Burghelea-Haller analytic torsion for manifolds with boundary

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Abstract

In this paper, we extend the complex-valued Ray-Singer torsion introduced by Burghelea-Haller to compact connected manifolds with boundary.

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1 Introduction

Let E be a unitary flat vector bundle on a closed Riemannian manifold M. In [19], Ray and Singer defined an analytic torsion associated to (M, E) and proved that it does not depend on the Riemannian metric on M. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on M (cf. [15]). This conjecture was later proved in the celebrated papers of Cheeger [10] and Müller [16]. Müller generalized this result in [17] to the case when E is a unimodular flat vector bundle on M. In [2], inspired by the considerations of Quillen [18], Bismut and Zhang reformulated the above Cheeger-Müller theorem as an equality between the Reidemeister and Ray-Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundle over M. The method used in [2] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [23] on the de Rham complex.

Braverman and Kappeler [4, 5] defined the refined analytic torsion for flat vector bundle over odd dimensional manifolds, and show that it equals to the Turaev torsion (cf. [11, 21]) up to a multiplication by a complex number of absolute value one. Burghelea and Haller [6, 7], following a suggestion of Müller, defined a generalized analytic torsion associated to a non-degenerate symmetric bilinear form on a flat vector bundle over an arbitrary dimensional manifold and make an explicit conjecture between this generalized analytic torsion and the Turaev torsion. This conjecture was proved up to sign by Burghelea-Haller [8] and in full generality by Su-Zhang [20].

Vertman [22] defined a different refinement of analytic torsion, similar to Braverman and Kappeler, which applied to compact manifolds with and without boundary. Inspired by this, in this paper, we extend the Burghelea-Haller analytic torsion to compact connected Riemannian manifolds with boundary.

The rest of this paper is organized as follows. In Section 2, we recall the definition of Hilbert complex and some properties of it. Particularly, the Hilbert complexes $(\mathscr{D}_{\min}, D_{\min})$ and $(\mathscr{D}_{\max}.D_{\max})$. In Section 3, we get some properties of the Hilbert complex $(\mathscr{D}_{\min}, D_{\min})$ and extend the Burghelea-Haller analytic torsion to the Hilbert complex $(\mathscr{D}_{\min}, D_{\min})$. In Section 4, we extend the Burghelea-Haller analytic torsion to the Hilbert complex $(\mathscr{D}_{\max}, D_{\max})$.

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2 Fredholm complexes for compact manifolds

Let (M, g^{TM}) be a smooth n-dimensional compact connected Riemannian manifold with boundary ∂M, which may be empty. Let (E, ∇) be a flat complex vector bundle over M. The flat connection ∇ extends to $\Omega_0^*(M, E)$, which is E-valued differential forms with compact support in the interior of the manifold M. Since $\nabla^2 = 0$, we have the de Rham complex $(\Omega_0^*(M, E), \nabla)$. Assume that there is a fiber wise non-degenerate symmetric bilinear form b on E. By [7, Theorem 5.10], there exists a complex anti-linear involution $\nu : E \to E$ such that

$$\nu^2 = \mathrm{id}_E, \quad b(\nu x, y) = \overline{b(x, \nu y)}, \quad b(x, \nu x) \geq 0, \quad x, y \in E.$$

Then

$$\mu : E \otimes E \to \mathbb{C}, \quad \mu(x, y) = b(x, \nu y) \tag{2.1}$$

is a fiber wise positive definite Hermitian structure on E. The Riemannian metric g^{TM} and together with the Hermitian metric μ define an inner product in $\Omega_0^*(M, E)$ which we denote it by $h_{g,\mu}$, we denote the L^2-completion of $\Omega_0^*(M, E)$ by $L_*^2(M, E)$. The Riemannian metric g^{TM} together with the fiber wise non-degenerate symmetric bilinear form b^E define a non-degenerate symmetric bilinear form $\beta_{g,b}$ on $\Omega_0^*(M, E)$,

$$\beta_{g,b}(\omega, \eta) = \int_M \omega \wedge (*_g \otimes b)\, \eta, \quad \omega, \eta \in \Omega_0^*(M, E). \tag{2.2}$$

Then $\beta_{g,b}$ extends to a non-degenerate symmetric bilinear form on $L_*^2(M, E)$, we still denote its extension on $L_*^2(M, E)$ by $\beta_{g,b}$. Then we have $h_{g,\mu}(\omega, \eta) = \beta_{g,b}(\omega, \nu\eta)$.

Consider the differential operator ∇ and its formal adjoint ∇^t with respect to the inner product. The associated minimal closed extensions $\nabla_{\min}$ and $\nabla^t_{\min}$ are defined as the graph-closures in $L_*^2(M, E)$. The maximal closed extension of ∇ is defined by

$$\nabla_{\max} = \left(\nabla^t_{\min}\right)^*, \tag{2.3}$$

where $*$ denote the adjoint operator with respect to the inner product in $L_*^2(M, E)$. We denote $\nabla^{\#}_{\min}$ and $\nabla^{\#}_{\max}$ to be the adjoint operators of $\nabla_{\min}$ and $\nabla_{\max}$ with respect to $\beta_{g,b}$ on $L_*^2(M, E)$. Let $\mathscr{D}_{\min}$, $\mathscr{D}^{\#}_{\min}$ be the domain of $\nabla_{\min}$, $\nabla^{\#}_{\min}$ respectively and let $\mathscr{D}_{\max}$, $\mathscr{D}^{\#}_{\max}$ denote the domain of $\nabla_{\max}$, $\nabla^{\#}_{\max}$ respectively. These extensions define Hilbert complexes in the following sense, as introduced in [9].

Definition 2.1. [9] Let the Hilbert spaces H_i, $i = 0, \cdots, m$, $H_{m+1} = \{0\}$ be mutually orthogonal. For each $i = 0, \cdots, m$, let $D_i \in C(H_i, H_{i+1})$ be a closed operator with domain $\mathscr{D}(D_i)$ dense in H_i and range in H_{i+1}. Put $\mathscr{D}_i = \mathscr{D}(D_i)$ and $R_i = D_i(\mathscr{D}_i)$ and assume

$$R_i \subseteq \mathscr{D}_{i+1}, \quad D_{i+1} \circ D_i = 0. \tag{2.4}$$

This defines a complex $(\mathscr{D}_*, D_*)$

$$0 \longrightarrow \mathscr{D}_0 \xrightarrow{D_0} \mathscr{D}_1 \xrightarrow{D_1} \cdots \xrightarrow{D_{m-1}} \mathscr{D}_m \longrightarrow 0. \tag{2.5}$$

Such a complex is called a Hilbert complex. If the homology of the complex is finite, i.e. if R_i is closed and $\ker D_i / \mathrm{im} D_{i-1}$ is finite-dimensional for all $i = 0, \cdots, m$, the complex is referred as a Fredholm complex.

For a Hilbert complex there is a dual Hilbert complex

$$0 \longrightarrow \mathscr{D}_m \xrightarrow{D^*_{m-1}} \mathscr{D}_{m-1} \xrightarrow{D^*_{m-2}} \cdots \xrightarrow{D^*_0} \mathscr{D}_0 \longrightarrow 0 \tag{2.6}$$

defined using the Hilbert space adjoints of the differentials D_i^* and Laplacian $\Delta_i = D_i^* D_i + D_{i-1} D^*_{i-1}$. We can compute the cohomology groups of the Hilbert complex (2.5) using the subcomplex $(\mathcal{D}^\infty \mathscr{D}_*, D_*)$, where $\mathcal{D}^\infty \mathscr{D}_i$ consisting of all elements x that are in the domain of Δ_i^l for all $l \geq 0$.

Proposition 2.2. [9, Theorem 2.12] *The cohomology of the complex* $(\mathscr{D}_*, D_*)$ *is equal to the cohomology of the complex* $(\mathcal{D}^\infty \mathscr{D}_*, D_*)$.

By [9, Lemma 3.1] we have the Hilbert complexes $(\mathscr{D}_{\min}, \nabla_{\min})$ and $(\mathscr{D}_{\max}, \nabla_{\max})$, where $\mathscr{D}_{\min} = \mathscr{D}(\nabla_{\min})$ and $\mathscr{D}_{\max} = \mathscr{D}(\nabla_{\max})$. The following theorem [22, Theorem 3.2] is the twisted setup of [9, Theorem 4.1].

Theorem 2.3. *The Hilbert complexes* $(\mathscr{D}_{\min}, \nabla_{\min})$ *and* $(\mathscr{D}_{\max}, \nabla_{\max})$ *are Fredholm with the associated Laplacians* Δ_{rel} *and* Δ_{abs} *being strongly elliptic in the sense of [12]. The de Rham isomorphism identifies the cohomology of the complexes with the relative and absolute cohomology with coefficients:*

$$H^*\left(\mathscr{D}_{\min}, \nabla_{\min}\right) \cong H^*(M, \partial M, E),$$

$$H^*(\mathscr{D}_{\max}, \nabla_{\max}) \cong H^*(M, E).$$

Furthermore the cohomology of the Fredholm complexes $(\mathscr{D}_{\min}, \nabla_{\min})$ *and* $(\mathscr{D}_{\max}, \nabla_{\max})$ *can be computed from the following smooth subcomplexes,*

$$(\Omega^*_{\min}(M, E), \nabla)\,,\ \Omega^*_{\min}(M, E) = \{\omega \in \Omega^*(M, E) | l^*(\omega) = 0\},$$

$$(\Omega^*_{\max}(M, E), \nabla)\,,\ \Omega^*_{\max}(M, E) = \Omega^*(M, E),$$

respectively, where we denote by $l : \partial M \to M$ *the natural inclusion of the boundary.*

3 Ray-Singer symmetric bilinear torsion for $(\mathscr{D}_{\min}, \nabla_{\min})$

In this section we define the Ray-Singer symmetric bilinear torsion for the Hilbert complex $(\mathscr{D}_{\min}, \nabla_{\min})$. This can be viewed as the extension of the Burghelea-Haller analytic torsion to compact manifolds with relative boundary condition.

Proposition 3.1. *The restriction of the non-degenerate symmetric bilinear form* $\beta_{g,b}$ *to* $\mathscr{D}_{\min}$ *is non-degenerate.*

Proof. Let $x \in \mathscr{D}_{\min}$, then there exist $\{x_n\} \subset \Omega^*_0(M, E)$, such that $x_n \to x$ in $L^2_*(M, E)$ and ∇x_n convergence in $L^2_*(M, E)$. Since ν is a bounded operator, we get $\nu(x_n) \to \nu x$ in $L^2_*(M, E)$. By

$$\nabla(\nu x_n) = (\nabla\nu)x_n + \nu(\nabla x_n) \tag{3.1}$$

and $\nabla\nu$ is a bounded operator, we get $\nabla(\nu x_n)$ convergence in $L^2_*(M, E)$. Then by definition of $\nabla_{\min}$, we get $\nu x \in \mathscr{D}_{\min}$. So that if for any $y \in \mathscr{D}_{\min}$, $\beta_{g,b}(x, y) = 0$, then by

$$h_{g,\mu}(x, x) = \beta_{g,b}(x, \nu x) = 0, \tag{3.2}$$

we get $x = 0$. Then the restriction of $\beta_{g,b}$ to $\mathscr{D}_{\min}$ is non-degenerate. □

Proposition 3.2. *The following identity holds*

$$\nabla^{\#}_{\min} = \nabla^*_{\min} + \left(\nu\left(\nabla\nu\right)\right)^*. \tag{3.3}$$

Particularly, the domain of $\nabla^*_{\min}$ *equals* $\mathscr{D}^{\#}_{\min}$.

Proof. Let $y \in \mathscr{D}^{\#}_{\min}$, then there exists $z \in L^2_*(M, E)$ such that for any $x \in \mathscr{D}_{\min}$, we have

$$\beta_{g,b}\left(\nabla_{\min}x, y\right) = \beta_{g,b}(x, z). \tag{3.4}$$

By Proposition 3.1, we have $\nu x \in \mathscr{D}_{\min}$. Then

$$\begin{aligned}
h_{g,\mu}\left(\nabla_{\min}(\nu x), y\right) &= h_{g,\mu}\left((\nabla\nu)x + \nu\nabla_{\min}x, y\right) \\
&= h_{g,\mu}\left((\nabla\nu)x, y\right) + h_{g,\mu}\left(\nu\nabla_{\min}x, y\right) = h_{g,\mu}\left((\nabla\nu)\nu\nu x, y\right) + \overline{\beta_{g,b}\left(\nabla_{\min}x, y\right)} \\
&= h_{g,\mu}\left(\nu x, \left((\nabla\nu)\nu\right)^{*} y\right) + \overline{\beta_{g,b}(x,z)} = h_{g,\mu}\left(\nu x, \left((\nabla\nu)\nu\right)^{*} y\right) + \beta_{g,b}(\nu x, \nu z) \\
&= h_{g,\mu}\left(\nu x, \left((\nabla\nu)\nu\right)^{*} y\right) + h_{g,\mu}(\nu x, z). \qquad (3.5)
\end{aligned}$$

Then by definition we have $y \in \mathscr{D}(\nabla^{*}_{\min})$ and

$$\nabla^{*}_{\min}y = \left((\nabla\nu)\nu\right)^{*} y + \nabla^{\#}_{\min}y. \qquad (3.6)$$

Since $\nu^2 = Id$, so that $\nu(\nabla\nu) = -(\nabla\nu)\nu$. Then by (3.6) we get

$$\nabla^{\#}_{\min} = \nabla^{*}_{\min} + \left(\nu\left(\nabla\nu\right)\right)^{*}. \qquad (3.7)$$

The proof of Proposition 3.2 is complete. □

We consider the operator

$$\Delta_{b,\mathrm{rel}} = \left(\nabla_{\min} + \nabla^{\#}_{\min}\right)^2 = \nabla_{\min}\nabla^{\#}_{\min} + \nabla^{\#}_{\min}\nabla_{\min}. \qquad (3.8)$$

The domain of $\Delta_{b,\mathrm{rel}}$ is the following,

$$\mathscr{D}(\Delta_{b,\mathrm{rel}}) = \left\{x \in \mathscr{D}_{\min} \cap \mathscr{D}^{\#}_{\min} | \nabla_{\min}x \in \mathscr{D}^{\#}_{\min} \text{ and } \nabla^{\#}_{\min}x \in \mathscr{D}_{\min}\right\}. \qquad (3.9)$$

By (3.3), we see that the domain of $\Delta^{l}_{b,\mathrm{rel}}$ equals the domain of $\Delta^{l}_{\mathrm{rel}}$ for all $l \geq 0$. By Proposition 3.2, $\Delta_{b,\mathrm{rel}}$ has same leading symbol with $\Delta_{\mathrm{rel}} = (\nabla_{\min} + \nabla^{*}_{\min})^2$. Then the spectral of $\Delta_{b,\mathrm{rel}}$ are discrete. Let $\lambda \in \mathrm{Spec}(\Delta_{b,\mathrm{rel}})$, denote by $P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ the spectral projection of $\Delta_{b,\mathrm{rel}}$ corresponding to λ, then

$$P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} = \frac{i}{2\pi}\int_{C(\lambda)} \left(\Delta_{b,\mathrm{rel}} - x\right)^{-1} dx, \qquad (3.10)$$

with $C(\lambda)$ being any closed counterclockwise circle surrounding λ with no other spectrum inside. The image of $P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ is finite dimensional. In particular $P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ is a bounded operator in $L^2_{*}(M,E)$. Then by [13, Section 4, p.155] the decomposition

$$L^2_{*}(M,E) = \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \oplus \mathrm{Im}\left(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}\right) \qquad (3.11)$$

is a direct sum decomposition into closed subspaces of the Hilbert space $L^2_{*}(M,E)$.

Proposition 3.3. *The decomposition $L^2_{*}(M,E) = \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \oplus \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ is $\beta_{g,b}$-orthogonal.*

Proof. Let N_λ be the multiplicity of the generalized eigenvalue λ. Then we have $(\Delta_{b,\mathrm{rel}} - \lambda)^{N_\lambda}|_{\mathscr{D}(\Delta_{b,\mathrm{rel}})\cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}} = 0$ and $\Delta_{b,\mathrm{rel}} - \lambda : \mathscr{D}(\Delta_{b,\mathrm{rel}}) \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}) \to \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ has an everywhere defined bounded inverse. By the decomposition (3.11), for $\Omega^{*}_{0}(M,E)$ we have

$$\Omega^{*}_{0}(M,E) = \Omega^{*}_{0}(M,E) \cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}} \oplus \Omega^{*}_{0}(M,E) \cap \mathrm{Im}\left(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}\right). \qquad (3.12)$$

In particular $(\Delta_{b,\mathrm{rel}} - \lambda)^{N_\lambda}|_{\Omega^{*}_{0}(M,E)\cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}} = 0$ and $\Delta_{b,\mathrm{rel}} - \lambda : \Omega^{*}_{0}(M,E) \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}) \to \Omega^{*}_{0}(M,E) \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ is bijective. So the decomposition (3.12) is $\beta_{g,b}$-orthogonal. In fact, for $\omega \in \Omega^{*}_{0}(M,E) \cap \mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ and $\eta \in \Omega^{*}_{0}(M,E) \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$, then there exists $\eta_{N_\lambda} \in \Omega^{*}_{0}(M,E) \cap \mathrm{Im}(1 - P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ such that

$$\left(\Delta_{b,\mathrm{rel}} - \lambda\right)^{N_\lambda} \eta_{N_\lambda} = \eta. \qquad (3.13)$$

Then we have

$$\beta_{g,b}(\omega,\eta)=\beta_{g,b}\left(\omega,(\Delta_{b,\mathrm{rel}}-\lambda)^{N_\lambda}\eta_{N_\lambda}\right)=\beta_{g,b}\left((\Delta_{b,\mathrm{rel}}-\lambda)^{N_\lambda}\omega,\eta_{N_\lambda}\right)=0. \tag{3.14}$$

For $x\in\mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ and $y\in\mathrm{Im}(1-P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$, there exist $\{x_n\}\subset\Omega_0^*(M,E)$ and $\{y_n\}\subset\Omega_0^*(M,E)$ such that $x_n\to x$ and $y_n\to y$. Since $\mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ and $\mathrm{Im}(1-P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ are closed subspaces of $L^2_*(M,E)$, so that for sufficient large n, $\beta_{g,b}(x_n,y_n)=0$. Then we have

$$\beta_{g,b}(x,y)=\lim_{n\to\infty}\beta_{g,b}(x_n,y_n)=0. \tag{3.15}$$

The proof of Proposition 3.3 is complete. □

By the decomposition (3.11), we can decompose $\mathscr{D}_{\min}$ as follows

$$\mathscr{D}_{\min}=\mathscr{D}_{\min}\cap\mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}\oplus\mathscr{D}_{\min}\cap\mathrm{Im}(1-P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}). \tag{3.16}$$

By Proposition 3.1 and Proposition 3.3, we get that the restrictions of $\beta_{g,b}$ to $\mathscr{D}_{\min}\cap\mathrm{Im}P_{\{\lambda\},\Delta_{b,\mathrm{rel}}}$ and $\mathscr{D}_{\min}\cap\mathrm{Im}(1-P_{\{\lambda\},\Delta_{b,\mathrm{rel}}})$ are all nondegenerate. For any $a\geq 0$, let $P_{[0,a],\Delta_{b,\mathrm{rel}}}$ be the spectral projection of $\Delta_{b,\mathrm{rel}}$ corresponding to the spectral with absolute value in $[0,a]$. Then we have the $\beta_{g,b}$-orthogonal decomposition

$$\mathscr{D}_{\min}=\mathscr{D}_{\min,[0,a]}\oplus\mathscr{D}_{\min,(a,\infty)}, \tag{3.17}$$

where $\mathscr{D}_{\min,[0,a]}=\mathscr{D}_{\min}\cap\mathrm{Im}P_{[0,a],\Delta_{b,\mathrm{rel}}}$ and $\mathscr{D}_{\min,(a,\infty)}=\mathscr{D}_{\min}\cap\mathrm{Im}(1-P_{[0,a],\Delta_{b,\mathrm{rel}}})$. By Proposition 3.1, we get that the restrictions of $\beta_{g,b}$ to $\mathscr{D}_{\min,[0,a]}$ and $\mathscr{D}_{\min,(a,\infty)}$ are all nondegenerate. Since $\nabla_{\min}$ commutes with $\Delta_{b,\mathrm{rel}}$, we get two subcomplexes $(\mathscr{D}_{\min,[0,a]},\nabla_{\min,[0,a]})$ and $(\mathscr{D}_{\min,(a,\infty)},\nabla_{\min,(a,\infty)})$ such that

$$(\mathscr{D}_{\min},\nabla_{\min})=\left(\mathscr{D}_{\min,[0,a]},\nabla_{\min,[0,a]}\right)\oplus\left(\mathscr{D}_{\min,(a,\infty)},\nabla_{\min,(a,\infty)}\right). \tag{3.18}$$

Proposition 3.4. *The inclusion $(\mathscr{D}_{\min,\{0\}},\nabla_{\min,\{0\}})\to(\mathscr{D}_{\min},\nabla_{\min})$ induces an isomorphism on cohomology. In particular, the subcomplex $(\mathscr{D}_{\min,(a,\infty)},\nabla_{\min,(a,\infty)})$ is acyclic for any $a\geq 0$, and*

$$H^*(\mathscr{D}_{\min,[0,a]},\nabla_{\min,[0,a]})\cong H^*(\mathscr{D}_{\min},\nabla_{\min}). \tag{3.19}$$

Proof. By Proposition 2.2, in order to compute the cohomology of $(\mathscr{D}_{\min,\{0\}},\nabla_{\min,\{0\}})$ and $(\mathscr{D}_{\min},\nabla_{\min})$ we need only to compute the cohomology of $(\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}},\nabla_{\min,\{0\}})$ and $(\mathcal{D}^\infty\mathscr{D}_{\min},\nabla_{\min})$. Then it only needs to prove the inclusion

$$(\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}},\nabla_{\min,\{0\}})\to(\mathcal{D}^\infty\mathscr{D}_{\min},\nabla_{\min})$$

induces an isomorphism of cohomology groups. Since $\Delta_{b,\mathrm{rel}}:\mathcal{D}^\infty\mathscr{D}_{\min}\to\mathcal{D}^\infty\mathscr{D}_{\min}$, then $\Delta_{b,\mathrm{rel}}$ induces an isomorphism

$$\Delta_{b,\mathrm{rel}}:\mathcal{D}^\infty\mathscr{D}_{\min}/\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}}\to\mathcal{D}^\infty\mathscr{D}_{\min}/\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}}. \tag{3.20}$$

So it induces an isomorphism on cohomology group

$$\Delta_{b,\mathrm{rel}}:H^*\left(\mathcal{D}^\infty\mathscr{D}_{\min}/\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}},\nabla_{\min}\right)\to H^*\left(\mathcal{D}^\infty\mathscr{D}_{\min}/\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}},\nabla_{\min}\right). \tag{3.21}$$

For $[x]\in H^*\left(\mathcal{D}^\infty\mathscr{D}_{\min}/\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}},\nabla_{\min}\right)$, we have $x=z+\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}}$ with $\nabla_{\min}z\in\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}}$. Then by $\Delta_{b,\mathrm{rel}}z=\nabla^\#_{\min}\nabla_{\min}z+\nabla_{\min}\nabla^\#_{\min}z$ and $\nabla^\#_{\min}\Delta_{b,\mathrm{rel}}=\Delta_{b,\mathrm{rel}}\nabla^\#_{\min}$, we get $\Delta_{b,\mathrm{rel}}z-\nabla_{\min}\nabla^\#_{\min}z\in\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}}$. Then by definition we get $\Delta_{b,\mathrm{rel}}[x]=0$. Since (3.21) is an isomorphism, we get

$$H^*(\mathcal{D}^\infty\mathscr{D}_{\min}/\mathcal{D}^\infty\mathscr{D}_{\min,\{0\}},\nabla_{\min})\cong\{0\}.$$

So that $H^*(\mathscr{D}_{\min,\{0\}},\nabla_{\min,\{0\}})\cong H^*(\mathscr{D}_{\min},\nabla_{\min})$. In particular for any $a\geq 0$, we have

$$H^*(\mathscr{D}_{\min,[0,a]},\nabla_{\min,[0,a]})\cong H^*(\mathscr{D}_{\min},\nabla_{\min})$$

and the subcomplex $(\mathscr{D}_{\min,(a,\infty)},\nabla_{\min,(a,\infty)})$ is acyclic. □

For a finite dimensional complex vector space V, we define

$$\det V = \Lambda^{\max} V. \tag{3.22}$$

Then for the complex $(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]})$, we define the complex determinant lines

$$\det\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) = \bigotimes_{k=0}^{n} \left(\det\left(\mathscr{D}_{\min,[0,a],k}\right)\right)^{(-1)^k} \tag{3.23}$$

and

$$\det H^*\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) = \bigotimes_{k=0}^{n} \left(\det H^k\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right)\right)^{(-1)^k}. \tag{3.24}$$

Let $\mathscr{D}_{\min,[0,a],k} = \mathscr{D}_{\min,[0,a]} \cap L^2_k(M,E)$ and the induced nondegenerate symmetric bilinear form denoted by $b_{\min,[0,a],k}$. Then by $b_{\min,[0,a],k}$ and (3.23), we get a nondegenerate symmetric bilinear form on $\det(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]})$ and denote it by $b_{\det(\mathscr{D}_{\min,[0,a]})}$. By the canonical isomorphism (cf. [14] and [1, Section 1a)])

$$\det H^*\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) \cong \det\left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) \tag{3.25}$$

and the isomorphism (3.19), we get a nondegenerate symmetric bilinear form on the determinant line $\det H^*(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}) = \det H^*(\mathscr{D}_{\min}, \nabla_{\min})$ and denote by $b_{\det H(\mathscr{D}_{\min,[0,a]})}$.

For the subcomplex $(\mathscr{D}_{\min,(a,\infty)}, \nabla_{\min,(a,\infty)})$, we define the Laplace operator by

$$\Delta_{b,\mathrm{rel},(a,\infty)} = \nabla_{\min,(a,\infty)} \nabla^{\#}_{\min,(a,\infty)} + \nabla^{\#}_{\min,(a,\infty)} \nabla_{\min,(a,\infty)}. \tag{3.26}$$

Where $\nabla^{\#}_{\min_{(a,\infty)}}$ is the adjoint of $\nabla_{\min,(a,\infty)}$ with respect to the induced nondegenerate symmetric bilinear form on $\mathscr{D}_{\min,(a,\infty)}$. For $0 \leq k \leq n$, let $\Delta_{b,\mathrm{rel},(a,\infty),k}$ be the restriction of $\Delta_{b,\mathrm{rel},(a,\infty)}$ to $\mathscr{D}(\Delta_{b,\mathrm{rel},(a,\infty)}) \cap L^2_k(M,E)$. Since $\Delta_{b,\mathrm{rel}}$ has the same leading symbol with Δ_{rel}, then the following regularized zeta determinant is well defined:

$$\det{}'\left(\Delta_{b,\mathrm{rel},(a,\infty),k}\right) = \exp\left(-\frac{\partial}{\partial s}\bigg|_{s=0} \mathrm{Tr}\left[\left(\Delta_{b,\mathrm{rel},(a,\infty),k}\right)^{-s}\right]\right). \tag{3.27}$$

Theorem 3.5. *The symmetric bilinear form on* $\det H^*(\mathscr{D}_{\min}, \nabla_{\min})$ *defined by*

$$b_{\det H^*(\mathscr{D}_{\min,[0,a]})} \prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel},(a,\infty),k}\right)\right)^{(-1)^k k} \tag{3.28}$$

is independent of the choice of $a \geq 0$.

Proof. Let $0 \leq a < c < \infty$. We have

$$\left(\mathscr{D}_{\min,[0,c]}, \nabla_{\min,[0,c]}\right) = \left(\mathscr{D}_{\min,[0,a]}, \nabla_{\min,[0,a]}\right) \oplus \left(\mathscr{D}_{\min,(a,c]}, \nabla_{\min,(a,c]}\right) \tag{3.29}$$

and

$$\left(\mathscr{D}_{\min,(a,\infty)}, \nabla_{\min,(a,\infty)}\right) = \left(\mathscr{D}_{\min,(a,c]}, \nabla_{\min,(a,c]}\right) \oplus \left(\mathscr{D}_{\min,(c,\infty)}, \nabla_{\min,(c,\infty)}\right). \tag{3.30}$$

Then we have

$$\det{}'\left(\Delta_{b,\mathrm{rel},(a,\infty),k}\right) = \det{}'\left(\Delta_{b,\mathrm{rel},(a,c],k}\right) \cdot \det{}'\left(\Delta_{b,\mathrm{rel},(c,\infty),k}\right). \tag{3.31}$$

Particularly,

$$\begin{aligned} &\prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel},(a,\infty),k}\right)\right)^{(-1)^k k} \\ &\qquad = \prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel},(a,c],k}\right)\right)^{(-1)^k k} \cdot \prod_{k=0}^{n} \left(\det{}'\left(\Delta_{b,\mathrm{rel},(c,\infty),k}\right)\right)^{(-1)^k k}. \end{aligned} \tag{3.32}$$

Applying [7, Lemma 3.3] to (3.29), we get

$$b_{\det H(\mathscr{D}_{\min,[0,a]})}\cdot\prod_{k=0}^{n}\left(\det{}'\left(\Delta_{b,\mathrm{rel},(a,c],k}\right)\right)^{(-1)^k k}=b_{\det H(\mathscr{D}_{\min,[0,c]})}. \tag{3.33}$$

The proof of Theorem 3.5 is complete. □

Definition 3.6. The symmetric bilinear form defined by (3.28) is called the Ray-Singer symmetric bilinear torsion on $\det H^*(\mathscr{D}_{\min},\nabla_{\min})$ and is denoted by $b^{\mathrm{RS}}_{(M,E,g^{TM},b^E),\mathrm{rel}}$.

Remark 3.7. By Theorem 2.3, we have

$$\det H^*\left(\mathscr{D}_{\min},\nabla_{\min}\right)\cong\det H^*(M,\partial M,E). \tag{3.34}$$

So that $b^{\mathrm{RS}}_{(M,E,g^{TM},b^E),\mathrm{rel}}$ can be viewed as the extension of the Burghelea-Haller analytic torsion to compact manifolds with relative boundary condition.

4 Ray-Singer symmetric bilinear torsion for $(\mathscr{D}_{\max},\nabla_{\max})$

In this section we define the Ray-Singer symmetric bilinear torsion for the Hilbert complex $(\mathscr{D}_{\max},\nabla_{\max})$. The steps are the same as Section 3. The proofs of main results in this section are also the same as in Section 3. It can be viewed as the extension of the Burghelea-Haller analytic torsion for compact manifolds with absolute boundary condition.

Proposition 4.1. *The restriction of $\beta_{g,b}$ to $\mathscr{D}_{\max}$ is nondegenerate.*

Proof. We first recall that for $\sigma\in L^2_*(M,E)$, if there exists $\eta\in L^2_*(M,E)$ such that for any $\phi\in\Omega^*_0(M,E)$,

$$h_{g,\mu}\left(\sigma,\nabla^t\phi\right)=h_{g,\mu}(\eta,\phi), \tag{4.1}$$

then $\sigma\in\mathscr{D}_{\max}$ and $\nabla_{\max}\sigma=\eta$. If $\sigma\in\mathscr{D}_{\max}$, then for $\nu\sigma\in L^2_*(M,E)$, and for any $\phi\in\Omega^*_0(M,E)$ then $\nu\phi\in\Omega^*_0(M,E)$, we have

$$\begin{aligned}
h_{g,\mu}&\left(\nu\sigma,\nabla^t(\nu\phi)\right)=\overline{h_{g,\mu}\left(\nabla^t(\nu\phi),\nu\sigma\right)}=\overline{\beta_{g,b}\left(\nabla^t(\nu\phi),\sigma\right)}\\
&=\overline{\beta_{g,b}\left(\nu\left(\nabla^t\phi\right)-\left(\nabla\nu^*\right)^*\phi,\sigma\right)}=\overline{\beta_{g,b}\left(\nu\left(\nabla^t\phi\right),\sigma\right)}-\overline{\beta_{g,b}\left(\left(\nabla\nu^*\right)^*\phi,\sigma\right)}\\
&=\beta_{g,b}\left(\left(\nabla^t\phi\right),\nu\sigma\right)-\overline{h_{g,\mu}\left(\left(\nabla\nu^*\right)^*\phi,\nu\sigma\right)}=h_{g,\mu}\left(\left(\nabla^t\phi\right),\sigma\right)-h_{g,\mu}\left(\nu\sigma,\left(\nabla\nu^*\right)^*\phi\right)\\
&=\overline{h_{g,\mu}\left(\sigma,\left(\nabla^t\phi\right)\right)}-h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma,\nu\phi\right)=\overline{h_{g,\mu}(\eta,\phi)}-h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma,\nu\phi\right)\\
&=h_{g,\mu}(\phi,\eta)-h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma,\nu\phi\right)=\beta_{g,b}(\phi,\nu\eta)-h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma,\nu\phi\right)\\
&=\overline{\beta_{g,b}(\nu\phi,\eta)}-h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma,\nu\phi\right)=\overline{h_{g,\mu}(\nu\phi,\nu\eta)}-h_{g,\mu}\left(\nu^*\left(\nabla\nu^*\right)\nu\sigma,\nu\phi\right)\\
&=h_{g,\mu}\left(\nu\eta-\nu^*\left(\nabla\nu^*\right)\nu\sigma,\nu\phi\right).
\end{aligned} \tag{4.2}$$

Then by definition we get $\nu\sigma\in\mathscr{D}_{\max}$. Then by the same reason in Proposition 3.1, we get that the restriction of $\beta_{g,b}$ to $\mathscr{D}_{\max}$ is nondegenerate.

The proof of Proposition 4.1 is complete. □

Proposition 4.2. *The following identity holds*

$$\nabla^{\#}_{\max}=\nabla^*_{\max}+\left(\nabla\nu^*\right)^*\nu. \tag{4.3}$$

Particularly, $\mathscr{D}^{\#}_{\max}$ equals the domain of $\nabla^*_{\max}$.

Proof. For $x \in \mathscr{D}_{\max}$, by (4.2) we have $\nu x \in \mathscr{D}_{\max}$ and

$$\nabla_{\max}(\nu x) = \nu\left(\nabla_{\max} x\right) - \nu^*\left(\nabla \nu^*\right)\nu x. \tag{4.4}$$

Let $y \in \mathscr{D}^{\#}_{\max}$, then there exists $z \in L^2_*(M,E)$ such that for any $x \in \mathscr{D}_{\max}$, we have

$$\beta_{g,b}\left(\nabla_{\max} x, y\right) = \beta_{g,b}(x,z). \tag{4.5}$$

Then

$$\begin{aligned} h_{g,\mu}\left(\nabla_{\max}(\nu x), y\right) &= h_{g,\mu}\left(\nu \nabla_{\max} x - \nu^*\left(\nabla \nu^*\right)\nu x, y\right) \\ &= h_{g,\mu}\left(\nu \nabla_{\max} x, y\right) - h_{g,\mu}\left(\nu x, \left(\nabla \nu^*\right)^* \nu y\right) = \overline{\beta_{g,b}\left(\nabla_{\max} x, y\right)} - h_{g,\mu}\left(\nu x, \left(\nabla \nu^*\right)^* \nu y\right) \\ &= h_{g,\mu}(\nu x, z) - h_{g,\mu}\left(\nu x, \left(\nabla \nu^*\right)^* \nu y\right). \end{aligned} \tag{4.6}$$

By definition we have $y \in \mathscr{D}(\nabla^*_{\max})$ and

$$\nabla^*_{\max} y = \nabla^{\#}_{\max} y - \left(\nabla \nu^*\right)^* \nu y. \tag{4.7}$$

The proof of Proposition 4.2 is complete. □

Let $\Delta_{b,\mathrm{abs}} = \nabla^{\#}_{\max}\nabla_{\max} + \nabla_{\max}\nabla^{\#}_{\max}$. By Proposition 4.2, $\Delta_{b,\mathrm{abs}}$ has the same leading symbol with $\Delta_{\mathrm{abs}} = (\nabla_{\max} + \nabla^*_{\max})^2$, so the spectral of $\Delta_{b,\mathrm{abs}}$ are discrete. For any $a \geq 0$, let $P_{[0,a],\Delta_{b,\mathrm{abs}}}$ be the spectral projection of $\Delta_{b,\mathrm{abs}}$ corresponding to the spectral with absolute value in $[0,a]$. Then we have the decomposition

$$L^2_*(M,E) = \mathrm{Im} P_{[0,a],\Delta_{b,\mathrm{abs}}} \oplus \mathrm{Im}\left(1 - P_{[0,a],\Delta_{b,\mathrm{abs}}}\right), \tag{4.8}$$

and $\mathrm{Im} P_{[0,a],\Delta_{b,\mathrm{abs}}}$, $\mathrm{Im}(1 - P_{[0,a],\Delta_{b,\mathrm{abs}}})$ are closed subspaces of $L^2_*(M,E)$. Then by the same proof in Proposition 3.3, the decomposition (4.8) is $\beta_{g,b}$-orthogonal. By (4.8), we have the $\beta_{g,b}$-orthogonal decomposition of $\mathscr{D}_{\max}$ as

$$\mathscr{D}_{\max} = \mathscr{D}_{\max,[0,a]} \oplus \mathscr{D}_{\max,(a,\infty)}, \tag{4.9}$$

where $\mathscr{D}_{\max,[0,a]} = \mathscr{D}_{\max} \cap \mathrm{Im} P_{[0,a],\Delta_{b,\mathrm{abs}}}$ and $\mathscr{D}_{\max,(a,\infty)} = \mathscr{D}_{\max} \cap \mathrm{Im}(1 - P_{[0,a],\Delta_{b,\mathrm{abs}}})$. By Proposition 4.1, we get that the restrictions of $\beta_{g,b}$ to $\mathscr{D}_{\max,[0,a]}$ and $\mathscr{D}_{\max,(a,\infty)}$ are nondegenerate. Since $\nabla_{\max}$ commutes with $\Delta_{b,\max}$, we have the following decomposition of the complex $(\mathscr{D}_{\max}, \nabla_{\max})$,

$$\left(\mathscr{D}_{\max}, \nabla_{\max}\right) = \left(\mathscr{D}_{\max,[0,a]}, \nabla_{\max,[0,a]}\right) \oplus \left(\mathscr{D}_{\max,(a,\infty)}, \nabla_{\max,(a,\infty)}\right). \tag{4.10}$$

By the same proof of Proposition of 3.4, we have

Proposition 4.3. *For any $a \geq 0$, we have*

$$H^*\left(\mathscr{D}_{\max,[0,a]}, \nabla_{\max,[0,a]}\right) \cong H^*\left(\mathscr{D}_{\max}, \nabla_{\max}\right). \tag{4.11}$$

Let $\mathscr{D}_{\max,[0,a],k} = \mathscr{D}_{\max,[0,a]} \cap L^2_k(M,E)$. Let

$$\det\left(\mathscr{D}_{\max,[0,a]}, \nabla_{\max,[0,a]}\right) = \bigotimes_{k=0}^{n}\left(\det\left(\mathscr{D}_{\max,[0,a],k}\right)\right)^{(-1)^k} \tag{4.12}$$

and

$$\det H^*\left(\mathscr{D}_{\max,[0,a]}, \nabla_{\max,[0,a]}\right) = \bigotimes_{k=0}^{n}\left(\det H^k\left(\mathscr{D}_{\max,[0,a]}, \nabla_{\max,[0,a]}\right)\right)^{(-1)^k} \tag{4.13}$$

be the determinant lines of $(\mathscr{D}_{\max,[0,a]},\nabla_{\max,[0,a]})$ and $H^*(\mathscr{D}_{\max,[0,a]},\nabla_{\max,[0,a]})$ respectively. Then we have a canonical isomorphism (cf. [14] and [1, Section 1a)])

$$\det\left(\mathscr{D}_{\max,[0,a]},\nabla_{\max,[0,a]}\right)\cong \det H^*\left(\mathscr{D}_{\max,[0,a]},\nabla_{\max,[0,a]}\right). \tag{4.14}$$

Let $b_{\mathscr{D}_{\max,[0,a]},k}$ denote the induced nondegenerate symmetric bilinear form from $\beta_{g,b}$, then by (4.12) and (4.14) it induces a symmetric bilinear form on $\det H^*(\mathscr{D}_{\max,[0,a]},\nabla_{\max,[0,a]})$ and denote it by $b_{\det H^*(\mathscr{D}_{\max,[0,a]})}$.

Let $\nabla^{\#}_{\max,(a,\infty)}$ be the adjoint of $\nabla_{\max,(a,\infty)}$ with respect to the induced symmetric bilinear form on $(\mathscr{D}_{\max,(a,\infty)},\nabla_{\max,(a,\infty)})$, and define

$$\Delta_{b,\mathrm{abs},(a,\infty)}=\nabla_{\max,(a,\infty)}\nabla^{\#}_{\max,(a,\infty)}+\nabla^{\#}_{\max,(a,\infty)}\nabla_{\max,(a,\infty)}. \tag{4.15}$$

For $0\leq k\leq n$, let $\Delta_{b,\mathrm{abs},(a,\infty),k}$ be the restriction of $\Delta_{b,\mathrm{abs},(a,\infty)}$ to $\mathscr{D}(\Delta_{b,\mathrm{abs},(a,\infty)})\cap L^2_k(M,E)$. Since $\Delta_{b,\mathrm{abs}}$ has the same leading symbol with Δ_{abs}, then the following regularized zeta determinant is well defined:

$$\det{}'\left(\Delta_{b,\mathrm{abs},(a,\infty),k}\right)=\exp\left(-\left.\frac{\partial}{\partial s}\right|_{s=0}\mathrm{Tr}\left[\left(\Delta_{b,\mathrm{abs},(a,\infty),k}\right)^{-s}\right]\right). \tag{4.16}$$

By the same proof of Theorem 3.5, we have

Theorem 4.4. *The symmetric bilinear form on* $\det H^*(\mathscr{D}_{\max},\nabla_{\max})$ *defined by*

$$b_{\det H^*(\mathscr{D}_{\max,[0,a]})}\prod_{k=0}^{n}\left(\det{}'\left(\Delta_{b,\mathrm{abs},(a,\infty),k}\right)\right)^{(-1)^k k} \tag{4.17}$$

is independent of the choice of $a\geq 0$.

Definition 4.5. The symmetric bilinear form defined by (4.17) is called the Ray-Singer symmetric bilinear torsion on $\det H^*(\mathscr{D}_{\max},\nabla_{\max})$ and is denoted by $b^{\mathrm{RS}}_{(M,E,g^{TM},b^E),\mathrm{abs}}$.

Remark 4.6. By Theorem 2.3, we have

$$\det H^*\left(\mathscr{D}_{\max},\nabla_{\max}\right)\cong \det H^*(M,E). \tag{4.18}$$

So that $b^{\mathrm{RS}}_{(M,E,g^{TM},b^E),\mathrm{abs}}$ can be viewed as the extension of the Burghelea-Haller analytic torsion to compact manifolds with absolute boundary condition.

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