Laurent Deformations of Algebraic Surfaces with an Isolated Singularity

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Abstract

We develop a theory of *Laurent deformations* of an algebraic surface X with an isolated singularity P. The deformation tensor of a Laurent deformation is a series of Laurent type on a desingularization Y of X with a singularity at the fundamental cycle Z. This series gives a deformation of the complex structure on the domain in X, which is the complement of a small neighborhood of P. It will be shown that formal families of Laurent deformations of X constitute an infinite dimensional space with a stratification by finite dimensional subspaces. Various properties of Laurent deformation are shown to be deeply related to algebro-geometric properties of (X, P).

1 Introduction

It is known from the Kodaira-Spencer theory that a deformation of a smooth compact complex surface S is described by a deformation tensor $\phi \in A^{0,1}(TS)$ satisfying the integrability condition

$$\bar{\partial}\phi - \frac{1}{2}[\phi,\, \phi] = 0. \qquad (1.1)$$

Now let us allow ϕ to have a singularity at a curve C on S. Roughly, we mean that ϕ is a series of Laurent type with a singularity at C satisfying (1.1). Then ϕ defines a deformation of the complement U of a tubular neighborhood of C in S, provided that U is the convergent domain of the series ϕ. In this paper, we apply this idea to construct formal deformations of the complement of a normal isolated singularity in an algebraic surface.

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To be precise, let us take an algebraic surafce X whose singular locus is a normal isolated singularity P. We assume that X is embedded in $\mathbf{P}^N$. Then the intersection of X with a small ball with the center P in $\mathbf{P}^N$ gives a small neighborhood V of P in X. As usual, by a deformation of P we mean a deformation of V. The deformation theory of a normal isolated singularity was established long ago by Tjurina [14] and Grauert [5], and there is a great number of works on this topic. By contrast, little attention has been paid so far to the structure of the space of deformations of the complement U of V in X.

In this paper, we develop a deformation theory of U based on the idea stated above. We will show that the deformation space of U has a rich structure which reflects algebro-geometric properties of the surface X as well as the singularity P.

Since U is a complex surface with a boundary, there is an uninteresting class of deformations of U, namely those induced by small deformations of the domain U in X. We refer to these deformations as *modifications of the domain* U. In this paper we introduce an interesting class of deformations of U, called *Laurent deformations* of X, which excludes modifications of U.

A Laurent deformation is given by a deformation tensor with a singularity. Namely, it is a series of Laurent type on a desingularization Y of X with a singularity at the fundamental cycle Z, which satisfies a certain integrability condition (Definition 6.1).

We will show that formal families of Laurent deformations of X constitute an infinite dimensional space with a natural stratification by finite dimensional subspaces (Proposition 6.1). This stratification reflects algebro-geometric properties of both X and P. If P is *of negative type* (see Definition 3.1), then this stratification is easy to understand (Corollary 4.1). Moreover, in §5 it is shown that, on the infinitesimal level, Laurent deformations change the complex structure of any curve C_m linearly equivalent to mH, for all sufficiently large m, where H is a hyperplane section of X which does not pass through the singularity P.

We remark that Laurent deformations cannot be generalized to the case $\dim_{\mathbf{C}} X > 2$, due to the existence of certain obstructions (see Remark 6.1). We also point out that Laurent deformations are closely related to another subject, namely CR-structures on 3-manifolds. We elaborate on this connection at the end of §6.

In the present paper we establish the *formal* deformation theory of Laurent deformations. Of course, it is important to investigate the domain of convergence of these deformations and we hope to discuss this problem in the near future.

This paper is organized as follows: In §2 we construct a filtration $\mathcal{F} = \{\mathcal{F}_n\}$ on $H^1(U, TU)$. The definition of an isolated singularity of negative type will be given in §3. Subsequently, in §3 and §4, we shall show the injectivity of some homomorphisms.

These results are related to the structure of the filtration $\mathcal{F}$. In §5 we investigate an important property of infinitesimal Laurent deformations. Finally, in §6 we construct a family of formal Laurent deformations

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2 Infinitesimal Laurent deformations

Suppose X is an algebraic variety of $\dim_{\mathbf{C}} X \geq 2$ with a normal isolated singularity P. We also assume that X does not have singularities other than P. In this section we shall define infinitesimal Laurent deformations of X, together with their filtrations.

We start with a resolution $f : Y \to X$ such that $A = f^{-1}(P)$ consists of divisors A_i $(i = 1, \cdots, k)$ with normal crossing. Let us take an effective divisor $Z = \sum_{i=1}^{k} a_i A_i$ $(a_i > 0)$. Let $U := X \setminus V$, with V a small closed neighborhood of P in X. Note that $U \cong Y \setminus f^{-1}(V)$. Now we consider the homomorphism $s_n : H^1(Y, TY(nZ)) \to H^1(U, TU)$ induced by restriction. Hence $TY(nZ) := TY \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(nZ)$ where TY is the holomorphic tangent bundle of Y and $\mathcal{O}_Y(nZ)$ is the line bundle defined by a divisor nZ.

We shall define a filtration $\mathcal{F} = \mathcal{F}_Z$ on the infinitesimal deformation space $H^1(U, TU)$ of U:

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset H^1(U, TU)$$

by setting $\mathcal{F}_n := \mathrm{Im}\{s_n : H^1(Y, TY(nZ)) \to H^1(U, TU)\}$.

An element of $\mathcal{F}_n$ is called an *infinitesimal Laurent deformations* of X *with a pole of order* n at Z. In §6 we shall construct the space F_n of *formal Laurent deformations* of X *with an initial pole of order* n at Z, in such a way that $\mathcal{F}_n$ is the tangent space of F_n.

Although infinitesimal Laurent deformations are defined regardless of $\dim_{\mathbf{C}} X$, we will see in §6 that they are obstructed and cannot be integrated to formal Laurent deformations if $\dim_{\mathbf{C}} X > 2$. Thus, we restrict our attention to the surface case.

Suppose X is an algebraic surface whose singular locus is a normal isolated singularity P. Recall that a resolution $f : Y \to X$ is called *good* if the following condition is satisfied:

(1) any A_i is non-singular.

(2) If $A_i \cap A_j \neq$, then A_i intersects A_j transversally.
(3) $A_i \cap A_j \cap A_k =$ for distinct i, j, k.

It is well-known that there exists a unique good resolution which is minimal. Hereafter we suppose that $f : Y \to X$ is the minimal good resolution of X.

In general, it is difficult to know $\dim \mathcal{F}_n$ explicitly, except for some special cases (see [13]). We will be able to estimate $\dim \mathcal{F}_n$, under the assumption that the self-intersection number Z^2 is negative. Later we shall take the fundamental cycle (see next section) as Z, in which case the condition $Z^2 < 0$ is satisfied.

For the remainder of this section, we suppose $Z^2 < 0$. We begin with the estimate of $h^1(Y, TY(nZ))$. Letting $V = TY(nZ)$, $V|_{nZ} := V \otimes_{\mathcal{O}_X} \mathcal{O}_{nZ}$ is a locally free $\mathcal{O}_{nZ}$-sheaf of rank 2. The Riemann-Roch theorem for $V|_{nZ}$ reads ([3] p.51)

$$h^0(V|_{nZ}) - h^1(V|_{nZ}) = \deg(V|_{nZ}) + 2\chi(\mathcal{O}_{nZ}),$$

where $h^i(V|_{nZ}) = \dim H^i(V|_{nZ})$.

Since $\det(V|_{nZ}) = -K_Y + 2nZ|_{nZ}$, we have $\deg(V|_{nZ}) = n(-K_Y \cdot Z + 2nZ^2)$. On the other hand, $\chi(\mathcal{O}_{nZ}) = 1 - g(nZ)$. Hence $g(nZ)$ is computed as ([3] p.68)

$$\begin{aligned} g(nZ) &= 1 + \tfrac{1}{2}\deg((K_Y + nZ)|_{nZ}) \\ &= 1 + \tfrac{1}{2}(nK_Y \cdot Z + n^2 Z^2). \end{aligned}$$

Hence $2\chi(\mathcal{O}_{nZ}) = -nK_Y \cdot Z - n^2 Z^2$ and

$$h^0(V|_{nZ}) - h^1(V|_{nZ}) = -2nK_Y \cdot Z + n^2 Z^2.$$

It follows from $Z^2 < 0$ that $h^1(TY(nZ)|_{nZ})$ is at least quadratic in n, for all sufficiently large n. Now we can show

Lemma 2.1 *$h^1(Y, TY(nZ))$ is at least quadratic in n for all sufficiently large n.*

Proof. From the exact sequence

$$0 \longrightarrow TY \longrightarrow TY(nZ) \longrightarrow TY(nZ)|_{nZ} \longrightarrow 0$$

we obtain

$$\cdots \longrightarrow H^1(Y, TY(nZ)) \xrightarrow{\alpha} H^1(TY(nZ)|_{nZ}) \xrightarrow{\beta} H^2(Y, TY) \longrightarrow \cdots$$

Then $h^1(TY(nZ)|_{nZ}) = \dim \mathrm{Im}\alpha + \dim \mathrm{Im}\beta$. Notice that the dimension of $\mathrm{Im}\beta$ is bounded by a constant $h^2(Y, TY)$. We have seen that $h^1(TY(nZ)|_{nZ})$ is at least quadratic in n for all sufficiently large n. Therefore $\dim \mathrm{Im}\alpha$ must also grow at least quadratically with n. In particular, the same is true for $h^1(Y, TY(nZ))$. ∎

Since X is algebraic we can take a hyperplane section H of X so that H does not meet a small neighborhood V of P in X. By Bertini's theorem, we may assume that H is connected and smooth. We also take a curve C_m, linearly equivalent to f^*mH, which does not meet $f^{-1}(V)$. Now we shall show

Lemma 2.2 *$h^1(TY(nZ)|_{C_m})$ grows at least quadratically with m, for all sufficiently large m.*

Proof. By the Riemann-Roch formula,

$$h^0(TY|_{C_m}) - h^1(TY|_{C_m}) = \deg(TY|_{C_m}) + 2\chi(\mathcal{O}_{C_m}). \qquad (2.1)$$

We laso have

$$\deg(TY|_{C_m}) = -K_Y \cdot mf^*H$$
$$2\chi(\mathcal{O}_{C_m}) = -mK_Y \cdot f^*H - m^2 f^*H^2$$

Thus the left hand side of (2.1) is $-2mK_Y \cdot f^*H - m^2 f^*H^2$. It follows from $H^2 > 0$ that $h^1(TY|_{C_m})$ is at least quadratic in n, for all sufficiently large n. Notice that $TY(nZ)|_{C_m} = TY|_{C_m}$ because the support of nZ is away from C_m. This completes the proof. ∎

Recall that $s_n : H^1(X, TX(nC)) \to H^1(U, TU)$ is the homomorphism induced by restriction. We will verify that the image $\mathcal{F}_n$ of s_n is non-trivial (indeed, it is quite large). When P is of negative type (Definition 3.1), this is not so difficult (see remark below Theorem 3.1). However, we need a special argument for the general case.

The following exact sequence is basic in the subsequent discussion:

$$\begin{array}{ccccc} \longrightarrow & H^1(Y, TY(nZ - mf^*H)) & \longrightarrow & H^1(Y, TY(nZ)) & \xrightarrow{r_n} \\ & H^1(Y, TY(nZ)|_{C_m}) & \longrightarrow & H^2(Y, TY(nZ - mf^*H)) & \longrightarrow . \end{array} \qquad (2.1)$$

This sequence is induced by

$$0 \longrightarrow TY(nZ - mf^*H) \longrightarrow TY(nZ) \longrightarrow TY(nZ)|_{C_m} \longrightarrow 0.$$

Lemma 2.3 *Let $m > 0$ be fixed. Then $H^2(Y, TY(nZ - mf^*H)) = 0$ for all sufficiently large n.*

Proof. By the Serre duality, we have

$$H^2(Y, TY(nZ - mf^*H)) \cong H^0(Y, \Omega^2 \otimes \Omega^1 \otimes (mf^*H - nZ)).$$

Notice that the right hand side is the set of holomorphic sections of the holomorphic vector bundle $\Omega^2 \otimes \Omega^1 \otimes (mf^*H)$ whose zeros have order at least n at Z. The finiteness of $h^0(\Omega^2 \otimes \Omega^1 \otimes (mf^*H))$ then implies that the right hand side must be zero for all sufficiently large n. ∎

Applying Lemma 2.3 to the exact sequence (2.1), we obtain

Lemma 2.4 *Let $m > 0$ be fixed. Then $r_n : H^1(Y, TY(nZ)) \to H^1(TY(nZ)|_{C_m})$ is surjective for all sufficiently large n.*

Now we can prove the main result of this section.

Theorem 2.1 *The image $\mathcal{F}_n$ of $s_n : H^1(Y, TY(nZ)) \to H^1(U, TU)$ is non-zero for all sufficiently large n. Moreover the dimension of $\mathcal{F}_n$ goes to infinity as $n \to \infty$.*

Proof. We fix a sufficiently large m and write r_n as a composite of homomorphisms induced by restriction:

$$r_n : H^1(Y, TY(nZ)) \xrightarrow{s_n} H^1(U, TU) \longrightarrow H^1(TY(nZ))|_{C_m}).$$

Hence $r_n : H^1(Y, TY(nZ)) \to H^1(TY(nZ)|_{C_m})$ is surjective for all sufficiently large n by Lemma 2.4, and $h^1(TY(nZ))|_{C_m})$ is at least quadratic in m by Lemma 2.2. Therefore the dimension of $\mathcal{F}_n$ is at least quadratic in m.

Moreover, if we choose a larger m at the beginning of the proof, and apply the same argument, the later part of the claim follows. ∎

Note $h^1(TY(nZ)|_{C_m})$ is at least quadratic in m (Lemma 2.2) and $h^1(Y, TY(nZ))$ is at least quadratic in n (Lemma 2.1). Observe that since m was already fixed, there is no contradiction in the proof of Theorem 2.1.

3 Singularity of negative type

Let X be a compact complex surface with a normal isolated singularity P. As before, we assume that X does not have singularities other than P. Suppose $f : Y \to X$ is the minimal good resolution of P and $A := f^{-1}(P)$ is the exceptional set consisting of irreducible curves $\{A_i\}$ on Y. It is known that there exists a unique effective divisor $Z = \sum a_i A_i \leq 0$ such that $Z \cdot A_i$ is for every curve A_i and, furthermore, Z is minimal with this property. We call Z the *fundamental cycle* of A (see [1]). In particular, the fundamental cycle Z satisfies $Z^2 = \sum\limits_i a_i Z \cdot A_i < 0$. We need the following

Definition 3.1 An isolated singularity P is said to be of *negative type* (or it is said to satisfy the *negativity condition*) provided that $Z \cdot A_i < 0$ for each A_i.

Of course, when the exceptional set has only one component, this condition is automatically satisfied. However, it should be remarked that a singularity which has low embedding dimension does not in general satisfy the negativity condition.

In fact, let us consider the weighted dual graph associated to a normal isolated singularity P. To each component of this graph there corresponds to a component of the exceptional set. The weight of a graph component is the self-intersection number of the corresponding component of the exceptional set.

If the weight of any graph component is high enough, then the weighted dual graph satisfies the negativity condition. On the other hand, if the weights are not too high, then in general there are many component A_i satisfying $Z \cdot A_i = 0$. For a singularity whose embedding dimension is low (e.g. a hypersurface singularity), the weighted dual graph usually contains component A_i satisfying $Z \cdot A_i = 0$.

For the remainder of this paper, Z always denotes the fundamental cycle of the desingularization. Now we assume that X is algebraic and H is a hyperplane section of X which does not pass through P. The reason why we choose the fundamental cycle among effective divisors with support on A is as follows. We want to take r_n in (2.1) as a homomorphism "close to being injective" though it is not injective in general. Alternatively, we want to choose Z in such a way that $H^1(Y, TY(nZ - mf^*H))$ is very small. Roughly speaking, from the Nakai-Moishezon ampleness criterion, when Z is the fundamental cycle, $mf^*H - nZ$ is the closest possible to being an ample divisor for sufficiently large m, among effective divisors with support on A (cf. Lemma 3.1 below).

A consequence of negativity is following;

Lemma 3.1 *If P is of negative type then $m(f^*H) - Z$ is ample for all sufficiently large m.*

Proof. Since H does not pass through P, we have

$$(f^*mH - Z) \cdot A_i = -Z \cdot A_i.$$

The right hand side is positive by our assumption. For a curve distinct from $\{A_i\}$, writing $f^*f(Z) = C + kZ$, we have

$$f^*(Z) \cdot f^*mH = C \cdot f^*mH + kZ \cdot f^*mH = C \cdot f^*mH$$

Now, $(f^*mH - Z) \cdot C = f^*mH \cdot C - Z \cdot C$, so that for all sufficiently large m the right hand side is positive. We may take curves $C_1, \cdots, C_k$ as generators of $\mathrm{Pic}(Y)$ in such a way that $mf^*H \cdot C_i > 0$ for each i and all sufficiently large m. Applying the Nakai-Moishezon ampleness criterion, the assertion follows. ∎

As a consequence of the above and the Kodaira vanishing theorem, we have

Lemma 3.2 *Suppose P is of negative type. We fix sufficiently large m such that $mf^*H - Z$ is ample. Then $H^1(Y, TY(nZ - nmf^*H)) = 0$ for all sufficiently large n.*

Next, we need

Lemma 3.3 *Suppose P is of negative type. Let m be so large that $mf^*H - Z$ is ample. Then the homomorphism $r_n : H^1(Y, TY(nZ)) \to H^1(TY(nZ))|_{C_{nm}})$ induced by restriction is injective for all sufficiently large n.*

Proof. Applying Lemma 3.2 to the exact sequence (2.1), the assertion follows. ∎

Now we can show

Theorem 3.1 *Assume X is algebraic. If P is of negative type then the homomorphism $s_n : H^1(Y, TY(nZ)) \to H^1(U, TU)$ induced by restriction is injective for all sufficiently large n.*

Proof. We note that the homomorphism r_n factorizes the composite of the restriction homomorphisms:

$$r_n : H^1(Y, TY(nZ - nmf^*H)) \xrightarrow{s_n} H^1(U, TU) \longrightarrow H^1(TY(nZ))|_{C_m}).$$

The injectivity of s_n follows from that of r_n. ∎

Together with Lemma 2.2, the above theorem shows that $\dim \mathcal{F}_n$ is at least quadratic in n for all sufficiently large n.

4 More on injectivity

For the work in this section, X does not need to be algebraic, only a compact complex surface with a normal isolated singularity at P. As before, we assume that X have no singularities other than P. Now we consider

$$H^0(TY((n+1)Z)|_Z) \longrightarrow H^1(Y, TY(nZ)) \xrightarrow{t_n} H^1(Y, TY((n+1)Z))$$

induced by

$$0 \longrightarrow TY(nZ) \longrightarrow TY((n+1)Z) \longrightarrow TY((n+1)Z)|_Z \longrightarrow 0.$$

In this section we proceed to investigate the kernel of the homomorphism

$$t_n : H^1(Y, TY(nZ)) \to H^1(Y, TY((n+1)Z)).$$

In general, t_n is not injective, but we have:

Theorem 4.1 *If P is of negative type, then $H^0(TY((n+1)Z)|_Z)$ vanishes for all sufficiently large n. In particular, t_n is injective.*

If X happen to be algebraic, then the above results, together with Theorem 3.1, allow us to describe the structure of the filtration $\mathcal{F}$ by means of the groups $H^1(Y, TY(nZ))$.

Corollary 4.1 *Suppose X is algebraic and P is of negative type. For sufficiently large n, s_n induces an isomorphism between the filtrations*

$$\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}_{n+2} \subset \cdots$$

and

$$H^1(Y, TY(nZ)) \subset H^1(Y, TY((n+1)Z) \subset H^1(Y, TY((n+2)Z)) \subset \cdots,$$

where inclusion is given by the injective homomorphisms t_n.

In order to prove Theorem 4.1 we need some lemmas. Recall that if $D = D' + D''$, where D', D'' are effective divisors, then we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_{D''}(-D') \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{D'} \longrightarrow 0.$$

Lemma 4.1 *Suppose E is a holomorphic vector bundle on Y. If C is a curve on Y with $Z \cdot C < 0$ and a is a positive integer, then $H^0(E((n+1)Z)|_{aC}) = 0$ for all sufficiently large n.*

Proof. By induction on a. If $a = 1$, the negativity of the line bundle $(n+1)Z$ restricted to C implies the desired result. If $a > 1$, we consider the exact sequence of cohomology

$$H^0(E((n+1)Z - C)|_{(a-1)C}) \longrightarrow H^0(E((n+1)Z)|_{aC}) \longrightarrow H^0(E((n+1)Z)|_C)$$

induced by

$$0 \longrightarrow E((n+1)Z - C)|_{(a-1)C} \longrightarrow E((n+1)Z)|_{aC} \longrightarrow E((n+1)Z)|_C \longrightarrow 0.$$

Note that $H^0(E((n+1)Z)|_C) = 0$ for all sufficiently large n (the case of $a = 1$). By the induction hypothesis applied to $E(-C)$, we have $H^0(E((n+1)Z - C)|_{(a-1)C}) = 0$ for all sufficiently large n. Thus our claim follows. ∎

Let $Z_r = \sum_i^r a_i A_i$ $(1 \leq r \leq k)$ and note that $Z = Z_k$.

Lemma 4.2 *Suppose E is a holomorphic vector bundle on Y. If P is of negative type, then $H^0(E((n+1)Z)|_{Z_r}) = 0$.*

Proof. By induction on r. If $r = 1$, this assertion is nothing other than Lemma 4.1, since $Z \cdot A_1 < 0$. Now let $r > 1$. From

$$0 \longrightarrow E((n+1)Z - a_r A_r)|_{Z_{r-1}} \longrightarrow E((n+1)Z)|_{Z_r} \longrightarrow E((n+1)Z)|_{a_r A_r} \longrightarrow 0,$$

we obtain

$$H^0(E((n+1)Z - a_r A_r)|_{Z_{r-1}}) \longrightarrow H^0(E((n+1)Z)|_{Z_r}) \longrightarrow H^0(E((n+1)Z)|_{a_r A_r}).$$

Applying the induction hypothesis to $E(-a_r A_r)$, we get $H^0(E((n+1)Z - a_r A_r)|_{Z_{r-1}}) = 0$. On the other hand, $H^0(E((n+1)Z)|_{a_r A_r}) = 0$ follows from the first step of induction hypothesis. Thus our claim is verified. ∎

In the particular case $E = TY$, we have $H^0(TY((n+1)Z)|_Z) = 0$ for all sufficiently large n. Thus

Theorem 4.1 The homomorphism $t_n : H^1(Y, TY(nZ)) \to H^1(Y, TY((n+1)Z))$ *is injective provided that* P *is of negative type and* n *is sufficiently large.*

It should be remarked that we assumed X to be algebraic in the proof of Theorem 3.1, while we did not in the proof of Theorem 4.1.

5 Properties of infinitesimal Laurent deformations

Suppose that X is an algebraic surface whose singular locus is a normal isolated singularity P. Let $U = X \setminus V$ as before. Obviously there is an uninteresting class of deformations of U. A small deformation of the domain U induces a deformation of the complex structure on U. This deformation is called a *modification of the domain* U. Notice that a modification of U changes the complex structure only near the boundary ∂U, while keeping unchanged the complex structure away from ∂U.

We shall show that any infinitesimal Laurent deformation is not a modification of U. Let H be a hyperplane section of X which does not pass through P. Then we shall verify that any non-trivial infinitesimal Laurent deformations preserves the curves C_m, linearly equivalent to mf^*H, on Y for sufficiently large m, while it induces a non-trivial infinitesimal deformation of them. It follows that any Laurent deformation is not a modification of U.

Let $TU(\log C_m)$ be the sheaf of infinitesimal automorphisms of U which send C_m to itself. Then the cohomology group $H^1(U, TU(\log C_m))$ is identified with the set of infinitesimal logarithmic deformations (Kawamata [7]). Recall that a *logarithmic deformation* of (U, C_m) is a deformation of U which preserves the marked divisor C_m.

The following short sequences are exact.

1. $0 \to TU(-C_m) \to TU(\log C_m) \to TC_m \to 0$

2. $0 \to TU(\log C_m) \to TU \to N_{C_m} \to 0$,

where N_{C_m} is the normal bundle of C_m.

Now we consider the exact sequence of cohomology

$$H^1(U, TU(\log C_m)) \xrightarrow{\psi} H^1(U, TU) \longrightarrow H^1(C_m, N_{C_m}) \qquad (5.1)$$

induced by the above sequence 2, where ψ is the forgetting map of marking C_m.

Since the normal bundle N_{C_m} is very ample for all sufficiently large m, we have $H^1(C_m, N_{C_m}) = 0$. Thus any infinitesimal deformation of U preserves the curve C_m as a submanifold (the Stability Theorem of Kodaira [8]).

For any nonzero $\alpha \in \mathcal{F}_n$, we shall show that α induces a non-trivial deformation of the complex structure on C_m infinitesimally.

Suppose that $\tilde{\alpha} \in H^1(U, TU(\log C_m))$ is a lift of α, i.e. $\psi(\tilde{\alpha}) = \alpha$ (ψ as in (5.1)). We consider the commutative diagram

$$\begin{array}{ccc} H^1(U, TU(\log C_m)) & \xrightarrow{\psi} & H^1(U, TU) \\ \downarrow \phi & & \downarrow \text{restriction} \\ H^1(C_m, TC_m) & \longrightarrow & H^1(TU|_{C_m}) \end{array}$$

where ϕ and the bottom arrow are respectively induced by the exact sequence 1 and a short exact sequence

$$0 \longrightarrow TC_m \longrightarrow TU|_{C_m} \longrightarrow N_{C_m} \longrightarrow 0.$$

The homomorphism ϕ associates each logarithmic deformation to the induced deformation of the marked divisor on the infinitesimal level. From the proof of Theorem 2.1, we see that the restriction $\alpha|_{C_m} \in H^1(TU|_{C_m})$ is nonzero. Therefore $\phi(\tilde{\alpha})$ must be nonzero by the commutativity of the diagram above. Consequently $\phi(\tilde{\alpha})$ induces a non-trivial deformation of C_m infinitesimally. We summarize

Theorem 5.1 *Any nonzero $\alpha \in \mathcal{F}_n$ induces a non-trivial deformation of the complex structure on curves C_m, for all sufficiently large m, on the infinitesimal level.*

Alternatively, if an infinitesimal Laurent deformation preserves an m-th order infinitesimal neighborhood of f^*H for any sufficiently large m, then the deformation must be trivial. This property is very similar to the formal principle property dicussed in [10] (see Remark 5.1).

We can of course move H to another hyperplane section H'. As long as H' is away from P, Theorem 5.1 holds for $C'_m := mf^*H'$. Namely, any Laurent deformation changes m-th order infinitesimal neighborhood of the inverse image of any hyperplane section for any sufficiently large m.

Remark 5.1 Suppose that A is a compact complex subspace in a complex space X. We say that the *formal principle* holds for (X, A) if the following condition is satisfied: given any embedding of A in a complex space Y, if the formal neighborhood of A in X is isomorphic to the formal neighborhood of A in Y, then there exist neighborhoods U of A in X and V of A in Y and a biholomorphic map $f : U \to V$ which induces the isomorphism of formal neighborhoods.

In our case, the role of formal neighborhood is played by neighborhoods of high enough order.

6 Formal Laurent deformations

In this section we shall construct formal Laurent deformations of X (Definition 6.1). Namely, we will construct deformation tensors with a singularity, which are given by series of Laurent type on Y with a singularity at the fundamental cycle Z.

For the construction of formal Laurent deformations, it does not matter whether X is algebraic or not. It suffices to assume X is a compact complex surface whose singular locus is a normal isolated singularity P. Let Y be the minimal good resolution of X, and Z its fundamental cycle.

The next lemma is the key to constructing formal Laurent deformations.

Lemma 6.1 *There exists $a > 0$ such that $H^2(Y, TY(nZ)) = 0$ for any $n \geq a$.*

Proof. By the Serre duality,

$$H^2(Y, TY(nZ)) \cong H^0(Y, \Omega^2 \otimes \Omega^1(-nZ)).$$

The right hand side is the space of the holomorphic sections of $\Omega^2 \otimes \Omega^1$ with a zero of order at least n at Z. From the finiteness of $h^0(Y, \Omega^2 \otimes \Omega^1)$, we conclude that the right hand side equals 0, for all sufficiently large n. ∎

As we will see below, the vanishing of $H^2(Y, TY(nZ))$ enables us to produce formal solutions of a certain differential equation. Thus the elements of $H^2(Y, TY(nZ))$ can be thought of as obstructions.

Fix a holomorphic section $s \in H^0(Y, \mathcal{O}_Y(Z))$ with the divisor $(s) = Z$. Note that s is uniquely determined up to constant.

In §2, we constructed a filtration $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset H^1(U, TU)$ with $\mathcal{F}_n = \operatorname{Im}\{s_n : H^1(Y, TY(nZ)) \to H^1(U, TU)\}$. We shall now construct formal

Laurent deformations whose tangent space is $\mathcal{F}_n$ for $n \geq a$, where a is as in Lemma 6.1.

To be precise, let $\omega \in H^{0,1}_{\bar\partial}(TY(nZ))$ and $\phi(t) = \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k + \cdots$ with $\alpha_k s^{n_k} \in A^{0,1}(TY(n_k Z))$ for some n_k and $\omega = \alpha_1 s^n$. Note that $H^1(Y, TY(nZ))$ is isomorphic to a Dolbeault cohomology $H^{0,1}_{\bar\partial}(TY(nZ))$.

Let us consider the following differential equation:

$$\bar\partial\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] = 0 \qquad (6.1)$$

where the bracket [,] is defined in a standard manner from the Lie bracket of vector fields with a singularity at Z, which are of the form β/s^k for some $k > 0$ and $\beta \in \Gamma(Y, TY(kZ))$.

Definition 6.1 $\phi(t)$ is called a *formal Laurent deformation* of X *with an initial pole of order* n at Z provided it satisfies (6.1).

It should be remarked that $\phi(t)$ is not a Laurent series on Y, because $\phi(t)$ is not analytic. However, it resembles a Laurent series, and we name it a formal *Laurent* deformation. Notice that $\bar\partial - \phi(t)\partial$ does not define a complex structure on Y since $\phi(t)$ has a singularity at Z. But $\bar\partial - \phi(t)\partial$ does give an actual deformation of its domain of convergence because (6.1) is nothing other than the integrability condition of the complex structure $\bar\partial - \phi(t)\partial$ there.

We also note that $n = n_1 \leq n_2 \leq \cdots \leq n_k \leq \cdots$, provided that $\phi(t)$ satisfies (6.1).

Theorem 6.1 *The differential equation (6.1) has a formal solution $\phi(t)$ provided that $n \geq n_0$.*

Proof. We shall construct a formal solution of the differential equation (6.1) in an inductive manner. First suppose we have constructed a formal solution $\phi_k(t) = \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k$ up to order k that satisfies

$$\bar\partial\phi_k(t) \equiv \frac{1}{2}[\phi_{k-1}(t), \phi_{k-1}(t)] \mod t^{k+1}, \qquad (6.2)$$

where this notation means polynomials in t coincide up to order k. We need to find α_{k+1} satisfying

$$\bar\partial\phi_{k+1}(t) \equiv \frac{1}{2}[\phi_k(t), \phi_k(t)] \mod t^{k+2}.$$

This is equivalent to

$$\bar\partial\alpha_{k+1}t^{k+1} \equiv \frac{1}{2}[\phi_k(t), \phi_k(t)] - \bar\partial\phi_k(t) \mod t^{k+2}. \qquad (6.3)$$

By the induction hypothesis, the right hand side does not contain any term which is of order lower than t^{k+1}. Writing

$$\frac{1}{2}[\phi_k(t), \phi_k(t)] - \bar{\partial}\phi_k(t) = b_{k+1}t^{k+1} + b_{k+2}t^{k+2} + \cdots, \tag{6.4}$$

the equation above is equivalent to

$$\bar{\partial}\alpha_{k+1} = b_{k+1}. \tag{6.5}$$

Since s is holomorphic, $\bar{\partial}(s^{n_{k+1}}\alpha_{k+1}) = \bar{\partial}s^{n_{k+1}} \cdot \alpha_{k+1} + s^{n_{k+1}} \cdot \bar{\partial}\alpha_{k+1} = s^{n_{k+1}} \cdot \bar{\partial}\alpha_{k+1}$. Multiplying the equation (6.5) by $s^{n_{k+1}}$, we have $\bar{\partial}(s^{n_{k+1}}\alpha_{k+1}) = s^{n_{k+1}}b_{k+1}$. We rewrite this as

$$\bar{\partial}\beta_{k+1} = c_{k+1}, \tag{6.6}$$

where $\beta_{k+1} := s^{n_{k+1}} \cdot \alpha_{k+1} \in A^{0,1}(TY(n_{k+1}Z))$ and $c_{k+1} := s^{n_{k+1}}b_{k+1} \in A^{0,2}(TY(n_{k+1}Z))$.

Once we show $\bar{\partial}c_{k+1} = 0$, it follows from $H^2(Y, TY(n_{k+1}Z)) = 0$ that we can find β_{k+1} with $\bar{\partial}\beta_{k+1} = c_{k+1}$. Consequently, we can get $\alpha_{k+1} := \beta_{k+1}/s^{n_{k+1}}$.

Thus, it suffices to show that $\bar{\partial}c_{k+1} = 0$. By (6.4), this is equivalent to

$$\bar{\partial}\frac{1}{2}[\phi_k(t), \phi_k(t)] \equiv 0 \mod t^{k+2}. \tag{6.7}$$

Notice that

$$\bar{\partial}\frac{1}{2}[\phi_k(t), \phi_k(t)] = [\bar{\partial}\phi_k(t), \phi_k(t)]. \tag{6.8}$$

Substitute (6.2) for $\bar{\partial}\phi_k(t)$ in (6.8). Then, it follows from (6.7) that we have

$$\bar{\partial}[\phi_k(t), \phi_k(t)] \equiv [[\phi_{k-1}(t), \phi_{k-1}(t)], \phi_k(t)] \equiv 0 \mod t^{k+2}.$$

Therefore we can find α_{k+1}. Clearly we can continue this process to obtain the desired formal solution. ∎

Remark 6.1 We made essential use of Lemma 6.1 for the construction of formal Laurent deformations as above. However Lemma 6.1 does not hold if $\dim_{\mathbf{C}}Y > 2$. Therefore Laurent deformations cannot be generalized to the case $\dim_{\mathbf{C}}X > 2$.

Recall that if X is algebraic, the image $\mathcal{F}_n$ of $s_n : H^1(Y, TY(nZ)) \to H^1(U, TU)$ is non-zero provided $n \geq b$ for some b (Theorem 2.1). Therefore if $n \geq \max(a, b)$, the formal solution $\phi(t)$ of (6.1) with non-zero $s_n(\omega) \in \mathcal{F}_n$ gives a family of non-trivial formal deformations of U. By Theorem 2.1, $\dim\mathcal{F}_n$ goes to infinity as $n \to \infty$. We summarize our main results,

Proposition 6.1 *Suppose X is an algebraic surface whose singular locus is a normal isolated singularity P. Then the formal families of Laurent deformations of X constitute an infinite dimensional space* $\mathbf{L}$. *Moreover,* $\mathbf{L}$ *has a natural stratification by finite dimensional subspaces F_n, and $\mathcal{F}_n$ is the tangent space of F_n at the distinguished point corresponding to the trivial deformation.*

On the other hand, for non-algebraic X we know little about the filtration $\{\mathcal{F}_n\}$. In principle, it may happen that $\mathcal{F}_n = 0$ for all n, in which case any Laurent deformation is trivial.

Laurent deformations are related to another subject, namely CR-structures on 3-manifolds. Let V be a small neighborhood of a normal isolated singularity P. Then the boundary M of V has a strongly pseudoconvex CR-structure induced from V. It is known that if $\dim_{\mathbf{C}} V = 2$, then the deformation space of the CR-structure on M is an infinite dimensional space, while it is finite dimensional in case $\dim_{\mathbf{C}} V > 2$ (see e.g. [4]).

From the work of M. Artin [2], one knows that any normal isolated singularity is equivalent to an algebraic one, that is, it is defined by a system of polynomials. Thus given a normal isolated singularity P, there exists an algebraic surface X with P as a unique singularity. To see this, observe that the defining polynomials of P give an algebraic surface with the isolated singularity P. Resolving the singularities on this surface other than P, we obtain the desired algebraic surface X.

Finally, we give some remark on the connection of Laurent deformations with CR-structures on a 3-manifold. Laurent deformations of X induce deformations of the CR-structure on the boundary M of U, where U is the convergent domain of Laurent deformations. It seems likely that the deformation space of CR-structures on M contains an infinite dimensional subspace, which consists of deformations induced by Laurent deformations.

On the other hand, if $\dim_{\mathbf{C}} V > 2$, there are no Laurent deformations of X. This fact may have connection with the finiteness of the dimension of the deformation space of CR-structures on a manifold of real dimension at least 5.

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