# ON MODULI OF POINTED REAL CURVES OF GENUS ZERO I

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Abstract. We introduce the moduli space of pointed real curves of genus 0 $\overline{M}^{\mathbb{R}}_{(2k,l)}$ and give its natural stratification. The strata of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ correspond to real curves with different degeneration types and may be encoded by trees with certain decorations. By using this stratification, we calculate the first Steifel-Whitney class of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ and construct the orientation double cover $\widetilde{M}_{(2k,l)}$ of $\overline{M}^{\mathbb{R}}_{(2k,l)}$.

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*Date*: October 24, 2005.

The author is partially supported by the Israeli Science Foundation grant 86/01 and by the Collège Doctoral Européen.

## 1. Introduction

The moduli space $\overline{M}_n$ of stable $n$-pointed (complex) curves of genus 0 has been extensively studied as one of the fundamental models of moduli problems in algebraic geometry (see [Ke, Knu, KM1, KM2, M1]). The moduli space $\overline{M}_n$ carries a set of antiholomorphic involutions. The fixed point sets of these involutions are the moduli spaces of pointed real curves of genus 0. These moduli spaces parameterize the isomorphism classes of pointed curves of genus 0 with a real structure. For each of these spaces, a different set of labelled points stay in the real parts of the curves and the different pairs of labelled points are conjugated by the real structures of the curves. The moduli spaces of pointed real curves have recently attracted attention in various contexts such as multiple $\zeta$-motives [GM], representations of quantum groups [EHKR, HK] and Welschinger invariants [W1, W2].

The aim of this note is to explore the topological properties of the moduli spaces of pointed real curves of genus 0. Hence, we first introduce a natural combinatorial stratification of the moduli spaces of pointed real curves of genus 0 through the stratification of $\overline{M}_n$. Each stratum is determined by the degeneration type of the real curve. They are identified with spaces of real point configurations in the projective line. The degeneration types of the pointed real curves are encoded by trees with corresponding decorations. Secondly, we calculate the first Steifel-Whitney classes of the moduli spaces in terms of their stratifications. The moduli spaces of pointed real curves are not orientable for $n \geq 5$. We construct the orientation double covers of the moduli spaces for the non-orientable cases. The double covering which is constructed in this work significantly differs from the 'double covering' in the recent literature on open Gromov-Witten invariants and moduli spaces of pseudoholomorphic discs (see [FOOO, Liu]): Our double covering has no boundaries.

Section 2 contains a brief overview of facts about the moduli space $\overline{M}_n$ and its intersection theory. In Section 3, we introduce real structures on $\overline{M}_n$ and real parts $\overline{M}^{\mathbb{R}}_{(2k,l)}$ as moduli spaces of $(2k,l)$-pointed real curves. The following section, the stratification of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ is given according to the degeneration types of pointed real curves of genus 0. In Section 5, the first Steifel-Whitney class of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ is calculated by using the stratification given in Section 4. Then in Section 6, we construct the orientation double coverings $\widetilde{M}_{(2k,l)} \to \overline{M}^{\mathbb{R}}_{(2k,l)}$.

**Acknowledgment.** I take this opportunity to express my deep gratitude to my supervisors V. Kharlamov and M. Polyak for their assistance and guidance in this work. I also wish to thank to Yu. I. Manin for his interest and suggestions which have been invaluable for this work.

I am thankful to E. Ha, A. Mellit, D. Radnell and A. Wand for their comments and suggestions.

Thanks are also due to Max-Planck-Institut für Mathematik, Israel Institute of Technology and Institut de Recherhe Mathématique Avancée de Strasbourg for their hospitality.

## 2. Preliminaries: Pointed Complex Curves and Their Moduli

This section reviews the basic facts on pointed complex curves of genus 0 and their moduli space. Moduli spaces of genus 0 curves with labelled points are stratified according to degeneration types of curves. The group of holomorphic automorphisms $Aut_\sharp(\overline{M}_n)$ respecting the stratification is discussed.

In this paper, the genus of the curves is 0 except when the contrary is stated explicitly. Therefore, we omit mentioning the genus of the curves.

### 2.1. **Pointed Curves and Their Trees.**

**Definition 2.1.** An *$n$-pointed curve* $(\Sigma;\mathbf{p})$ is a connected complex algebraic curve $\Sigma$ with distinct, smooth, *labelled points* $\mathbf{p}=(p_1,\cdots,p_n)\subset\Sigma$, satisfying the following conditions:

- $\Sigma$ has only nodal singularities.
- The arithmetic genus of $\Sigma$ is equal to 0.

A *family of $n$-pointed curves* over a complex manifold $S$ is a proper, holomorphic map $\pi_S:\mathcal{U}_S\to S$ with $n$ sections $p_1,\cdots,p_n$ such that each geometric fiber $(\Sigma(s),\mathbf{p}(s))$ is an $n$-pointed curve.

Two such curves $(\Sigma;\mathbf{p})$ and $(\Sigma',\mathbf{p}')$ are *isomorphic* if there exists a biholomorphic equivalence $\Phi:\Sigma\to\Sigma'$ mapping $p_s$ to $p'_s$.

An $n$-pointed curve is *stable* if its automorphism group is trivial (i.e, on each irreducible component, the number of singular points plus the number of labelled points is at least three).

#### 2.1.1. *Graphs.*

**Definition 2.2.** A *graph* $\Gamma$ is a collection of finite sets of *vertices* $V_\Gamma$ and *flags (or half edges)* $F_\Gamma$ with a boundary map $\partial_\Gamma:F_\Gamma\to V_\Gamma$ and an involution $j_\Gamma:F_\Gamma\to F_\Gamma$ ($j_\Gamma^2=id$). We call $E_\Gamma=\{(f_1,f_2)\in F_\Gamma^2\mid f_1=j_\Gamma f_2\ \&\ f_1\neq f_2\}$ the set of *edges*, and $T_\Gamma=\{f\in F_\Gamma\mid f=j_\Gamma f\}$ the set of *tails*. For a vertex $v\in V_\Gamma$, let $F_\Gamma(v)=\partial^{-1}(v)$ and $|v|=|F_\Gamma(v)|$ be the *valency* of $v$.

We think of a graph $\Gamma$ in terms of its following *geometric realization* $||\Gamma||$: Consider the disjoint union of closed intervals $\bigsqcup_{f_i\in F_\Gamma}[0,1]\times f_i$ and identify $(0,f_i)$ with $(0,f_j)$

if $\partial_\Gamma f_i = \partial_\Gamma f_j$, and identify $(t, f_i)$ with $(1-t, j_\Gamma f_i)$ for $t \in [0,1]$ and $f_i \neq f_j$. The geometric realization of $\Gamma$ has a piecewise linear structure.

**Definition 2.3.** A *tree* $\tau$ is a graph whose geometric realization is connected and simply-connected. If $|v| > 2$ for all $v \in V_\tau$ then such a tree is called *stable*.

We associate a subtree $\tau_v$ for each vertex $v \in V_\tau$ which is given by $V_{\tau_v} = \{v\}$, $F_{\tau_v} = F_\tau(v)$, $j_{\tau_v} = id$, and $\partial_{\tau_v} = \partial_\tau$.

**Definition 2.4.** Let $\tau$ and $\gamma$ be trees with $n$ tails. A *morphism* between these trees $\phi : \tau \to \gamma$ is a pair of maps $\phi_F : F_\gamma \to F_\tau$ and $\phi_V : V_\tau \to V_\gamma$ satisfying the following conditions:

- $\phi_F$ is injective and $\phi_V$ is surjective.
- The following diagram commutes

$$\begin{array}{ccc} F_\tau & \xrightarrow{\partial_\tau} & V_\tau \\ {\scriptstyle \phi_F}\uparrow & & \downarrow{\scriptstyle \phi_V} \\ F_\gamma & \xrightarrow{\partial_\gamma} & V_\gamma . \end{array}$$

- $\phi_F \circ j_\gamma = j_\tau \circ \phi_F$.
- $\phi_T$ is a bijection.

An *isomorphism* $\phi : \tau \to \gamma$ is a morphism where $\phi_F$ and $\phi_V$ are bijections. We denote the isomorphic trees by $\tau \approx \gamma$.

Each morphism induces a piecewise linear map on geometric realizations.

**Lemma 2.5.** *Let $\tau$ and $\gamma$ be stable trees with $n$ tails. Any isomorphism $\phi : \tau \to \gamma$ is uniquely defined by its restriction on tails $\phi_T : T_\gamma \to T_\tau$.*

*Proof.* Let $\phi, \varphi : \tau \to \gamma$ be two isomorphisms such that their restriction on tails are the same. Consider the path ${}_{t_1}P_{t_2}$ in $||\tau||$ that connects a pair of tails $t_1, t_2$. The automorphism $\varphi^{-1} \circ \phi$ of $\tau$ maps ${}_{t_1}P_{t_2}$ to itself; otherwise, the union of the ${}_{t_1}P_{t_2}$ and its image $\varphi^{-1} \circ \phi({}_{t_1}P_{t_2})$ gives a loop in $||\tau||$, which contradicts simply-connectedness. Moreover, the restriction of $\varphi^{-1} \circ \phi$ to the path ${}_{t_1}P_{t_2}$ is identity map since it preserves distances of vertices to tails. This follows from the compatibility of the automorphism $\varphi^{-1} \circ \phi$ with $\partial_\tau$ and $j_\tau$.

The geometric realization $||\tau||$ of $\tau$ can be covered by paths that connects pairs of tails of $\tau$. We conclude that the automorphism $\varphi^{-1} \circ \phi$ is identity since it is identity every such path. $\square$

There are only finitely many isomorphisms classes of stable trees whose set of tails is $T_\tau = \{1, \cdots, n\}$. We call the isomorphism classes of such trees *$n$-trees*. We denote the set of all $n$-trees by $\mathcal{T}ree$.

2.1.2. *Dual Trees of Pointed Curves.* Let $(\Sigma;\mathbf{p})$ be an $n$-pointed curve and $\hat{\Sigma}\to\Sigma$ be its normalization. Let $(\hat{\Sigma}_v;\hat{\mathbf{p}}_v)$ be the following $|v|$-pointed stable curve: $\hat{\Sigma}_v$ is a component of $\hat{\Sigma}$, and the labelled points $\hat{\mathbf{p}}_v$ be the set of points consisting of the preimages of *special* (i.e. labelled and nodal) points on $\Sigma_v$ under the normalization map. The points $\hat{\mathbf{p}}_v=(p_{f_1},\cdots,p_{f_{|v|}})$ on $\hat{\Sigma}_v$ are ordered by the flags $f_*\in F_\tau(v)$.

**Definition 2.6.** The *dual tree* $\tau$ of an $n$-pointed curve $(\Sigma;\mathbf{p})$ is the tree consisting of following data:

- $V_\tau$ is the set of components of $\hat{\Sigma}$.
- $F_\tau$ is the set consisting of the preimages of special points.
- $\partial_\tau: f\mapsto v$ if and only if $p_f\in\hat{\Sigma}_v$.
- $j_\tau: f\mapsto f$ if and only if $p_f$ is a labelled point, and $j_\tau: f_1\mapsto f_2$ if and only if $p_{f_1}\in\hat{\Sigma}_{v_1}$ and $p_{f_2}\in\hat{\Sigma}_{v_2}$ are the preimages of the nodal point $\Sigma_{v_1}\cap\Sigma_{v_2}$.

**Lemma 2.7.** *Let $\Phi$ be an isomorphism between the $n$-pointed stable curves $(\Sigma;\mathbf{p})$ and $(\Sigma',\mathbf{p}')$.*

*(i) $\Phi$ induces an isomorphism $\phi$ between their dual trees $\tau,\gamma$.*

*(ii) $\Phi$ is uniquely defined by its restriction on labelled points.*

*Proof.* (i) The result follows from the decomposition of $\Phi$ into its restriction to each irreducible component and the Def. 2.4.

(ii) Due to Lemma 2.5, the isomorphism $\phi:\tau\to\gamma$ is determined by the restriction of $\Phi$ to the labelled points. The isomorphism $\phi$ determines which component of $\Sigma$ is mapped to which component of $\Sigma'$ as well as the restriction of $\Phi$ to the special points. Each component of $\Sigma$ is rational and has at least three special points. Therefore, the restriction of $\Phi$ to a component uniquely given by the images of the three special points. □

2.2. **Deformations of Pointed Curves.** Let $\tau$ be the dual tree of $(\Sigma;\mathbf{p})$ and $\hat{\Sigma}\to\Sigma$ be the normalization. Let $(\hat{\Sigma}_v;\hat{\mathbf{p}}_v)$ be the following $|v|$-pointed stable curve corresponding to the irreducible component $\Sigma_v$ of $(\Sigma;\mathbf{p})$. Let $\Omega^1_\Sigma$ be the sheaf of Kähler differentials.

The infinitesimal deformations of a nodal curve $\Sigma$ with divisor $D_{\mathbf{p}}=p_1+\cdots+p_n$ is canonically identified with the complex vector space

$$Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\mathbf{p}}),\mathcal{O}_\Sigma), \tag{1}$$

and the obstruction lies in

$$Ext^2_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_{\mathbf{p}}),\mathcal{O}_\Sigma).$$

In this case, it is known that there is no obstructions (see, for example [Liu]).

The space of infinitesimal deformations is the tangent space of the space of deformations at $(\Sigma;\mathbf{p})$. It can be written explicitly in the following form:

$$\bigoplus_{v\in V_\tau} H^1(\hat{\Sigma}_v,T_{\hat{\Sigma}_v}(-D_{\hat{\mathbf{p}}_v}))\oplus\bigoplus_{(f_e,f^e)\in E_\tau} T_{p_{f_e}}\hat{\Sigma}\otimes T_{p_{f^e}}\hat{\Sigma}. \tag{2}$$

The first part corresponds to the equisingular deformations of $\Sigma$ with the divisor $D_{\hat{\mathbf{p}}_v} = \sum_{f_i \in F_\tau(v)} p_i$, and the second part corresponds to the smoothing of nodal points $p_e$ of the edges $e = (f_e, f^e)$ (see [HM]).

2.2.1. *Combinatorics of Degenerations.* Let $(\Sigma; \mathbf{p})$ be an $n$-pointed curve with the dual tree $\tau$. Consider the deformation of a nodal point of $(\Sigma; \mathbf{p})$. Such a deformation of $(\Sigma; \mathbf{p})$ gives a *contraction* of an edge of $\tau$ : Let $e = (f_e, f^e) \in E_\gamma$ be the edge corresponding to the nodal point and $\partial_\tau(e) = \{v_e, v^e\}$, and consider the equivalence relation $\sim$ on the set of vertices, defined by: $v \sim v$ for all $v \in V_\gamma \setminus \{v_e, v^e\}$, and $v_e \sim v^e$. Then, there is a tree $\tau/e$ whose vertices are $V_\tau/\sim$ and whose flags are $F_\tau \setminus \{f_e, f^e\}$. The boundary map and involution of $\tau/e$ are the restrictions of $\partial_\tau$ and $j_\tau$.

We denote by $\tau > \gamma$, if $\tau$ is obtained by contracting some edges of $\gamma$.

2.3. **Stratification of the Moduli Space $\overline{M}_n$.** The moduli space $\overline{M}_n$ is the space of isomorphism classes of $n$-pointed stable curves. This space is stratified according to degeneration types of $n$-pointed stable curves which are given by $n$-trees. The principle stratum $M_n$ corresponds to the one-vertex $n$-tree and it is the quotient of the product $(\mathbb{CP}^1)^n$ minus the diagonals $\Delta = \bigcup_{k<l}\{(p_1, \cdots, p_n) | p_k = p_l\}$ by $Aut(\mathbb{CP}^1) = PSL_2(\mathbb{C})$.

**Theorem**(Knudsen & Keel, [Knu, Ke]). *(i) For any $n \geq 3$, $\overline{M}_n$ is a smooth projective algebraic variety of (real) dimension $2n - 6$.*

*(ii) Any family of $n$-pointed stable curves over $S$ is induced by a unique morphism $\kappa : S \to \overline{M}_n$. The universal family of curves $\overline{\mathcal{U}}_n$ of $\overline{M}_n$ is isomorphic to $\overline{M}_{n+1}$.*

*(iii) For any $n$-tree, there exists a quasi-projective subvariety $D_\tau \subset \overline{M}_n$ parameterizing the curves whose dual tree is given by $\tau$. $D_\tau$ is isomorphic to $\prod_{v \in V_\tau} M_{|v|}$. Its (real) codimension is $2|E_\tau|$.*

*(iv) $\overline{M}_n$ is stratified by pairwise disjoint subvarieties $D_\tau$. The closure of any stratum $D_\tau$ is stratified by $\{D_\gamma \mid \tau > \gamma\}$.*

**Example 2.8.** (i) From the definition it follows that $\overline{M}_n$ is empty for $n < 3$. $\overline{M}_3$ is simply a point, and its universal curve $\overline{\mathcal{U}}_3$ is $\mathbb{CP}^1$ endowed with three points.

(ii) The moduli space $\overline{M}_4$ is $\mathbb{CP}^1$ with three points. These points $D_{\tau_1}, D_{\tau_2}$ and $D_{\tau_3}$ correspond to the curves with two irreducible components, and $M_4$ is the complement of these three points (see Fig. 1). The universal family $\overline{\mathcal{U}}_4$ is a del Pezzo surface of degree 5 which is obtained by blowing up three points of $\mathbb{CP}^1 \times \mathbb{CP}^1$.

(iii) The moduli space $\overline{M}_5$ is isomorphic to $\overline{\mathcal{U}}_4$. It has ten divisors and each of these divisors contains three codimension two strata. The corresponding 5-trees are shown in Fig. 1.

2.4. **Forgetful morphism.** We say that $(\Sigma; p_1, \cdots, p_{n-1})$ is obtained by forgetting the labelled point $p_n$ of the $n$-pointed curve $(\Sigma; p_1, \cdots, p_n)$. However, the resulting pointed curve may well be unstable. This happens when the component $\Sigma_v$ of $\Sigma$

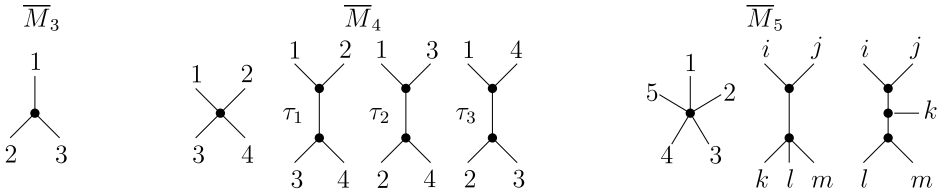


FIGURE 1. All strata of $\overline{M}_n$ for $n = 3, 4$, and 5.

supporting $p_n$ has only two additional special points. In this case, we contract this component to its intersection point(s) with the components adjacent to $\Sigma_v$. With this *stabilization* we extend this map to whole space and obtain $\pi_n : \overline{M}_n \to \overline{M}_{n-1}$. The forgetfull morphism $\pi_n$ is a fibration whose fiber over $(\Sigma; \mathbf{p}) \in \overline{M}_{n-1}$ is $\Sigma$. In other words, $\pi_n : \overline{M}_n \to \overline{M}_{n-1}$ is the universal family of curves. For details, see [Ke, Knu].

2.5. **Automorphisms of $\overline{M}_n$.** The open stratum $M_n$ of the moduli space $\overline{M}_n$ can be identified with the orbit space $((\mathbb{CP}^1)^n \setminus \Delta)/PSL_2(\mathbb{C})$. The latter orbit space may be viewed as the configuration space of $(n-3)$ ordered distinct points of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$:

$$M_n \cong \{\mathbf{p} = (z_1, \cdots, z_n) \in \mathbb{C}^{n-3} \mid z_i \neq z_j \ \forall i \neq j, \ z_{n-2} = 0, z_{n-1} = 1, z_n = \infty\}.$$

where $z_i := [z_i : 1]$ are the coordinates of labelled points $p_i$ in an affine chart of $\mathbb{CP}^1$.

Let $\psi = (\psi_1, \ldots, \psi_{n-3}) : M_n \to M_n$ be a non-constant holomorphic map. In [K], Kaliman discovered the following fact:

**Theorem** (Kaliman, [K])**.** *For $n \geq 4$, every non-constant holomorphic endomorphism $\psi = (\psi_1, \cdots, \psi_{n-3})$ of $M_n$ is an automorphism and its components $\psi_r$ are of the form*

$$\psi_r(\mathbf{p}) = \frac{z_{\varrho(r)} - z_{\varrho(n-2)}}{z_{\varrho(r)} - z_{\varrho(n)}} \Big/ \frac{z_{\varrho(n-2)} - z_{\varrho(n-1)}}{z_{\varrho(n)} - z_{\varrho(n-1)}}, \ 1 \leq r \leq n-3$$

*where $\varrho \in S_n$ is a permutation not depending on $r$.*

Kaliman's theorem implies the following corollary.

**Theorem** (Kaliman & Lin, [K, Lin])**.** *Every holomorphic automorphism of $M_n$ is produced by a certain permutation $\varrho \in S_n$. Hence, $Aut(M_n) \cong S_n$.*

On the other hand, the permutation $S_n$ acts on the compactification $\overline{M}_n$ of $M_n$ via relabeling: for $\varrho \in S_n$, there is an holomorphic map $\psi_\varrho$ which is given by

$$\psi_\varrho : (\Sigma; \mathbf{p}) \mapsto (\Sigma; \varrho(\mathbf{p})) := (\Sigma; p_{\varrho(1)}, \cdots, p_{\varrho(n)}). \tag{3}$$

Therefore, the permutation action given in (3) forms a subgroup of holomorphic automorphisms $Aut(\overline{M}_n)$.

Let $Aut_\sharp(\overline{M}_n)$ be the group of holomorphic automorphisms of $\overline{M}_n$ that respect the stratification: $\psi \in Aut_\sharp(\overline{M}_n)$ maps $D_\tau$ onto $D_\gamma$ where $\dim D_\tau = \dim D_\gamma$. Kaliman's theorem leads us to the following immediate corollary.

**Theorem 1.** *The group $Aut_\sharp(\overline{M}_n)$ is $S_n$.*

*Proof.* Let $\psi \in Aut_\sharp(\overline{M}_n)$. The restriction of $\psi$ to the open stratum gives the permutation action on $M_n$ since the automorphism group of the open stratum $M_n$ contains only permutations $\psi_\varrho$. The unicity theorem of holomorphic maps implies that $\psi = \psi_\varrho$ since they coincide on the open stratum $\psi|_{M_n} = \psi_\varrho|_{M_n}$. □

*Remark* 2.9. Note that the whole group of holomorphic automorphisms $Aut(\overline{M}_n)$ is not necessarily isomorphic to $S_n$. For example, the automorphism groups of $\overline{M}_4$ is $PSL_2(\mathbb{C})$. According to our knowledge, there is no systematic exposition of $Aut(\overline{M}_n)$ for $n > 5$.

## 3. Moduli of Pointed Real Curves of Genus 0

A *real structure* on complex variety $X$ is an anti-holomorphic involution $c_X : X \to X$. The fixed point set $\mathbb{R}X = \mathrm{Fix}(c_X)$ of the involution is called *real part* of the variety (or of the real structure).

In this section, the moduli spaces of pointed real curves of genus 0 are introduced as the fixed point sets of real structures on $\overline{M}_n$.

### 3.1. Real Structures on $\overline{M}_n$.

The moduli space $\overline{M}_n$ comes equipped with a natural real structure. The involution $c : (\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$ acts on $\overline{M}_n$. Here a complex curve $\Sigma$ is regarded as a pair $\Sigma = (C, J)$, where $C$ is the underlying 2-dimensional manifold and $J$ is a complex structure on it, and $\overline{\Sigma} = (C, -J)$ is its complex conjugated pair.[1]

**Lemma 3.1.** *The map $c$ is a real structure on $\overline{M}_n$.*

*Proof.* The differentiability of $c$ follows form the Kodaira-Spencer construction of infinitesimal deformations. We need to show that the differential of $c$ is anti-linear at each $(\Sigma; \mathbf{p}) \in \overline{M}_n$. It is sufficient to show that it is anti-linear without taking the quotient with respect to $PSL_2(\mathbb{C})$.

The infinitesimal deformations of a nodal curve $\Sigma$ with divisor $D_\mathbf{p} = p_1 + \cdots + p_n$ is canonically identified with the complex vector space $Ext^1_{\mathcal{O}_\Sigma}(\Omega^1_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma)$, (see Section 2.2). By reversing the complex structure on $\Sigma$, we reverse the complex structure on the tangent space $Ext^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma)$ at $(\Sigma; \mathbf{p})$. The differential of the map $(\Sigma; \mathbf{p}) \mapsto (\overline{\Sigma}; \mathbf{p})$

$$Ext^1_{\mathcal{O}_\Sigma}(\Omega_\Sigma(D_\mathbf{p}), \mathcal{O}_\Sigma) \to Ext^1_{\mathcal{O}_{\overline{\Sigma}}}(\Omega_{\overline{\Sigma}}(D_\mathbf{p}), \mathcal{O}_{\overline{\Sigma}}),\ \mathbf{v} \mapsto \bar{\mathbf{v}}$$

[1] There is some notational ambiguity. The bar over $\overline{M}_n$ and that over $\overline{\Sigma}$ refer to different structures on underlying manifolds: the first one refers to the compactification of $\overline{M}_n$ and second refer to the manifold with reverse complex structure. Both of these notations are widely used, we use the bar for both cases. The context should make it clear which structure is referred to.

is clearly anti-linear. □

The group of holomorphic automorphism $Aut_\sharp(\overline{M}_n)$ respecting the stratification is the permutation group $S_n$ (see Theorem 1). The permutation group $S_n$ acts on $\overline{M}_n$ via relabeling as given in (3). So that, the set of anti-holomorphic automorphisms that respect the stratification can be obtain by composing the principle real structure $(\Sigma;\mathbf{p}) \mapsto (\overline{\Sigma},\mathbf{p})$ with a permutation $\psi_\varrho$ of $\varrho \in S_n$.

For each involution $\sigma \in S_n$, we have an additional real structure on $\overline{M}_n$:

$$c_\sigma := c \circ \psi_\sigma : (\Sigma;\mathbf{p}) \mapsto (\overline{\Sigma};\sigma(\mathbf{p})). \tag{4}$$

**Lemma 3.2.** *Every real structure of $\overline{M}_n$ preserving the stratification are of the form (4) where $\sigma \in S_n$ is an involution.*

*Proof.* The holomoprhic and anti-holomorphic automorphisms preserving the stratification are in $\mathbb{Z}_2$-extension of the subgroup of automorphisms preserving the stratification $Aut_\sharp(\overline{M}_n) \subset Aut(\overline{M}_n)$ by the principle real structure $c$. Therefore, every real structure preserving the stratification of $\overline{M}_n$ are given by a certain involution $\sigma \in S_n$ and are of the form (4). □

### 3.2. $\sigma$-Invariant Pointed Curves and $\sigma$-Equivariant Families of Pointed Curves.

**Definition 3.3.** An $n$-pointed stable curve $(\Sigma;\mathbf{p})$ is called *$\sigma$-invariant* if it admits a real structure $conj : \Sigma \to \Sigma$ such that $conj(p_i) = p_{\sigma(i)}$ for all $i \in \{1,\cdots,n\}$.

A family of $n$-pointed stable curves $\pi_S : \mathcal{U}_S \to S$ is called *$\sigma$-equivariant* if a pair of maps $c_S : s \mapsto c_S(s)$, $c_\mathcal{U} : s \times (\Sigma(s),\mathbf{p}(s)) \mapsto c_S(s) \times (\overline{\Sigma}(s),\sigma(\mathbf{p}(s)))$ act as real structures respectively on $S$ and $\mathcal{U}$, and the following diagram commutes

$$\begin{array}{ccc} \mathcal{U}_S & \xrightarrow{c_\mathcal{U}} & \mathcal{U}_S \\ {\scriptstyle\pi_S}\downarrow & & \downarrow{\scriptstyle\pi_S} \\ S & \xrightarrow{c_S} & S. \end{array}$$

Notice that, if $(\Sigma;\mathbf{p})$ is $\sigma$-invariant, then the real structure $conj$ is uniquely determined by its restriction on the labelled points due to Lemma 2.7.

**Lemma 3.4.** *Let $\pi : \mathcal{U}_S \to S$ be a $\sigma$-equivariant family. Each $(\Sigma;\mathbf{p}) \in \mathbb{R}S$ is $\sigma$-invariant.*

*Proof.* If $c_S((\Sigma;\mathbf{p})) = (\Sigma;\mathbf{p})$ then there exists a unique biholomorphic equivalence $conj : \Sigma \to \overline{\Sigma}$ (in other words, anti-holomorphic $conj : \Sigma \to \Sigma$) such that $conj(p_i) = p_{\sigma(i)}$. The restriction of $conj$ on labelled points is an involution. By applying Lemma 2.7 to $conj^2$, we determine that $conj$ is an involution on $\Sigma$. □

3.3. **The Moduli Space of Pointed Real Curves $\overline{M}^{\mathbb{R}}_{(2k,l)}$.** Let $\mathrm{Fix}(\sigma)$ be the fixed point set of the action of $\sigma$ on the labeling set $\{1, \cdots, n\}$, and let $\mathrm{Perm}(\sigma)$ be its complement.

Let $\varrho \in S_n$ and $\psi_\varrho$ be the corresponding automorphism of $\overline{M}_n$. The conjugation of real structure $c_\sigma$ with $\psi_\varrho$ provide a conjugate real structure $c_{\sigma'} = \psi_\varrho \circ c_\sigma \circ \psi_{\varrho^{-1}}$. The conjugacy classes of real structures are determined by the cardinalities $|\mathrm{Fix}(\sigma)|$ and $|\mathrm{Perm}(\sigma)|$. Therefore, from now on, we only consider $c_\sigma$ where

$$\sigma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & 2k & 2k+1 & \cdots & n \\ k+1 & \cdots & 2k & 1 & \cdots & k & 2k+1 & \cdots & n \end{pmatrix}. \tag{5}$$

**Definition 3.5.** For $\sigma$ as above in (5), $\sigma$-invariant curves are called $(2k, l)$*-pointed real curves.*

The fixed point set $\mathrm{Fix}(c_\sigma)$ is called the *moduli space of* $(2k, l)$*-pointed real curves* and denoted by $\overline{M}^{\mathbb{R}}_{(2k,l)}$.

**Theorem 2.** *(i) For any $n \geq 3$, $\overline{M}^{\mathbb{R}}_{(2k,l)}$ is a smooth projective manifold of real dimension $n-3$.*

*(ii) The universal family of curves $\pi : \overline{\mathcal{U}}_n \to \overline{M}_n$ is a $\sigma$-equivariant family.*

*(iii) Any $\sigma$-equivariant family of $n$-pointed stable curves over $\pi_S : \mathcal{U}_S \to S$ is induced by a unique pair of real morphisms*

$$\begin{array}{ccc} \mathcal{U}_S & \xrightarrow{\hat{\kappa}} & \overline{\mathcal{U}}_n \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\kappa} & \overline{M}_n. \end{array}$$

*Proof.* (i) The smoothness of the real part of $c_\sigma$ is a consequence of the implicit function theorem, and $dim_{\mathbb{R}}\, \overline{M}^{\mathbb{R}}_{(2k,l)} = dim_{\mathbb{C}}\, \overline{M}_n = n-3$ since the real part $\overline{M}^{\mathbb{R}}_{(2k,l)}$ is not empty.
(ii) The fiber over $(\Sigma; \mathbf{p}) \in \overline{M}_n$ is $\pi^{-1}((\Sigma; \mathbf{p})) = \Sigma$. We define real structures on $\overline{M}_n$ and $\overline{\mathcal{U}}_n$ as follows;

$$\begin{array}{llcl} c_\sigma : & (\Sigma; \mathbf{p}) & \mapsto & (\overline{\Sigma}, \sigma(\mathbf{p})), \\ \hat{c}_\sigma : & z \in \pi^{-1}((\Sigma; \mathbf{p})) & \mapsto & z \in \pi^{-1}((\overline{\Sigma}, \sigma(\mathbf{p}))), \end{array}$$

The real structures $c_\sigma, \hat{c}_\sigma$ satisfy the conditions of $\sigma$-equivariant families in Definition 3.3.
(iii) Due to Knudsen's theorem (see Section 2.3), each of the morphisms $\kappa : S \to \overline{M}_n$ and $\hat{\kappa} : \mathcal{U}_S \to \overline{\mathcal{U}}_n$ are unique. Therefore, they are the same as $c_\sigma \circ \kappa \circ c_S : S \to \overline{M}_n$ and $\hat{c}_\sigma \circ \hat{\kappa} \circ c_{\mathcal{U}} : \mathcal{U}_S \to \overline{\mathcal{U}}_n$. Hence, the morphisms $\kappa, \hat{\kappa}$ are real. ☐

**Corollary 3.** *Let $\mathfrak{M}_\sigma$ be the contravariant functor that sends a real variety $(S, c_S)$ to the $\sigma$-equivariant family of curves over $S$. The real variety $(\overline{M}_n, c_\sigma)$ represents the moduli functor $\mathfrak{M}_\sigma$.*

*Proof.* The statement is immediate consequence of the definition of moduli functor and Theorem 2. □

## 4. Stratification of $\overline{M}^{\mathbb{R}}_{(2k,l)}$

A stratification for $\overline{M}^{\mathbb{R}}_{(2k,l)}$ can be obtained by the stratification of $\overline{M}_n$ given in Section 2.3.

**Lemma 4.1.** *Let $\tau$ and $\overline{\tau}$ be the dual trees of $(\Sigma;\mathbf{p})$ and $(\overline{\Sigma};\sigma(\mathbf{p}))$ respectively.*

*(i) If $\tau$ and $\overline{\tau}$ are not isomorphic, then the restriction of $c_\sigma$ on the union of complex strata $D_\tau \bigcup D_{\overline{\tau}}$ gives a real structure with empty real part.*

*(ii) If $\tau$ and $\overline{\tau}$ are isomorphic, then the restriction of $c_\sigma$ on $D_\tau$ gives a real structure on it and the real part $\mathbb{R}D_\tau$ is the intersection of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ with $D_\tau$.*

*Proof.* (i) Since $\tau$ and $\overline{\tau}$ are not isomorphic, $D_\tau$ and $D_{\overline{\tau}}$ are disjoint complex strata. The restriction of $c_\sigma$ on $D_\tau \bigcup D_{\overline{\tau}}$ swaps the strata, therefore the real part of this real structure is empty.

(ii) Since $\tau$ and $\overline{\tau}$ are isomorphic, the $n$-pointed curves $(\Sigma;\mathbf{p})$ and $(\overline{\Sigma};\sigma(\mathbf{p}))$ are in $D_\tau$. Therefore, $D_\tau = D_{\overline{\tau}}$ and the restriction of $c_\sigma$ on $D_\tau$ is a real structure. The real part $\mathbb{R}D_\tau$ of the $\sigma$-equivariant family $D_\tau$ is $\overline{M}^{\mathbb{R}}_{(2k,l)} \bigcap D_\tau$ since $\overline{M}^{\mathbb{R}}_{(2k,l)} = \mathrm{Fix}c_\sigma$. □

A tree $\tau$ is called *$\sigma$-invariant* if it is isomorphic to $\bar{\tau}$, and the set of $\sigma$-invariant $n$-trees are denoted by $\mathcal{T}ree(\sigma)$.

**Theorem 4.** *$\overline{M}^{\mathbb{R}}_{(2k,l)}$ is stratified by real analytic subsets $\mathbb{R}D_\tau$ where $\tau \in \mathcal{T}ree(\sigma)$.*

*Proof.* The statement is a direct consequence of Lemma 4.1. □

Although the notion of $\sigma$-invariant trees leads us a combinatorial stratification of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ given in Theorem 4, it does not give a stratification in terms of connected strata. For a $\sigma$-invariant $\tau$, the real part of the strata $\mathbb{R}D_\tau$ has many connected components. In the next subsection, we refine this stratification by using the spaces of $\mathbb{Z}_2$-equivariant point configurations in projective line $\mathbb{CP}^1$.

### 4.1. **Spaces of $\mathbb{Z}_2$-Equivariant Point Configurations in $\mathbb{CP}^1$.**

Let $z := [z:1]$ be the affine coordinate on $\mathbb{CP}^1$. Consider the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{CP}^1 \mid \Im(z) > 0\}$ (resp. lower half plane $\mathbb{H}^- = \{z \in \mathbb{CP}^1 \mid \Im(z) < 0\}$) as a half of the $\mathbb{CP}^1$ with respect to $z \mapsto \bar{z}$, and the real part $\mathbb{RP}^1$ as its boundary. Denote by $\mathbb{H}$ the compactified disc $\mathbb{H}^+ \cup \mathbb{RP}^1$.

#### 4.1.1. *Irreducible $(2k,l)$-Pointed Real Curves and Configuration Spaces.*

Let $(\Sigma;\mathbf{p})$ be an irreducible $(2k,l)$-pointed real curve. As a real curve, $\Sigma$ is isomorphic to $\mathbb{CP}^1$ with real structure which is either $z \mapsto \bar{z}$ or $z \mapsto -1/\bar{z}$ (see, for example [M2]). The real structure $z \mapsto -1/\bar{z}$ has empty real part. Notice that a $(2k,l)$-pointed curve with empty real part is possible only when $\mathrm{Fix}(\sigma) = \emptyset$.

We will consider the configuration spaces of real curves with non-empty and empty real parts as separate cases.

**Case I.** ***Pointed Real Curves with Non-empty Real Part.*** Each finite subset $\mathbf{p}$ of $\mathbb{CP}^1$ which is invariant under the real structure $z \mapsto \bar{z}$ inherits additional structures:

- **I.** *An oriented cyclic ordering on* $\mathrm{Fix}(\sigma)$: For any point $p \in (\mathbf{p} \bigcap \mathbb{RP}^1)$ there is unique $q \in (\mathbf{p} \bigcap \mathbb{RP}^1)$ which follows the point $p$ in the positive direction of $\mathbb{RP}^1$ (the direction in which the coordinate $x := [x : 1]$ on $\mathbb{RP}^1$ increases). The elements of $\mathbf{p}$ are labelled, therefore the cyclic ordering can be seen as a linear ordering on $(\mathbf{p} \bigcap \mathbb{RP}^1) \setminus \{p_n\}$. This linear ordering gives an *oriented cyclic ordering* on $\mathrm{Fix}(\sigma) = \{2k+1, \cdots, n\}$ which we denote by $\{i_1\} < \cdots < \{i_{l-1}\} < \{i_l\}$ where $i_l = n$.
- **II.** *A 2-partition on* $\mathrm{Perm}(\sigma)$: The subset $\mathbf{p} \bigcap (\mathbb{CP}^1 \setminus \mathbb{RP}^1)$ of $\mathbf{p}$ admits a partition into two disjoint subsets $\{p_i \in \mathbb{H}^{\pm}\}$. This partition gives an *ordered 2-partition* $\mathrm{Perm}^{\pm} := \{i \mid p_i \in \mathbb{H}^{\pm}\}$ of $\mathrm{Perm}(\sigma)$. The subsets $\mathrm{Perm}^{\pm}$ are swapped by the permutation $\sigma$.

The set of data $o := \{(\mathbb{CP}^1, z \mapsto \bar{z}); \mathrm{Perm}^{\pm}; \mathrm{Fix}(\sigma) = \{\{i_1\} < \cdots < \{i_l\}\}\}$ is called *oriented combinatorial type* of the $\mathbb{Z}_2$-equivariant point configuration $\mathbf{p}$ on $(\mathbb{CP}^1, z \mapsto \bar{z})$.

The oriented combinatorial types of equivariant point configurations on $(\mathbb{CP}^1, z \mapsto \bar{z})$ enumerate the connected components of the space $\widetilde{Conf}_{(2k,l)}$ of $k$ distinct pairs of conjugate points on $\mathbb{H}^+ \bigcup \mathbb{H}^-$ and $l$ distinct points on $\mathbb{RP}^1$:

$$\begin{aligned}\widetilde{Conf}_{(2k,l)} \;:=\; & \{(p_1, \cdots, p_{2k}; q_1, \cdots, q_l) \mid p_i \in \mathbb{CP}^1 \setminus \mathbb{RP}^1,\ p_i = p_j \Leftrightarrow i = j, \\ & p_i = \bar{p}_j \Leftrightarrow i = \sigma(j)\ \&\ q_i \in \mathbb{RP}^1,\ q_i = q_j \Leftrightarrow i = j\}.\end{aligned}$$

The number of connected components is $2^k(l-1)!$.[2] They are all diffeomorphic to each other; natural diffeomorphisms are given by $\sigma$-invariant relabeling.

Let $z := [z : 1]$ be the affine coordinate on $\mathbb{CP}^1$ and $x := [x : 1]$ be affine coordinate on $\mathbb{RP}^1$. The action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ is given by

$$SL_2(\mathbb{R}) \times \mathbb{H} \to \mathbb{H}, \quad (\Lambda, z) \mapsto \Lambda(z) = \frac{az+b}{cz+d}, \quad \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$$

in affine coordinates. It induces an isomorphism $SL_2(\mathbb{R})/\pm I \to Aut(\mathbb{H})$. The automorphism group $Aut(\mathbb{H})$ acts on $\widetilde{Conf}_{(2k,l)}$

$$\Lambda : (z_1, \cdots, z_{2k}; x_1, \cdots, x_l) \mapsto (\Lambda(z_1), \cdots, \Lambda(z_{2k}); \Lambda(x_1), \cdots, \Lambda(x_l)).$$

It preserves each of the connected components of $\widetilde{Conf}_{(2k,l)}$. This action is free when $2k + l \geq 3$ and it commutes with diffeomorphisms given by $\sigma$-invariant relabelings. Therefore, the quotient space $\widetilde{C}_{(2k,l)} := \widetilde{Conf}_{(2k,l)}/Aut(\mathbb{H})$ is a manifold of dimension $2k + l - 3$ whose connected components are pairwise diffeomorphic.

[2] Here we use the convention $n! = 1$ whenever $n \leq 0$.

In addition to the automorphisms considered above, there is a diffeomorphism $-\mathbb{I}$ of $\widetilde{Conf}_{(2k,l)}$ which is given in affine coordinates as follows.

$$-\mathbb{I} : (z_1, \cdots, z_{2k}; x_1, \cdots, x_l) \mapsto (-z_1, \cdots, -z_{2k}; -x_1, \cdots, -x_l).$$

Consider the quotient space $Conf_{(2k,l)} = \widetilde{Conf}_{(2k,l)}/(-\mathbb{I})$. Note that, $-\mathbb{I}$ interchanges components with *reverse* combinatorial types. Namely, the combinatorial type $\overline{o}$ of $-\mathbb{I}(\mathbf{p})$ is obtained from the combinatorial type $o$ of $\mathbf{p}$ by reversing the cyclic ordering on $\mathrm{Fix}(\sigma)$ and swapping $\mathrm{Perm}^+$ and $\mathrm{Perm}^-$. The equivalence classes of oriented combinatorial types with respect to the action of $-\mathbb{I}$ are called *un-oriented combinatorial types* of $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto \overline{z})$. The un-oriented combinatorial types enumerate the connected components of $Conf_{(2k,l)}$.

The diffeomorphism $-\mathbb{I}$ commutes with each $\sigma$-invariant relabeling and normalize the action of $Aut(\mathbb{H})$. Therefore, the quotient space $C_{(2k,l)} := Conf_{(2k,l)}/Aut(\mathbb{H})$ is a manifold of dimension $2k+l-3$, its connected components are diffeomorphic to the components of $\widetilde{C}_{(2k,l)}$, and, moreover, the quotient map $\widetilde{C}_{(2k,l)} \to C_{(2k,l)}$ is a trivial double covering.

**Case II.** ***Pointed Real Curves with Empty Real Part.*** Let $(\Sigma; \mathbf{p})$ be an irreducible $(2k, 0)$-pointed curve and let $\mathbb{R}\Sigma = \emptyset$. Such a pointed real curve is isomorphic to $(\mathbb{CP}^1, \mathbf{p})$ with real structure $conj : z \mapsto -1/\bar{z}$.

The group of automorphisms of $\mathbb{CP}^1$ which commutes with $conj$ is

$$Aut(\mathbb{CP}^1, conj) := \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SL_2(\mathbb{C}) \right\}.$$

Thus, the group $Aut(\mathbb{CP}^1, conj)$ acts naturally on the space

$$Conf^{\emptyset}_{(2k,0)} := \{(z_1, \cdots, z_{2k}) \mid z_i = -1/\bar{z}_{i+k}\}$$

of $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$. For $k \geq 2$, the action is free and the quotient $B_{(2k,0)} := Conf^{\emptyset}_{(2k,0)}/Aut(\mathbb{CP}^1, conj)$ is a $2k-3$ dimensional connected manifold. For $\mathbb{Z}_2$-equivariant point configurations on $(\mathbb{CP}^1, z \mapsto -1/\overline{z})$, the data $\{(\mathbb{CP}^1, z \mapsto -1/\bar{z})\}$ is called the *combinatorial type* of the $\mathbb{Z}_2$-equivariant point configurations.

4.2. **O/U-Planar Trees: One-vertex Case.** An *oriented planar (o-planar) structure* on the one-vertex $n$-tree $\tau$ is one of the two possible set of data

$$o := \begin{cases} \{(\mathbb{CP}^1, z \mapsto \bar{z}); \text{a } \sigma-\text{equivariant two}-\text{partition } \mathrm{Perm}^{\pm} \text{ of } \mathrm{Perm}(\sigma); \\ \text{an oriented cyclic ordering on } \mathrm{Fix}(\sigma)\}, \\ \{(\mathbb{CP}^1, z \mapsto -1/\bar{z})\}. \end{cases}$$

We denote the o-planar trees by $(\tau, o)$.

An *un-oriented planar (u-planar) structure* $u$ on the one-vertex $n$-tree $\tau$ is a pair of reverse o-planar structure $\{o, \overline{o}\}$. We denote the u-planar trees by $(\tau, u)$.

*Remark* 4.2. For the case $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$, the o-planar and u-planar structures have the same set of data. However, this convention above is suitable for our presentation.

As it is shown in the Section 4.1.1, each connected component of $C_{(2k,l)}$ for $l > 0$ (resp. $C_{(2k,0)} \bigcup B_{(2k,0)}$ for $l = 0$) is associated to a unique u-planar tree since the group $Aut(\mathbb{H})$ is connected and the un-oriented combinatorial types of $\mathbb{Z}_2$-equivariant point configurations are encoded by the same set of data. We denote the connected components of $C_{(2k,l)}$ (and $C_{(2k,0)} \bigcup B_{(2k,0)}$) by $C_{(\tau,u)}$. Similarly, each connected component of $\widetilde{C}_{(2k,l)}$ is associated to a unique o-planar tree. We denote the connected components of $\widetilde{C}_{(2k,l)}$ by $C_{(\tau,o)}$.

4.2.1. *A Normal Position of $\mathbb{Z}_2$-Equivariant Point Configurations on* $\mathbb{CP}^1$. By using the automorphisms we can make the following choices for the representatives of the points in $\widetilde{C}_{(2k,l)}$ and $B_{(2k,0)}$.

**Case I. *Pointed Real Curves with Non-empty Real Part.*** Every element in $\widetilde{C}_{(2k,l)}$ is represented by $(\mathbb{CP}^1, \mathbf{p})$ with $\mathbf{p} \in \widetilde{Conf}_{(2k,l)}$. Calibrating the choice by $Aut(\mathbb{H})$, consider an isomorphism $(\mathbb{CP}^1, \mathbf{p}) \mapsto (\mathbb{CP}^1, \mathbf{p}')$ which puts the labelled points in the following normal position $\mathbf{p}' \in \mathbb{CP}^1$.

**A.** In the case $l \geq 3$, the three consecutive labelled points $(p'_{i_{l-1}}, p'_n, p'_{i_1})$ in $\mathbb{RP}^1$ are put in the position $x'_{i_{l-1}} = 1, x'_n = \infty, x'_{i_1} = 0$. We then obtain

$$\mathbf{p}' = (z_1, \cdots, z_k, \overline{z}_{k+1}, \cdots, \overline{z}_{2k}, x_{2k+1}, \cdots, x_{2k+l-1}, \infty).$$

**B.** In the case $l = 1, 2$, the three labelled points $\{p_k, p_{2k}, p_n\}$ are put in the position $\{\pm\sqrt{-1}, \ \infty\}$. Then,

$$\mathbf{p}' = \begin{cases} (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_{k+1}, \cdots, \overline{z}_{2k-1}, -\epsilon\sqrt{-1}, x_{2k+1}, \infty) & \text{if } l = 2, \\ (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_{k+1}, \cdots, \overline{z}_{2k-1}, -\epsilon\sqrt{-1}, \infty) & \text{if } l = 1 \end{cases}$$

where $\epsilon = \pm$.

**C.** In the case $l = 0$,the labelled points $\{p_k, p_{2k}\}$ are fixed at $\{\pm\sqrt{-1}\}$ and $p_i$ where $\{i\} = \{k-1, 2k-1\} \bigcap \text{Perm}^+$ is placed on the interval $]0, \sqrt{-1}[\subset \mathbb{H}^+$. Then,

$$\mathbf{p}' = (z_1, \cdots, z_{k-2}, \epsilon_1\lambda\sqrt{-1}, \epsilon_2\sqrt{-1}, \overline{z}_{k+1}, \cdots, \overline{z}_{2k-2}, -\epsilon_1\lambda\sqrt{-1}, -\epsilon_1\sqrt{-1}).$$

where $\lambda \in ]0, 1[$ and $\epsilon_i = \pm, i = 1, 2$.

*Remark* 4.3 (**A′.**). In case of $k > 1$ and $l > 3$, we can consider the alternative map which puts the labelled points $\mathbf{p}'$ into the following normal positions. In this case, three labelled points $\{p_k, p_{2k}, p_n\}$ can be put at $\{z_k, z_{2k}, x_n\} = \{\pm\sqrt{-1}, \infty\}$ by the action of $Aut(\mathbb{H})$. We then obtain

$$\mathbf{p}' = (z_1, \cdots, z_{k-1}, \epsilon\sqrt{-1}, \overline{z}_{k+1}, \cdots, \overline{z}_{2k-1}, -\epsilon\sqrt{-1}, x_{2k+1}, \cdots, x_{2k+l-1}, \infty)$$

where $\epsilon = \pm$.

**Case II.** ***Pointed Real Curves with Empty Real Part.*** Every element of $B_{(2k,0)}, k \geq 2$, is represented by $(\mathbb{CP}^1, \mathbf{p})$ with $\mathbf{p} \in Conf^{\emptyset}_{(2k,0)}$. Calibrating the choice by $Aut(\mathbb{CP}^1, conj)$, consider an isomorphism $(\mathbb{CP}^1, \mathbf{p}) \mapsto (\mathbb{CP}^1, \mathbf{p}')$ which puts the labelled points of $(\Sigma; \mathbf{p})$ in the following normal position $\mathbf{p}' \in \mathbb{CP}^1$.

**D.**

$$\mathbf{p} = (z_1, \cdots, z_{k-2}, \lambda, 1, \frac{-1}{\overline{z}_{k+1}}, \cdots, \frac{-1}{\overline{z}_{2k-2}}, -\frac{1}{\lambda}, -1), \text{where } \lambda \in ]-1, 1[.$$

4.2.2. Every $\mathbb{Z}_2$-equivariant point configuration defines a $(2k, l)$-pointed real curve. Hence, we define

$$\Xi : \bigsqcup_{(\tau,u)} C_{(\tau,u)} \rightarrow M^{\mathbb{R}}_{(2k,l)} \tag{6}$$

which maps $\mathbb{Z}_2$-equivariant point configurations to the corresponding isomorphisms classes of irreducible $(2k, l)$-pointed curves.

**Lemma 4.4.** *(i) The map $\Xi$ is a diffeomorphism.*

*(ii) The configuration space $C_{(\tau,u)}$ is diffeomorphic to*

- $((\mathbb{H}^+)^k \setminus \Delta) \times \square^{l-3}$ *when* $l > 2$,
- $((\mathbb{H}^+ \setminus \{\sqrt{-1}\})^{k-1} \setminus \Delta) \times \square^{l-1}$ *when* $l = 1, 2$,
- $((\mathbb{H}^+ \setminus \{\sqrt{-1}, \sqrt{-1}/2\})^{k-2} \setminus \Delta) \times \square^1$ *when $l = 0$ and the type of real structure is* $(\mathbb{CP}^1, z \mapsto \bar{z})$,
- $((\mathbb{CP}^1 \setminus \{-1, -1/2, 1/2, 1\})^{k-2} \setminus (\Delta \cup \Delta^c)) \times \square^1$ *when $l = 0$ and the type of the real structure is* $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$.

*Here, $\Delta$ is the union of all diagonals $z_i \neq z_j (i \neq j)$, $\Delta^c$ is the union of all cross-diagonals $z_i \neq -\frac{1}{\bar{z}_j} (i \neq j)$, and $\square^l$ is the $l$-dimensional open simplex.*

*Proof.* (i) The map $\Xi$ is clearly smooth. It is surjective since any $(2k, l)$-pointed irreducible curve is isomorphic either to $(\mathbb{CP}^1, z \mapsto \bar{z})$ or $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$ with a $\mathbb{Z}_2$-equivariant point configuration $\mathbf{p}$ on it. It is injective since the group of holomorphic automorphisms commuting with the real structure $z \mapsto \bar{z}$ is generated by $Aut(\mathbb{H})$ and $-\mathbb{I}$, and the group of holomorphic automorphisms commuting with the real structure $z \mapsto -1/\bar{z}$ is $Aut(\mathbb{CP}^1, conj)$. These automorphism are taken into account during construction of the configuration spaces.

(ii) As it is shown in Section 4.1.1 $C_{(\tau,u)}$ is the quotient $C_{(\tau,o)} \bigsqcup C_{(\tau,\bar{o})}/(-\mathbb{I})$. The spaces $C_{(\tau,u)}$ and $C_{(\tau,o)}$ are clearly diffeomorphic. To replace $C_{(\tau,u)}$ by $C_{(\tau,o)}$, we choose an o-planar representative for each one-vertex u-planar tree $(\tau, u)$ among $(\tau, o), (\tau, \bar{o})$ as follows.

- $l \geq 3$ case: Let $(\tau, o)$ be the representative of $(\tau, u)$ for which $p_{2k+1} < p_{n-1} < p_n$ with respect to the cyclic ordering on $\mathrm{Fix}(\sigma)$.
- $l = 0, 1, 2$ case: Let $(\tau, o)$ be the representative of $(\tau, u)$ for which $k \in \mathrm{Perm}^+$.

We put the $\mathbb{Z}_2$-equivariant point configurations into a normal position as in 4.2.1. The parameterizations stated in (ii) for $l > 0$ cases follow from (A) and (B) of

4.2.1. In the case of $l = 0$ and $(\mathbb{CP}^1, z \mapsto \bar{z})$, according to (C) the configuration space $C_{(\tau,o)}$ is a locally trivial fibration over $\square^1 = ]0,1[$ whose fibers over $\lambda \in \square^1$ are $(\mathbb{H}^+ \setminus \{\sqrt{-1}, \lambda\sqrt{-1}\})^{k-2} \setminus \Delta$. Similarly, in the case of $l = 0$ and $(\mathbb{CP}^1, z \mapsto -1/\bar{z})$, according to (D) the configuration space $C_{(\tau,u)}$ is a locally trivial fibration over $\square^1 = ]-1,1[$ whose fibers over $\lambda \in \square^1$ are $(\mathbb{CP}^1 \setminus \{-1, -1/\lambda, \lambda, 1\})^{k-2} \setminus \Delta$. Since the bases of these locally trivial fibrations are contractible, they are trivial fibrations, and the result follows. $\square$

4.3. **Pointed Real Curves and O-Planar Trees.** This section extends the notions of o/u-planar structures to the all $\sigma$-invariant trees.

4.3.1. *Notations.* Let $(\Sigma; \mathbf{p}) \in \mathbb{R}D_\tau$ for some $\tau \in \mathcal{T}ree(\sigma)$ and $c_\Sigma : \Sigma \to \Sigma$ be the real structure on $\Sigma$. We denote the set of real components $\{v \in V_\tau \mid c_\Sigma(\Sigma_v) = \Sigma_v\}$ of $\Sigma$ by $V_\tau^{\mathbb{R}}$. If this set is empty, then $\mathbb{R}\Sigma$ is an isolated real node; we call the edge of $\tau$ representing the isolated real node *special invariant edge.*

Two vertices $v, \bar{v} \in V_\tau \setminus V_\tau^{\mathbb{R}}$ are said to be *conjugate* if the real structure $c_\Sigma$ maps the components $\Sigma_v$ and $\Sigma_{\bar{v}}$ onto each other. Similarly, we call the flags $f, \bar{f} \in F_\tau \setminus \{f_* \in F_\tau \mid p_{f_*} \in \mathbb{R}\Sigma\}$ conjugate if $c_\Sigma$ swaps the corresponding special points $p_f, p_{\bar{f}}$.

4.3.2. *O/U-Planar Trees: General Case.* Let $\tau \in \mathcal{T}ree(\sigma)$. Each $(2k, l)$-pointed curve $(\Sigma; \mathbf{p}) \in \mathbb{R}D_\tau$ with $\mathbb{R}\Sigma \neq \emptyset$ admits additional structures:

- **I′.** If $V_\tau^{\mathbb{R}} \neq \emptyset$, one can fix an oriented combinatorial type of point configurations on each real component $\Sigma_v$. Namely, for $v \in V_\tau^{\mathbb{R}}$ with $\mathbb{R}\Sigma \neq \emptyset$, an oriented combinatorial type $o_v$ is given by an oriented cyclic ordering on the set of invariant flags $F_\tau^{\mathbb{R}}(v) = \{f \mid p_f \in \mathbb{R}\Sigma_v\}$ and an ordered 2-partition $F_\tau^{\pm}(v) = \{i \mid p_i \in \mathbb{H}^{\pm}\}$ of $F_\tau(v) \setminus F_\tau^{\mathbb{R}}(v)$. In the case of $V_\tau^{\mathbb{R}} = \{v^*\}$ and $\mathbb{R}\Sigma = \emptyset$, the o-planar structure is simply given by the type of the real structure i.e, $\mathbb{R}\Sigma_{v^*} = \emptyset$ (as in 4.1.1).
- **II′.** If $V_\tau^{\mathbb{R}} = \emptyset$, then one can fix an ordering of the flags of the special invariant edge: $f_e \mapsto \pm$, $f^e \mapsto \mp$.

This additional structures motivate the following definition.

An *o-planar* structure on $\tau$ is the following set of data,

$$o := \begin{cases} \{(\tau_v, o_v) \mid v \in V_\tau^{\mathbb{R}}\} & \text{when } V_\tau^{\mathbb{R}} \neq \emptyset, \\ \{\text{ordering } \{f_e, f^e\} \to \{\pm\}\}, & \text{when } V_\tau^{\mathbb{R}} = \emptyset, \end{cases}$$

where $(\tau_v, o_v)$ is an o-planar structure on the one-vertex tree $\tau_v$ for $v \in V_\tau^{\mathbb{R}}$, and $e = (f_e, f^e)$ is the special invariant edge of $\tau$ when $V_\tau^{\mathbb{R}} = \emptyset$.

Similarly, an *u-planar* structure on $\tau$ is the following set of data,

$$u := \begin{cases} \{(\tau_v, u_v) \mid v \in V_\tau^{\mathbb{R}}\} & \text{when } V_\tau^{\mathbb{R}} \neq \emptyset, \\ \{\emptyset\} & \text{when } V_\tau^{\mathbb{R}} = \emptyset, \end{cases}$$

where $(\tau_v, u_v)$ is an u-planar structure on the one-vertex tree $\tau_v$ for $v \in V_\tau^{\mathbb{R}}$. We denote u-planar trees by $(\tau, u)$.

*Remark* 4.5. The real structures on real components are not independent; either all are conjugate to $z \mapsto \bar{z}$ or there is unique real component with a real structure conjugate to $z \mapsto -1/\bar{z}$. In the latter case we denote this unique real component by $\Sigma_{v^*}$.

4.3.3. *Notations.* We associate the subsets of vertices $V_\tau^{\pm}$ and flags $F_\tau^{\pm}$ to every o-planar tree $(\tau, o)$ when $V_\tau^{\mathbb{R}} \neq \emptyset$ as follows. Let $v_1 \in V_\tau \setminus V_\tau^{\mathbb{R}}$ and let $v_2 \in V_\tau^{\mathbb{R}}$ be the closest invariant vertex to $v_1$ in the geometric realization $||\tau||$ of $\tau$. Let $f \in F_\tau(v_2)$ be in the shortest path connecting the vertices $v_1$ and $v_2$. The vertex $v_1 \in V_\tau^{\pm}$ whenever the flag $f \in F_\tau^{\pm}(v_2)$. The subset of flags $F_\tau^{\pm}$ is defined as $\partial_\tau^{-1}(V_\tau^{\pm})$.

Similarly, we associate the subsets of vertices $V_\tau^{\pm}$ and flags $F_\tau^{\pm}$ to every o-planar tree $(\tau, o)$ when $\mathbb{R}\Sigma$ is isolated real node: Let $f_e \mapsto +$ and $\partial_\tau(f_e) = v_e$. The vertex $v \in V_\tau$ is an element of the subset $V_\tau^{+}$ whenever it is closer to $v_e$ than $v^e$ in $||\tau||$. The complement $V_\tau \setminus V_\tau^{+}$ is denoted by $V_\tau^{-}$. The subset of flags $F_\tau^{\pm}$ is defined as $\partial_\tau^{-1}(V_\tau^{\pm})$.

4.4. **O-Planar Trees and Their Configuration Spaces.** We associate a space to each o-planar tree as follows.

$$C_{(\tau,o)} \quad := \quad \begin{cases} \prod_{v\in V_\tau^{\mathbb{R}}} C_{(\tau_v, o_v)} \times \prod_{v \in V_\tau^{+}} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} \neq \emptyset \text{ and } (\mathbb{CP}^1, z \mapsto \bar{z}) \\ C_{(\tau_{v^*}, o_{v^*})} \times \prod_{\{v,\bar{v}\} \subset V_\tau \setminus V_\tau^{\mathbb{R}}} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} \neq \emptyset \text{ and } (\mathbb{CP}^1, z \mapsto -1/\bar{z}) \\ \prod_{v \in V_\tau^{+}} M_{|v|} & \text{when } V_\tau^{\mathbb{R}} = \emptyset \end{cases}$$

Here the product for $z \mapsto -1/\overline{z}$ case runs over the un-ordered pairs of conjugate vertices belonging to $V_\tau \setminus V_\tau^{\mathbb{R}}$ i.e, $\{v, \bar{v}\} = \{\bar{v}, v\}$.

For each u-planar tree $(\tau, u)$, we first choose an o-planar representative and then put $C_{(\tau,u)} = C_{(\tau,o)}$. Note that, so defined space $C_{(\tau,u)}$ does not depend on the o-planar representatives up to an isomorphism.

**Lemma 4.6.** *Let $\tau \in \mathcal{T}ree(\sigma)$. The real part $\mathbb{R}D_\tau$ is diffeomorphic to $\bigsqcup_{(\tau,u)} C_{(\tau,u)}$ where the disjoint union is taken over all possible u-planar structures of $\tau$.*

*Proof.* The complex strata $D_\tau$ is diffeomorphic to the product $\prod_{v\in V_\tau} \overline{M}_{|v|}$. The real structure $c_\sigma : D_\tau \to D_\tau$ maps the factor $\overline{M}_{|v|}$ onto $\overline{M}_{|\bar{v}|}$ when $v$ and $\bar{v}$ are conjugate vertices, and maps the factor $\overline{M}_{|v|}$ onto itself when $v \in V_\tau^{\mathbb{R}}$. Therefore the real part

$\mathbb{R}D_\tau$ of $c_\sigma$ is given by

$$\begin{aligned}
&\prod_{v\in V_\tau^{\mathbb{R}}} C_{(2k_v,l_v)} \times \prod_{\{v,\bar v\}\subset V_\tau\setminus V_\tau^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when } |V_\tau^{\mathbb{R}}|>1,\\
&(C_{(|v^*|,0)} \bigsqcup B_{(|v^*|,0)}) \times \prod_{\{v,\bar v\}\subset V_\tau\setminus V_\tau^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when} |V_\tau^{\mathbb{R}}|=1,\\
&\prod_{\{v,\bar v\}\subset V_\tau\setminus V_\tau^{\mathbb{R}}} \overline{M}_{|v|} \quad \text{when} |V_\tau^{\mathbb{R}}|=0,
\end{aligned}$$

where $k_v = |F_\tau^+(v)|$ and $l_v = |F_\tau^{\mathbb{R}}(v)|$. The decompositions of $C_{(2k,l)}$ and $C_{(2k,0)} \bigsqcup B_{(2k,0)}$ into their connected components are given in Lemma 4.4. $\square$

**Theorem 5.** *$\overline{M}^{\mathbb{R}}_{(2k,l)}$ is stratified by $C_{(\tau,u)}$.*

*Proof.* The moduli space $\overline{M}^{\mathbb{R}}_{(2k,l)}$ can be stratified by $\mathbb{R}D_\tau$ due to Theorem 4. The claim directly follows from the stratification of open strata $\mathbb{R}D_\tau$ given in Lemma 4.6. $\square$

*Remark* 4.7. Above stratification gives a cell decomposition in the case of $\overline{M}^{\mathbb{R}}_{(0,l)}$. This cell decomposition has been examined in [De, DJS, GM, Ka]. In [De], Devadoss gave the cell decomposition of $\overline{M}^{\mathbb{R}}_{(0,l)}$ and also calculated the Euler characteristic of this space. The hyperbolicity and the fundamental group of $\overline{M}^{\mathbb{R}}_{(0,l)}$ are discussed in [DJS]. A double covering of $\overline{M}^{\mathbb{R}}_{(0,l)}$ is given in [Ka].

4.5. **Boundaries of the Strata.** In this section we investigate the adjacency of the strata $C_{(\tau,u)}$ in $\overline{M}^{\mathbb{R}}_{(2k,l)}$. We start with considering the complex situation in order to introduce natural coordinates near each codimension one stratum $C_{(\tau,u)}$.

4.5.1. *Intermezzo: Coordinates at the Codimension One Strata.* Let $\gamma$ be a 2-vertex $n$-tree with $V_\gamma = \{v_e, v^e\}$, $F_\gamma(v^e) = \{i_1,\cdots,i_s, f^e\}$ and $F_\gamma(v_e) = \{f_e, i_{s+1}, \cdots, i_{n-1}, n\}$. Let $(z,w) := [z:1]\times[w:1]$ be affine coordinates on $\mathbb{CP}^1\times\mathbb{CP}^1$.

Consider a neighborhood $V \subset D_\gamma$ of a nodal $n$-pointed curve $(\Sigma^o, \mathbf{p}^o) \in D_\gamma$. Any $(\Sigma;\mathbf{p}) \in V$ can be identified with a nodal curve $\{(z-z_{f_e})\cdot(w-w_{f^e}) = 0\}$ in $\mathbb{CP}^1\times\mathbb{CP}^1$ with special points $\mathbf{p}_{v_e} = (a_{f_e}, a_{i_{s+1}}, \cdots, a_{i_{n-1}}, a_n) \subset \{w - w_{f^e} = 0\}$ and $\mathbf{p}_{v^e} = (b_{f^e}, b_{i_1}, \cdots, b_{i_s}) \subset \{z - z_{f_e} = 0\}$. In order to determine a nodal curve in $\mathbb{CP}^1\times\mathbb{CP}^1$ and the position of its special points uniquely defined by $(\Sigma;\mathbf{p})$, we make the following choice. Firstly, we fix three labelled points $a_{i_{s+1}}, a_{i_{n-1}}, a_n$ on the line $\{w-w_{f^e}=0\}$ whenever $|v_e|>3$, and three special points $a_{f_e}, a_{i_{n-1}}, a_n$ whenever $|v_e|=3$. Secondly, we fix three special points $b_{f^e}, b_{i_1}, b_{i_s}$ on $\{z-z_{f_e}=0\}$. Finally, we choose: $a_{i_{s+1}} = (0,0)$, $a_{i_{n-1}} = (1,0)$, $a_n = (\infty,0)$ for $|v_e|>3$; $a_{f_e} = (0,0)$, $a_{i_{n-1}} = (1,0)$, $a_n = (\infty,0)$ for $|v_e| = 3$; and $b_{i_1} = (z_{f_e},1)$, $b_{i_s} = (z_{f_e},\infty)$, $b_{f^e} = (z_{f_e},0)$. Then the components $z$ and $w$ of the special points provide a coordinate system in $V$; in particular, for $|v_e|>3$ such a coordinate system is formed by $z_{f^e}$, $z_{i_*}$ with $i_* = i_{s+2},\cdots,i_{n-2}$, and $w_{j_*}$ with $j_* = i_2,\cdots,i_{s-1}$.

We now consider a family of $n$-pointed curves over $V$ times the $\epsilon$-ball $B_\epsilon = \{|t| < \epsilon\}$. It is given by a family curves $\{(z - z_{f_e}) \cdot w + t = 0 \mid t \in B_\epsilon\}$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$. The labelled points $(z_i, w_i), i = 1, \cdots n$ on these curves are chosen in the following way. If $|v_e| > 3$, we put $(z_{i_1}, w_{i_1}) = (z_{f_e} - t, 1)$, $(z_{i_s}, w_{i_s}) = (z_{f_e}, \infty)$, $(z_{i_{s+1}, w_{i_{s+1}}}) = (0, t/z_{f_e})$, $(z_{i_{n-1}}, w_{i_{n-1}}) = (1, -t/(1 - z_{f_e}))$ and $(z_n, w_n) = (\infty, 0)$. Similarly, for $|v_e| = 3$, $(z_{i_1}, w_{i_1}) = (-t, 1)$, $(z_{i_{n-2}}, w_{i_{n-2}}) = (0, \infty)$, $(z_{i_{n-1}}, w_{i_{n-1}}) = (1, -t)$ and $(z_n, w_n) = (\infty, 0)$. The other labelled points are taken in an arbitrary position. The component $z$ of the special points and the parameter $t$ provide a coordinate system in $V \times B_\epsilon$.

Due to Knudsen's theorem there exists a unique $\kappa : V \times B_\epsilon \to \overline{M}_n$ which gives the family of $n$-pointed curves given above.

**Lemma 4.8.** $\det(d\kappa) \neq 0$ *at* $(\Sigma^o, \mathbf{p}^o) \in D_\gamma$. *Hence,* $\kappa$ *gives a local isomorphism.*

*Proof.* The parameter $t$ gives a regular function on $\kappa(V \times B_\epsilon)$ which is 0 along $D_\gamma \bigcap \kappa(V \times B_\epsilon)$. The differential $d\kappa(\vec{v}) = \vec{v}$ for $\vec{v} \in T_{(\Sigma^o, \mathbf{p}^o)}V$ since the restriction of $\kappa$ on $V \times \{0\}$ is identity map. On the other hand, $d\kappa(\partial_t) \neq 0$ since the curves are non-isomorphic for different values of parameter $t$. ☐

*Remark* 4.9. Due to Lemma 4.8, the coordinates on $V \times B_\epsilon$ provides a coordinates system at $(\Sigma^o, \mathbf{p}^o) \in D_\gamma$. There is a natural coordinate projection $\rho : V \times B_\epsilon \to V$ that is given by the projections of labelled points of the curves $\{(z - z_{f_e}) \cdot w + t = 0\}$ onto $\{w = 0\}$ axes.

For a $\sigma$-invariant $\gamma$ and $c_\sigma$-invariant $V$, above coordinates and the local isomorphism $\kappa$ are equivariant with respect to a suitable real structure on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Therefore, the real part $\mathbb{R}V \times ]-\epsilon, \epsilon[$ of $V \times B_\epsilon$ provides a neighborhood for a $(\Sigma^o, \mathbf{p}^o)$ in $\mathbb{R}D_\gamma$ with a set of coordinates on it.

4.5.2. **Contraction Morphisms for O-Planar Trees.** Let $(\gamma, \hat{o})$ be an o-planar tree and $\phi : \gamma \to \tau$ be a morphism of $n$-trees contracting an invariant set of edges $E_{con} = E_\gamma \setminus \phi_E(E_\tau)$. In such a situation, we associate a particular o-planar structure $o$ on $\tau$, as it is described below in separate cases **A** and **B**, and speak on a *contraction morphism* $\varphi : (\gamma, \hat{o}) \to (\tau, o)$. In all the cases, except **A**-(2), the o-planar structure $o$ is uniquely defined by $\hat{o}$.

**A.** Let $E_{con} = \{e = (f_e, f^e)\}$ and $e$ be an invariant edge.

(1) If $\partial_\gamma(e) = \{v_e, v^e\} \subset V_\gamma^{\mathbb{R}}$, we then convert the o-planar structures at the vertices $v_e$ and $v^e$ given by

$$\begin{aligned} \hat{o}_{v_e} &= \{\mathbb{R}\Sigma_{v_e} \neq \emptyset; F_\gamma^\pm(v_e); F_\gamma^{\mathbb{R}}(v_e) = \{\{i_1\} < \cdots < \{i_m\} < \{f_e\}\}\} \\ \hat{o}_{v^e} &= \{\mathbb{R}\Sigma_{v^e} \neq \emptyset; F_\gamma^\pm(v^e); F_\gamma^{\mathbb{R}}(v^e) = \{\{i'_1\} < \cdots < \{i'_{m'}\} < \{f^e\}\}\}. \end{aligned}$$

to an o-planar structure at vertex $v = \phi_V(\{v_e, v^e\})$ of $(\tau, o)$ defining it by

$$\begin{aligned} o_v &= \{\mathbb{R}\Sigma_v \neq \emptyset; F_\tau^\pm(v) = F_\gamma^\pm(v_e) \bigcup F_\gamma^\pm(v^e); \\ &\quad F_\tau^{\mathbb{R}}(v) = \{\{i_1\} < \cdots < \{i_m\} < \{i'_1\} < \cdots < \{i'_{m'}\}\}\}. \end{aligned}$$

The o-planar structures are kept unchanged at all other invariant vertices.

(2) If $e$ is a special invariant edge, then we convert the o-planar structure $\hat{o} = \{f_e \mapsto +, f^e \mapsto -\}$ of $\gamma$ into an o-planar structure at the vertex $v = \phi_V(\{v_e, v^e\})$ of $\tau$ defining it by

$$o_v = \{\mathbb{R}\Sigma_v \neq \emptyset; F_\tau^+(v) = F_\gamma(v_e) \setminus \{f_e\}, F_\tau^-(v) = F_\gamma(v^e) \setminus \{f^e\}; F_\tau^{\mathbb{R}}(v) = \emptyset\}.$$

or by

$$o_v = \{\mathbb{R}\Sigma_v = \emptyset\}.$$

**B.** Let $E_{con} = \{e_i = (f_{e_i}, f^{e_i}) \mid i = 1, 2\}$ where $f_{e_i}, i = 1, 2$ and $f^{e_i}, i = 1, 2$ are conjugate pairs of flags.

(1) If $\partial_\gamma(e_i) = \{\hat{v}, v^{e_i}\}$, and $\hat{v} \in V_\gamma^{\mathbb{R}}$, $v^{e_i} \notin V_\gamma^{\mathbb{R}}$, then we convert the o-planar structure at the vertex $\hat{v}$ given by

$$o_{\hat{v}} = \{\mathbb{R}\Sigma_{\hat{v}} \neq \emptyset; F_\gamma^{\pm}(\hat{v}); F_\gamma^{\mathbb{R}}(\hat{v}) = \{\{i_1\} < \cdots < \{i_m\}\}\}.$$

to an o-planar structure at vertex $v = \phi(\{\hat{v}, v^{e_1}, v^{e_2}\})$ of $\tau$ defining it by

$$\begin{aligned} o_v \quad = \quad & \{\mathbb{R}\Sigma \neq \emptyset; F_\tau^+(v) = F_\gamma^+(\hat{v}) \bigcup F_\gamma^+(v^{e_1}) \setminus \{f_{e_1}, f^{e_1}\}, \\ & F_\tau^-(v) = F_\gamma^-(\hat{v}) \bigcup F_\gamma^-(v^{e_2}) \setminus \{f_{e_2}, f^{e_2}\}; \\ & F_\tau^{\mathbb{R}}(v) = \{\{i_1\} < \cdots < \{i_m\}\}\}. \end{aligned}$$

(2) If $E_{con} = \{e_i = (f_{e_i}, f^{e_i}) \mid i = 1, 2\}$ and $\partial_\gamma(e_i) \bigcap V_\gamma^{\mathbb{R}} = \emptyset$, then we define the o-planar structure at each vertex $v$ in $\tau$ to be the same o-planar structure with the o-planar structure at same vertex $v$ of $(\gamma, \hat{o})$.

*Remark* 4.10. Let $(\gamma, \hat{o})$ be an o-planar tree with $\mathbb{R}\Sigma \neq \emptyset$, and let $\varphi_e : (\gamma, \hat{o}) \to (\tau, o)$ be the contraction of an edge $e \in E_\gamma$. If the o-planar tree $(\tau, o)$ and the u-planar tree $(\gamma, \hat{u})$ underlying $(\gamma, \hat{o})$ are given, then the o-planar structure $\hat{o}$ can be reconstructed. For this reason, when a stratum $C_{(\gamma,\hat{u})}$ contained in the boundary of $\overline{C}_{(\tau,u)}$ is given, we denote the corresponding o-planar structure $\hat{o}$ by $\delta(o)$.

**Proposition 4.11.** *A stratum $C_{(\gamma,\hat{u})}$ is contained in the boundary of $\overline{C}_{(\tau,u)}$ if and only if the u-planar structures $u, \hat{u}$ can be lifted to o-planar structures $o, \hat{o}$ such a way that $(\tau, o)$ is obtained by contracting an invariant set of edges of $(\gamma, \hat{o})$.*

*Proof.* We need to consider the statement only for the strata of codimension one and two. These cases correspond to the contraction morphisms from two/three-vertex o-planar (sub)trees to one-vertex o-planar (sub)trees given in **A** and **B**. For a stratum of higher codimension, the statement can be proved by applying the elementary contractions **A** and **B** inductively. Here, we consider only the case (**A**)-(1). The proof for other cases is the same.

We first assume that $(\tau, o)$ is obtained by contracting the edge $e$ of $(\gamma, \hat{o})$, where $(\gamma, \hat{o})$ is an o-planar two-vertex tree with $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$. An element $(\Sigma; \mathbf{p}) \in$

$C_{(\gamma,\hat{o})}$ can be represented by the nodal curve $\{(z-z_{f_e})\cdot w=0\}$ in $\mathbb{CP}^1\times\mathbb{CP}^1$ with special points $a_f=(z_f,0)$ and $b_f=(z_{f_e},w_f)$ such that

$$\begin{array}{ll} a_f\in\{w=0\ \&\ \Im(z)>0\} & \text{for } f\in F_\gamma^+(v_e)\\ a_{\bar f}\in\{w=0\ \&\ \Im(z)<0\} & \text{for } \bar f\in F_\gamma^-(v_e)\\ \{a_{i_1}<\cdots<a_{i_m}\}\subset\{w=0\ \&\ \Im(z)=0\} & \text{for } i_*\in F_\gamma^{\mathbb{R}}(v_e)\end{array}$$

on the axis $w=0$, and $b_*=(z_{f_e},w_*)$

$$\begin{array}{ll} b_f\in\{z-z_{f_e}=0\ \&\ \Im(w)>0\} & \text{for } f\in F_\gamma^+(v^e)\\ b_{\bar f}\in\{z-z_{f_e}=0\ \&\ \Im(w)<0\} & \text{for } \bar f\in F_\gamma^-(v^e)\\ \{b_{i'_1}<\cdots<b_{i'_{m'}}\}\subset\{z-z_{f_e}=0\ \&\ \Im(w)=0\} & \text{for } i'_*\in F_\gamma^{\mathbb{R}}(v^e)\end{array}$$

When we include the curve $\{(z-z_{f_e})\cdot w=0\}$ into the family $\{(z-z_{f_e})\cdot w+t=0\}$, the complex orientation defined on the irreducible components $w=0$ and $z-z_{f_e}=0$ by the halves $\Im(z)>0$ and, respectively, $\Im(w)>0$ extends continuously to a complex orientation of $\{(z-z_{f_e})\cdot w+t=0\}$ with $t\in[0,\epsilon[$ defined by, say, $\Im(z)>0$. As a result, the curves $\{(z-z_{f_e})\cdot w+t=0\}$ with $t\in[0,\epsilon[$ acquire an o-planar structure given by

$$\begin{array}{ll} (z_f,w_f)\in\{(z-z_{f_e})\cdot w+t=0\ \&\ \Im(z)>0\} & \text{for } f\in F_\gamma^+(v_e)\bigcup F_\gamma^+(v^e)\\ (z_{\overline{f}},w_{\overline{f}})\in\{(z-z_{f_e})\cdot w+t=0\ \&\ \Im(z)<0\} & \text{for } \bar f\in F_\gamma^-(v_e)\bigcup\in F_\gamma^-(v^e)\\ (z_f,w_f)\in\{(z-z_{f_e})\cdot w+t=0\ \&\ \Im(z)=0\} & \text{for } f\in F_\gamma^{\mathbb{R}}(v_e)\bigcup F_\gamma^{\mathbb{R}}(v^e)\end{array}$$

where the points on the real part of the curves $\{(z-z_{f_e})\cdot w+t=0\}$ are cyclicly ordered by

$$z_{i_1}<\cdots<z_{i_m}<z_{i'_1}<\cdots<z_{i'_{m'}}.$$

This is exactly the o-planar structure $(\tau,o)$ defined in (**A**)-(1) of 4.5.2.

Now assume that $C_{(\gamma,\hat{u})}$, where $(\gamma,\hat{u})$ is an u-planar tree with $V_\gamma=V_\gamma^{\mathbb{R}}=\{v_e,v^e\}$, is contained in the boundary of $\overline{C}_{(\tau,u)}$. There are four different o-planar representatives of $(\gamma,\hat{u})$, and any two non reverse to each other representatives $\hat{o}_1,\hat{o}_2$ provide by contraction two different o-planar structures $(\tau,o_i), i=1,2$. By the already proved part of the statement, $C_{(\gamma,\hat{u})}$ is contained in the boundary of $\overline{C}_{(\tau,o_i)}$ for each $i=1,2$. It remains to notice that any codimension one stratum is adjacent to at most two main strata. $\square$

**Example 4.12.** (i) The first nontrivial example is $\overline{M}_4$. There are 3 types of real structures: $c_{id}$, $c_{\sigma_1}$, $c_{\sigma_2}$, where $\sigma_1=(1\ 2)$ transposes labelled points $p_1$, and $p_2$ and $\sigma_2=(1\ 3)(2\ 4)$ transposes also labelled points $p_3$ and $p_4$. These then give $\overline{M}^{\mathbb{R}}_{(2k,l)}$, where $(2k,l)=(0,4),(2,2)$, and $(4,0)$ respectively.

In the case $\sigma=id$, $M^{\mathbb{R}}_{(0,4)}$ is the space of configuration of 4 distinct points on $\mathbb{RP}^1$ up to the action of $PSL_2(\mathbb{R})$. The 4-pointed curve $(\Sigma;\mathbf{p})\in M^{\mathbb{R}}_{(0,4)}$ can be identified with $(0,x_2,1,\infty)$ where $x_2\in\mathbb{RP}^1\setminus\{0,1,\infty\}$ by the $PSL_2(\mathbb{R})$ action. Hence $M^{\mathbb{R}}_{(0,4)}=\mathbb{RP}^1\setminus\{0,1,\infty\}$ and its compactification is $\overline{M}^{\mathbb{R}}_{(0,4)}=\mathbb{RP}^1$. The three intervals of $M^{\mathbb{R}}_{(0,4)}$ are the three configuration spaces $C_{(\tau,u_i)}$ and the three points are the configuration spaces $C_{(\gamma_i,u_i)}$. The u-planar trees $(\tau, u_i)$ and $(\gamma_i, u_i)$ are given in Fig. 2.

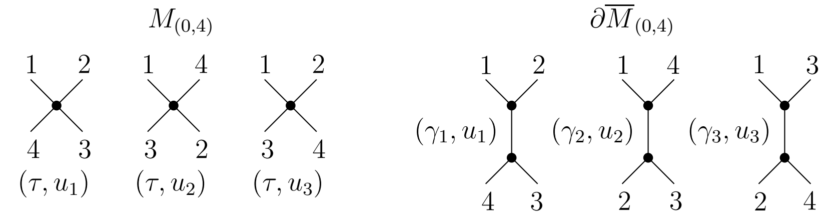


FIGURE 2. All strata of $\overline{M}_{(0,4)}$.

In the case $\sigma = \sigma_1$, $M^{\mathbb{R}}_{(2,2)}$ is the space of distinct configurations of 2 points in $\mathbb{RP}^1$ and a pair of complex conjugate points in $\mathbb{CP}^1$. $(\Sigma; \mathbf{p}) \in M^{\mathbb{R}}_{(2,2)}$ is identified with $(\sqrt{-1}, -\sqrt{-1}, x_3, \infty) \in C_{(\tau,u)}, -\infty < x_3 < \infty$. Hence, $M^{\mathbb{R}}_{(2,2)} = \mathbb{RP}^1 \setminus \{\infty\}$ and its compactification is $\overline{M}^{\mathbb{R}}_{(2,2)} = \mathbb{RP}^1$. The interval $M^{\mathbb{R}}_{(2,2)}$ is $C_{(\tau,u)}$ and the point at its closure is $C_{(\gamma,u)}$.

In the case $\sigma = \sigma_2$, the space $M^{\mathbb{R}}_{(4,0)}$ has different pieces parameterizing real curves with non-empty and empty real parts: The subspace of $M^{\mathbb{R}}_{(4,0)}$ parameterizing the $(4,0)$-pointed real curve with $\mathbb{R}\Sigma \neq \emptyset$ is identified with $(\lambda\sqrt{-1}, \sqrt{-1}, -\lambda\sqrt{-1}, -\sqrt{-1})$ where $\lambda \in ]-1, 1[\setminus\{0\}$. The subspace of $M^{\mathbb{R}}_{(4,0)}$ parameterizing the real curves with $\mathbb{R}\Sigma = \emptyset$ is $(\lambda, 1, -\lambda, -1)$, where $\lambda \in ]-1, 1[$. Note that, the pieces parameterizing $\mathbb{R}\Sigma \neq \emptyset$ and $\mathbb{R}\Sigma = \emptyset$ are joined through the boundary points corresponding to curves with isolated real singular points. The compactification $\overline{M}^{\mathbb{R}}_{(4,0)}$ is $\mathbb{RP}^1$.

(ii) The moduli space $\overline{M}_5$ has 3 different types of real structures that are given by $\sigma =$id$, (1\ 2)$ and $(1\ 3)(2\ 4)$.

The space $M^{\mathbb{R}}_{(0,5)}$ is identified with the configuration space of 5 distinct points on $\mathbb{RP}^1$ modulo $PSL_2(\mathbb{R})$. It is $(\mathbb{RP}^1 \setminus \{0, 1, \infty\})^2 \setminus \Delta$, where $\Delta$ is union of all diagonals. Each connected component of $M^{\mathbb{R}}_{(0,5)}$ is isomorphic to a two dimensional simplex. The closure of each cell can be obtained by adding the boundaries given in Sec. 4.5; for an example see Fig. 3a. It gives the compactification of $\overline{M}^{\mathbb{R}}_{(0,5)}$ which is a torus with 3 points blown up: the cells corresponding to u-planar trees $(\tau, u_1)$ and $(\tau, u_2)$ are glued along the face corresponding to $(\gamma, u)$ which gives $(\tau, u_i), i = 1, 2$ by contracting some edges, see Fig. 3b.

The space $M^{\mathbb{R}}_{(2,3)}$ is isomorphic to configurations of a conjugate pair of points on $\mathbb{CP}^1$. The automorphisms allows us to identify such configurations with $(z, \bar{z}, 0, 1, \infty)$ where $z \in \mathbb{C} \setminus \mathbb{R}$. Hence, it can be given as $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. The $\overline{M}^{\mathbb{R}}_{(2,3)}$ is obtained as a sphere with 3 points blown up according to the stratification given in Sec. 4.5.

Finally, elements of $M^{\mathbb{R}}_{(4,1)}$ can be identified with $(z, \sqrt{-1}, \bar{z}, -\sqrt{-1}, \infty)$. Hence it can be identified with $\mathbb{CP}^1 \setminus (\mathbb{RP}^1 \cup \{\sqrt{-1}, -\sqrt{-1}\})$. Therefore, connected components are isomorphic to $\mathbb{H}^+ \setminus \{\sqrt{-1}\}$. The $\overline{M}^{\mathbb{R}}_{(4,1)}$ is a sphere with a point blown up.

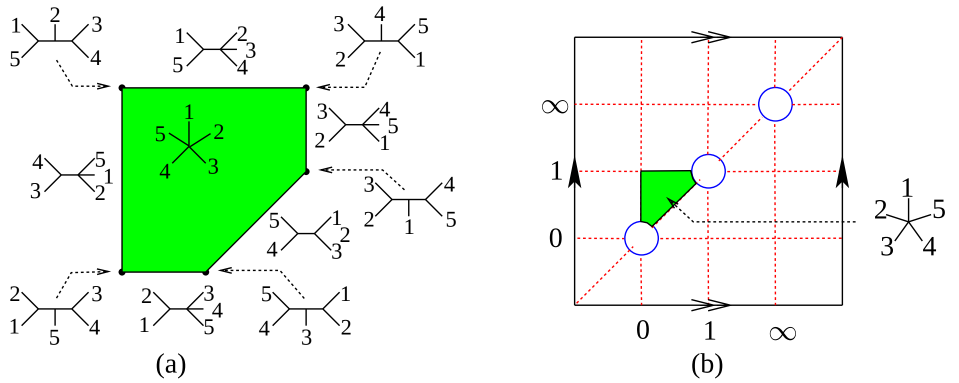

FIGURE 3. (a) Stratification of $\widetilde{C}_{(\tau,u)}$. (b) The stratification of $\overline{M}^{\mathbb{R}}_{(0,5)}$.

In fact, these are all the possible real parts of the del Pezzo surfaces of degree 5 (see [DIK]).

## 5. The First Steifel-Whitney Class of $\overline{M}^{\mathbb{R}}_{(2k,l)}$

In this section we calculate the first Steifel-Whitney class of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ by using the stratification given in Theorem 5.

5.1. **Orientations of Top-Dimensional Strata.** For a one-vertex o-planar tree $(\tau, o)$, the coordinates on $C_{(\tau,o)}$, that are introduced in Sec. 4.2.1, determine an orientation on the stratum $C_{(\tau,o)}$. For instance, let $\mathrm{Fix}(\sigma) \geq 3$ and let the o-planar structure on $(\tau, o)$ be given by $\mathrm{Perm}^{\pm}$ and the linear ordering $x_{i_1} = 0 < x_{i_2} < \cdots < x_{i_{l-1}} = 1 < x_{i_l} := x_n = \infty$ on $\mathrm{Fix}(\sigma)$. The coordinates in (A) of 4.2.1 gives the following top-dimensional differential form on $C_{(\tau,o)}$:

$$\omega_{(\tau,o)} := \left(\frac{\sqrt{-1}}{2}\right)^k \bigwedge_{\alpha_* \in \mathrm{Perm}^+} dz_{\alpha_*} \wedge d\overline{z}_{\alpha_*} \bigwedge dx_{i_2} \wedge \cdots \wedge dx_{i_{l-2}}. \tag{7}$$

The multiplication of top-dimensional forms with a positive valued function $\Theta : C_{(\tau,o)} \to \mathbb{R}_{>0}$ defines an equivalence relation on sections of $\det(TC_{(\tau,o)})$. An *orientation* is an equivalence class of nowhere zero top-dimensional form with repsect this equivalence relation. We denote equivalence class of the form $\omega_{(\tau,o)}$ by $[\omega_{(\tau,o)}]$.

Similarly, using the coordinates given in (A′), (B), (C) and (D) in 4.2.1 and their ordering, we determine differential forms $\omega_{(\tau,o)}$ and corresponding orientation $[\omega_{(\tau,o)}]$ on $C_{(\tau,o)}$ for all $(\tau, o)$ with $|V_\tau| = 1$.

5.1.1. *Orientations on Codimension One Strata.* Let $(\gamma, o)$ be a two-vertex o-planar tree. Let $V_\gamma = \{v_e, v^e\}$ and $e = (f_e, f^e)$ be the edge where $\partial_\gamma(f_e) = v_e$ and $\partial_\gamma(f^e) = v^e$. Let $v_e = \partial_\gamma(n)$.

By choosing three flags at vertices $v_e, v^e$ and using the calibrations as in 4.2.1 we obtain a set of coordinates on $C_{(\gamma_v,o_v)}$ for each $v \in \{v_e, v^e\}$. More precisely, we use the following choice.

**I.** Let $\mathbb{R}\Sigma_v \neq \emptyset$ for $v \in \{v_e, v^e\}$ and let $\mathrm{Fix}(\sigma) \neq \emptyset$. If $|F^{\mathbb{R}}_\gamma(v_e)| \geq 3$ (resp. $|F^{\mathbb{R}}_\gamma(v^e)| \geq 3$), then we fix three consecutive labelled points $(p_{i_{l-1}}, p_n, p_{i_1})$ on $\Sigma_{v_e}$ (resp. three consecutive special points $(p_{i_{q+1}}, p_{i_{q+r}}, p_{f^e})$ on $\Sigma_{v^e}$ ) at $(1, \infty, 0)$. If $|F^{\mathbb{R}}_\gamma(v_e)| < 3$ (resp. $|F^{\mathbb{R}}_\gamma(v^e)| < 3$), then in addition to $p_n = \infty$ (resp. $p_{f^e} = 0$), we fix a conjugate pair of labelled points $(p_\alpha, p_{\overline{\alpha}})$ at $(\sqrt{-1}, -\sqrt{-1})$ where $\alpha$ is the maximal element in $F^+_\gamma(v_e)$ (resp. $F^+_\gamma(v^e)$).

**II.** Let $\mathbb{R}\Sigma_v \neq \emptyset$ for $v \in \{v_e, v^e\}$ and let $\mathrm{Fix}(\sigma) = \emptyset$. We put the special point $p_{f^e}$ and $p_{f_e}$ to 0 on the respective components. We put the conjugate pairs of special points $\{p_n, p_{\overline{n}}\}$ (resp. $\{p_\alpha, p_{\overline{\alpha}}\}$) at $\{\pm\sqrt{-1}\}$ (where $\alpha$ is the maximal element in $F^+_\gamma(v^e)$).

**III.** Let $\mathbb{R}\Sigma_v$ be an isolated real node. We put the special point $p_{f^e}$ and $p_{f_e}$ to 0 on the respective components. We put the conjugate pairs of special points $\{p_n, p_{\alpha_{k-1}}\}$ (resp. $\{p_{\overline{n}}, p_{\overline{\alpha}_{k-1}}\}$) at $\{\pm\sqrt{-1}\}$ (where $\alpha_{k-1}$ is the maximal element in $F^+_\gamma(v_e) \setminus \{n\}$).

The o-planar structures with above calibrations determine a set of coordinates of $C_{(\gamma_{v^e}, o_{v^e})}$ and $C_{(\gamma_{v_e}, o_{v_e})}$. If the o-planar structure is $o_v = \{\mathbb{R}\Sigma_v \neq \emptyset; F^\pm_\gamma(v); F^{\mathbb{R}}_\gamma = \{\{i_1\} < \cdots < \{i_{l_v}\}\}\}$ at a vertex $v \in \{v_e, v^e\}$, then we give the coordinates by

$$(z_{\alpha_1}, \cdots, z_{\alpha_{k_v}}, z_{\overline{\alpha}_1}, \cdots, z_{\overline{\alpha}_{k_v}}, x_{i_1}, \cdots, x_{i_{l_v}})$$

where $\alpha_* \in F^+_\gamma(v)$ and three of the special points are fixed according to (**I**) or (**II**). If o-planar structure is $o = \{f_e \mapsto \pm, f^e \mapsto \mp\}$ then we give the coordinates by

$$(z_{\alpha_1}, \cdots, z_{\alpha_{k_v}})$$

where $\partial_\gamma(\alpha_1), \cdots, \partial_\gamma(\alpha_{k_v}) \in F^+_\gamma$ and three of the special points are fixed according to (**III**).

These set of coordinates and their ordering determine orientations $[\omega_{(\gamma_{v^e}, o_{v^e})}]$ and $[\omega_{(\gamma_{v_e}, o_{v_e})}]$ as in 5.1, therefore an orientation on $C_{(\gamma,o)}$. The orientation determined by these coordinates is

$$\begin{array}{ll} [\omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\omega_{(\gamma_{v_e}, o_{v_e})}] & \text{when } V_\gamma = V^{\mathbb{R}}_\gamma \\ [\omega_{(\gamma_v, o_v)}] & \text{when } V^{\mathbb{R}}_\gamma = \emptyset \text{ and } v \in V^+_\gamma. \end{array}$$

5.2. **Induced Orientations on Codimension One Strata.** Let $\overline{C}_{(\tau,u)}$ be a top dimensional stratum and $C_{(\gamma,\hat{u})}$ be a codimesion one stratum contained in the boundary of $\overline{C}_{(\tau,u)}$. We lift the u-planar structures $u, \hat{u}$ to o-planar representatives $o, \hat{o} = \delta(o)$ such that $(\tau, o)$ is obtained by contracting the edge of $(\gamma, \delta(o))$ (see Prop. 4.11). Then we pick a point $(\Sigma^o, \mathbf{p}^o) \in C_{(\gamma,\delta(o))}$ and the tubular neighborhood $\mathbb{R}V \times [0, \epsilon[$ of $(\Sigma^0, \mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$.

The orientation $[\omega_{(\tau,o)}]$, introduced in 5.1, induces some orientation on $C_{(\gamma,\delta(o))}$ which we denote by $[\omega_{(\gamma,\delta(o))}]$. It means that the differential form $\omega_{(\gamma,\delta(o))}$ satisfies the following equation

$$-dt \wedge \omega_{(\gamma,\delta(o))} = \omega_{(\tau,o)}, \tag{8}$$

since traditionally the outward normal direction chosen to be normal vector (see, for example [DFN]), and on the codimension one stratum $C_{(\gamma,\delta(o))}$ of $\overline{C}_{(\tau,o)}$ such normal vector is $-\partial_t$.

In what follows we compare the induced orientation $[\omega_{(\gamma,\delta(o))}]$ with $[\omega_{(\gamma_{v^e},o_{v^e})}] \wedge [\omega_{(\gamma_{v_e},o_{v_e})}]$.

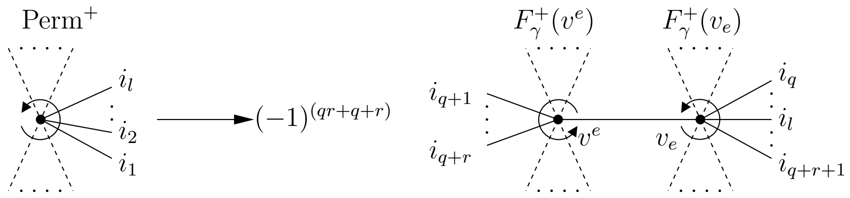


FIGURE 4. Codimension 1 boundaries of $\overline{C}_{(\tau,\hat{o})}$ where $l-r\geq 3$ & $r\geq 2$.

5.2.1. *Case I: $l \geq 1$.*

**Lemma 5.1.**

$$[\omega_{(\gamma,\delta(o))}] = (-1)^{\aleph}\ [\omega_{(\gamma_{v^e},o_{v^e})}] \wedge [\omega_{(\gamma_{v_e},o_{v_e})}]$$

*where the values of $\aleph$ for separate cases are given in the following table.*

| $\aleph$ | $l-r\geq 3$ | $l-r=2$ | | $l-r=1$ |
|---|---|---|---|---|
| | | $x_{l-1}<x_{f_e}<x_l$ | $x_{f_e}<x_{l-1}<x_l$ | |
| $r\geq 2$ | $(q+1)(r+1)$ | $0$ | $l+1$ | $l+1$ |
| $r=1$ | $1$ | $1$ | $1$ | $1$ |
| $r=0$ | $q+1$ | $0$ | $0$ | $0$ |

*Proof.* We will illustrate the statement only the first case where $l-r\geq 3$, $r\geq 2$. The calculations for other cases are almost identical.

Let $(\Sigma^o,\mathbf{p}^o)\in C_{(\gamma,\delta(o))}$. We set $\Sigma^o_{v_e}$ to be $\{w=0\}$ and $\Sigma^o_{v^e}$ to be $\{z-z_{f_e}=0\}$. According to the convention in 5.1.1, the consecutive special points $(p_{i_{l-1}},p_n,p_{i_1})$ (resp. $(p_{i_{q+1}},p_{i_{q+r}},p_{f^e})$) on the component $\Sigma_{v_e}$ (resp. $\Sigma_{v^e}$) are fixed at $(1,\infty,0)$. A tubular neighborhood $\mathbb{R}V\times[0,\epsilon[$ of $(\Sigma^o,\mathbf{p}^o)$ in $\overline{C}_{(\tau,o)}$ can be given by the family $\{(z-z_{f_e})\cdot w+t=0\}$ with labelled points which is the deformation of nodal point $(\Sigma^o,\mathbf{p}^o)$.

The boundary orientation $[\omega_{(\gamma_{v^e},o_{v^e})}]\wedge[\omega_{(\gamma_{v_e},o_{v_e})}]$ is given by the following differential forms that is determined by the ordering of the coordinates on $\mathbb{R}V$ according to 5.1.1.

$$\begin{aligned}
\omega_{(\gamma_{v_e},o_{v_e})} &= \left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma(v_e)|} \bigwedge\nolimits_{\alpha\in F^+_\gamma(v_e)} dz_\alpha\wedge d\overline{z}_\alpha\wedge \\
&\quad dx_{i_2}\wedge\cdots\wedge dx_{i_q}\wedge dx_{f_e}\wedge\widehat{dx_{i_{q+1}}}\wedge\cdots\wedge\widehat{dx_{i_{q+r}}}\wedge dx_{i_{q+r+1}}\wedge\cdots\wedge dx_{i_{l-2}}, \\
\omega_{(\gamma_{v^e},o_{v^e})} &= \left(\frac{\sqrt{-1}}{2}\right)^{|F^+_\gamma(v^e)|} \bigwedge\nolimits_{\beta\in F^+_\gamma(v^e)} dw_\beta\wedge d\overline{w}_\beta \bigwedge dy_{i_{q+2}}\wedge\cdots\wedge dy_{i_{q+r-1}}.
\end{aligned}$$

The coordinate projection $\mathbb{R}V\times[0,\epsilon[\to\mathbb{R}V$ is given by the projections of the labelled points lying on the curves $\{(z-z_{f_e})\cdot w+t=0\}$ onto $\{w=0\}$ axes (see

Remark 4.9). By using the identities $w_\beta = -t/(z_\beta - x_{f_e})$ for $\beta \in F_\gamma^+(v^e)$ and $y_i = -t/(x_i - x_{f_e})$ for $i \in F_\gamma^{\mathbb{R}}(v^e)$, we obtain the following equalities:

$$dt = -dx_{i_{q+1}} + dx_{f_e}, \quad dx_{f_e} = dx_{i_{q+r}}, \text{ and}$$

$$\begin{array}{ll} dw_\beta = -\frac{dt}{z_\beta - x_{f_e}} + \frac{tdz_\beta}{(z_\beta - x_{f_e})^2} - \frac{tdx_{f_e}}{(z_\beta - x_{f_e})^2} & \text{for } \beta \in F_\gamma^\pm(v^e), \\ dy_i = -\frac{dt}{x_i - x_{f_e}} + \frac{tdx_i}{(x_i - x_{f_e})^2} - \frac{tdx_{f_e}}{(x_i - x_{f_e})^2} & \text{for } i = q+2, \cdots, q+r-1. \end{array}$$

By simple manuplation $-dt \wedge \omega_{(\gamma_{v^e}, o_{v^e})} \wedge \omega_{(\gamma_{v^e}, o_{v^e})}$ gives us

$$(-1)^{(r-1)(q-1)} \left(\frac{\sqrt{-1}}{2}\right)^{|F_\gamma^+|} \Theta \bigwedge_{\alpha \in F_\gamma^+} dz_\alpha \wedge d\overline{z}_\alpha \bigwedge dx_{i_2} \wedge \cdots \wedge dx_{l-2}$$

where $\Theta = \prod_{\beta \in F_\gamma^+(v^e)} t(z_\beta - x_{f_e})^{-2} \prod_{i_{q+2}, \cdots, i_{q+r-1}} t(x_i - x_{f_e})^{-2}$. Since the function $\Theta > 0$, we obtain that the orientation defined by $-dt \wedge \omega_{(\gamma_{v^e}, o_{v^e})} \wedge \omega_{(\gamma_{v^e}, o_{v^e})}$ is equal to $(-1)^{\aleph}[\omega_{(\tau,o)}]$. □

5.2.2. *Case II.* $l = 0$. The different cases for boundaries of $C_{(\tau,o)}$ and $B_{(2k,0)}$ are treated separately. The proofs are essentially same with the Lemma 5.1. We will not repeat the detailed calculations.

**Lemma 5.2.** *Let $(\tau, o)$ be a one-vertex o-planar tree with* $\mathrm{Fix}(\sigma) = \emptyset$ *and* $\mathbb{R}\Sigma \neq \emptyset$, *and let $[\omega_{(\tau,o)}]$ be the orientation defined by the differential form*

$$(\frac{\sqrt{-1}}{2})^{k-2} \bigwedge_{\mathrm{Perm}^+ \setminus (\mathrm{Perm}^+ \bigcap \{\alpha_{k-1}, \alpha_k, \overline{\alpha}_{k-1}, \overline{\alpha}_k\})} dz_{\alpha_*} \wedge d\overline{z}_{\alpha_*} \bigwedge d\lambda$$

*which given by the coordinates in (C) of 4.2.1. Here $\lambda = \Im(z_{\alpha'})$ and $\alpha' \in \{\alpha_{k-1}, \overline{\alpha}_{k-1}\}$.*

*(i) Let $(\gamma, \delta(o))$ be a two-vertex o-planar tree where $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$, and $\partial_\gamma(n) = v_e$. The induced orientation on codimension one strata $C_{(\gamma,\delta(o))}$ of $\overline{C}_{(\tau,o)}$ is*

$$[\omega_{(\gamma,\delta(o))}] = -[\omega_{(\gamma_{v^e}, o_{v^e})}] \wedge [\omega_{(\gamma_{v_e}, o_{v_e})}].$$

*(ii) Let $(\gamma, \delta(o))$ be a two-vertex o-planar tree where $V_\gamma = \{v_e, v^e\}$, $V_\gamma^{\mathbb{R}} = \emptyset$, and let $v$ be the vertex in $V_\gamma^+$. The induced orientation $[\omega_{(\gamma,\delta(o))}]$ on codimension one strata $C_{(\gamma,\delta(o))}$ of $\overline{C}_{(\tau,o)}$ is given by*

$$\omega_{(\gamma,o)} = - \bigwedge_{\alpha_* \in F_\gamma^+(v) \setminus (F_\gamma^+(v) \bigcap \{\alpha_{k-1}, \alpha_k, \overline{\alpha}_{k-1}, \overline{\alpha}_k\})} dz_{\alpha_*} \wedge d\bar{z}_{\alpha_*}.$$

The orientation on the boundary of the strata parametering the $(\Sigma; \mathbf{p})$ with $\mathbb{R}\Sigma = \emptyset$ is given similar way:

**Lemma 5.3.** *Let $(\tau, o)$ be a one-vertex o-planar tree where $o = \{\mathbb{R}\Sigma = \emptyset\}$ and let $[\omega_{(\tau,o)}]$ be the orientation on $B_{(2k,0)}$ defined by*

$$(\frac{\sqrt{-1}}{2})^{k-2} \bigwedge_{\alpha_1, \cdots, \alpha_{k-2}} dz_{\alpha_*} \wedge d\overline{z}_{\alpha_*} \bigwedge d\lambda$$

*which is given by the coordinates in (D) of 4.2.1. Here* $\lambda = \Im(z_{\alpha_{k-1}})$*.*

*Let* $(\gamma, \delta(o))$ *be a two-vertex o-planar tree where* $V_\gamma = \{v_e, v^e\}$*,* $V_\gamma^{\mathbb{R}} = \emptyset$*, and let* $v$ *be the vertex in* $V_\gamma^+$ *The induced orientation* $[\omega_{(\gamma,\delta(o))}]$ *on codimension one strata* $C_{(\gamma,\delta(o))}$ *of* $C_{(\tau,o)}$ *is given by*

$$\omega_{(\gamma,o)} = - \bigwedge_{\alpha_* \in F_\gamma^+(v) \setminus (F_\gamma^+(v) \bigcap \{\alpha_{k-1}, \alpha_k, \overline{\alpha}_{k-1}, \overline{\alpha}_k\})} dz_{\alpha_*} \wedge d\bar{z}_{\alpha_*}.$$

5.3. **Conventions.** Let $(\tau, o_\star)$ be the one-vertex o-planar tree given by $F_\tau^+ = \{1, 2, \cdots, k\}$ and $F_\tau^- = \{k+1, \cdots, 2k\}$, and let the cyclic ordering on $F_\tau^{\mathbb{R}}$ be given by $\{2k+1\} < \{2k+2\} < \cdots < \{2k+l := n\}$. All possible o-planar structures with $\mathbb{R}\Sigma \neq \emptyset$ on $\tau$ are obtained by the permutation actions on $(\tau, o_\star)$ as follows.

Let $\varrho \in S_n$ and let $\phi$ be the morphism of $n$-trees where the restriction over the set of tails $\phi_T = \varrho$ and assume that $\phi$ commutes with the morphism $\varphi$ where $\varphi_T = \sigma$.

The image of the o-planar tree $(\tau, o_\star)$ under $\phi$ is given by action of the permutation $\varrho$ on the o-planar structure $o$ since $\tau$ is an one-vertex tree and all one-vertex trees are isomorphic to each other. The o-planar structure of the image is given by $\varrho(o_\star) := \{\mathbb{R}\Sigma \neq \emptyset; \varrho(\mathrm{Perm}^{\pm}); \mathrm{Fix}(\sigma) = \{\{\varrho(2k+1)\} < \cdots < \{\varrho(2k+l-1)\} < \{\varrho(n)\}\}\}$. The *parity* $|o|$ of the o-planar structure $o = \varrho(o_\star)$ is the parity of the permutation $\varrho$.

5.3.1. To be able to give a unique orientation to each component of $M_{(2k,l)}^{\mathbb{R}}$, we have to choose an o-planar representative for each one-vertex u-planar tree with $\mathbb{R}\Sigma \neq \emptyset$.

- $l \geq 3$ case: Let $(\tau, o)$ be the representative of $(\tau, u)$ on which $p_1 < p_{n-1} < p_n$ with respect to the cyclic ordering on $F_\tau^{\mathbb{R}}$.
- $l = 0, 1, 2$ case: Let $(\tau, o)$ be the representative of $(\tau, u)$ on which $k \in \mathrm{Perm}^+$.

We denote the set of o-planar representatives of u-planar trees by $\mathcal{U}Tree(\sigma)$.

5.4. **The First Steifel-Whitney Class.** The Poincare dual of the first Steifel-Whitney class of $T\overline{M}_{(2k,l)}^{\mathbb{R}}$ is given by

$$W_1 = \frac{1}{2} \sum_{(\tau,o) \in \mathcal{U}Tree(\sigma)} \left( \sum_{(\gamma,\delta(o))} \varepsilon_{(\gamma,\delta(o))} [\overline{C}_{(\gamma,\delta(o))}] \right) \mod 2, \tag{9}$$

where $[\overline{C}_{(\gamma,\delta(o))}]$ are the fundamental cycles of the codimension one strata $\overline{C}_{(\gamma,\delta(o))}$ (relative to their boundary) of $\overline{C}_{(\tau,o)}$. Here summation runs over all codimension one strata of $\overline{C}_{(\tau,o)}$ for all $(\tau, o) \in \mathcal{U}Tree(\sigma)$. The parity $\varepsilon_{(\gamma,\delta(o))} = \pm 1$ is the sign of the induced orientation given in Lemmata 5.1, 5.2 and 5.3.

Let $(\gamma, \delta(o_1))$ and $(\gamma, \delta(o_2))$ be a pair of o-planar representatives of a two-vertex u-planar tree $(\gamma, u)$. Let $C_{(\gamma,\delta(o_i))}, i = 1, 2$ be codimesion one strata of the top dimensional strata $\overline{C}_{(\tau,o_i)}$ for $(\tau, o_i) \in \mathcal{U}Tree(\sigma)$. Note that, if $C_{(\gamma,\delta(o_i))}$ carry opposite orientation then $[\overline{C}_{(\gamma,\delta(o_1))}]$ and $[\overline{C}_{(\gamma,\delta(o_2))}]$ cancel each other, and if not, $([\overline{C}_{(\gamma,\delta(o_2))}] + [\overline{C}_{(\gamma,\delta(o_2))}])/2$ gives $\mathbb{Z}_2$-fundamental cycles of the corresponding codimension one strata $\overline{C}_{(\gamma,u)}$ relative to its boundary. In other words, the sum (9) detects where the orientation on $M^{\mathbb{R}}_{(2k,l)}$ can not be extended to whole moduli space $\overline{M}^{\mathbb{R}}_{(2k,l)}$.

**Proposition 5.4.** *For $l > 0$, the Poincare dual of the first Steifel-Whitney class of the tangent bundle $T\overline{M}^{\mathbb{R}}_{(2k,l)}$ is given by*

$$W_1 = \sum_{(\gamma,u):\ |v^e|=0 \mod 2} [\overline{C}_{(\gamma,u)}] \ = \sum_{\gamma:\ |v^e|=0 \mod 2} [\mathbb{R}\overline{D}_\gamma] \mod 2. \tag{10}$$

First, consider the following technical lemma;

**Lemma 5.5.** *Let $(\tau, o), (\tau, \overline{o})$ be one-vertex o-planar trees with reverse o-planar structures. Let $(z_{\alpha_1}, \cdots, z_{\alpha_k}, \overline{z}_{\alpha_1}, \cdots, \overline{z}_{\alpha_k}, x_{i_1}, \cdots, x_{i_l})$ be coordinates on $C_{(\tau,o)}$. Then, the coordinate transformation $w_{\alpha_*} := \overline{z}_{\alpha_*}$, $y_{i_*} := 1 - x_{i_{l-*}}$ gives coordinates on $C_{(\tau,\overline{o})}$, and the orientations can be given as*

$$[\omega_{(\tau,o)}] = (-1)^\mu [\omega_{(\tau,\overline{o})}],$$

*where $\mu = |F^+_\tau| + (|F^{\mathbb{R}}_\tau| - 2)(|F^{\mathbb{R}}_\tau| - 3)/2$.*

The proof the Lemma follows from the simple calculation of coordinate change.

*Proof.* (of the Proposition 5.4) Firstly, we introduce an orientation $[\omega_{(\tau,o)}]$ on each top dimensional strata $\overline{C}_{(\tau,o)}$ that are determined uniquely according to the convention in 5.1.1 for all $(\tau, o) \in \mathcal{U}Tree(\sigma)$.

Let $(\gamma, \delta(o_1))$ and $(\gamma, \delta(o_2))$ be a pair of o-planar representatives of a two-vertex u-planar tree $(\gamma, u)$ such that $C_{(\gamma,\delta(o_i))}, i = 1, 2$ are codimension one strata of $\overline{C}_{(\tau,o_i)}$ for $(\tau, o_i) \in \mathcal{U}Tree(\sigma)$. If so, $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at vertex $v^e$ where $\partial_\gamma(n) \neq v^e$. These two o-planar structures correspond to the combinatorial types of the deformation $\{(z - z_{f_e}) \cdot w + t = 0\}$ of the central fiber $\{(z - z_{f_e}) \cdot w = 0\}$ respectively by positive and negative values of the parameter $t$.

Next, we calculate $\varepsilon_{(\gamma,\delta(o_1))} \left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \varepsilon_{(\gamma,\delta(o_2))} \left[\overline{C}_{(\gamma,\delta(o_2))}\right]$. In above cases, the ratio $\varepsilon_{(\gamma,\delta(o_1))}/\varepsilon_{(\gamma,\delta(o_2))}$ is given by the difference of the parities $(-1)^{|o_1|-|o_2|}$ which is simply $|o_1| - |o_2| = |F^+_\gamma(v^e)| + (|F^{\mathbb{R}}_\gamma(v^e)| - 1)(|F^{\mathbb{R}}_\gamma(v^e)| - 2)/2$. Here $|F^+_\gamma(v^e)|$ is the parity of permutation swaps $F^\pm_\gamma(v^e)$, and $(|F^{\mathbb{R}}_\gamma(v^e)| - 1)(|F^{\mathbb{R}}_\gamma(v^e)| - 2)/2$ is the parity of reversing the linear ordering on $F^{\mathbb{R}}_\gamma(v^e)$. On the other hand, reversing the o-planar structure at vertex $v^e$ gives the parity $\mu(v^e)$ which is calculated in Lemma 5.5. It is easy to check that $(|o_1| - |o_2|) + \mu(v^e) = |F^{\mathbb{R}}_\gamma(v^e)| \mod 2$. Hence, we have

$$\varepsilon_{(\gamma,\delta(o_1))} \left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \varepsilon_{(\gamma,\delta(o_2))} \left[\overline{C}_{(\gamma,\delta(o_2))}\right] = \varepsilon_{(\gamma,\delta(o_1))}(1 + (-1)^{|F^{\mathbb{R}}_\gamma(v^e)|}) \left[\overline{C}_{(\gamma,\delta(o_1))}\right].$$

This sum vanishes when $|v^e|$ is odd. The coefficient $(1 + (-1)^{|F^{\mathbb{R}}_\gamma(v^e)|})$ is equal to 2 when $|v^e|$ is even. Finally, the equation (9) can be rewritten as

$$W_1 \ = \ \frac{1}{2} \sum_{(\tau,o)\in\mathcal{U}Tree(\sigma)} \ \sum_{(\gamma,\delta(o)):|v^e|=0 \mod 2} \left[\overline{C}_{(\gamma,\delta(o))}\right].$$

since we consider the expression in modulo two. Moreover, $([\overline{C}_{(\gamma,\delta(o_1))}]+[\overline{C}_{(\gamma,\delta(o_2))}])/2$ gives us the $\mathbb{Z}_2$-fundamental cycle $[\overline{C}_{(\gamma,u)}]$, and then, the above sum gives $W_1$ as it is stated. Since the sum $\sum_{(\tau,u)}[\overline{C}_{(\gamma,u)}]$ is taken over all u-planar structures on $\gamma$, we obtain the second sum in the statement. $\square$

**Example 5.6.** Due to Proposition 5.4, the Poincare dual of the first Steifel-Whitney class $W_1$ of $\overline{M}^{\mathbb{R}}_{(0,5)}$ can be represented by $\sum_\gamma[\mathbb{R}\overline{D}_\gamma] = \sum_{(\gamma,u)}[\overline{C}_{(\gamma,u)}]$ where $\gamma$ are 5-trees with $|v^e| = 0 \mod 2$. These 5-trees are given in Figure 5a, and the union corresponding strata $\bigcup_\tau \mathbb{R}\overline{D}_\gamma$ is given as highlighted diagonals in Figure 6a.

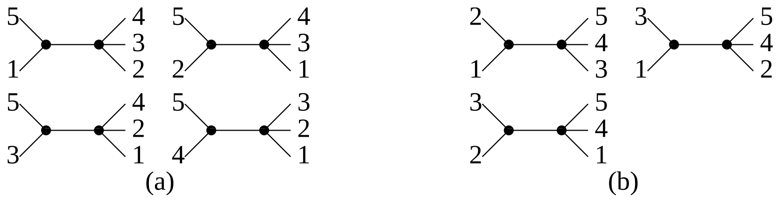


FIGURE 5. The 5-trees of (a) Steifel-Whitney class of $\overline{M}^{\mathbb{R}}_{(0,5)}$ due Prop. 5.4, (b) blown-up locus in $\overline{M}^{\mathbb{R}}_{(0,5)}$

The moduli space $\overline{M}^{\mathbb{R}}_{(0,5)}$ is a torus with 3 points blow-up (see Figure 6b). Hence, the Poincare dual of $W_1$ can also be represented by the (real part of the) exceptional divisors $\sum_{\gamma'}[\mathbb{R}\overline{D}_{\gamma'}]$ where the 5-trees $\gamma'$ are given in Figure 5b.

These two chains are homologous. The sum these chains give a contractible chain (see Figure 6c) plus three disjoint intervals $\overline{M}^{\mathbb{R}}_{(0,5)}$.

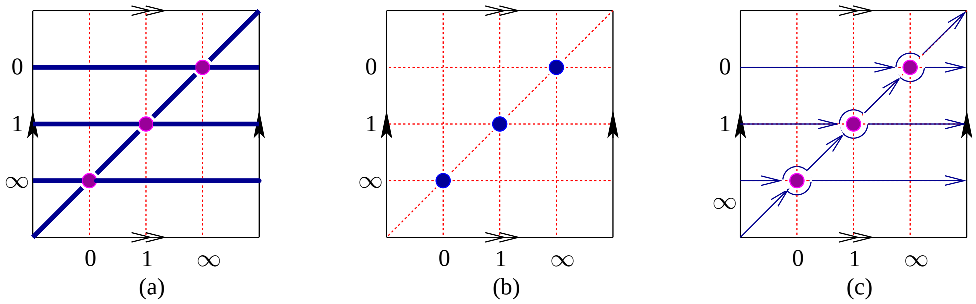


FIGURE 6

**Proposition 5.7.** *The Poincare dual of the first Steifel-Whitney class of the tangent bundle $T\overline{M}^{\mathbb{R}}_{(2k,0)}$ vanishes.*

*Proof.* A similar reasoning with Proposition 5.4 gives that $W_1 = 0$. We check consistency of the induced orientations on the codimension one strata. We consider two different types of codimension one strata which parameterize:(i) real curves with two real components; and (ii) real curves with isolated real nodes.

We introduce an orientation $[\omega_{(\tau,o)}]$ on each top dimensional strata $\overline{C}_{(\tau,o)}$, that are determined uniquely according to the convention in 5.1.1 for all $(\tau,o)\in\mathcal{U}Tree(\sigma)$ with $\mathbb{R}\Sigma\neq\emptyset$, and $B_{(2k,0)}$ with $[\omega_{(2k,0)}]$ which is defined by

$$-(\frac{\sqrt{-1}}{2})^{k-2}\bigwedge_{\alpha_1,\cdots,\alpha_{k-2}} dz_{\alpha_*}\wedge d\overline{z}_{\alpha_*}\bigwedge d\lambda=(-1)^{|o|}(\frac{\sqrt{-1}}{2})^{k-2}\bigwedge_{\alpha_*|\Im(z_{\alpha_*})>0} dz_{\alpha_*}\wedge d\overline{z}_{\alpha_*}\bigwedge d\lambda$$

where $\lambda=\Re(z_{\alpha_{k-1}})$.

(i) Let $(\gamma,\delta(o_i)), i=1,2$ be two-vertex o-planar trees representing the same $(\gamma,u)$ with $V_\gamma=V_\gamma^{\mathbb{R}}$. If so, $(\gamma,\delta(o_1))$ produces $(\gamma,\delta(o_2))$ when we reverse the o-planar structure at the vertex $v^e\neq\partial_\gamma(n)$. These two o-planar structures correspond to the combinatorial types of the deformation $\{(z-z_{f_e})\cdot w+t=0\}$ of the central fiber $\{(z-z_{f_e})\cdot w=0\}$ respectively by positive and negative values of parameter $t$. By using the same parity reasoning in the proof of Proposition 5.4, we obtain $\varepsilon_{(\gamma,o_1)}\left[\overline{C}_{(\gamma,\delta(o_1))}\right]+\varepsilon_{(\gamma,o_2)}\left[\overline{C}_{(\gamma,o_2)}\right]=0$, since $v^e$ supports only one invariant flag which corresponds to the real node.

(ii) Let $(\gamma,o)$ be an o-planar tree such that $V_\gamma^{\mathbb{R}}=\emptyset$ and $C_{(\gamma,o)}$ be a codimension one strata of $\overline{C}_{(\tau,o)}$ and $\overline{B}_{(2k,0)}$.

The induced orientations on $C_{(\gamma,\delta(o))}$ by orientations of $\overline{C}_{(\tau,o)}$ and $B_{(2k,0)}$ are given in Lemmata 5.2 and 5.3. They are respectively given by the following differential forms

$$(-1)^{|o|+1}\bigwedge_{\alpha_*\in F_\gamma^+(v)\setminus(F_\gamma^+(v)\bigcap\{\alpha_{k-1},\alpha_k,\overline{\alpha}_{k-1},\overline{\alpha}_k\})} dz_{\alpha_*}\wedge d\bar{z}_{\alpha_*},$$

$$(-1)^{|o|}\bigwedge_{\alpha_*\in F_\gamma^+(v)\setminus(F_\gamma^+(v)\bigcap\{\alpha_{k-1},\alpha_k,\overline{\alpha}_{k-1},\overline{\alpha}_k\})} dz_{\alpha_*}\wedge d\bar{z}_{\alpha_*},$$

where $v$ is the vertex in $V_\gamma^+$. Therefore, the orientations induced from different sites are opposite and the sum $(-1)^{|o|}\left[\overline{C}_{(\gamma,\delta(o))}\right]+(-1)^{|o|+1}\left[\overline{C}_{(\gamma,o)}\right]$ vanishes for all such $(\gamma,o)$. $\square$

## 6. The Orientation Double Covering of $\overline{M}^{\mathbb{R}}_{(2k,l)}$

In Section 5.4, the first Steifel-Whitney class of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ is determined in terms of its strata. As a consequence, it is observed that $\overline{M}^{\mathbb{R}}_{(2k,l)}$ is not orientable when $l\neq 0$ and $n\geq 5$. Therefore, in these cases, the $\overline{M}^{\mathbb{R}}_{(2k,l)}$ admits a unique orientation double covering. In this section, we will construct such coverings.

6.1. **Construction of $\widetilde{M}_{(2k,l)}$.** Consider the disjoint union $\overline{C}_{(2k,l)}=\bigsqcup_{(\tau,o)}\overline{C}_{(\tau,o)}$ where $|V_\tau|=1$ and $(\tau,o)$ runs over all possible o-planar structures on $\tau$. The union $\overline{C}_{(2k,l)}$ gives a compactification of $\widetilde{C}_{(2k,l)}$. We have already seen that the map $\widetilde{C}_{(2k,l)}\to M^{\mathbb{R}}_{(2k,l)}$, which identifies the o-planar with the corresponding u-planar structure, is trivial double covering. The orientation double cover of $\overline{M}^{\mathbb{R}}_{(2k,l)}$ must be a manifold with no boundaries. We need to get rid of the faces by gluing them together in order to obtain the orientation double cover. We use the following simple recipe:

Let $(\gamma,\delta(o_i)), i=1,2$ be two-vertex o-planar trees with $V_\gamma = V_\gamma^{\mathbb{R}} = \{v_e, v^e\}$ and let $\partial_\gamma(n) = v_e$. Let $(\tau, o_i), = 1,2$ be one-vertex o-planar trees obtained by contracting the invariant edges of $(\tau,\delta(o_i)), i=1,2$ respectively.

We glue the codimension one boundaries $\overline{C}_{(\gamma,\delta(o_i))}, i=1,2$ of $\overline{C}_{(\tau,o_i)}$ if

(1) $(\gamma,\delta(o_1))$ produces $(\gamma,\delta(o_2))$ by reversing the o-planar structure at $v^e$, and $|v^e| = 1 \mod 2$,
(2) $(\gamma,\delta(o_1))$ produces $(\gamma,\delta(o_2))$ by reversing the o-planar structure at $v_e$, and $|v^e| = 0 \mod 2$.

We denote the manifold obtained by gluing by $\widetilde{M}_{(2k,l)}$.

**Theorem 6.** *$\widetilde{M}_{(2k,l)}$ is the orientation double cover of $\overline{M}^{\mathbb{R}}_{(2k,l)}$.*

*Proof.* From the construction, it is clear that the map $\widetilde{M}_{(2k,l)} \to \overline{M}^{\mathbb{R}}_{(2k,l)}$ forgetting the o-planar structures is two-to-one everywhere.

We are going prove that the first Steifel-Whitney class of $\widetilde{M}_{(2k,l)}$ vanishes. We orient the configuration space $\overline{C}_{(\tau,o)}$ of one-vertex $\tau$ as follows: If $(\tau,o) \in \mathcal{U}Tree(\sigma)$, we orient it by $[\omega_{(\tau,o)}]$ according to 4.2.1, and if $(\tau,o) \notin \mathcal{U}Tree(\sigma)$ (i.e, $(\tau,\overline{o}) \in \mathcal{U}Tree(\sigma)$) we orient it by $(-1)^{\varsigma+1}[\omega_{(\tau,o)}]$ where $\varsigma = k + (l-2)(l-3)/2$. It needs to be checked that the Poincare dual of first Steifel-Whitney class

$$W_1 = \frac{1}{2}\sum_{(\tau,o)}\sum_{(\gamma,\delta(o))} \varepsilon_{(\gamma,\delta(o))}\ [\overline{C}_{(\gamma,\delta(o))}]$$

vanishes. Let $(\gamma,\delta(o_i)), i=1,2$ be a pair of two-vertex o-planar satistying the condition glueing (1) or (2) given above. Then $W_1$ can be rewritten in terms of such pairs:

$$W_1 = \frac{1}{2}\sum_{(\gamma,\delta(o_i)):|v^e|=1 \mod 2} \left(\varepsilon_{(\gamma,\delta(o_1))}\ [\overline{C}_{(\gamma,\delta(o_1))}] + \varepsilon_{(\gamma,\delta(o_2))}\ [\overline{C}_{(\gamma,\delta(o_2))}]\right) \tag{11}$$

$$+\frac{1}{2}\sum_{(\gamma,\delta(o_i)):|v^e|=0 \mod 2} \left(\varepsilon_{(\gamma,\delta(o_1))}\ [\overline{C}_{(\gamma,\delta(o_1))}] + \varepsilon_{(\gamma,\delta(o_2))}\ [\overline{C}_{(\gamma,\delta(o_2))}]\right). \tag{12}$$

The first part of the sum in (11), where $|v^e| = 1 \mod 2$, vanishes due to the parity arguments used in Proposition 5.4.

Similarly, we calculate the difference of the parity of o-planar structures $(-1)^{|o_1|-|o_2|} = \varepsilon_{(\gamma,\delta(o_1))}/\varepsilon_{(\gamma,\delta(o_2))}$ when $|v^e| = 1 \mod 2$. Let $(\gamma,\delta(\hat{o}_2))$ be the o-planar tree produced by reversing the o-planar structure at $v^e$ for $(\gamma,\delta(o_1))$. Let $[\omega_{(\tau,\hat{o}_2)}]$ be orientation on $\overline{C}_{(\tau,\hat{o}_2)}$. If we reverse the o-planar structure of $(\tau,\hat{o}_2)$, we obtain the o-planar tree $(\tau,o_2)$ and the orientation on $\overline{C}_{(\tau,o_2)}$ is given by $[\omega_{(\tau,o_2)}] = (-1)^{\varsigma}[\omega_{(\tau,\hat{o}_2)}]$ due to Lemma 5.5 i.e, the orientation that we have fixed is $-[\omega_{(\tau,\hat{o}_2)}]$.

Therefore, the second sum turns into

$$\varepsilon_{(\gamma,\delta(o_1))}\ \left[\overline{C}_{(\gamma,\delta(o_1))}\right] + \varepsilon_{(\gamma,\delta(o_2))}\ \left[\overline{C}_{(\gamma,\delta(o_2))}\right] = \varepsilon_{(\gamma,\delta(o_1))}\left(\left[\overline{C}_{(\gamma,\delta(o_1))}\right] - \left[\overline{C}_{(\gamma,\delta(\hat{o}_2))}\right]\right).$$

We have already seen in Proposition 5.4, the strata $C_{(\gamma,\delta(o_1))}$, $C_{(\gamma,\delta(\hat{o}_2))}$ have the same induced orientation. Hence the sum vanishes. □

*Remark* 6.1. In [Ka], Kapranov constructed a different double covering of $\overline{M}^{\mathbb{R}}_{(0,l)}$ without any boundaries. Let $\overline{C}_{(0,l)}$ be the disjoint union $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$ where $(\tau, o)$ runs over all possible o-planar structures on $\tau$ with $|V_\tau| = 1$. He has applied the following recipe to obtain the double covering: Let $(\gamma, \delta(o_i)), i = 1, 2$ be two o-planar trees corresponding to codimension 1 boundaries of $(\tau, o_i)$. The faces $\overline{C}_{(\gamma,\delta(o_i))}, i = 1, 2$ are glued if $(\gamma, \delta(o_1))$ produces $(\gamma, \delta(o_2))$ by reversing the o-planar structure at vertex $v^e \neq \partial_\gamma(n)$. The first Steifel-Whitney class can be explicitly obtained as above.

$$W_1 = \frac{1}{2}\sum_{(\tau,o)}\ \sum_{(\gamma,\delta(o)):|v^e|=0 \mod 2} \left[\overline{C}_{(\gamma,o)}\right] \mod 2.$$

These spaces are not orientable when $l \geq 5$.

*Remark* 6.2. In [FOOO, Liu], a different 'orientation double covering' is considered. It can be given as the disjoint union $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)}$ where $F^+_\tau = \{1, \cdots, n\}$, and $F^{\mathbb{R}}_\tau$ carries all possible oriented cyclic ordering. It is a disjoint union of manifolds with corners. The covering map $\bigsqcup_{(\tau,o)} \overline{C}_{(\tau,o)} \to \overline{M}^{\mathbb{R}}_{(2k,l)}$ is two-to-one only over the open space $M^{\mathbb{R}}_{(2k,l)}$.

6.1.1. *Examples.* The construction in Section 6.1 can also be applied when $2k + l = 4$ and $l \neq 0$, although in such cases, the moduli space $\overline{M}^{\mathbb{R}}_{(2k,l)}$ is orientable. For such cases, the above construction excludes the trivial coverings and gives the stratification of $\widetilde{M}_{(0,4)}$ shown on Figure 7. The picture of $\widetilde{M}_{(2,2)}$ is similar.

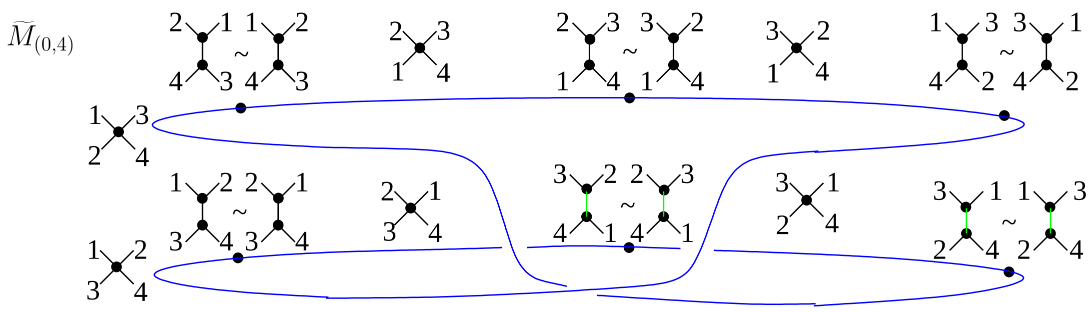

FIGURE 7. Stratification of $\widetilde{M}_{(0,4)}$

In Example 4.12, we obtained that $\overline{M}_{(0,5)}, \overline{M}_{(2,3)}$, and $\overline{M}_{(4,1)}$ are respectively, a torus with 3 points blown up, a sphere with 3 points blown up, and a sphere with one point blown up. The coverings $\widetilde{M}_{(0,5)}, \widetilde{M}_{(2,3)}$ and $\widetilde{M}_{(4,1)}$ are obtained by taking the two copies of the corresponding moduli space of real curves and replacing the blown up loci by annuli. Therefore, $\widetilde{M}_{(0,5)}, \widetilde{M}_{(2,3)}$ and $\widetilde{M}_{(4,1)}$ are surfaces of genus 4, genus 2 and genus 0, respectively (see Figure 8 which illustrates the case $(k,l) = (0,5)$).

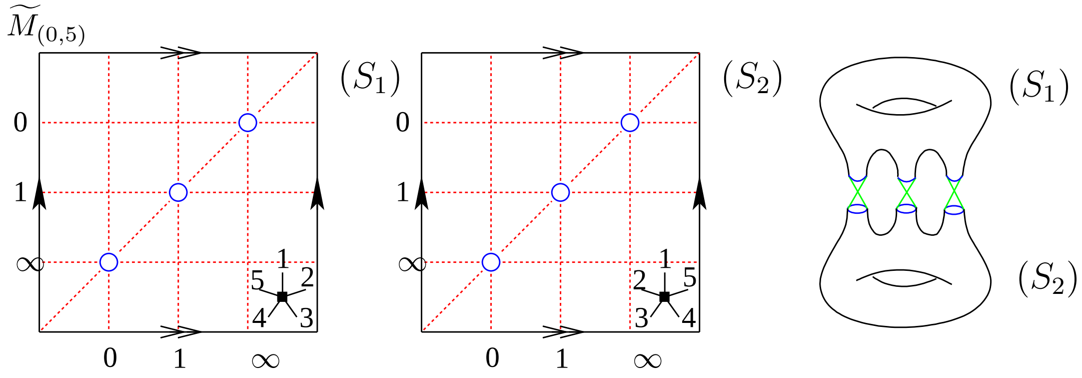


FIGURE 8. Stratification of $\widetilde{M}_{(0,5)}$

6.2. **Combinatorial Types of Strata of $\widetilde{M}_{(2k,l)}$.** While constructing $\widetilde{M}_{(2k,l)}$, the closure of the each codimension one strata are glued in a consistent way. This identification gives an equivalence relation among the o-planar trees when $l \neq 0$.

We define the notion of *R-equivalence* on the set of such o-planar trees by treating different cases separately. Let $(\gamma_1, o_1), (\gamma_2, o_2)$ be o-planar trees.

(1) If $|V^{\mathbb{R}}_{\gamma_i}| = 1$, then we say that they are R-equivalent whenever the o-planar structures are the same.
(2) If $\gamma_i$ have an edge corresponding to real node (i.e. $E^{\mathbb{R}}_{\gamma_i} = \{e\}$ and $V^{\mathbb{R}}_{\gamma_i} = \partial_\gamma(e) = \{v^e, v_e\}$), we first obtain $(\gamma_i(e), \hat{o}_i)$ by contracting conjugate pairs of edges until there will be none. We say that $(\gamma_1, o_1)$ and $(\gamma_2, o_2)$ are R-equivalent whenever $\gamma_1, \gamma_2$ are isomorphic and
   - $(\gamma_1(e), \hat{o}_1)$ produces $(\gamma_2(e), \hat{o}_2)$ by reversing o-planar structure at $v^e$ and $|v^e|$ is odd,
   - $(\gamma_1(e), \hat{o}_1)$ produces $(\gamma_2(e), \hat{o}_2)$ by reversing o-planar structure at $v_e$ and $|v^e|$ is even.
(3) Otherwise, if $\gamma_i$ have more than one edge corresponding to real nodes (i.e. $|E_{\gamma_i}|^{\mathbb{R}} > 1$), we say that $(\gamma_1, o_1)$, $(\gamma_2, o_2)$ are R-equivalent whenever $\gamma_1, \gamma_2$ are isomorphic and there exists an edge $e \in E^{\mathbb{R}}_{\gamma_i}$ such that the o-planar trees $(\gamma_i(e), \hat{o}_i)$ which are obtained by contracting all edges but $e$ are R-equivalent in the sense of case 2.

We call the maximal set of pairwise R-equivalent o-planar trees by *R-equivalence classes* of o-planar trees.

**Theorem 7.** *A stratification of the orientation double cover $\widetilde{M}_{(2k,l)}$ is given by*

$$\widetilde{M}_{(2k,l)} = \bigsqcup_{\substack{\mathrm{R-equivalance\ classes} \\ \mathrm{of\ o-planar\ } (\gamma,o)}} C_{(\gamma,o)}. \tag{13}$$

*Proof.* The statement directly follows from the construction of $\widetilde{M}_{(2k,l)}$ and definition of R-equivalence classes. □

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