X)=0$. We refer informally to this situation as *Deligne's good case* [8, 12.1]. A fair amount of [8] relies on this hypothesis.

In this article, we circumvent the fact that an elliptic curve does not satisty (6) by using the analytic description of elliptic curves as quotients of type $X=\mathbb{C}^*/q^{\mathbb{Z}}$ and pulling everything back as trivial bundles on the affine $\mathbb{C}^*$, with q-action. The correspondence between q-difference equations and vector bundles on elliptic curves is very classical, and becomes quite simple in the nilpotent case, as well as the condition that a connection on $\mathbb{C}^*$ has to fulfill to be a pullback from X.

1.3. **Plan of the article.** In section 2, we provide the description of the fundamental De Rham torsor and fundamental group for a given punctured elliptic curve $X=\mathbb{C}/\Lambda\setminus e$, by means of a full analytic description of the category of vector bundles with meromorphic connection on X and simple poles at the unit e. This part is very elementary.

The end result is as close as one could hope to Deligne's good case: we obtain a canonical abstract basepoint (ie., a fiber functor) ω_{DR}, at which the fundamental Hopf algebra is canonically identified with the free Lie algebra in two generators. Furthermore, the universal torsor $\mathcal{P}$ at ω_{DR} and its connection are completely explicit. This could be thought of as the elliptic analog of (3).

In section 3, we define a vector bundle with flat connection $\mathfrak{P}$ on the standard family $\mathbb{E}$ of elliptic curves over Poincaré's upper-half plane, such that the fiber at any τ is the former $\mathcal{P}$ of the corresponding elliptic curve. Furthermore, $\mathfrak{P}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivariant.

In section 4, we turn to arbitrary smooth families of elliptic curves and to relatively nilpotent connections with simple poles at the unit section. For this, we merge the results of the previous sections: $\mathfrak{P}$ descends to a vector bundle $\mathfrak{P}_{X/S}$

on each family $X \to S$. The main result, theorem II, then mostly says that $\mathfrak{P}_{X/S}$ is the relative fundamental torsor at some abstract fixed fiber functor ω_{DR}, in the sense of [21]. In short, this is an explicit analytic description of the De Rham torsor of paths on the moduli stack $\mathcal{M}_{1,2}$, relative to $\mathcal{M}_{1,1}$.

We then treat the case of geometric base points, *i.e.*, sections of our family of punctured elliptic curves and derive a differential equation for parallel transports along families of paths have to fulfill. We believe this to be also a nice and concrete illustration of basic 2-categorical aspects of the relative tannakian theory.

In the last section, we exhibit natural $\mathbb{Q}$-structures on the previously constructed analytic fundamental torsors and their connections. The construction goes mostly through the rewriting of the previously obtained formulas in terms of elliptic functions and modular forms.

1.4. **Conventions and notations.** Wherever possible without leading to confusion, we'll use the word "connection" to actually mean a vector bundle with connection.

The letter X will usually denote an elliptic curve, or a smooth family of these on a base S. Since we use alternatively additive and multiplicative conventions, we'll always denote the unit (section) by e.

Straight letters usually refer to objects living over a point: vector spaces, algebras, a typical example would be the algebra R from 2.2.5. We'll use curly letters for objects living over a space, for instance the fundamental torsor $\mathcal{P}$ of 2.2.7 or the algebra bundle with connection $\mathcal{R}$ of 4.2.2. We'll use capital gothic letters for sheaves over a family, like the $\mathfrak{P}$ from 4.1.3.

Exceptions to these conventions are Poincaré's upper half-plane $\mathbb{H}$, the family on elliptic curves $\mathbb{E} \to \mathbb{H}$.

1.4.1. *Eisenstein functions and series.* We recall the notations from [22, chap. III & IV] that we'll throughout this article.

Consider a lattice in $\mathbb{C}$ generated by τ and 1, with $\Im(\tau) > 0$. We take ξ as coordinate on $\mathbb{C}$ and set $z = \exp(2\pi i\xi)$, $q = \exp(2\pi i\tau)$

The symbol $\sum'$ denotes the summation over nonzero elements of a lattice and $\sum'_e$ denotes some variant of regularization of the divergent sums, known as *Eisenstein summation.* With this, we set:

$$e_k(\tau) = \sum_e{}'(n\tau + m)^{-k} \quad \text{and} \quad E_k(\xi, \tau) = \sum_e{}'(\xi + n\tau + m)^{-k}$$

The e_k are modular functions for $k \geq 4$, and E_k are elliptic functions for $k \geq 2$. Moreover one can express the Weierstrass elliptic functions and modular forms as $\wp = E_2 - e_2$, $\wp' = -2E_3$, $g_2 = 60e_4$ and $g_3 = 140e_6$. It is known that the e_k are rational polynomials in e_4 and e_6 for $k \geq 8$, while the E_k are rational polynomials in $E_2 - e_2$, E_3, e_4 and e_6 for $k \geq 4$. Moerover the rings $\mathbb{Q}[e_k]$ and $\mathbb{Q}[E_k, e_k]$ are stable under the derivation $(2\pi i)\partial_\tau$.

The basic theta function is defined by the Jacobi product formula

$$\theta(\xi, \tau) = -iq^{1/8}(z^{1/2} - z{-}1/2)\prod_{j=0}^{\infty}(1 - zq^j)(1 - z^{-1}q^j)(1 - q^j).$$

We have $E_1 = \partial_\xi \theta$.

1.4.2. *Lie algebras.* Recall the notation for the adjoint action of a Lie algebra on itself $\mathrm{ad}_a = (x \mapsto [a, x])$.

2. The De Rham fundamental groupoid of a punctured elliptic curve

In this section, we consider an elliptic curve $X = \mathbb{C}/\Lambda$, where Λ is the lattice of $\mathbb{C}$ generated by τ and 1, for some fixed $\tau \in \mathbb{H}$. As usual, we'll denote by ξ the coordinate on $\mathbb{C}$ and $z = \exp(2\pi i\xi)$ the coordinate on $\mathbb{C}^*$. Since we'll switch frequently between this additive description and the multiplicative $X = \mathbb{C}^*/q^{\mathbb{Z}}$, we'll denote the unit of X by e.

2.1. **Nilpotent vector bundles.** Before starting to investigate connections on X, we have to describe in analytic terms the nilpotent vector bundles themselves. Needless to say, apart perhaps from the formulation, there is definitely nothing original in this subsection.

2.1.1. Let V be a $\mathbb{C}$-vector space and $m \in \mathrm{End}\,(V)$. It is a well known fact that the equation

$$v(qz) = mv(z), \tag{7}$$

where $m \in \mathsf{GL}(V)$, gives rise to a vector bundle $\mathcal{V}$ on X, whose sections over an open subvariety U are the analytic solutions of (7) defined over the preimage of U in $\mathbb{C}^*$. We'll frequently call $\mathcal{V}$ the *bundle with multiplier m.* An equivalent considerations can be made in the additive situation, *i.e.*, for $X = \mathbb{C}/\Lambda$ that we shall use liberally when it's more convenient, for instance for modular considerations.

2.1.2. The pairs (V, m) form a category in the usual way: an arrow from (V_1, m_1) to (V_2, m_2) is simply a linear map f which intertwines m_1 and m_2, *i.e.*, statisfies $fm_1 = m_2 f$. It can be further equipped with a tensor structure by the standard rule:

$$(V_1, m_1) \otimes (V_2, m_2) := (V_1 \otimes V_2, m_1 \otimes m_2)$$

and inner homomorphisms:

$$\underline{\mathrm{Hom}}\,((V, m_1), (V, m_2)) := (\mathrm{Hom}_{\mathbb{C}\text{–}\mathrm{Vect}}(V_1, V_2), f \mapsto m_2 f m_1^{-1})$$

Altogether, we have a rigid tensor category, which is nothing but the category $\mathbb{Z}-\mathrm{Rep}$ of representations of $\mathbb{Z}$.

The trivial object is simply $(\mathbb{C}, \mathrm{Id})$. Therefore the nilpotent objects are the pairs (V, m) such that m is unipotent.

Given two objects (V_1, m_1) and (V_2, m_2), any linear map u that interwines m_1 and m_2 maps obviously solutions of (7) to solutions (7), hence our construction of bundles extends to a functor $\mathcal{G}$.

Proposition 2.1.3. *The functor $\mathcal{G}\colon (V, m) \mapsto \mathcal{V}$ is an equivalence of categories*

$$\text{(vector spaces with unipotent endomorphism)} \longrightarrow \mathrm{NBdl}(X)$$

In particular, $\mathrm{NBdl}(X)$ *is a sub-abelian category of the category of coherent $\mathcal{O}_X$-modules and is tannakian.*

2.1.4. *Global sections.* Before proceeding to the proof of the above theorem, we gather here some very elementary yet useful properties of global sections of nilpotent bundles.

Proposition 2.1.5. *Let $m \in \mathsf{GL}(V)$ be unipotent. The global holomorphic sections of the bundle with multiplier m are the constants fixed by m.*

Proof. Thanks to the functoriallity of $\mathcal{G}$, it is enough to prove this for $(V, m) = (\mathbb{C}^n, \mathrm{Id} + J_n)$, where J_n is the lower Jordan block of rank n. We prove by induction on n that, for any global section $s = (s_1, \ldots, s_n)$, we have $s_j = 0$ for $j < n$ and s_n is a constant.

The equation (7) can be expressed as:

$$s_{j+1}(qz) = s_j(z) + s_{j+1}(z), \quad \text{for} \quad j = 0, \dots, n-1, \tag{8}$$

where we put $s_0 = 0$.

For $j = 0$, this equation means that s_1 is an entire elliptic function, hence must be constant. If $n = 1$, we are done. Otherwise, for $j = 1$, the set of meromorphic solutions of (8) is easily seen to be $\{(2\pi i)^{-1}s_1E_1 + \varphi\}$, where φ runs over elliptic functions. Since E_1 has simple poles at lattice points only, which can't be the case for φ, we must have $s_1 = 0$ for s_2 to be entire. Now $(s_2, \dots, s_n)$ is a global section of $\mathcal{G}(\mathbb{C}^{n-1}, J_{n-1})$. □

We'll have more solutions if we allow logarithmic poles, but there is some rigidity which will prove useful in the course of the proof of theorem I:

Proposition 2.1.6. *Let m be a unipotent endomorphism of V. There is at most one meromorphic section of the bundle with multiplier m having a simple pole at e and given degree 0 component in its Laurent series expansion near e.*

Proof. It's enough to prove that any such section s with zero constant term vanishes. Let $s_1, \dots, s_n$ be the coordinates of s in a basis such that the matrix of the multiplier is lower triangular.

Again, we have $s_1(qz) = s_1(z)$, hence s_1 is elliptic. Since it has at most one simple pole, it must be constant, hence identically zero, by the assumption on the constant term. Now applying $s_1 = 0$ in our triangular system of q-difference equations yields the induction. □

Proof of proposition 2.1.3. The unit object of the source category is $(\mathbb{C}, \mathrm{Id})$. Its image is the sheaf of local elliptic functions, *i.e.*, the trivial bundle $\mathcal{O}_X$. On the other hand, $\mathcal{G}$ is exact, since exactness can be checked on fibres, and $\mathcal{G}_{|z}$ is simply the forgetting of multipliers.

Tensor structure

Fully faithfullness The faithfullness is obvious, and can be checked on any fibre. For fullness, we have to prove that any morphism $\mathcal{G}(V_1, m_1) \to \mathcal{G}(V_2, m_2)$ comes from a constant $u : V_1 \to V_2$ such that $u = m_2 u m_1^{-1}$. Since homomorphisms of bundles are global sections of $\underline{\mathrm{Hom}}(\mathcal{V}_1, \mathcal{V}_2,)$, this follows from proposition 2.1.5.

Essential surjectivity By Atiyah's theorem 5 of [2], there is up to isomorphism exactly one indecomposable nilpotent bundle on X of rank n. Using the fully faithfullness, we see that the bundle associated to $\mathbb{C}^n$ and $m = \mathrm{Id} + J_n$, where J_n is the Jordan block of rank n, has those required properties. □

2.1.7. *Remark.* Let S be a space (in some sense) with action of a group G and assume S/G makes sense in such a way that $p\colon S \to S/G$ is a G-bundle. he pullback by p induces an equivalence from the category of bundles on S/G to the category of G-equivariant bundles on S. In our case, where S is the multiplicative group, any vector bundle is trivial. The multiplier can be thought as the expression of the incompatibility between an arbitrary trivialisation and the G action. The additional information we obtained is that *constant* multipliers are enough.

2.2. **Nilpotent connections with simple poles at the origin.**

2.2.1. *A two variable Jacobi form.* The following function was introduced by Kronecker [17], rediscovered by Zagier [24], and considered by the first author in the context of elliptic polylogarithms [18]:

$$F(\xi, \alpha; \tau) := (2\pi i)\left(1 - \frac{1}{1-z} - \frac{1}{1-w} - \sum_{m,n=1}^{\infty}(z^m w^n - z^{-m}w^{-n})q^{mn}\right),$$

where $q = \exp(2\pi i\tau)$, $\quad z = \exp(2\pi i\xi)$, $\quad w = \exp(2\pi i\alpha)$. We recall here some of the statements of [24, 3. Theorem]

The double series F converges in the domain $\{\Im\tau > \Im\xi > 0\, \Im\tau > \Im\alpha > 0\}$ and extends meromorphically to all values of ξ and α. It has simple poles at divisors $\xi = m + n\tau$ and $\alpha = m' + n'\tau$ and can be expressed by means of the Jacobi theta function:[1]

$$F(\xi, \alpha; \tau) = \frac{\theta'(0;\tau)\theta(\xi+\alpha;\tau)}{\theta(\xi;\tau)\theta(\alpha;\tau)}. \tag{9}$$

The residue of F at $\xi = 0$ is α^{-1}.

The elliptic and modular properties of F are rather nice:

$$F(\xi+1, \alpha; \tau) = F(\xi, \alpha; \tau); \tag{10}$$

$$F(\xi+\tau, \alpha; \tau) = \exp(-2\pi i\alpha)F(\xi, \alpha; \tau); \tag{11}$$

$$F(\frac{\xi}{c\tau+d}, \frac{\alpha}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) = (c\tau+d)\exp(2\pi i\frac{c\xi\alpha}{c\tau+d})F(\xi, \alpha; \tau). \tag{12}$$

F can also be expressed as the exponential of the generating series in one variable of Eisenstein functions of the other variable:

$$F(\xi, \alpha; \tau) = \exp\left(-\sum_{k\geqslant 1}\frac{(-1)^k\xi^k}{k}(E_k(\alpha;\tau) - e_k(\tau))\right) \tag{13}$$

This statement can easily be deduced from Zagier's "logarithmic formula" for F (loc. cit., *(viii)*) and the power series expansion formula for E_n [22, III, (10)]

2.2.2. Although they won't be needed before section 3, we gather here further properties of F, also easily deduced from classical properties of theta series.

It satisfies the "mixed heat" equation:

$$2\pi i\frac{\partial F(\xi,\alpha;\tau)}{\partial\tau} = \frac{\partial^2 F(\xi,\alpha;\tau)}{\partial\xi\partial\alpha} \tag{14}$$

and the following identity:

$$\begin{aligned} F(\xi, \alpha_1; \tau)F_2'(\xi, \alpha_2; \tau) - F(\xi, \alpha_2; \tau)F_2'(\xi, \alpha_1; \tau) \\ = F(\xi, \alpha_1+\alpha_2; \tau)(\wp(\alpha_1) - \wp(\alpha_2)), \end{aligned} \tag{15}$$

where $F_2'(\xi, \alpha; \tau) = \frac{\partial F(\xi,\alpha;\tau)}{\partial\alpha}$ denotes the derivative with respect to the second argument α and $\wp$ denotes the Weierstraß function $\wp = E_2 - e_2$.

In this section about single elliptic curves, τ will always be a constant. We will therefore omit it in the notation.

2.2.3. *Connections and multipliers.* Let $\mathcal{V}$ be the vector bundle on X defined by (V, m). In the same way as before, connections on $\mathcal{V}$ are just the same as Λ-equivariant connections on $\mathbb{C}$.

So let ∇ be a connection on $\mathcal{V}$. Since the exterior derivative d on the affine space $\mathbb{C}$ is itself a Λ-equivariant connection, we can write, slightly abusing notation:

$$\nabla = d + \omega,$$

where ω is a section of the bundle $\underline{\mathrm{End}}(\mathcal{V}) \otimes \Omega^1(X)$.

More explicitely, a connection on $\mathcal{V}$ is given by its pullback to $\mathbb{C}$, which takes the form:

$$\nabla = d + \omega, \quad \text{with} \quad \omega(\xi+\tau) = m\omega(\xi)m^{-1},\ \omega(\xi+1) = \omega(\xi) \tag{16}$$

since $\underline{\mathrm{End}}(\mathcal{V})$ is the bundle with fibre $\mathrm{End}(\mathcal{V})$ and multiplier the conjugation by m. In the general framework of [1], this should be thought as an integrability condition.

[1]This is actually the definition Kronecker started with.

We'll work mostly with multipliers given in exponential form $m = \exp(n)$. In this setup, the conditions in (16) become:

$$\omega(\xi+\tau) = \exp(\mathrm{ad}_n)\omega(\xi),\ \omega(\xi+1) = \omega(\xi) \tag{17}$$

Definition-Proposition 2.2.4. *Let V be a vector space with two simultaneously nilpotent operators $\mathbf{t}$ and A. The formula*

$$\nabla = d - \mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}})(A) d\xi, \tag{18}$$

where F has to be understood as a Laurent series in the second variable, defines a nilpotent meromorphic connection on the vector bundle $\mathcal{V}$ on X with fibre V and multiplier $\exp(-2\pi i\mathbf{t})$.

Proof. We have to check that $\mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}})(A)d\xi$ satisfies (17) for $n = -2\pi i\mathbf{t}$. This is nothing but the quasi-periodicity of F, as expressed in equations (10) and (11). □

Notation 2.2.5. *In the sequel, R will denote the Hopf algebra $\mathbb{C}\langle\langle \mathbf{t}, A\rangle\rangle$ of formal non-commutative series in $\mathbf{t}$ and A over $\mathbb{C}$, equipped with the standard coproduct*

$$\Delta(\mathbf{t}) = 1\otimes\mathbf{t} + \mathbf{t}\otimes 1; \Delta(A) = 1\otimes A + A\otimes 1$$

As a Hopf algebra, R is also the completed universal enveloping algebra of $\mathfrak{fr}_{\mathbb{C}}(\mathbf{t}, A)$, the free Lie algebra generated by $\mathbf{t}$ and A.

Vector spaces with simultaneous nilpotent operators $\mathbf{t}$ and A are objects of the tensor category $R-$NMod of nilpotent modules over R.

We can now formulate the main result of this section:

Theorem I. *The assignment given by the formula (18) extends to an equivalence of rigid tensor categories*

$$\mathcal{F}\colon R-\mathrm{NMod} \longrightarrow \mathrm{NConn}(X; e)$$

Before proceeding to the proof in 2.2.10 and followings, we'd like to expand a bit further on the meaning of this result.

2.2.6. *Tannakian reformulation.* Let us choose once and for all a quasi-inverse of $\mathcal{F}$ and denote its composition with the forgetful functor $(V, \mathbf{t}, A) \mapsto V$ by ω_{DR}. This is a fibre functor, that depends a priori only on the choice of the uniformisation $\mathbb{C}^* \to X$, up to a canonical functorial isomorphism.

The following corollary is then just a restatement of the theorem:

Corollary 2.2.7. *The fundamental group $\pi_1^{DR}(X\setminus(e), \omega_\tau)$ is $\exp\mathfrak{fr}(\mathbf{t}, A)$.*

The (pro)vector bundle $\mathcal{P} := \mathcal{F}(R)$, in which R is considered as a left module over itself, is the fundamental torsor $P^{DR}(X\setminus(e), \omega_A, -)$.

The connection of $\mathcal{P}$ is given by the formula (18), in which $\mathbf{t}$ and A have to be interpreted as left multiplications in R.

2.2.8. *Semi-canonical De Rham paths.* The trivialisation of the pull-back of $\mathcal{P}$ to $\mathbb{C}^*$ means moreover that, for any $z \in \mathbb{C}^*\setminus\{1\}$, we have a canonical De Rham path between its image x in $X\setminus\{e\}$ and ω_A: that would be the 1 of $\mathcal{P}_x$; it is simply induced by the canonical isomorphism between $z^*(V\otimes\mathcal{O}_{\mathbb{C}^*})$ and V. This applies to tangential base points at e as well. By composition, we get also semi-canonical paths between any two points, obeying to similar ambiguities upon changes of liftings.

The existence of canonical paths is the anchoring needed to interpret the parallel transport along a topological path γ for $\mathcal{P}$ as a De Rham path, and assign a well-defined element of R to by comparing it to the canonical path, as in [8, 12.15].

These paths really depends on the choice of the lifting: changing z in qz amounts to a left multiplication by $\exp(2\pi i\mathbf{t})$.

2.2.9. Although ω_{DR} and $\mathcal{P}$ seem to depend on the choice of uniformisation, they actually don't. Since this is a special case of the more general statement for families of section 4, we won't repeat it here. Of course, the explicit description of $\mathcal{P}$ via its pullback depend on this choice as well as of the lifting of base points.

2.2.10. *Plan of the proof.* The remainder of this subsection is devoted to the proof of theorem I. Most of the steps are routine checks and will therefore be only sketched.

We then turn to the essential surjectivity, which just means that any connection can be put in the form (18). To prove this, we remark that A can be recovered from the degree 0 component in ξ of ∇, and use proposition 2.1.6 above.

As in the proof of proposition 2.1.3, we reduce the question of fully faithfullness to the special case of global sections, which is treated first in a separate proposition.

2.2.11. *Tensor structure.* It is obvious that $\mathrm{ad}_{\mathbf{t}}F(\xi, \mathrm{ad}_{\mathbf{t}})(A)$ is a Lie series in the variables $\mathbf{t}$ and A. Therefore, it is a primitive element with respect to the Hopf algebra structure of R. This ensures that our functor is compatible with tensor product and duality.

2.2.12. *Essential surjectivity.* Let $(V, \mathbf{t})$ represent a vector bundle on X and let ∇ be a covariant connection with simple poles at lattice points. We thus have $\nabla = d - \nu d\xi$ where ν is a global section of $\underline{\mathrm{End}}_{\mathrm{NBdl}(X)}(\mathcal{V})$ with simple poles at lattice points.

$$\nu(\xi+\tau) = \exp(-2\pi i \mathrm{ad}_{\mathbf{t}})\nu(\xi); \quad \nu(\xi+1) = \nu(\xi)$$

Let's denote by ν_0 the constant term of ν in its Laurent series expression at $\xi = 0$. By proposition 2.2.13, if we find an endomorphism A of V such that the degree 0 term in ξ of $\mathrm{ad}_{\mathbf{t}}F(\xi, \mathrm{ad}_{\mathbf{t}})(A)$ is equal to ν_0, then both sections $\mathrm{ad}_{\mathbf{t}}F(\xi, \mathrm{ad}_{\mathbf{t}})(A)$ and ν will have to coincide. We have:

$$\begin{aligned} \xi F(\xi, \alpha) &= \exp(-\sum_{k \geqslant 1} \frac{(-1)^k \xi^k}{k}(E_k(\alpha) - e_k) \\ &= 1 + \xi E_1(\alpha) \mod \xi^2, \quad \text{hence} \\ \alpha F(\xi, \alpha) &= \xi^{-1} + \alpha E_1(\alpha) \mod \xi \end{aligned}$$

Therefore, the constant term in ξ of $\mathrm{ad}_{\mathbf{t}}F(\xi, \mathrm{ad}_{\mathbf{t}})$ is $F_0(\mathrm{ad}_{\mathbf{t}})$, for $F_0(\alpha) := \alpha E_1(\alpha)$. Since $F_0(\alpha) = 1 \bmod \alpha$, the operator $F_0(\mathrm{ad}_{\mathbf{t}})$ is invertible, and $F_0^{-1}(\mathrm{ad}_{\mathbf{t}})(\nu_0)$ is the A we sought. □

Proposition 2.2.13 (Horizontal sections)**.** *Let V be an R-module. The horizontal sections of $\mathcal{F}(V)$ are the constants annihilated by $\mathbf{t}$ and A.*

Proof. Let s be an horizontal section. By proposition 2.1.5 above, we already know that s must be constant and annihilated by $\mathbf{t}$. Let us express the horizontality:

$$\begin{aligned} \nabla s = ds + \alpha F(\xi, \alpha)A \cdot s d\xi &= 0, \quad \textit{i.e.} \\ (\alpha F(\xi, \alpha)A) \cdot s &= 0, \quad \text{hence} \\ F_0(\alpha)(A)s &= 0, \end{aligned}$$

where F_0 is as in the proof of essential surjectivity above. Since $F_0(\alpha)$ is invertible, we thus obtain the wished $As = 0$. □

2.2.14. *Fully faithfullness.* Let V, W be R-modules. An horizontal morphism $f : \mathcal{F}(V) \to \mathcal{F}(W)$ is an horizontal section of

$$\underline{\mathrm{Hom}}_{\mathrm{NConn}(X;e)}(\mathcal{F}(V_1), \mathcal{F}(V_2)) = \mathcal{F}(\underline{\mathrm{Hom}}_{R\text{–Mod}}(V_1, V_2))$$

By proposition 2.2.13 above, f is a constant linear map $V \to W$ satisfying $\mathbf{t}f = f\mathbf{t}$ and $Af = fA$, hence a morphism of R-modules. The faithfullness is obvious. □

2.2.15. *The Hodge structure.* The weight and Hodge filtrations on the fundamental De Rham Hopf algebra R can be defined in complete analogy with the genus 0 case [8, §12].

$$\begin{aligned} W_n &:= \bigoplus_{i \leqslant -n} R_n \\ F^p &:= \bigoplus_{i \geqslant -p} \{x \in R, \deg_A(x) = i\}, \end{aligned}$$

where R_n is the component of degree n of R for the total degree in $\mathbf{t}$ and A.

It is easy to check that they are compatible with the $\mathbb{Q}$-structure of 5.1. and are invariant with respect to the action of $\mathrm{SL}_2(\mathbb{Z})$ (see 3.3 or definition 4.2.2), hence are well-defined.

2.2.16. *Simple elliptic polylogarithms.* Although we won't detail it here, it is possible to show that a regularized version of the generating series Λ of simple elliptic polylogarithms introduced in [18] is horizontal for $\nabla_{\mathcal{P}}$ in the quotient of R by the second term of its derived series. This is in complete analogy with the genus 0 situation.

3. A universal flat connection over the upper-half plane

Treating the parameter τ of the elliptic curve as a variable, we get the standard family $\mathbb{E}$ of elliptic curves over the upper-half-plane $\mathbb{H} = \{\tau | \Im\tau > 0\}$. Furthermore, we can define a bundle $\mathfrak{P}$ with fiber $R = \mathbb{C}\langle\langle \mathbf{t}, A\rangle\rangle$ on $\mathbb{E}$ by the same rule as in the case of an individual curve.

In this section, we tie the fiberwise bundles with connection $(\mathcal{P}_\tau, \nabla_\tau)$ from section 2 together into a *flat* connection $\nabla_{\mathfrak{P}}$ on $\mathfrak{P}$ over the *surface* $\mathbb{E}$.

3.1. **Combinatorial conventions and lemmas.** We shall make a frequent use of the following special notation:

Notation 3.1.1. *Let $\mathfrak{g}$ a pronilpotent Lie algebra and t in $\mathfrak{g}$. For $B, C \in \mathfrak{g}$ and a formal* commutative *power series $f(X,Y) = \sum f_{ij} X^i Y^j$, we set:*

$$f(X,Y)[\![B, C]\!]_t := \sum_{i,j \geqslant 0} f_{ij} [\mathrm{ad}_t^i B, \mathrm{ad}_t^j C]$$

Most of the times, t will be clear from the context, so we'll omit it.

Lemma 3.1.2 (Jacobi identity)**.** *For any formal power series $f(X,Y)$, we have:*

$$\mathrm{ad}_t f(X,Y)[\![B,C]\!]_t = (X+Y) f(X,Y)[\![B,C]\!]_t$$

3.1.3. Notice that for a symmetric f, i.e., $f(X,Y) = f(Y,X)$, and $C = B$, we have $f(X,Y)[\![B,B]\!] = 0$, hence in this case one may replace f by its antisymmetrisation.

Lemma 3.1.4. *Let D be a derivation of R. For any power series f in one variable, we have the following identity:*

$$D\left(f(\mathrm{ad}_t)B\right) = \frac{f(X+Y) - f(Y)}{X} [\![D(t), B]\!]_t + f(\mathrm{ad}_t) D(B)$$

Proof. Induction on monomials using the Jacobi identity 3.1.2 above. □

Let us also recall this well-known fact [20, corollary 3.23]:

Lemma 3.1.5. *Let D be a derivation of R. The following equality holds:*

$$D(\exp(t)) = \exp(t) \frac{\exp(-\mathrm{ad}_t) - 1}{-\mathrm{ad}_t} D(t).$$

3.2. **The formula for the connection.** Now we are able to define the connection $\nabla_{\mathfrak{P}}$. As in section 2, it will be convenient to write Laurent series in $\mathrm{ad}_{\mathbf{t}}$ within computations, as long as the polar parts in $\mathrm{ad}_{\mathbf{t}}$ do eventually vanish.

$$\begin{aligned}
\psi_A(\tau) &:= -\frac{1}{2}\frac{1}{2\pi i}\frac{XY}{X+Y}\left(\left(\wp(X,\tau)-\frac{1}{X^2}\right)-\left(\wp(Y,\tau)-\frac{1}{Y^2}\right)\right)[\![A,A]\!]d\tau\\
\psi_{\mathbf{t}}(\tau) &:= -\frac{1}{2\pi i}A d\tau,\\
\nu(\xi,\tau) &:= -\left(\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)d\,\xi+\frac{1}{2\pi i}\left(\mathrm{ad}_{\mathbf{t}}F_2'(\xi,\mathrm{ad}_{\mathbf{t}};\tau)+\frac{1}{\mathrm{ad}_{\mathbf{t}}}\right)d\,\tau\right)A,
\end{aligned}$$

and, finally,

$$\nabla_{\mathfrak{P}} := d+\nu+\psi_A\frac{\partial}{\partial A}+\psi_{\mathbf{t}}\frac{\partial}{\partial \mathbf{t}}, \tag{19}$$

where $f\frac{\partial}{\partial A}$ (resp. $f\frac{\partial}{\partial \mathbf{t}}$) has to be understood as the derivation which maps A to f and $\mathbf{t}$ to 0 (resp. A to 0 and $\mathbf{t}$ to f). Hence the commutator $[f\frac{\partial}{\partial A}, r]$ of $f\frac{\partial}{\partial A}$ and r is just $f\frac{\partial r}{\partial A}$

Proposition 3.2.1. *The operator $\nabla_{\mathfrak{P}}$ above descends to a connection on the vector bundle $\mathfrak{P}$ on $\mathbb{E}$.*

Proof. We must check that $\nabla_{\mathfrak{P}}$ is invariant with respect to the shift $\xi\to\xi+1$ and that it transforms in correct way under $\xi\to\xi+\tau$. The first property follows immediately from the invariance of F with respect to this shift. So we shall check the quasiperiodicity with respect to τ:

$$\nabla_{\mathfrak{P}}|_{\xi+\tau} = \exp(-2\pi i\mathbf{t})\nabla_{\mathfrak{P}}|_{\xi}\exp(2\pi i\mathbf{t}). \tag{20}$$

The ν summand in the left hand side of (20) is equal to

$$\begin{aligned}
&- \left(\mathrm{ad}_{\mathbf{t}}F(\xi+\tau,\mathrm{ad}_{\mathbf{t}};\tau)d\,(\xi+\tau)+\tfrac{1}{2\pi i}\left(\mathrm{ad}_{\mathbf{t}}F_2'(\xi+\tau,\mathrm{ad}_{\mathbf{t}};\tau)+\tfrac{1}{\mathrm{ad}_{\mathbf{t}}}\right)d\,\tau\right)A\\
&- \Big(\exp(-2\pi i\mathrm{ad}_{\mathbf{t}})(\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)d\,\xi+\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)d\,\tau)\\
&+ \tfrac{1}{2\pi i}\left(\exp(-2\pi i\mathrm{ad}_{\mathbf{t}})(-2\pi i\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)+\mathrm{ad}_{\mathbf{t}}F_2'(\xi,\mathrm{ad}_{\mathbf{t}};\tau))+\tfrac{1}{\mathrm{ad}_{\mathbf{t}}}\right)d\,\tau\Big)A\\
&= -\exp(-2\pi i\mathrm{ad}_{\mathbf{t}})\Big(\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)d\,\xi\\
&\qquad + \left(\tfrac{1}{2\pi i}\left(\mathrm{ad}_{\mathbf{t}}F_2'(\xi,\mathrm{ad}_{\mathbf{t}};\tau)+\tfrac{1}{\mathrm{ad}_{\mathbf{t}}}\right)+\tfrac{\exp(2\pi i\mathrm{ad}_{\mathbf{t}})-1}{2\pi i\mathrm{ad}_{\mathbf{t}}}\right)d\,\tau\Big)A
\end{aligned}$$

The other terms in the left hand side of (20) do not depend on ξ; they are therefore invariant. In the right hand side, the term $\exp(-2\pi i\mathbf{t})\nu\exp(2\pi i\mathbf{t})$ is equal to

$$-\exp(-2\pi i\mathrm{ad}_{\mathbf{t}})\left(\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)d\,\xi+\frac{1}{2\pi i}\left(\mathrm{ad}_{\mathbf{t}}F_2'(\xi+\tau,\mathrm{ad}_{\mathbf{t}};\tau)+\frac{1}{\mathrm{ad}_{\mathbf{t}}}\right)d\,\tau\right)A,$$

whereas the term $\exp(-2\pi i\mathbf{t})\psi_A\frac{\partial}{\partial A}\exp(2\pi i\mathbf{t})$ equals $\psi_A\frac{\partial}{\partial A}$ since $\psi_A\partial_A(\tau)=0$.

The last remaining term $\exp(-2\pi i\mathbf{t})\psi_{\mathbf{t}}\frac{\partial}{\partial \mathbf{t}}\exp(2\pi i\mathbf{t})$ is equal to

$$\exp(-2\pi i\mathbf{t})\left(-\frac{1}{2\pi i}A\frac{\partial}{\partial\mathbf{t}}\right)\exp(2\pi i\mathbf{t})=\frac{\exp(-2\pi i\mathrm{ad}_{\mathbf{t}})-1}{-2\pi i\mathrm{ad}_{\mathbf{t}}}(-A)-\frac{1}{2\pi i}A\frac{\partial}{\partial\mathbf{t}},$$

by lemma 3.1.5. So, both sides of (20) are indeed equal. □

Proposition 3.2.2. *The connection $\nabla_{\mathfrak{P}}$ is flat.*

Proof. Denote by ω the differential form $\nu+\psi_A\frac{\partial}{\partial A}+\psi_{\mathbf{t}}\frac{\partial}{\partial\mathbf{t}}$.

We shall prove that $d\,\omega + \omega\wedge\omega = 0$. The term $d\omega$ is equal to $d\nu$, as other terms are lifted from the one dimensional $\mathbb{H}$, and therefore are closed.

$$\begin{aligned}
d\omega &= -d\left(\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)d\xi + \frac{1}{2\pi i}\left(\mathrm{ad}_{\mathbf{t}}F_2'(\xi,\mathrm{ad}_{\mathbf{t}};\tau) + \frac{1}{\mathrm{ad}_{\mathbf{t}}}\right)d\,\tau\right)A \\
&= \left(\frac{\partial}{\partial\tau}(\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)) - \frac{1}{2\pi i}\frac{\partial}{\partial\xi}\left(\mathrm{ad}_{\mathbf{t}}F_2'(\xi,\mathrm{ad}_{\mathbf{t}};\tau) + \frac{1}{\mathrm{ad}_{\mathbf{t}}}\right)\right)Ad\xi\wedge d\tau \\
&= \mathrm{ad}_{\mathbf{t}}\left(\frac{\partial}{\partial\,\tau}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau) - \frac{1}{2\pi i}\frac{\partial}{\partial\,\xi}F_2'(\xi,\mathrm{ad}_{\mathbf{t}};\tau)\right)Ad\xi\wedge d\tau = 0,
\end{aligned}$$

as the function F satisfies the mixed heat equation (14).

Let's now turn our attention to the $\omega\wedge\omega$ term. We have

$$\begin{aligned}
\omega\wedge\omega &= (2\pi i)^{-1}(\Sigma_1+\Sigma_2+\Sigma_3)d\xi\wedge d\tau, \quad \text{with} \\
\Sigma_1 &:= \left[\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)A, \left(\mathrm{ad}_{\mathbf{t}}F_2'(\xi,\mathrm{ad}_{\mathbf{t}};\tau) + \frac{1}{\mathrm{ad}_{\mathbf{t}}}\right)A\right] \\
\Sigma_2 &:= -\left[\frac{1}{2}\frac{XY}{X+Y}\left(\left(\wp(X)-\frac{1}{X^2}\right) - \left(\wp(Y)-\frac{1}{Y^2}\right)\right)[\![A,A]\!]\frac{\partial}{\partial\,A},\right. \\
&\qquad \left.\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)A\right] \\
\Sigma_3 &:= \left[A\frac{\partial}{\partial\mathbf{t}}, \mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)A\right]
\end{aligned}$$

Till the end of this computation, it will be safe to omit τ from the notation. Using notation 3.1.1 and remark 3.1.3 on the first summand, we have:

$$\Sigma_1 = XF(\xi,X)\left(YF_2'(\xi,Y)+\frac{1}{Y}\right)[\![A,A]\!]$$

$$= \frac{1}{2}\left(XF(\xi,X)\left(YF_2'(\xi,Y;\tau)+\frac{1}{Y}\right) - YF(\xi,Y)\left(XF_2'(\xi,X)+\frac{1}{X}\right)\right)[\![A,A]\!]$$

Applying Jacobi identity in the form of lemma 3.1.2 to Σ_2, we obtain:

$$\Sigma_2 = -\frac{1}{2}(X+Y)F(\xi,X+Y)\frac{XY}{X+Y}\left(\left(\wp(X)-\frac{1}{X^2}\right) - \left(\wp(Y)-\frac{1}{Y^2}\right)\right)[\![A,A]\!]$$

For the third summand, we apply lemma 3.1.4 and get:

$$\begin{aligned}
\Sigma_3 &= -\frac{(X+Y)F(\xi,X+Y) - YF(\xi,Y)}{X}[\![A,A]\!] \\
&= -\frac{1}{2}\left(\frac{(Y^2-X^2)F(\xi,X+Y) + X^2F(\xi,X) - Y^2F(\xi,Y)}{XY}\right)[\![A,A]\!]
\end{aligned}$$

Putting the three summands together again, we have:

$$\begin{aligned}
\omega\wedge\omega &= 2^{-1}(2\pi i)^{-1}\Xi(X,Y,\tau)[\![A,A]\!]d\xi\wedge d\tau, \quad \text{with} \\
\Xi(X,Y,\tau) &:= XF(\xi,X)\left(YF_2'(\xi,Y)+\frac{1}{Y}\right) - YF(\xi,Y)\left(XF_2'(\xi,X)+\frac{1}{X}\right) \\
&\quad -XYF(\xi,X+Y)\left(\left(\wp(X)-\frac{1}{X^2}\right) - \left(\wp(Y)-\frac{1}{Y^2}\right)\right) \\
&\quad -\frac{(Y^2-X^2)F(\xi,X+Y) + X^2F(\xi,X) - Y^2F(\xi,Y)}{XY} \\
&= XF(\xi,X)YF_2'(\xi,Y) - YF(\xi,Y)XF_2'(\xi,X) \\
&\quad -XYF(\xi,X+Y)\left(\wp(X)-\wp(Y)\right).
\end{aligned}$$

This latter expression vanishes, according to (15); so does the curvature of $\nabla_{\mathfrak{P}}$. $\square$

3.3. Modularity.

3.3.1. Consider the standard action of the modular group $SL_2(\mathbb{Z})$ on the upper-half-plane $\mathbb{H}$ and the relative curve $\mathbb{E}$ over $\mathbb{H}$: $\tau \to \tau' = \frac{a\tau+b}{c\tau+d}$; $\xi \to \xi' = \frac{\xi}{c\tau+d}$. Denote by $\mathcal{R}$ the (pro)bundle on $\mathbb{H}$ with fiber R. Define an action of $SL_2(\mathbb{Z})$ on $\mathcal{R}$ by the rule: $\mathbf{t} \to \mathbf{t}' = \frac{\mathbf{t}}{c\tau+d}$, $A \to A' = (c\tau+d)A + 2\pi i c\mathbf{t}$; and lift this action to an action on $\mathfrak{P}$ by multiplier $\exp(2\pi i \frac{c\xi\mathbf{t}}{c\tau+d})$. Thus, a $SL_2(\mathbb{Z})$-invariant section of $\mathfrak{P}$ is an R-valued function f in τ and ξ such that

(21)
$$f\left(\frac{a\tau+b}{c\tau+d}, \frac{\xi}{c\tau+d}; \frac{\mathbf{t}}{c\tau+d}, (c\tau+d)A + 2\pi i c\mathbf{t}\right) = \exp\left(2\pi i \frac{c\xi\mathbf{t}}{c\tau+d}\right) f(\tau,\xi;\mathbf{t},A).$$

This action is well defined (compatible with the group structure on $SL_2(\mathbb{Z})$ and compatible with fiberwise action of $\mathbb{Z}^2$). We leave these as an exercise to the reader.

As before, the condition that a global endomorphism has to fulfill is obtained by replacing $\mathbf{t}$ with $\mathrm{ad}_{\mathbf{t}}$.

Proposition 3.3.2. *The connection $\nabla_{\mathfrak{P}}$ is equivariant for the action of $SL_2(\mathbb{Z})$ on the vector bundle $\mathfrak{P}$.*

Proof.
$$\nabla_{\mathfrak{P}}|_{\tau',\xi';\mathbf{t}',A'} = \exp\left(2\pi i \frac{c\xi\mathbf{t}}{c\tau+d}\right) \nabla_{\mathfrak{P}}|_{\tau,\xi;\mathbf{t},A} \exp\left(-2\pi i \frac{c\xi\mathbf{t}}{c\tau+d}\right).$$

The term ν in the l.h.s. equals

$$-\left(\frac{\mathrm{ad}_{\mathbf{t}}}{c\tau+d}F\left(\frac{\xi}{c\tau+d}, \frac{\mathrm{ad}_{\mathbf{t}}}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) d\frac{\xi}{c\tau+d} + \right.$$
$$\left.\frac{1}{2\pi i}\left(\frac{\mathrm{ad}_{\mathbf{t}}}{c\tau+d}F_2'(\frac{\xi}{c\tau+d}, \frac{\mathrm{ad}_{\mathbf{t}}}{c\tau+d}; \frac{a\tau+b}{c\tau+d}) + \frac{c\tau+d}{\mathrm{ad}_{\mathbf{t}}}\right) d\frac{a\tau+b}{c\tau+d}\right)((c\tau+d)A + 2\pi i c\mathbf{t}) =$$
$$-\exp\left(2\pi i\frac{c\xi\mathrm{ad}_{\mathbf{t}}}{c\tau+d}\right)\left(\mathrm{ad}_{\mathbf{t}}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau)\left(d\xi - \frac{c\xi}{c\tau+d}d\tau\right) - 2\pi i c\mathbf{t}\left(\frac{d\xi}{c\tau+d} - \frac{c\xi d\tau}{(c\tau+d)^2}\right) + \right.$$
$$\left.\frac{1}{2\pi i}\left(\mathrm{ad}_{\mathbf{t}}\left(2\pi i\frac{c\xi}{c\tau+d}F(\xi,\mathrm{ad}_{\mathbf{t}};\tau) + F_2'(\xi,\mathrm{ad}_{\mathbf{t}};\tau)\right) + \exp\left(-2\pi i\frac{c\xi\mathrm{ad}_{\mathbf{t}}}{c\tau+d}\right)\frac{1}{\mathrm{ad}_{\mathbf{t}}}\right)d\tau\right)A =$$
$$\exp\left(2\pi i\frac{c\xi\mathrm{ad}_{\mathbf{t}}}{c\tau+d}\right)\nu + \frac{\exp\left(2\pi i\frac{c\xi\mathrm{ad}_{\mathbf{t}}}{c\tau+d}\right) - 1}{\mathrm{ad}_{\mathbf{t}}}d\tau A - 2\pi i c\mathbf{t}\left(\frac{d\xi}{c\tau+d} - \frac{c\xi d\tau}{(c\tau+d)^2}\right).$$

In this calculations we use the modular property of the function F and the evident equality $f(\mathrm{ad}_{\mathbf{t}})\mathbf{t} = f(0)\mathbf{t}$.

We have
$$\frac{\partial}{\partial A'} = \frac{1}{c\tau+d}\frac{\partial}{\partial A}, \quad \frac{\partial}{\partial \mathbf{t}'} = (c\tau+d)\frac{\partial}{\partial \mathbf{t}} - 2\pi i c\frac{\partial}{\partial A}.$$

As ψ_A is a modular form of weight 1:

$$\psi_A(\tau',\xi';\mathbf{t}',A') = -\frac{1}{2}\frac{1}{2\pi i}(c\tau+d)\frac{XY}{X+Y}\left(\left(\wp(X) - \frac{1}{X^2}\right) - \left(\wp(Y) - \frac{1}{Y^2}\right)\right) \times$$
$$((c\tau+d)A + 2\pi i c\mathbf{t}, (c\tau+d)A + 2\pi i c\mathbf{t})\frac{d\tau}{(c\tau+d)^2} = (c\tau+d)\psi_A(\tau,\xi;\mathbf{t},A),$$

the term $\psi_A\frac{\partial}{\partial A}$ is $SL_2(\mathbb{Z})$ invariant.

We have also:
$$\psi_{\mathbf{t}}(\tau',\xi';\mathbf{t}',A') = -\frac{1}{2\pi i}((c\tau+d)A + 2\pi i c\mathbf{t})\frac{d\tau}{(c\tau+d)^2} = \frac{1}{c\tau+d}\psi_{\mathbf{t}} - c\mathbf{t}\frac{d\tau}{(c\tau+d)^2}.$$

Note that A' and $\mathbf{t}'$ depend in τ explicitly as function in A and $\mathbf{t}$, so

$$\left.\frac{d}{d\tau}\right|_{\mathbf{t}',A'}=\frac{\partial}{\partial\tau}-\frac{cA'}{(c\tau+d)^2}\frac{\partial}{\partial A}+c\mathbf{t}'\frac{\partial}{\partial\mathbf{t}}$$

Therefore

$$d|_{\mathbf{t}',A'}+\psi_{\mathbf{t}}(\tau',\xi';\mathbf{t}',A')\frac{\partial}{\partial\mathbf{t}'}=d|_{\mathbf{t},A}+d\tau\left(-\frac{cA'}{(c\tau+d)^2}\frac{\partial}{\partial A}+c\mathbf{t}'\frac{\partial}{\partial\mathbf{t}}\right)$$

$$\left(\frac{1}{c\tau+d}\psi_{\mathbf{t}}-c\mathbf{t}\frac{d\tau}{(c\tau+d)^2}\right)(c\tau+d)\frac{\partial}{\partial\mathbf{t}}+$$

$$\left(-\frac{1}{2\pi i}A'\frac{d\tau}{(c\tau+d)^2}\right)\left(-2\pi ic\frac{\partial}{\partial A}\right)=d|_{\mathbf{t},A}+\psi_{\mathbf{t}}\frac{\partial}{\partial\mathbf{t}}.$$

In the r.h.s. we have:

$$\exp\left(2\pi i\frac{c\xi\mathbf{t}}{c\tau+d}\right)\left(d-\frac{1}{2\pi i}Ad\tau\frac{\partial}{\partial\mathbf{t}}\right)\exp\left(-2\pi i\frac{c\xi\mathbf{t}}{c\tau+d}\right)=$$

$$d-2\pi ic\mathbf{t}\left(\frac{d\xi}{c\tau+d}-\frac{c\xi d\tau}{(c\tau+d)^2}\right)+\frac{\exp\left(2\pi i\frac{c\xi\mathrm{ad}_{\mathbf{t}}}{c\tau+d}\right)-1}{\mathrm{ad}_{\mathbf{t}}}d\tau A,$$

the second summand comes from the differentiation by ξ and τ, the last can be calculated using Lemma 3.1.4.

Evidently,

$$\exp\left(2\pi i\frac{c\xi\mathbf{t}}{c\tau+d}\right)\nu\exp\left(-2\pi i\frac{c\xi\mathbf{t}}{c\tau+d}\right)=\exp\left(2\pi i\frac{c\xi\mathrm{ad}_{\mathbf{t}}}{c\tau+d}\right)\nu;$$

and $\psi_A\frac{\partial}{\partial A}$ commutes with $\exp\left(-2\pi i\frac{c\xi\mathbf{t}}{c\tau+d}\right)$. This finishes the proof. □

Note that such a lifting of the bunch of fiberwise connections to a flat connection is not unique. We shall return to this topic in 4.4.

Indeed, one can twist the connection $\nabla_{\mathfrak{P}}$ by any R-valued $SL_2(\mathbb{Z})$-invariant form μ on $\mathbb{H}$ by the right multiplication on μ:

$$\nabla^{\mu}_{\mathfrak{P}}(G)=\nabla_{\mathfrak{P}}(G)+G\mu=\nabla_{\mathfrak{P}}(G)+\mu G-\mathrm{ad}_{\mu}G.$$

This connection is flat as the right and the left multiplications commute.

4. Equivalence for the relative case

In this section, we generalize the results of the previous section to the case of a smooth family $p:X\to S$ of elliptic curves. We refer to the case treated in the previous section as the "fiberwise case". We denote by e the unit section of X.

We are mostly interested in the following families:

(i)) S is Poincaré's upper-half plane $\mathbb{H}$ with coordinate τ and X is the standard family $\mathbb{E}$.

(ii)) S is the punctured unit disc D^* with coordinate q and X is $(\mathbb{C}^*\times D)/q^{\mathbb{Z}}$.

(iii)) S is some modular curve, or even the moduli stack $\mathscr{M}_{1,1}$ of elliptic curves (smooth genus 1 curves with one marked point)

We provide here an explicit description of the *relative* nilpotent de Rham fundamental group and torsor of X. These objects are defined in 4.1. A precise statement is given as theorem II.

4.1. **Generalities.**

Definition 4.1.1. *A connection will be said to be relatively trivial on X if it is a pullback from S by p.*

A relatively nilpotent connection on X (resp., on $X \setminus e$, on X with simple poles at e) is an iterated extension of relatively trivial connections, in the relevant category of flat connections.

We denote by $\mathrm{NConn}(X/S; e)$ the category of relatively nilpotent flat meromorphic connections on X wich are holomorphic on $X \setminus e(S)$ and have simple poles at the unit section, and by $\mathrm{Conn}(S)$ the category of *all* flat connections on S. These are tannakian categories, and the pullback functor

$$p^*\colon \mathrm{Conn}(S) \longrightarrow \mathrm{NConn}(X/S; e)$$

respects the tannakian structure.

4.1.2. *Relative tannakian theory.* We give here a short account of some results of Spitzweck [21], generalizing the theory of tannakian categories to such relative situations.

In general, one considers a tensor functor $i\colon \mathcal{S} \to \mathcal{T}$ between two tannakian categories. *Relative fiber functors are* defined similarly as the usual ones, except that they take values in categories of vector bundles over $\mathcal{S}$-schemes, which are also defined in loc.cit., and have to be compatible with i.

The category $\mathcal{T}$ is said to be *neutral* relative to i if there exists a relative fixed fiber functor over $\mathcal{S}$. Such an object boils down to a pair (ω, φ) where ω is a tensor functor from $\mathcal{T}$ to $\mathcal{S}$ and $\varphi : \omega i \xrightarrow{\sim} \mathrm{Id}$. In this case, there is an $\mathcal{S}$-affine group scheme $\pi_1(\mathcal{T}/\mathcal{S}, \omega)$ and an equivalence of categories:

$$\mathcal{T} \xrightarrow{\sim} \pi_1(\mathcal{T}/\mathcal{S}, \omega) - \mathrm{Mod} \tag{22}$$

Further, there exists a pro-coalgebra in $\mathcal{T}$, denoted by $\mathcal{P}(\mathcal{T}/\mathcal{S}, \omega, -)$, called the *fundamental torsor coalgebra relative to i at ω.* It comes equipped with a right action of $i\pi_1(\mathcal{T}/\mathcal{S}, \omega)$, and is characterized by the fact that the above equivalence is given by the twisting:

$$T \longmapsto \mathcal{P}(\mathcal{T}/\mathcal{S}, \omega, -) \underset{i\pi_1(TC/SC, \omega)}{\otimes} i(T), \tag{23}$$

exactly as in the usual case.

As a side remark, let us observe that any tannakian category $\mathcal{S}$ over a field k is relative over k–Vect. There is indeed [9, 2.9] a canonical tensor functor k–Vect $\to \mathcal{S}$. In geometric categories such as $\mathrm{Conn}(S)$, it is given by trivial objects, hence is simply the pullback functor.

Definition 4.1.3. *For a fixed relative fiber functor ω on $\mathrm{NConn}(X/S; e)$ with values in Conn(S), we will denote by $\pi_1^{DR}(U/S, \omega)$ and $\mathfrak{P}^{DR}(U/S, \omega)$ the fundamental group and torsor of $\mathrm{NConn}(X/S, e)$ at ω, relative to the pullback functor p^*.*

These objects are respectively a pro-Hopf algebra in $\mathrm{Conn}(S)$ and a pro-coalgebra objects in $\mathrm{NConn}(X/s; e)$: flat connections equipped with structure laws which are expressed by diagrams of horizontal morphisms.

4.1.4. *Remarks.* A typical example of a relative fiber functor over $\mathrm{Conn}(S)$ would be the pullback functor by a section of $p : X \setminus \{e\} \to S$. It is therefore in general not a priori granted that relative fiber functors exist, i.e. that $\mathrm{NConn}(X/S; e)$ is neutral over $\mathrm{Conn}(S)$.

On the other hand, for an étale covering $S' \to S$, any section of $X_{S'}$ provides a fixed fiber functor over S'. A relative tannakian theory restricted to such geometric fiber functors has been formulated and used by Wildeshaus [16, I.3].

4.2. **The main statement.**

4.2.1. *A connection algebra on $\mathscr{M}_{1,1}$.* In this subsection, we define for any S a Hopf algebra in Conn(S) which will turn out to be the fundamental Lie algebra of NConn($X/S; e$). We start with the case of the upper-half plane.

Definition 4.2.2. *Let $\mathcal{R}_{\mathbb{H}}$ be the trivial (pro)bundle on $\mathbb{H}$ of Hopf algebras with fiber $R = \mathbb{C}\langle\langle \mathbf{t}, A\rangle\rangle$, equipped with the connection*

$$\nabla_{\mathcal{R}} := d + \psi_A \partial_A + \psi_{\mathbf{t}} \partial_{\mathbf{t}},$$

where we put, using notation 3.1.1:

$$\psi_A := -\frac{1}{2}\frac{1}{2\pi i}\frac{XY}{X+Y}\left(\left(\wp(X) - \frac{1}{X^2}\right) - \left(\wp(Y) - \frac{1}{Y^2}\right)\right)[\![A, A]\!]d\tau, \tag{24}$$

$$\psi_{\mathbf{t}} := -\frac{1}{2\pi i} A d\tau, \tag{25}$$

This bundle comes further equiped with the following action of $\mathrm{SL}_2(\mathbb{Z})$*:*

$$\mathbf{t} \to \mathbf{t}' = \frac{\mathbf{t}}{c\tau + d} \tag{26}$$

$$A \to A' = (c\tau + d)A + 2\pi i c\mathbf{t} \tag{27}$$

This $\mathcal{R}_{\mathbb{H}}$ is actually a Hopf algebra in the category of connections on $\mathbb{H}$:

Proposition 4.2.3. *The connection $\nabla_{\mathcal{R}}$ is compatible with the Hopf algebra structure of $\mathcal{R}_{\mathbb{H}}$ and $\mathrm{SL}_2(\mathbb{Z})$ equivariant.*

Proof. This follows directly from the fact that $\psi_{\mathbf{t}}\partial_{\mathbf{t}} + \psi_A \partial_A$ is a derivation which maps $\mathbf{t}$ and A to primitive elements and the freeness of R as an algebra.

The $\mathrm{SL}_2(\mathbb{Z})$ equivariance is immediate. □

4.2.4. *Local uniformisations.* Let $p : X \longrightarrow S$ be a smooth family of elliptic curves. Any local symplectic basis of $R^1 p_*(\mathbb{Z}_X)$ over an open 1-connected subvariety U of S defines a morphism $\tau : U \to \mathbb{H}$, and an isomorphism $X_{|U} \xrightarrow{\sim} \tau^*\mathbb{E}$, where $\mathbb{E}$ denotes the standard family over $\mathbb{H}$. We call such a pair (U, τ) a local uniformisation of X.

Proposition 4.2.5. *Let $X \longrightarrow S$ be a familly of elliptic curves. There is a well-defined Hopf algebra $\mathcal{R}_S$ in Conn(S) such that, for any local uniformisation (U, τ):*

$$\mathcal{R}_S|U \xrightarrow{\sim} \tau^*\mathcal{R} \tag{28}$$

Sketch of proof. The $\mathrm{SL}_2(\mathbb{Z})$ invariance of $\mathcal{R}$ precisely provides a canonical isomorphism between the pullbacks by two local uniformisations. The cocycle condition is nothing but the associativity of the action. □

Of course, this is nothing but a reassertion of the well-known fact that the analytic stack $\mathscr{M}_{1,1}$ is the quotient of $\mathbb{H}$ by $\mathrm{SL}_2(\mathbb{Z})$ in the sense of stacks, together with the description of sheaf-like objects on quotient stacks by equivariance.

4.2.6. *Remarks.* Note that the term $\psi_{\mathbf{t}}\frac{\partial}{\partial \mathbf{t}}$ corresponds to the connection on the tensor algebra of the first homology group equipped with Gauß-Manin connection.

One can also consider the (sub)bundle $\mathcal{L}$ of free Lie algebras generated by $\mathbf{t}$ and A. It is invariant under $SL_2(\mathbb{Z})$ and the connection $\nabla_{\mathcal{R}}$.

Proposition 4.2.7. *On any family $p : X \to S$ of elliptic curves, there is a well-defined relatively nilpotent connection $\mathfrak{P}_S$, such that, for any local uniformisation (U, τ),*

$$\mathfrak{P}_S \xrightarrow{\sim} \tau^*\mathfrak{P}_{\mathbb{H}}.$$

Moreover, $\mathfrak{P}_S$ comes equipped with a right $p^*\mathcal{R}_S$ module structure, which is locally the pullback of the right $p^*\mathcal{R}_{\mathbb{H}}$-module structure on $\mathfrak{P}_{\mathbb{H}}$.

Proof. As in proposition 4.2.5, this is a simple consequence of the $\mathrm{SL}_2(\mathbb{Z})$ equivariance of $\mathfrak{P}$ and of the action morphism $\mathfrak{P}\otimes p^*\mathcal{R}\to\mathfrak{P}$. □

We can now state the main result of this section.

Theorem II. *For any smooth family of elliptic curves $p: X\to S$, the functor*

$$\mathcal{F}_S\colon \begin{array}{lcl} \mathcal{R}-\mathrm{Mod} & \longrightarrow & \mathrm{NConn}(X/S;e) \\ \mathcal{V} & \longmapsto & \mathfrak{P}\underset{p^*\mathcal{R}}{\otimes} p^*\mathcal{V} \end{array} \tag{29}$$

is an equivalence of categories.

One may find the following reformulation in the relative tannakian terms of 4.1.2 to be more telling:

Corollary 4.2.8. *there is a well-defined and fixed fiber functor*

$$\omega_S\colon \mathrm{NConn}(X/S;e)\to \mathit{Conn}(S),$$

which makes $\mathrm{NConn}(X/S;e)$ *neutral relative to p^*, and such that:*

(i)) *the fundamental Hopf algebra at ω_S is $\mathcal{R}_S$.*
(ii)) *the relative fundamental torsor $P^{DR}((X\setminus e)/S,\omega_S)$ is $\mathfrak{P}_S$.*

As before, ω_S is the composite of an inverse of $\mathcal{F}_S$ and the forgetful functor $\mathcal{R}-\mathrm{Mod}\to\mathrm{Conn}(S)$.

4.2.9. *Explicit form.* The pullback condition for a local uniformisation (U,τ) provides directly an explicit description of $\mathfrak{P}$ on U. Since the formulas involving the function F can all be written in terms of q, things stay explicit under the milder assumption that $X\to S$ admits a Schottky uniformisation, i.e., a function $q: S\to D^*$, such that $X\to S$ is the pullback of $*\mathbb{C}^*\times S)/q^{\mathbb{Z}}$ by q, which amounts to the choice of a section of $R^1p_*(\mathbb{Z}_X)$.

Indeed, to a pair $(\mathcal{V},\mathbf{t})$, where $\mathcal{V}$ is a vector bundle on S and $\mathbf{t}$ is an endomorphism of $\mathcal{V}$, we can, as in 2.1, associate a nilpotent bundle $\mathcal{G}(\mathcal{V},\mathbf{t})$ on X:

$$\Gamma(U,\mathcal{G}(\mathcal{V},\mathbf{t})) := \{s\in\Gamma(\mathrm{pr}_2^{-1}U,\mathrm{pr}_2^*\mathcal{V}), s(qz)=\exp(-2\pi i\mathbf{t})s(z)\}, \tag{30}$$

where pr_2 is the projection to the second factor of $\mathbb{C}^*\times S$. More precisely, we have a functor

$$\mathcal{G}: (\text{vector bundles with nilpotent endomorphism on } S)\longrightarrow \mathrm{NBdl}(X/S),$$

As before, we'll refer to $\mathcal{G}(\mathcal{V},\mathbf{t})$ as *the bundle with multiplier* $\exp(-2\pi i\mathbf{t})$.

The fundamental torsor $\mathfrak{P}_S$ can then be described as the bundle with fiber R and multiplier $\exp(-2\pi i\mathbf{t})$, equipped with the connection whose pullback to $\mathbb{C}^*\times S$ is:

$$\nabla := d-\nu+\psi_{\mathbf{t}}\partial_{\mathbf{t}}+\psi_A\partial_A, \quad \text{where} \tag{31}$$

$$\nu := \mathrm{ad}_{\mathbf{t}}F(q,\mathrm{ad}_{\mathbf{t}};q)\frac{dz}{z}+\frac{1}{2\pi i}\left(\mathrm{ad}_{\mathbf{t}}F_2'(q,\mathrm{ad}_{\mathbf{t}};q)+\frac{1}{\mathrm{ad}_{\mathbf{t}}}\right)\frac{dq}{q}A, \tag{32}$$

and $\psi_{\mathbf{t}},\psi_A$ are as in definition 4.2.2.

4.2.10. *Reformulation for moduli stacks.* The theorem can be summarized in the following way:

There is a canonical fixed fiber functor ω on $\mathrm{NConn}((\mathscr{X}_{1,1}\setminus e)$, relative to $\mathrm{Conn}(\mathscr{M}_{1,1})$, where $\mathscr{X}_{1,1}$ is the universal family of elliptic curves.

The relative fundamental Hopf algebra $\mathsf{U}\pi_1^{\mathrm{DR}}((\mathscr{X}_{1,1}\setminus e)/\mathscr{M}_{1,1},\omega)$ is $\mathcal{R}$ and the fundamental torsor is $\mathfrak{P}$. Note also that $\mathscr{X}_{1,1}\setminus e$ can be identified with $\mathscr{M}_{1,2}$, the moduli stack of smooth curves of genus 1 with two marked points.

4.3. **Proof of theorem II.** We treat the local situation in 4.3.1 to 4.3.4. We explain how to glue those equivalences in 4.3.5, which is nothing but the well-known statement that isomorphicity for stacks can be checked locally.

4.3.1. *Essential surjectivity in the local case.* In this part, we work over a local uniformisation (U, τ), where U is contractible. We'll provide full details for the 1-dimensional case, i.e, essentially $U = \mathbb{H}$.

Let $(\mathfrak{V}, \nabla)$ be a object of $\mathrm{NConn}(X/U; e)$. According to a relative version of Atiyah theorem, the category of vector bundles on X, relatively nilpotent over U is equivalent to the category of modules over $\mathcal{O}_U[\![\mathbf{t}]\!]$, and we have a description of the underlying bundle of $\mathfrak{V}$ as a trivial bundle $\mathcal{V}$ on U with fiber V and multiplier $exp(\mathbf{t}(\tau))$ in the ξ direction.

By using the same formal inverting procedure as in 2.2.12, we get a section $A(\tau)$ such that each fiber of $\mathcal{F}_U(\mathcal{V}, \mathbf{t}, A)$ coincides with our $(\mathfrak{V}, \nabla)$. In other words, its pull-back to $\mathbb{C}^* \times U$ can be written as:

$$\nabla = d - \nu(\mathbf{t}, A) - \varphi(\xi, \tau) d\tau \tag{33}$$

Note that φ must be holomorphic, since the poles of ∇ have dlogξ form.

Proposition 4.3.2. *The expression $\varphi(\xi, \tau)$ does not depends in ξ. We have $\frac{d\mathbf{t}}{d\tau} - [\varphi, \mathbf{t}] = -\frac{A}{2\pi i}$.*

Proof. Writing down the quasiperiodicity with respect to $\xi \to \xi + \tau$ of ∇ and taking into account the computations from 3.2.1, we get:

$$\varphi(\xi + \tau, \tau) - \exp(-2\pi i \mathrm{ad}_{\mathbf{t}}) \varphi(\xi, \tau) = -\frac{\exp(-2\pi i \mathrm{ad}_{\mathbf{t}}) - 1}{-\mathrm{ad}_{\mathbf{t}}} \left(\frac{d}{d\tau} \mathbf{t} + \frac{A}{2\pi i} \right) \tag{34}$$

Denoting for a while by K the expression $\left(\frac{d\mathbf{t}}{d\tau} + \frac{A}{2\pi i}\right)$, let us observe that function $-(F(\xi, \mathrm{ad}_{\mathbf{t}}) - \frac{1}{\mathrm{ad}\,\mathbf{t}})K$ also satisfies to (34), so $\varphi(\xi, \tau) + (F(\xi, \mathrm{ad}_{\mathbf{t}}) - \frac{1}{\mathrm{ad}\,\mathbf{t}})K$ satisfies to the corresponding homogeneous equation. By 2.2.12, it can therefore be written as $\mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}}) C(\tau)$ for some $C(\tau)$. This gives us:

$$\varphi(\xi, \tau) = \mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}}) C(\tau) - (F(\xi, \mathrm{ad}_{\mathbf{t}}) - \frac{1}{\mathrm{ad}\,\mathbf{t}})K \tag{35}$$

Since φ is holomorphic, the singular part at $\xi = 0$ of the latter expression must vanish. This provides first $K = \mathrm{ad}_{\mathbf{t}} C(\tau)$, which is the expected expression for $dt/d\tau$, and (35) boils down to $\varphi(\xi, \tau) = C(\tau)$. □

Proposition 4.3.3. *The following formula holds:*

$$\frac{\partial A}{\partial \tau} = [\varphi(\tau), A] + \psi_A$$

Proof. We'll get this by asserting the flatness of ∇. The calculations are parallel to the proof of flatness of $\nabla_{\mathcal{P}}$ of 3.

The 1-dimension assumption gives us $d(\varphi(\tau) d\tau) = 0$. Let's compute the other terms.

$$\begin{aligned}
d\nu &= -\left(\frac{\partial \mathbf{t}}{\partial \tau} \frac{\partial}{\partial \mathbf{t}} + \frac{\partial A}{\partial \tau} \frac{\partial}{\partial A} \right) \nu \wedge d\tau \\
&= \left(\left(-\frac{A}{2\pi i} + [\varphi, \mathbf{t}] \right) \frac{\partial}{\partial \mathbf{t}} + \frac{\partial A}{\partial \tau} \frac{\partial}{\partial A} \right) \mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}}; \tau) A d\xi \wedge d\tau \\
&= \frac{(X+Y)F(\xi, X+Y; \tau) - Y F(\xi, Y; \tau)}{X} \left(\left(-\frac{A}{2\pi i} + [\varphi, \mathbf{t}] \right), A \right) d\xi \wedge d\tau \\
&\quad + \mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}}; \tau) \frac{\partial A}{\partial \tau} d\xi \wedge d\tau
\end{aligned}$$

The second summand, $\nu \wedge \nu$, equals, up to the factor $2^{-1}(2\pi i)^{-1} d\xi \wedge d\tau$,

$$\left(XYF(\xi, X+Y;\tau)\left(\wp(X) - \wp(Y)\right) + \frac{XF(\xi,X;\tau)}{Y} - \frac{YF(\xi,Y;\tau)}{X} \right) [\![A, A]\!]$$

The last term, $\nu \wedge (\varphi(\tau) d\tau) + (\varphi(\tau) d\tau) \wedge \nu$, is equal to

$$[\mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}};\tau)A, \varphi(\tau)] d\xi \wedge d\tau = XF(\xi, X;\tau)[\![A, \varphi(\tau)]\!] d\xi \wedge d\tau.$$

Finally, up to factor $\frac{1}{2}\frac{d\xi \wedge d\tau}{2\pi i}$, the curvature is equal to:

$$\begin{aligned} &\mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}};\tau)\left(\frac{XY}{X+Y}\left(\left(\wp(X) - \frac{1}{X^2}\right) - \left(\wp(Y) - \frac{1}{Y^2}\right)\right)[\![A, A]\!] \right. \\ &\qquad \left. +4\pi i \left(\frac{\partial A}{\partial \tau} - [\![\varphi(\tau), A]\!] \right)\right) \\ = \quad &\mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}};\tau)\left(4\pi i \psi_A + 4\pi i \left(\frac{\partial A}{\partial \tau} - [\![\varphi(\tau), A]\!] \right)\right). \end{aligned}$$

By invertibility of $\mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}};\tau)$, the vanishing of this curvature thus yields the wished formula. □

4.3.4. *Conclusion for the local case.* Let's consider on $\mathcal{V}$ the connection:

$$\nabla_{\mathcal{V}} := d - \varphi(\tau) d\tau$$

The operators $\mathbf{t}(\tau)$ and $A(\tau)$ give the $\mathcal{R}$-algebra structure on $\mathcal{V}$ and propositions 4.3.2 and 4.3.3 state its horizontality. Applying $\mathcal{F}_U$, we obtain the original $\mathfrak{V}$, thanks to formula (33).

In the higher dimensional case, there are more factors to take into account in the computations above. It turns out that they express nothing more than the flatness of $\nabla_{\mathcal{V}}$. We'll spare the reader those details.

As for fully faithfullness, the proof in 2.2.14 carries over transparently.

4.3.5. *General bases.* According to the previous paragraph, there is an open covering $(U_i)_{i\in I}$ of S such that each $\mathcal{F}_{U_i}$ is an equivalence. Then one can construct an essential inverse u to $\mathcal{F}$ in the following way:

Let us choose an essential inverse u_i of each $\mathcal{F}_{U_i}$. The functors $u_i|U_i \cap U_j$ and $u_j|U_i \cap U_j$ are both inverses of $\mathcal{F}_{U_i \cap U_j}$ and therefore canonically isomorphic through φ_{ij}. The canonicity implies in particular that the descent condition $\varphi_{ik} = \varphi_{ij}\varphi_{jk}$ holds for any triple $\{i, j, k\}$ For any object $\mathfrak{V}$ of $\mathrm{NConn}(X/S; e)$, the various $u_i(\mathfrak{V})$ therefore define an object $u(\mathfrak{V})$ of $\mathcal{R}-\mathrm{Mod}$, and it goes in the same way for morphims, which can be defined locally. It's easy to see that u and $\mathcal{F}$ are inverse of each others, again because arrows can be defined locally in both categories. □

4.4. **The Fundamental Hopf algebras for geometrical fiber functors.** We now consider a fixed family $X \to S$ and a section σ of $X \setminus e(S) \to S$. The pullback σ^* is a fiber functor of $\mathrm{NConn}(X/S; e)$ over $\mathrm{Conn}(S)$, fixed by the canonical isomorphism $\sigma^* p^* \simeq \mathrm{Id}$. In this subsection, we want to identify the corresponding fundamental Hopf algebra in the target category $\mathrm{Conn}(S)$ at σ^*.

Since the base S is fixed, we will silently drop it from notations.

4.4.1. It is also worthwhile to consider the following generalization of such pullbacks, known as tangential base points. Let z be a 1-germ of transversal coordinate at the section e. Then it defines a fiber functor as follows: let ∇ be a flat connection on X with simple pole along e; denote by K its residue at e. Then the local connection $\nabla - Kdz/z$ has no singularities and it is simple to check that $e_z^*(\nabla) := e^*(\nabla - Kdz/z)$ is a flat connection on S. Note that this functor depends on the choice of 1-germ of z. A more canonical version would involve the relative normal bundle of e in X.

4.4.2. *Twistings.* As was earlier mentioned, a continuation of the fiberwise functors $\mathcal{F}_s\colon R-\mathrm{Mod} \to \mathrm{NConn}(X_s;e), s \in S$ to the relative functor $\mathcal{F}_S\colon \mathcal{R}-\mathrm{Mod} \to \mathrm{NConn}(X/S;e)$ is not unique. Indeed, let μ be a $\mathcal{R}$-valued differential 1-form on S that satisfies the Maurer-Cartan equation:

$$\nabla_{\mathcal{R}}\mu + \mu \wedge \mu = 0 \tag{MC}$$

Then we can define a new connection $\nabla^{\mu}_{\mathcal{R}} := \nabla_{\mathcal{R}} + \mathrm{ad}_{\mu}$ on the S-bundle $\mathcal{R}$. It is easy to check that (MC) yields the flatness of $\nabla^{\mu}_{\mathcal{R}}$. We'll denote by $\mathcal{R}^{\mu}$ the bundle $\mathcal{R}$ equipped with this connection. Further, if μ takes values in primitive elements of $\mathcal{R}$, then $\mathcal{R}^{\mu}$ is again a Hopf algebra in $\mathrm{Conn}(S)$,

Analogously, one can also define a new flat connection on $\mathfrak{P}$ as $\nabla^{\mu}_{\mathfrak{P}} := \nabla_{\mathfrak{P}} - p^*\mu$, where μ acts from the right. This bundle with connection will be denoted by $\mathfrak{P}^{\mu}$. It is a $p^*\mathfrak{R}^{\mu}$ right module and induces a functor

$$\mathcal{F}^{\mu}_S\colon \mathcal{R}^{\mu}-\mathrm{Mod} \to \mathrm{NConn}(X/S;e)$$

by the same rule (29) that $\mathfrak{P}_{X/S}$ induces $\mathcal{F}_S$.

4.4.3. *Twisted equivalence.* Note that the categories $\mathcal{R}-\mathrm{Mod}$ and $\mathcal{R}^{\mu}-\mathrm{Mod}$ are equivalent under the functor $F_{0,\mu}$ sending $(\mathcal{V}, \nabla_{\mathcal{V}})$ to $(\mathcal{V}, \nabla_{\mathcal{V}} + \mu)$. In Morita form, this functor is given by the bimodule $\mathcal{P}_{0,\mu}$ of $\mathcal{R}^{\mu}-\mathrm{Mod}-\mathcal{R}$, which is defined as the bundle $\mathcal{R}$ with connection $\nabla_{\mathcal{R}} + l_{\mu}$ (left multiplication).

Evidently, $\mathcal{F}$ is isomorphic to $\mathcal{F}^{\mu} \circ F_{0,\mu}$, so $\mathcal{F}^{\mu}_S$ is an equivalence of categories. This defines a new fiber functor $\omega^{\mu}_{\mathrm{DR}}$, again fixed by $\sigma^* p^* \simeq \mathrm{Id}$. We can think of $\mathcal{P}_{0,\mu}$ as the torsor of paths from ω_{DR} to $\omega^{\mu}_{\mathrm{DR}}$. More generally, the torsor of paths from $\omega^{\mu}_{\mathrm{DR}}$ to $\omega^{\mu'}_{\mathrm{DR}}$ would be the bimodule $\mathcal{P}_{\mu,\mu'}$ in $\mathcal{R}^{\mu'}-\mathrm{Mod}-\mathcal{R}^{\mu}$, whose connection is $\nabla_{\mathcal{R}} + l_{\mu'} - r_{\mu}$ (left and right multiplications).

4.4.4. *Equivalence at σ.* We now turn to the composition $\sigma^* \circ \mathcal{F}^{\mu}_S$. By the very definition, we get that it maps $(\mathcal{V}, \nabla_{\mathcal{V}})$ to $(\mathcal{V}, \nabla_{\mathcal{V}} + \nu_{\sigma} - \mu)$, where ν_{σ} is defined as $\nabla_{\mathcal{V}} - p^*\nabla_{\mathfrak{P}}$. More precisely, this is true up to $\sigma^* p^* \simeq \mathrm{Id}$.

In the local $S = \mathbb{H}$ situation, this ν_{σ} is nothing but $\sigma^*\nu$, with ν as in (19). This motivates the notation and shows that ν_{σ} is actually independent of $\nabla_{\mathcal{V}}$.

For $\mu = \nu_{\sigma}$, we obtain that the fixing isomorphism $\sigma^* p^* \simeq \mathrm{Id}$ induces a canonical isomorphism from this composition to the forgetful functor from $\mathcal{R}-\mathrm{Mod}^{\mu}_S$ to $\mathrm{Conn}(S)$. Together with the fact that $\mathcal{F}^{\mu}_S$ is an equivalence of categories, this allows us to conclude that $\mathcal{R}^{\nu_{\sigma}}_S$ is the fundamental Hopf algebra at the fiber functor σ^*, fixed by $\sigma^* p^* \simeq Id$. In the same way, for two sections σ_1 and σ_2, the bimodule $\mathcal{P}_{\mu_1,\mu_2}$ with $\mu_i = \nu_{\sigma_i}$ is the torsor of paths from σ_1 to σ_2.

4.4.5. *A differential equation for parallel transports.* Since the Riemann-Hilbert correspondence commutes with geometric functors like p^*, σ^* and respects natural isomorphisms as $\sigma^* p^* \simeq \mathrm{Id}$ we conclude from the previous paragraph that the parallel transport Φ with respect to the connection $\mathfrak{P}_{X/S}$ along a S-family of paths from σ_1 to σ_2 satisfies the equation

$$\nabla\Phi + \nu_{\sigma_1}\Phi - \Phi\nu_{\sigma_2} = 0.$$

Let us stress that for $\mathbb{H} = S$, it takes the following very explicit form:

$$\nabla\Phi + \sigma_1^*(\nu)\Phi - \Phi\sigma_2^*(\nu) = 0.$$

This would remain true in the case of tangential base points.

5. Algebraic $\mathbb{Q}$-structure

In this section, we exhibit the natural algebraic $\mathbb{Q}$-structures of the previously constructed fundamental groups and torsors. By "natural", we mean here with respect to the fact [8, 10.41] that the algebraic De Rham fundamental group(oid) (defined by means of algebraic connections over the base field) commutes with the extension of scalars to $\mathbb{C}$. We should stress that the nilpotency assumptions we've made all along are crucial for this to be true [8, 10.35].

5.1. **Over a single curve.** We first treat the algebraic $\mathbb{Q}$-structures on $\mathcal{P}$ and R for a single elliptic curve defined over $\mathbb{Q}$.

5.1.1. *Reminders.* Let us remind some standard facts about algebraic elliptic curves, mostly to fix terminology and notations.

Let (X, O) be an elliptic curve defined over some field k of characteristic zero. Then it can be represented as a plane cubic $y^2 = 4x^3 - g_2x - g_3$, with $g_2, g_3 \in k$ and the marked point O lies at infinity. This form is unique up to dilatations $g_2 \to \lambda^4 g_2, \quad g_3 \to \lambda^6 g_3, \quad x \to \lambda^2 x, \quad y \to \lambda^3 y$, for $\lambda \in k$.

If k is a subfield of $\mathbb{C}$, e.g., $k = \mathbb{Q}$, one can represent the analytic curve $X(\mathbb{C})$ as the quotient of $\mathbb{C}$ by some lattice Λ. The map from $X(\mathbb{C})$ to $\mathbb{C}/\Lambda$ is defined by integration of the form $\omega = dx/y$ from the marked point to the variable one $p \to \int_O^p \omega$. The ambiguity in choosing of the path of integration belongs to the topological first homology group $H_1(X(\mathbb{C}), \mathbb{Z})$ which corresponds to the lattice Λ.

Choose some basis (u, v) of Λ in such a way that $\Im(v/u) > 0$, and put $\tau = v/u$. Then we have $g_2 = 60u^{-4}e_4(\tau)$ and $g_3 = 140u^{-4}e_6(\tau)$. Moreover, the map

$$\xi \to (x(\xi) = u^{-2}(E_2(\xi, \tau) - e_2(\tau)), \quad y(\xi) = -2u^{-3}E_3(\xi, \tau))$$

produces an analytic isomorphism between $X_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ and $X(\mathbb{C})$ and we have $\omega = dx/y = ud\xi$. So, u can be retrieved as the period (elliptic integral) of the algebraic form ω against a topological chain. If we choose another equation of X, twisted, say, by λ, then ω becomes $\lambda^{-1}\omega$ and $(u, v) \to (\lambda^{-1}u, \lambda^{-1}v)$.

5.1.2. *Some analytic preparations.* Note that over the complement U_τ of the marked point in X_τ, the bundle $\mathcal{P}$ with multiplier $\exp(-2\pi i\mathbf{t})$ of section 2 is trivialized by the left multiplication by $g(\xi) = \exp(-E_1(\xi, \tau)\mathbf{t})$.

Indeed, sections of $\mathcal{P}$ are the quasiperiodic functions f on $\mathbb{C}$: $f(\xi + 1) = f(\xi); \quad f(\xi + \tau) = \exp(-2\pi i\mathbf{t})f(\xi)$. For each such, $g(\xi)f$ is elliptic, since we have $E_1(\xi + 1, \tau) = E_1(\xi, \tau)$ and $E_1(\xi + \tau, \tau) = E_1(\xi, \tau) - 2\pi i$.

5.1.3. The connection $\nabla = d - \mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}})A d\xi$ transforms under the gauge transformation $s \to g(\xi)s$ into the operator

$$\nabla_{\mathrm{alg}} := g(\xi)\nabla g^{-1}(\xi) = d - dg(\xi)g^{-1}(\xi) - g(\xi)\mathrm{ad}_{\mathbf{t}} F(\xi, \mathrm{ad}_{\mathbf{t}})Ag^{-1}d\xi,$$

which we expand in terms of Eisenstein series using (13):

$$= d - E_2(\xi, \tau)\mathbf{t}d\xi - \exp(-E_1(\xi, \tau)\mathrm{ad}_{\mathbf{t}}) \exp\left(-\sum_{k=1}^{\infty} \frac{(-\mathrm{ad}_{\mathbf{t}})^k}{k}(E_k(\xi, \tau) - e_k(\tau))\right) A\, d\xi$$

$$= d - \left((E_2(\xi, \tau) - e_2(\tau))\mathbf{t} + \exp\left(-\sum_{k=2}^{\infty} \frac{(-\mathrm{ad}_{\mathbf{t}})^k}{k}(E_k(\xi, \tau) - e_k(\tau))\right)(A + e_2(\tau)\mathbf{t})\right) d\xi.$$

5.1.4. *The $\mathbb{Q}$-structure.* Let us introduce the following new generators of the ring R:

$$\mathbf{t}_{\rm alg} = u\mathbf{t}, \qquad \mathbf{s}_{\rm alg} = u^{-1}(A + e_2(\tau)\mathbf{t})$$

In terms of these, $\nabla_{\rm alg}$ takes the form $d - K(\xi,\tau)ud\xi$, where $K(\xi,\tau)$ is:

$$u^{-1}(E_2(\xi,\tau) - e_2(\tau))\mathbf{t}_{\rm alg} + \exp\left(-\sum_{k=2}^{\infty}\frac{(-\mathrm{ad}_{\mathbf{t}_{\rm alg}})^k}{k}u^{-k}(E_k(\xi,\tau) - e_k(\tau))\right)\mathbf{s}_{\rm alg}$$

As the $e_{2k}(\tau)$, $k \geq 4$ are polynomials in $e_4(\tau)$ and $e_6(\tau)$ with rational coefficients, the $u^{-2k}e_{2k}(\tau)$ are polynomials in $60u^{-4}e_4(\tau) = g_2$ and $140u^{-6}e_6(\tau) = g_3$ with rational coefficients.

As the $E_k(\xi,\tau)$ are polynomials in $E_2(\xi,\tau) - e_2(\tau)$, $E_3(\xi,\tau)$, $e_4(\tau)$ and $e_6(\tau)$ with rational coefficients, the $u^{-k}E_k(\xi,\tau)$ are polynomials in $u^{-2}(E_2(\xi,\tau) - e_2(\tau) = x$, $-2u^{-3}E_3(\xi,\tau) = y$, g_2 and g_3 with rational coefficients.

So, we see that for generators $\mathbf{t}_{\rm alg}$ and $\mathbf{s}_{\rm alg}$, the transformed connection $\nabla_{\rm alg}$ is algebraic over $\mathbb{Q}$; hence the ring $R_{alg} = \mathbb{Q}\langle\langle \mathbf{t}_{\rm alg}, \mathbf{s}_{\rm alg}\rangle\rangle$ is the natural algebraic $\mathbb{Q}$-structure on R. Note that this rational structure does not depend in the choice of the equation of the elliptic curve, as a change of the equation multiplies u by some *rational* number λ^{-1}.

5.2. **Relative case.** We start by a short account in our notations of the realisation of the analytic stacks $\mathbb{H}/SL_2(\mathbb{Z})$ and $\mathbb{X}/SL_2(\mathbb{Z})$ as stacks of $\mathbb{C}$-points of some algebraic stacks. All the results mentionned there are very classical and belong to the mathematics of 19th century. We then apply the same gauge transformation as for single elliptic curves.

5.2.1. Consider the product $\mathbb{H}\times\mathbb{C}^*$ and denote the coordinate on $\mathbb{C}^*$ by u. Define the action of $SL_2(\mathbb{Z})$ by the following formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (\tau, u) \to \left(\frac{a\tau + b}{c\tau + b}, (c\tau + d)u\right).$$

The map $(\tau, u) \to (g_2 = 60u^{-4}e_4(\tau), \quad g_3 = 140u^{-4}e_6(\tau))$ is $SL_2(\mathbb{Z})$-invariant and provides an isomorphism of the quotient and the complement $\mathcal{B}_{\mathbb{C}}$ of the curve $\Delta := g_2^3 - 27g_3^2 = 0$ in $\mathbb{C}^2$. As $\mathbb{H}/SL_2(\mathbb{Z})$ is the quotient of $(\mathbb{H}\times\mathbb{C}^*)/SL_2(\mathbb{Z})$ by $\mathbb{C}^*$ acting on the second factor, this quotient stack is equal to $\mathbb{C}$-points of the quotient stack of $\mathcal{B} = \mathbb{A}^2\setminus\{\Delta = 0\}$ by the following action of $\mathbb{G}_m$: $(g_2, g_3) \to (\lambda^{-4}g_2, \lambda^{-6}g_3)$. Evidently, this algebraic stack is defined over $\mathbb{Q}$.

In the same way, we can consider the quotient $(\mathbb{X}\times\mathbb{C}^*)/SL_2(\mathbb{Z})$ with the same action of the group $SL_2(\mathbb{Z})$ on the second factor. Furthermore, the map $(\tau, \xi, u) \to (g_2 = 60u^{-4}e_4(\tau), \quad g_3 = 140u^{-4}e_6(\tau)x = u^{-2}(E_2(\xi,\tau) - e_2(\tau)), y = -2u^{-3}E_3(\xi,\tau), z = 1)$ is an isomorphism of this quotient with the complex projective cubic $\mathcal{Q}_{\mathbb{C}} = \{y^2z = 4x^3 - g_2xz^2 - g_3z^3\}$ over $\mathcal{B}_{\mathbb{C}}$.

We get that $\mathbb{X}/SL_2(\mathbb{Z})$ is the stack of $\mathbb{C}$-points of the quotient of the cubic $\mathcal{Q} = \{y^2z = 4x^3 - g_2xz^2 - g_3z^3\}$ over $\mathcal{B}$ by the action of $\mathbb{G}_m$ defined as follows:

$$(g_2, g_3, x, y) \xrightarrow{\lambda} (\lambda^{-4}g_2, \lambda^{-6}g_3, \lambda^{-2}x, \lambda^{-3}y)$$

The variety $\mathcal{Q}$ and its $\mathbb{G}_{\rm m}$-action on it are defined over $\mathbb{Q}$. The complement of the neutral section corresponds to the affine curve $\mathcal{Q}^{\rm aff} = \{y^2 = 4x^3 - g_2x - g_3\}$.

5.2.2. *Analytic preparation.* As in the case of individual curve we apply the gauge transformation of the left multiplication by $g(\xi,\tau) = \exp(-E_1(\xi,\tau)\mathbf{t})$. This transforms the connection ∇ on $\mathfrak{P}$ in

$$\begin{aligned}\nabla_{\rm alg} &= d - dg(\xi,\tau)g^{-1}(\xi,\tau) + g(\xi,\tau)\nu(\xi,\tau)g^{-1}(\xi,\tau) \\ &\quad - g(\xi,\tau)\psi_{\mathbf{t}}\partial_{\mathbf{t}}g^{-1}(\xi,\tau) - g(\xi,\tau)\psi_A\partial_A g^{-1}(\xi,\tau)\end{aligned}$$

$$= d - \left(E_2 d\xi - \frac{1}{2\pi i}(E_3 - E_1E_2)d\tau\right)\mathbf{t} - \left(\exp\left(-\sum_{k=2}^{\infty}\frac{(-\mathrm{ad}_{\mathbf{t}})^k}{k}(E_k - e_k)\right)d\xi\right.$$
$$+\left(\left(-\frac{1}{\mathrm{ad}_{\mathbf{t}}} + \sum_{k=1}^{\infty}(-\mathrm{ad}_{\mathbf{t}})^{k-1}(E_k - e_k)\right)\exp\left(-\sum_{k=2}^{\infty}\frac{(-\mathrm{ad}_{\mathbf{t}})^k}{k}(E_k - e_k)\right)\right.$$
$$\left.\left.+\frac{\exp(-E_1\mathrm{ad}_{\mathbf{t}})}{\mathrm{ad}_{\mathbf{t}}}\right)\frac{d\tau}{2\pi i}\right)A + \left(\psi_{\mathbf{t}}\partial_{\mathbf{t}} - \frac{1}{2\pi i}\frac{\exp(-E_1\mathrm{ad}_{\mathbf{t}}) - 1}{-\mathrm{ad}_{\mathbf{t}}}Ad\tau\right) + \psi_A\partial_A$$
$$= d - \left(E_2\mathbf{t} + \exp\left(-\sum_{k=2}^{\infty}\frac{(-\mathrm{ad}_{\mathbf{t}})^k}{k}(E_k - e_k)\right)A\right)\left(d\xi + \frac{1}{2\pi i}E_1 d\tau\right) + E_3\frac{\mathbf{t}d\tau}{2\pi i}$$
$$-\left(\left(-\frac{1}{\mathrm{ad}_{\mathbf{t}}} + \sum_{k=2}^{\infty}(-\mathrm{ad}_{\mathbf{t}})^{k-1}(E_k - e_k)\right)\exp\left(-\sum_{k=2}^{\infty}\frac{(-\mathrm{ad}_{\mathrm{asymp}})^k}{k}(E_k - e_k)\right) + \frac{1}{\mathrm{ad}_{\mathbf{t}}}\right)\frac{Ad\tau}{2\pi i}$$
$$+\psi_{\mathbf{t}}\partial_{\mathbf{t}} + \psi_A\partial_A$$
$$= d + \nu_{\mathrm{alg}} + \psi_{\mathbf{t}}\partial_{\mathbf{t}} + \psi_A\partial_A.$$

In this expansion, we used the equality $2\pi i\partial E_1/\partial\tau = E_3 - E_1E_2$.

If we replace generators $\mathbf{t}, A$ by $\mathbf{t}_{\mathrm{alg}} = u\mathbf{t}, \mathbf{s}_{\mathrm{alg}} = u^{-1}(B + e_2\mathbf{t})$, the coefficients of the differential form ν_{alg} become polynomials in $u^{-2}(E_2 - e_2) = x$, $-2u^{-3}E_3 = y$, $60u^{-4}e_4 = g_2$ and $140u^{-6}e_6 = g_3$. What remains to be done for ν_{alg} is summarized as the following proposition.

Proposition 5.2.3. *The differential forms*

$$\frac{u^2 d\tau}{2\pi i}, \quad \left(e_2\frac{d\tau}{2\pi i} + \frac{du}{u}\right), \quad \text{and} \quad u^{-1}\left(d\xi + \frac{1}{2\pi i}E_1 d\tau\right)$$

can be expressed as rational polynomials in terms of x, y, z, g_2 and g_3.

Proof. Note that $d\Delta = -12\Delta(e_2(2\pi i)^{-1}d\tau + u^{-1}du)$, so $e_2(2\pi i)^{-1}d\tau + u^{-1}du$ is the algebraic differential form $\kappa := 12^{-1}\Delta^{-1}d\Delta$. We can treat the operator $d + j\kappa$ as a $\mathbb{G}_m$-equivariant connection on the trivial bundle on $\mathcal{B}$, equipped with the action $\lambda \to \lambda^{-j}$ of $\mathbb{G}_m$. This explains the coefficient of κ in formulas below.

From $2\pi i de_4 = (14e_6 - 4e_2e_4)d\tau$ follows $dg_2 = 6g_3(2\pi i)^{-1}u^2d\tau - 4\kappa g_2$, hence $(2\pi i)^{-1}u^2 d\tau$ is algebraic.

As $2\pi i\partial e_2/\partial\tau = (5e_4 - e_2^2)$ and $2\pi i\partial E_2/\partial\tau = 3((E_2 - e_2)^2 - 5e_4) - E_2^2 - 2E_1E_3$, we have

$$dx = y\left(u(d\xi + \frac{1}{2\pi i}E_1 d\tau)\right) + \left(2x^2 - \frac{1}{3}g_2\right)\frac{u^2 d\tau}{2\pi i} - 2x\kappa,$$

□

5.2.4. As we perform the change of generators $(\mathbf{t}, A) \to (\mathbf{t}_{\mathrm{alg}}, \mathbf{s}_{\mathrm{alg}})$ the differential term $d + \psi_{\mathbf{t}}\partial_{\mathbf{t}} + \psi_A\partial_A$ transforms. The generator $\mathbf{s}_{\mathrm{alg}} = u^{-1}(A + e_2\mathbf{t})$ depends directly in τ, and both $\mathbf{t}_{\mathrm{alg}} = u\mathbf{t}$ and $\mathbf{s}_{\mathrm{alg}}$ depend in u, so the differentiation d transforms into $d + \mathbf{t}_{\mathrm{alg}}\partial_{\mathbf{t}_{\mathrm{alg}}}u^{-1}du + \mathbf{t}_{\mathrm{alg}}\partial_{\mathbf{s}_{\mathrm{alg}}}u^{-2}de_2 - \mathbf{s}_{\mathrm{alg}}\partial_{\mathbf{s}_{\mathrm{alg}}}u^{-1}du$; furthermore $\psi_{\mathbf{t}}\partial_{\mathbf{t}} = -\frac{Ad\tau}{2\pi i}\partial_{\mathbf{t}}$ transforms into

$$\left(u^{-1}e_2\mathbf{t}_{\mathrm{alg}} - u\mathbf{s}_{\mathrm{alg}}\right)\frac{d\tau}{2\pi i}\left(u\partial_{\mathbf{t}_{\mathrm{alg}}} + u^{-1}e_2\partial_{\mathbf{s}_{\mathrm{alg}}}\right).$$

As $de_2 = (2\pi i)^{-1}(5e_4 - e_2^2)d\tau$, the term $d + \psi_{\mathrm{asymp}}\partial_{\mathrm{asymp}}$ becomes:

$$d + \left(\kappa\mathbf{t}_{\mathrm{alg}} + \frac{u^2 d\tau}{2\pi i}\mathbf{s}_{\mathrm{alg}}\right)\partial_{\mathbf{t}_{\mathrm{alg}}} + \left(\frac{1}{12}g_2\frac{u^2 d\tau}{2\pi i}\mathbf{t}_{\mathrm{alg}} - \kappa\mathbf{s}_{\mathrm{alg}}\right)\partial_{\mathbf{s}_{\mathrm{alg}}};$$

which is $\mathbb{Q}$-algebraic.

Finally, the summand $\psi_A\partial_A$ transforms into

$$\psi_{\mathbf{s}_{\mathrm{alg}}}\partial_{\mathbf{s}_{\mathrm{alg}}} = -\frac{1}{2}u\frac{u^{-2}XY}{u^{-1}X+u^{-1}Y}$$
$$\times\left(\left(\wp(u^{-1}X)-\frac{1}{(u^{-1}X)^2}\right)-\left(\wp(u^{-1}Y)-\frac{1}{(u^{-1}Y)^2}\right)\right)[\![\mathbf{s}_{\mathrm{alg}},\mathbf{s}_{\mathrm{alg}}]\!]_{\mathbf{t}_{\mathrm{alg}}}\frac{d\tau}{2\pi i}\partial_{\mathbf{s}_{\mathrm{alg}}},$$

and the coefficient of each term $[\mathrm{ad}^i_{\mathbf{t}_{\mathrm{alg}}}\mathbf{s}_{\mathrm{alg}},\mathrm{ad}^j_{\mathbf{t}_{\mathrm{alg}}}\mathbf{s}_{\mathrm{alg}}]\partial_{\mathbf{s}_{\mathrm{alg}}}$ in this expression can be calculated in terms of the Taylor expansion at zero of $\wp(X)-X^2$. This coefficient is equal to $(-1)^{j-1}(i+j)(2\pi i)^{-1}u^{i+j+1}e_{i+j+1}u^2d\tau$, and is therefore a polynomials in g_2 and g_3 multiplied by the algebraic form $(2\pi i)^{-1}u^2d\tau$.

So we constructed $\mathbb{Q}$-algebraic connections $(\mathcal{R}_{\mathrm{alg}},\nabla_{\mathcal{R}_{\mathrm{alg}}})$ on $\mathcal{B}$ and $(\mathfrak{P}_{\mathrm{alg}},\nabla_{\mathfrak{P}\mathrm{alg}})$ on $\mathcal{Q}^{\mathrm{aff}}$ which are equal to the analytic ones after tensoring by $\mathbb{C}$. These algebraic connections are moreover $\mathbb{G}_m$-equivariant.

5.2.5. *Pull-back to a family.* Let $X\to S$ be a smooth algebraic family of elliptic curves defined over $\mathbb{Q}$. Then we have a well defined map from the family $X(\mathbb{C})\to B(\mathbb{C})$ of $\mathbb{C}$-points to $\mathbb{X}/SL_2(\mathbb{Z})$, and this map determines the analytic connections $\mathcal{R}_S$ and $\mathfrak{P}_{X/S}$, on which we shall now define an algebraic $\mathbb{Q}$-structure.

Denote the generic point of S by η. The elliptic curve X_η. Since η is the spectrum of the field $\mathbb{Q}(\eta)$, the elliptic curve X_η over η is isomorphic to a plane cubic $y^2=4x^3-g_2x-g_3$, whose equation is unique up to dilatations $g_2\to\lambda^4g_2,\quad g_3\to\lambda^6g_3,\quad x\to\lambda^2x,\quad y\to\lambda^3y$, with $\lambda\in\mathbb{Q}(\eta)$.

Fix some choice of g_2 and g_3, let $\tilde{S}$ be the open subscheme of S determined by inequalities $g_2\neq\infty$, $g_3\neq\infty$ and $\Delta\neq 0$, and consider the induced family $\tilde{X}\to\tilde{S}$. We have a map $g=(g_2,g_3)$ from $\tilde{S}$ to $\mathcal{B}$ and $\tilde{X}$ is induced by g.

The connections $\mathcal{R}_{\tilde{S}}$ and $\mathfrak{P}_{\tilde{X}/\tilde{S}}$ are equal to the pull-back of the corresponding connections on $\mathcal{Q}\to\mathcal{B}$, so the pull-back of $\mathcal{R}_{\mathrm{alg}}$ and $\mathfrak{P}_{\mathrm{alg}}$ provide a $\mathbb{Q}$-algebraic structure on them. Note that $g^*\mathcal{R}_{alg}$ and $g^*\mathfrak{P}_{alg}$ can be continued as regular algebraic connections to S and X respectivly, as after tensoring by $\mathbb{C}$ they become regular analytic connections.

These algebraic $\mathbb{Q}$-structures don't depend in the choice of g_2 and g_3 as they can be changed by dilated one and $\mathcal{R}_{\mathcal{B}}$ and $\mathfrak{P}_{\mathcal{Q}/\mathcal{B}}$ are $\mathbb{G}_m$-equivariant.

5.2.6. *Twistings.* Note that if X/S be an algebraic family over $\mathbb{Q}$ and μ is $\mathbb{Q}$-algebraic $\mathcal{R}_{alg}$-valued 1-form, then $\mathcal{R}^\mu$ and $\mathfrak{P}^\mu$ can be equipped by an algebraic $\mathbb{Q}$-structure. Hence, for a σ, defined over $\mathbb{Q}$, the fundamental Hopf algebra at σ has also a natural algebraic $\mathbb{Q}$-structure.

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