THE HOMEOMORPHISM TYPES OF CONTRACTIBLE PLANAR POLYHEDRA

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Abstract. Invariants are constructed to classify all homeomorphism types of contractible planar polyhedra. This result relies on a new classification of contractible 2-manifolds via "cyclic Cantor sets".

A polyhedron is planar if it admits an embedding into the plane $\mathbf{R}^2$. In this paper we classify all contractible planar polyhedra. In particular, locally finite trees and contractible 2-manifolds are such polyhedra. We show that contractible 2-manifolds are classified by "cyclic Cantor sets" and that homeomorphism types of contractible planar polyhedra are in 1-1 correspondence with equivalence classes of "Cantorian trees". A Cantorian tree essentially is a tree and a collection of cyclic Cantor sets.

Brown-Messer [B-M] classified all 2-manifolds by using "abstract 2-manifold diagrams". However, such diagrams even for contractible 2-manifolds are quite intricate and therefore we replace them by the natural notion of a cyclic Cantor set. We give some examples of contractible planar polyhedra defined by the universal cover of 2-dimensional polyhedra and we decribe their associated Cantorian trees.

In the literature very little is known on the classification of homeomorphism types of polyhedra. Whittlesey ([W1], [W2], [W3) classifies the homeomorphism types of finite 2-dimensional polyhedra. Our classification here seems to be the first in the literature considering a class of infinite polyhedra which need not to be manifolds.

§1 On the classification of contractible 2-dimensional manifolds

We consider contractible 2-dimensional topological manifolds M. It is well known that M is the open unit disk $\overset{\circ}{D^2} \cong \mathbf{R}^2$ if the boundary is empty. Moreover M is the closed unit disk D^2 in $\mathbf{R}^2$ if M is compact. There are, however, further examples of such manifolds which are obtained from D^2 by removing a closed subset of the boundary ∂D^2. We show that homeomorphism types of contractible topological 2-manifolds $\neq D^2$, $\mathbf{R}^2$ are in 1-1 correspondence with "cyclic Cantor sets".

A Cantor set C is a compact totally disconnected metrizable space. We always assume that C is non-empty. Each Cantor set is homeomorphic to a subspace of the classical "middle third" Cantor space of the real line $\mathbf{R}$. An ordered Cantor set $C = (C, \leq)$ is a Cantor set C together with a total ordering $\leq$ such that the topology given by the ordering coincides with the topology of C. Here the basic open sets of the topology of the ordering $\leq$ are the "open intervals" $C(a,b) = \{x \in C; a < x < b\}$ for $a, b \in C$. Let $C[a,b] = \{x \in C; a \leq x \leq b\}$ be the "closed interval". Since C is compact we know that the maximun $max \in C$ and the minimun $min \in C$ are defined so that $C = C[min, max]$. A closed subset of an ordered Cantor set is again an ordered Cantor set.

(1.1) Definition: A small closed interval in an ordered Cantor set C is an interval $C[a,b]$ with $C(a,b) = \emptyset$ and $a < b$. Two ordered Cantor sets C and C' are cyclic equivalent if there exist small closed intervals $C[a,b]$ and $C'[a',b']$ such that there are order preserving homeomorphisms

$$\psi_1 : C[min, a] \cong C'[b', max]$$

and

$$\psi_2 : C[b, max] \cong C'[min, a']$$

Let $\psi : C \cong C'$ be the homeomorphism given by the union of ψ_1 and ψ_2. The cyclic equivalence class of C, $\overline{C}$, is called a cyclic ordered Cantor set. The opposite $-\overline{C}$ of $\overline{C}$ is represented by $(C, \leq^{op})$ where we define $a \leq^{op} b$ if and only if $b \leq a$. The cyclic Cantor set $\overline{\overline{C}}$ is the equivalence class $\{\overline{C}, -\overline{C}\}$.

As usual a 1-1 correspondence is a function which is injective and surjective.

(1.2) Theorem: *There is a 1-1 correspondence between homeomorphism types of contractible 2-dimensional manifolds $\neq D^2$, $\mathbf{R}^2$ and cyclic Cantor* *sets. Given a cyclic Cantor set $\overset{=}{C}$ represented by $(C, \leq)$ we can choose an order preserving embedding $C \subset \mathbf{R}$ with $C \subset \mathbf{R} \subset \mathbf{R} \cup \infty = \partial D^2$ and we obtain a contractible 2-manifold by the complement $D^2 - C$. The 1-1 correspondence carries $\overset{=}{C}$ to the homeomorphism type of $D^2 - C$. The inverse of the correspondence is described in theorem (1.6)*

We remark that there is a similar 1-1 correspondence between orientation preserving homeomorphism types of contractible oriented 2-manifolds $\neq D^2$, $\mathbf{R}^2$ and cyclic ordered Cantor sets.

For a contractible 2-manifold $\neq D^2$, $\mathbf{R}^2$ we consider the diagram

(1.3)
$$\mathcal{E}(M) = \{\ \pi_0(\partial M) \underset{\theta}{\overset{e}{\leftrightarrows}} End(\partial M) \overset{\epsilon}{\hookrightarrow} End(M)\ \}$$

Here π_0 is the set of path components and End is the space of Freudenthal ends [F]. The map e is defined by $x \in End(e(x))$, and ϵ is induced by the inclusion $\partial M \subset M$. Moreover an orientation of M yields the section θ of e since ∂M is a disjoint union of oriented open intervals in $\mathbf{R}$, θ carries the path component $C \cong (-\infty_C, \infty_C)$ of ∂M to $+\infty_C$. The diagram (1.3) leads to the following notion of Cantor diagrams.

(1.4) **<u>Definition</u>:** A <u>Cantor diagram</u>

$$E \underset{\theta}{\overset{e}{\leftrightarrows}} I \overset{\epsilon}{\hookrightarrow} C$$

consists of a Cantor set C, countable sets E and I and functions ϵ, e, and θ such that θ is a section of e and e is surjective and two to one. The image of ϵ is dense in C. An isomorphism between Cantor diagrams is a diagram

$$\begin{tikzcd}
E \arrow[d, "\alpha"] & I \arrow[l, shift right, "e"'] \arrow[l, leftarrow, shift left, "\theta"] \arrow[r, hook, "\epsilon"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
E' & I' \arrow[l, shift right, "e"'] \arrow[l, leftarrow, shift left, "\theta"] \arrow[r, hook, "\epsilon"] & C'
\end{tikzcd}$$

with $\alpha e = e\beta$, $\beta\theta = \theta\alpha$, $\gamma\epsilon = \epsilon\beta$, for which α and β are bijections and γ is a homeomorphism.

It is easy to check that (1.3) is a Cantor diagram and that the isomorphism type of (1.3) is well defined by the homeomorphism type of M.

(1.5) Definition: Let $(C, \leq)$ be an ordered Cantor set. We define the Cantor diagram

$$D(C, \leq) = \{\ E \underset{\theta}{\overset{e}{\leftrightarrows}} I \xrightarrow{\epsilon} C\ \}$$

as follows. We introduce the "outside interval" $[max, min]$ of $(C, \leq)$ which is not an interval of C, given by the maximun and the minimun of C in reverse order. In case C consists of a single point there are no small closed intervals but the outside interval is still defined with $max = min$. Let E be the set consisting of the outside interval and all small closed intervals in $(C, \leq)$. Let I be the disjoint union of all boundary sets ∂L with $L \in E$. The map e is defined by $e(x) = L$ if and only if $x \in \partial L$. The map ϵ is the union of all inclusions $\partial L \subset C$. Moreover the section θ of e carries $[max, min]$ to $min \in I$ and carries a small interval $C[a, b] \in E$ to $b \in I$.

(1.6) Theorem: *The function which carries a cyclic Cantor set $\bar{\bar{C}}$ represented by $(C, \leq)$ to the isomorphism class, $D(\bar{\bar{C}})$, of the Cantor diagram $D(C, \leq)$ is well defined and injective. Moreover the correspondence in Theorem (1.2) carries the homeomorphism type of a contractible 2-manifold $M \neq D^2, \mathbf{R}^2$ to the cyclic Cantor set $\bar{\bar{C}}$ for which $D(C, \leq)$ is isomorphic to $\mathcal{E}(M)$.*

(1.7) Remark: Brown-Messer [B-M] classified homeomorphism types of all 2-manifolds by "abstract 2-manifold diagrams" which are actually equivalent to Cantor diagrams above if one considers only contractible 2-manifolds $\neq D^2, \mathbf{R}^2$. However the realizability condition for such diagrams (in particular (i),(ii),(iii) on page 393 of [B-M]) is fairly complicated even in the case of contractible 2-manifolds. In this case we can replace the realizability condition of Brown-Messer by the following corollary of (1.6)

(1.8) Corollary: *A Cantor diagram D satisfies $D \cong \mathcal{E}(M)$ for a contractible 2-manifold $M \neq D^2, \mathbf{R}^2$ if and only if there is a cyclic Cantor set $\bar{\bar{C}}$ with $D \in D(\bar{\bar{C}})$.*

This realizability condition via ordered Cantor sets could be generalized to obtain an alternative realizability condition for the "abstract 2-manifold diagrams" of Brown-Messer in the case of arbitrary 2-manifolds. We consider the following examples of contractible 2-manifolds.

(2.9) Example: Let M be a discrete and closed subset of the plane and let X_M be the universal covering space of $\mathbf{R}^2 - U(M)$ where $U(M)$ is a small open neighbourhood of M. Moreover, let N be a finite subset of $\overset{\circ}{D}{}^2$, and let X'_N be the universal covering space of $D^2 - U(N)$. Then X_M, X'_N are contractible 2-manifolds. Moreover all manifolds X_M, X'_N with $\#M, \#N \geq 2$ are of the same homeomorphism type corresponding via (1.2) to the cyclic Cantor set $\overset{=}{C}$ where C is the whole "middle third" Cantor space.

§2 Cantorian trees.

We here define the notion of Cantorian trees. Equivalence classes of such trees are in $1-1$ correspondence with homeomorphism types of contractible planar polyhedra $\neq D^2, \mathbf{R}^2$. This is the main theorem of this paper.

We use the following notation. Let $(C \leq)$ be an ordered Cantor set and let $D(C, \leq) = \{E, I, C, e, \theta, \epsilon\}$ be defined as in (1.5). For $L \in E$ we have $\theta(L) \in \partial L$ and we define $\theta'(L) \in \partial L$ by $\{\theta'(L)\} = \partial L - \{\theta(L)\}$. An interval $\mathbf{Z}[a, b]$ in $\mathbf{Z}$ with $-\infty \leq a < b \leq \infty$ is the subset $\{x \in \mathbf{Z}; a \leq x \leq b\}$. In the following we only use the special intervals $\mathbf{Z}[0, b]$ with $0 < b \leq \infty$, $\mathbf{Z}[-\infty, 0]$, and $\mathbf{Z}[-\infty, \infty]$. A <u>sequence</u> S in $(C, \leq)$ is a collection of elements $L_i \in E$ where i is an element in a special interval $\mathbf{Z}[a, b]$ such that for $i, i+1 \in \mathbf{Z}[a, b]$

$$\theta(L_i) = \theta'(L_{i+1})$$

we also write $S = S(a, b)$. The subset of C consisting of the points $\theta(L_i)$ with $i \in \mathbf{Z}[a, b]$ and $i < b$ is called the <u>interior</u> $\overset{o}{S}$ of S. Moreover, let $|S|$ with $\overset{o}{S} \subset |S| \subset C$ be the set of all elements $\theta(L_i), \theta'(L_i)$ with $i \in \mathbf{Z}[a, b]$.

An <u>ordered Cantor set with sequences</u> $(C, \leq, \mathcal{S})$ is an ordered Cantor set $(C, \leq)$ together with a set of sequences $\mathcal{S}$ in $(C, \leq)$ satisfying

$$\overset{o}{S} \cap |S'| = \emptyset, \quad S, S' \in \mathcal{S} \textit{ and } S \neq S'$$

The opposite $(C, \leq^{op}, \mathcal{S}^{op})$ of $(C, \leq, \mathcal{S})$ is the ordered Cantor set with sequences given by the opposite ordering of C and by $\mathcal{S}^{op} = \{S^{op}; S \in \mathcal{S}\}$ where S^{op} is the canonical reverse sequence in $(C, \leq^{op})$ determined by S.

The <u>interior</u> of $\mathcal{S}$ is the subset of C given by the disjoint union

$$\overset{o}{\mathcal{S}} = \bigcup \{\overset{o}{S}; S \in \mathcal{S}\}$$

Clearly the interior of $\mathcal{S}^{op}$ coincides with the interior of $\mathcal{S}$. Let $\psi : (C, \leq) \cong (C', \leq)$ be a cyclic equivalence as in (1.1). Then ψ carries a sequence S in $(C, \leq)$ to a sequence ψS in $(C', \leq)$ so that for $\psi\mathcal{S} = \{\psi S; S \in \mathcal{S}\}$ one has the bijection of interiors $\psi : \overset{o}{\mathcal{S}} \cong (\psi\mathcal{S})^o$ induced by the homeomorphism $\psi : C \cong C'$.

An <u>equivalence</u> $\psi : (C, \leq \mathcal{S}) \cong (C', \leq, \mathcal{S}')$ is either a cyclic equivalence $\psi : (C, \leq) \cong (C', \leq)$ or a cyclic equivalence $\psi : (C, \leq) \cong (C', \leq^{op})$ such that $\mathcal{S}' = \psi\mathcal{S}$ or $\mathcal{S}' = (\psi\mathcal{S})^{op}$

(2.1) <u>Definition</u>: A <u>Cantorian tree</u> $\mathcal{T}$ is a tuple

$$\mathcal{T} = (T,\ P,\ A,\ \sigma, \lambda)$$

where T is a tree, P and A are subsets of vertices of T, that is $A, P \subset T^0$, and where σ is a function which carries $t \in P$ to an ordered Cantor set with sequences $\sigma(t) = (C_t, \leq, \mathcal{S}_t)$ as above. Moreover λ is a collection of bijections

$$\lambda_t : \overset{o}{\mathcal{S}}_t \cong link(t;T); \quad t \in P$$

As usual $link(t;T)$ is the subset of vertices of T which are connected with t by an edge. We assume that T^0 is countable and that $T - P$ is locally finite. Moreover $A \cap P = \emptyset$, and

$$a \in A, \ then \ \ link(a;T) \cap P \neq \emptyset,$$

$$t \in P, \ then \ \ link(t;T) \subset A$$

Note that this is equivalent to $A = \bigcup\{link(t;T), t \in P\}$.

Two Cantorian trees, $\mathcal{T}$ and $\mathcal{T}'$, are equivalent if there exists a homeomorphism

$$\tau : (T, P, A) \cong (T', P', A')$$

together with a collection of equivalences $\psi_t : \sigma(t) \cong \sigma'(\tau(t))$ for $t \in P$ such that the diagram

$$\begin{array}{ccc} \overset{o}{\mathcal{S}}_t & \xrightarrow{\lambda_t} & link(t;T) \\ \downarrow{\scriptstyle \psi_t} & & \downarrow{\scriptstyle \tau} \\ \overset{o}{\mathcal{S}}_{\tau(t)} & \xrightarrow{\lambda'_{\tau(t)}} & link(\tau(t);T') \end{array}$$

commutes for all $t \in P$. Notice that if $T = P = \{*\}$ is a point then the equivalence class of the Cantorian tree is just a cyclic Cantor set.

(2.2) **Theorem:** *There is a 1-1 correspondence between homeomorphism types of contractible planar polyhedra $\neq D^2, \mathbf{R}^2$ and equivalence classes of Cantorian trees.*

The 1-1 correspondence in this theorem yields for each Cantorian tree $\mathcal{T}$ the contractible planar polyhedron $X(\mathcal{T})$ constructed as follows. For each $t \in P$ we choose a contractible 2-manifold M_t associated to $\bar{\bar{C}}_t$ by (1.2); that is $M_t = D^2 - C_t$. We define for $t \in P$ the open cone

$$C(\overset{o}{\mathcal{S}}_t) \subset D^2 - C_t$$

Here $C(\overset{o}{\mathcal{S}}_t)$ consists of all points λx, $x \in \overset{o}{\mathcal{S}}_t \subset C_t \subset \partial D^2$, $\lambda \in [0,1) \subset \mathbf{R}$. For $t \in T$ let $star(t;T)$ be the subtree of T generated by $\{t\} \cup link(t;T)$. Then λ_t defines a homeomorphism

$$\overline{\lambda}_t : \overset{o}{star}\,(t;T) \cong C(\overset{o}{\mathcal{S}}_t)$$

where $\overset{o}{star}\,(t;T)$ is the open star.

Now $X(\mathcal{T})$ is the union of T and all 2-manifolds $D^2 - C_t$, $t \in P$, where we identify $x \in \overset{o}{star}\,(t;T) \subset T$ with $\overline{\lambda}_t(x) \in D^2 - C_t$ for $t \in P$. More precisely $X(\mathcal{T})$ is the pushout

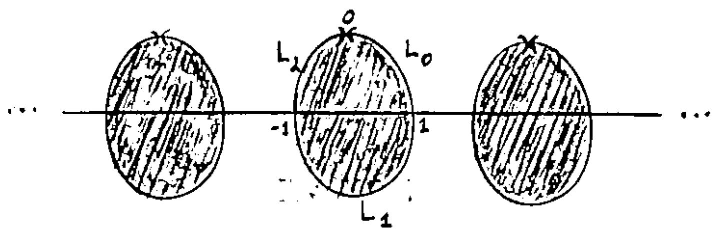
$$\begin{array}{ccc}
\coprod_{t\in P} \overset{o}{star}\,(t;T) & \longrightarrow & \coprod_{t\in P} D^2 - C_t \\
\downarrow & & \downarrow \\
T & \longrightarrow & X(\mathcal{T})
\end{array}$$

in the category of topological spaces, where "$\coprod$" denotes the disjoint union. As part of theorem (2.2) we, in particular, obtain the following result.

(2.3) **Theorem:** *Let X be a contractible planar $\neq D^2, \mathbf{R}^2$. Then there exists a homeomorphism $X \cong X(\mathcal{T})$ where $\mathcal{T}$ is a Cantorian tree.*

(2.4) **Example:** For the convenience of the reader we describe the following example which illustrates the Cantorian tree associated to a contractible planar polyhedron. Let X be the following subset of $\mathbf{R}^2$,

$$X = \mathbf{R} \times \{0\} \bigcup \{(x,y) \in \mathbf{R}^2 | (x,y) \neq (3t,1),\ and\ (x-3t)^2 + y^2 \leq 1; t \in \mathbf{Z}\}$$

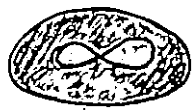


Then the Cantorian tree T for X is given as follows. The tree $T = \mathbf{R}$ is the real line and $P = 3\mathbf{Z}$, $A = \mathbf{Z} - 3\mathbf{Z}$. Moreover for all $t \in P$ we have $\sigma(t) = (C, \leq, \mathcal{S})$ where $C = \{-1, 0, 1\}$ and where $\mathcal{S} = \{S\}$ consists of the single sequence $S = (L_0, L_1, L_2)$ with $L_0 = [0,1]$, $L_1 = [1,-1]$ the outside interval, $L_2 = [-1, 0]$, and $\theta(L_0) = 1$, $\theta(L_1) = -1$, $\theta(L_2) = 0$. We have $\overset{o}{\mathcal{S}} = \{-1, 1\}$, and $\lambda_t : \overset{o}{\mathcal{S}} \cong link(t; T)$ carries -1 to $t-1$ and 1 to $t+1$ with $t \in 3\mathbf{Z} = P$.

(2.5) __**Notation**__: A) For any set M we define the __M-tree__ $T[M]$ generated by M as follows. The set of vertices is the free monoid generated by M, that is $T[M]^0 = Mon(M)$. The edges are all pairs of vertices of the form (a, ax) with $a \in Mon(M), x \in M$. Here ax is defined by the multiplication in $Mon(M)$. Let $\emptyset$ be the empty word in $Mon(M)$ which is the unit.

B) Given a tree T and $k \in \mathbf{N}$ we define the tree $(1/k)T$ to be the tree obtained by introducing $k-1$ subdivision points in each edge of T.

(2.6) __**Examples:**__ Let X be the universal covering space of the space



We describe the Cantorian tree T with $X = X(T)$ as follows. Let T be the tree $T = (1/2)T[M]$ with $M = \mathbf{Z} - \{0\}$ given by notation in (2.5). Let P be the set of vertices in $T[M]$ and let A be the set of subdivision points denoted by $(1/2)(a, ax)$. For each $t \in P$ let $C_t = C$ be the ordered Cantor set

$$C = \{0, 3\} \cup \{1/n,\ 3 - 1/n; n \in \mathbf{N}\} \subset \mathbf{R}$$

The set $\mathcal{S}_t = \mathcal{S}$ of sequences in C consists of the single sequence S of intervals L_j, $j \in \mathbf{Z}[-\infty, \infty]$, with

$$L_j = \begin{cases} [1/n + 1, 1/n] & j = -n \\ [1, 2] & j = 0 \\ [3 - 1/n, 3 - 1/n + 1] & j = n \end{cases}$$

where $n \in \mathbf{N}$. Here the interior is

$$\overset{o}{\mathcal{S}} = \{1/n,\ 3 - 1/n; n \in \mathbf{N}\} \subset C$$

and for $t \in P = Mon(\mathbf{Z} - \{0\})$ we define

$$\lambda_t : \overset{o}{\mathcal{S}} = \{1/n,\ 3 - 1/n; n \in \mathbf{N}\} \cong link(t;T)$$

as follows. Fot $t = \emptyset$ we have

$$link(\emptyset;T) = \{1/2(\emptyset, m); m \in M\} = M = \mathbf{Z} - \{0\}$$

and for $t \neq \emptyset$ with $t = am$, $m \in M$, we get

$$link(t;T) = \{1/2(a,t)\} \cup \{1/2(t,tm), m \in M\} = \mathbf{Z}$$

where $1/2(a,t)$ corresponds to $0 \in \mathbf{Z}$. Choosing order preserving bijections $\mathbf{Z} - \{0\} \cong \overset{o}{\mathcal{S}} \cong \mathbf{Z}$ we obtain λ_t above. This completes the definition of the Cantorian tree $\mathcal{T}$.

(2.7) Example: Let X be the universal covering space of the space

We describe the Cantorian tree $\mathcal{T}$ with $X = X(\mathcal{T})$ as follows. Let T be the tree

$$T = (1/3)T[N]\ with\ N = (M - \{0\}) \cup (M' - \{0\})$$

where M and M' are two copies of $\mathbf{Z}$. Let P be the set of vertices of $T[N]$ and let A be the set of subdivisdion points denoted by $1/3(a, ax)$ and $2/3(a, ax)$. For each $t \in P$ let $C_t = C$ be the ordered Cantor set

$$C = \{0, 3\} \cup \{\pm 1/n,\ 3 \pm 1/n; n \in \mathbf{N}\} \subset \mathbf{R}$$

The set $\mathcal{S}_t = \mathcal{S}$ of sequences in C consists of two sequences S, S' given as follows. Let S be the same sequence as in (2.6) and let S' be given by the intervals L'_j, $j \in \mathbf{Z}[-\infty, \infty]$, with

$$L'_j = \begin{cases} [3 + 1/(n+1), 3 + 1/n] & j = -n \\ [max, min] & j = 0 \\ [-1/n, -1/(n+1)] & j = n \end{cases}$$

where $n \in \mathbf{N}$. Here $max = 4$ and $min = -1$ yield the outside interval. Now the interior of $\mathcal{S}$ is

$$\overset{o}{\mathcal{S}} = \{\pm 1/n,\ 3 \pm 1/n; n \in \mathbf{N}\} \subset C$$

and for $t \in P = Mon(N)$ we define

$$\lambda_t : \overset{o}{\mathcal{S}} = \{\pm 1/n, 3 \pm 1/n; n \in \mathbf{N}\} \cong link(t;T)$$

as follows. For $t = \emptyset$ we have

$$link(\emptyset;T) = \{1/3(\emptyset, n); n \in N\} = N$$

and for $t \neq \emptyset$ with $t = an$, $n \in N$, we get

$$link(t;T) = \{2/3(a,t)\} \cup \{1/3(t,tn), n \in N\} = N_t$$

with

$$N_t = \begin{cases} M \cup (M' - \{0\}) & t = am,\ m \in M - \{0\} \\ (M - \{0\}) \cup M' & t = am',\ m' \in M' - \{0\} \end{cases}$$

Here $2/3(a,t)$ corresponds to $0 \in M$ and $0' \in M'$ respectively. We choose an order preserving bijection $\epsilon : \mathbf{Z} = \mathbf{Z} - \{0\}$ which induces $N_t = N$ by $\epsilon \cup 1$ and $1 \cup \epsilon$ respectively.

Let $\lambda_\emptyset$ be the disjoint union of the unique order preserving bijections

$$\{1/n; n \in \mathbf{N}\} = [-\infty, -1] \subset M, \quad \{3 - 1/n; n \in \mathbf{N}\} = [1, \infty] \subset M$$

$$\{3 + 1/n; n \in \mathbf{N}\} = [-\infty, -1] \subset M', \quad \{-1/n; n \in \mathbf{N}\} = [1, \infty] \subset M'$$

Then λ_t is the composition

$$\lambda_t : \overset{o}{\mathcal{S}} \overset{\lambda_\emptyset}{=\!=} N = N_t = link(t;T)$$

This completes the definition of the Cantorian tree $\mathcal{T}$.

§3 <u>Contractible 2-manifolds</u>

We here prove the results in §1. We need the following lemmas. Let $[0,1]$ denote the unit interval in $\mathbf{R}$.

(3.1) **<u>Lemma</u>:** *For each ordered Cantor set $(C, \leq)$ there is an order preserving embedding $C \subset [0,1]$ which carries min to 0 and max to 1.*

<u>Proof</u>: As C is Hausdorff and totally disconnected we can find two disjoint open and closed subsets A_0, and A_1 with $minC \in A_0$, $maxC \in A_1$ and

$C = A_0 \cup A_1$. Since A_1 is an ordered Cantor set take $C_1 = \{x \in C; x \geq minA_1\}$. As $A_0 \cap A_1 = \emptyset$, it is easily checked that C_1 is an open and closed subset of C. Then $C_0 = C - C_1$ is an ordered Cantor set. Notice that $minA_1 = minC_1$. In addition $maxC_0 \in A_0$, and $maxC_0 < minC_1$. Moreover, $P = C[maxC_0, minC_1]$ is a small interval in C. We can repeat this procedure inside both C_0 and C_1 if $C_0, C_1 \neq \{*\}$, and we get two disjoint decompositions $C_0 = C_{0,0} \cup C_{0,1}$, $C_1 = C_{1,0} \cup C_{1,1}$ and two small intervals $P_0 = C[maxC_{0,0}, minC_{0,1}]$, and $P_1 = C[maxC_{1,0}, minC_{1,1}]$. When $C_0 = \{*\}$ we take $C_{0,0} = C_0$ and $C_{0,1} = \emptyset$. Similarly for C_1.

We define inductively two families of intervals in C $(n \geq 1)$

$$\{C_{i_1,i_2,\ldots,i_n}; (i_1, i_2, \ldots, i_n) \in \{0,1\}^n\}$$

$$\{P_{i_1,i_2,\ldots,i_{n-1}}); (i_1, i_2, \ldots, i_{n-1}) \in \{0,1\}^{n-1}\}$$

such that $P_{i_1,i_2,\ldots,i_{n-1}} \subset C_{i_1,i_2,\ldots,i_{n-1}}$ is the small interval

$$C[maxC_{i_1,i_2,\ldots,i_{n-1},0}, minC_{i_1,i_2,\ldots,i_{n-1},1}]$$

when $C_{i_1,i_2,\ldots,i_{n-1}} \neq \{*\}$. Otherwise we define $C_{i_1,i_2,\ldots,i_{n-1},0} = C_{i_1,i_2,\ldots,i_{n-1}}$, $C_{i_1,i_2,\ldots,i_{n-1},1} = \emptyset$, and $P_{i_1,i_2,\ldots,i_{n-1}} = \emptyset$. Furthermore, the above intervals satisfy the following properties

$$(1)\quad C_{i_1,i_2,\ldots,i_{n-1},0} \cup C_{i_1,i_2,\ldots,i_{n-1},1} = C_{i_1,i_2,\ldots,i_{n-1}}$$

$$(2)\quad C = \bigcup_{n=1}^{\infty} \{\, C_{i_1,i_2,\ldots,i_{n-1},i_n}; (i_1, i_2, \ldots, i_n) \in \{0,1\}^n\}$$

Moreover, each element $x \in C$ defines a unique sequence $(i_1^x, i_2^x, \ldots, i_n^x, \ldots) \in \{0,1\}^{\mathbf{N}}$ with $\{x\} = \bigcap_{n=1}^{\infty} C_{i_1^x,i_2^x,\ldots,i_n^x}$

By using this fact, we define the function $f : C \to [0,1]$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{2i_n^x}{3^n}$$

If $x < y$ are two elements in C, take $n_0 = max\{n; x, y \in C_{i_1,i_2,\ldots,i_{n-1},i_n}\}$. Then $x \in C_{i_1,i_2,\ldots,i_{n_0-1},0}$, and $y \in C_{i_1,i_2,\ldots,i_{n_0-1},1}$. Therefore

$$f(y) - f(x) \geq \frac{2}{3^{n_0+1}} - \sum_{n=n_0+2}^{\infty} \frac{2}{3^n} = \frac{2}{3^{n_0+1}} - \frac{1}{3^{n_0+1}} = \frac{1}{3^{n_0+1}}$$

This shows that f is an order preserving map. It is clear from the definition that $f(minC) = 0$ and $f(maxC) = 1$.

q.e.d.

(3.2) Lemma: *Let $(C, \leq)$ and $(C', \leq)$ be ordered Cantor sets and let $\phi : C \to C'$ be a homeomorphism with $\phi(min) = min$ and $\phi(max) = max$, and such that for each small interval $C[a,b]$ in C also $C'[\phi(a), \phi(b)]$ is a small interval in C'. Then ϕ is an order preserving homeomorphism.*

Proof: Using (3.1) we can assume that C and C' are closed subspaces of the unit interval $[0,1]$, and also that $minC = minC' = 0$ as well as $maxC = maxC' = 1$. Furthermore, the small intervals $C[a,b]$ define closed intervals $[a,b] \subset [0,1]$ such that $[a,b] \cap C = \{a,b\}$. Similarly for C'. Since the homeomorphism ϕ carries $C[a,b]$ to $C'[\phi(a), \phi(b)]$, we can extend ϕ to a bijection $\tilde{\phi} : [0,1] \to [0,1]$ by setting $\tilde{\phi}(\lambda a + (1-\lambda)b) = \lambda\phi(a) + (1-\lambda)\phi(b)$ in each interval $[a,b]$ $(0 \leq \lambda \leq 1)$.

We shall next show that $\tilde{\phi}$ is actually a homeomorphism of $[0,1]$ onto itself. As $\tilde{\phi}(0) = 0$, and $\tilde{\phi}(1) = 1$ it is clear that $\tilde{\phi}$ is an order preserving homeomorphism, and hence $\phi : C \to C'$ will be an order preserving homeomorphism By compactness, we only need to check that $\tilde{\phi}$ is continuous. Let $\{x_n\}$ be a sequence in $[0,1]$ converging to x_0. We claim that $\tilde{\phi}(x_n)$ converges to $\tilde{\phi}(x_0)$. By compactness, it suffices to check that any convergent subsequence $\{\tilde{\phi}(x_{n_k})\}$ of $\{\tilde{\phi}(x_n)\}$ converges to $\tilde{\phi}(x_0)$. Assume $\{\tilde{\phi}(x_{n_k})\}$ converges to y, and assume in addition that $C[\phi(a_0), \phi(b_0)]$ is a small interval which contains infinitely many elements of $\{\tilde{\phi}(x_{n_k})\}$, as $\tilde{\phi}$ is a bijection we get a subsequence of $\{x_{n_k}\}$ in the closed interval $[a_0, b_0] \subset [0,1]$. Therefore $\tilde{\phi}(x_0) = y$ since the restriction $\tilde{\phi}|_{[a,b]}$ obviously is a homeomorphism. Otherwise, we can assume without loss of generality that there exists an infinite family of small intervals $\{C'[\phi(a_k), \phi(b_k)]\}$ such that $\phi(a_k) \leq \tilde{\phi}(x_{n_k}) \leq \phi(b_k)$ in $[0,1]$. Moreover, we can also assume that the sequences $\{|b_k - a_k|\}$ and $\{|\phi(b_k) - \phi(a_k)|\}$ converge to 0. Then one can readily check when $k \to \infty$, $lim\{b_k\} = lim\{a_x\} = lim\{x_{n_k}\} = x$, and $lim\{\phi(b_k)\} = lim\{\phi(a_k)\} = lim\{\tilde{\phi}(x_{n_k})\} = y$. Since C is closed in $[0,1]$, one gets $x \in C$, and the continuity of ϕ yields $\tilde{\phi}(x) = \phi(x) = y$.

q.e.d.

(3.3) Proof of (1.6): Let $\bar{\bar{C}}$ and $\bar{\bar{K}}$ be cyclic Cantor sets represented by $C = (C, \leq)$, and $K = (K, \leq)$ respectively. We have to show that $D(\bar{\bar{C}}) =$

$D(\bar{\bar{K}})$ implies $\bar{\bar{C}}=\bar{\bar{K}}$. The isomorphism is given by the diagram

$$\begin{tikzcd} E \arrow[r, shift right, "\theta"'] \arrow[d, "\alpha"] & I \arrow[l, shift right, "e"'] \arrow[r, "c"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ E' \arrow[r, shift right, "\theta'"'] & I' \arrow[l, shift right, "e'"'] \arrow[r, "c'"] & K \end{tikzcd}$$

where rows are defined by $(C, \leq)$ and $(K, \leq)$ respectively, see (1.5). A priori the bijection α carries the outside interval $[max, min]$ not to the outside interval. Let $K[a, b] = \alpha([max, min])$. Then there exists a cyclic equivalence $K \cong K'$ such that $a = minK'$ and $b = maxK'$. Since K' also represents $\bar{\bar{K}}$ we therefore can replace K by K'. Hence we can assume that α above carries the outside interval to the outside interval. This implies that the homeomorphism γ satisfies the asumptions in (3.2) and hence γ is an order preserving isomorphism which again yields a cyclic equivalence $C \cong K$, therefore $\bar{\bar{C}}=\bar{\bar{K}}$.

The second part of (1.6) is an easy consequence of the definition of the correspondence in (1.2) which carries $\bar{\bar{C}}$ to the 2-manifold $D^2 - C$ for which $\mathcal{E}(D^2 - C) = D(\bar{\bar{C}})$ as can be readily seen.

q.e.d.

(3.4) Proof of the injectivity in (1.2): First, we have to show that the construction of $D^2 - C$ yields a well defined correspondence in (1.2). That is, the homeomorphism type of $D^2 - C$ only depends on the cyclic Cantor set $\bar{\bar{C}}$. Assume that $(C', \leq)$ is another ordered Cantor set which is cyclic equivalent to $(C, \leq)$. By (1.1) we can find small intervals $C[a, b]$ and $C'[a', b']$ such that we have $C[min, a] = C'[b', max]$, and $C[b, max] = C'[min, a']$. Let $\psi : C \to C'$ denote the obvious homeomorphism defined by the equalities above

We now consider $S^1 = \mathbf{R} \cup \infty$ oriented by the usual ordering of $\mathbf{R}$. By using (3.1) we can assume that C and C' are embedded in $\mathbf{R} \subset S^1$ by an order preserving embedding. The homeomorphism ψ above can be extended to an orientation preserving homeomorphism $\tilde{\psi} : S^1 \to S^1$ by setting $\tilde{\psi}([a, b]) = [max, min]$ and $\tilde{\psi}([max, min]) = [a', b']$. Here the intervals $[max, min] \subset S^1$ are defined by the orientation of S^1.

Similarly we can get an orientation reversing homeomorphism $S^1 \cong S^1$ when $(C', \leq)$ is cyclic equivalent to $(C \leq^{op})$. Thus the correspondence (1.2) is well defined. By (1.6) we know that this correspondence is injective.

q.e.d.

In order to show the surjectivity in (1.2) we shall need the following lemma whose proof is a consequence of the triangulability of 2-manifolds ([Mo; 8.3]) which allows us to choose suitable increasing sequences of regular neighbourhoods.

(3.5) <u>Lemma</u>: *Given a non-compact 2-manifold M there exists an increasing sequence $M_i \subset M_{i+1}$ $(i \geq 1)$ of compact connected 2-manifolds with $M = \bigcup\{M_i; i \geq 1\}$, and each clousure $\overline{M_{i+1} - M_i}$ is a family of disjoint 2-manifolds.*

In addition, for each non-compact component $C \subset \partial M$, each non-trivial intersection $M_i \cap C$ is an arc in ∂M_i, and the intersections $M_i \cap \overline{M_{i+1} - M_i}$ is a family of disjoint arcs. Furthermore, if M is contractible the 2-manifolds M_i are 2-disks.

(3.6) <u>Proof of the surjectivity in (1.2)</u>: Let M be a contractible non-compact 2-manifold $\neq D^2, \mathbf{R}^2$. We choose an orientation on M, and so each component $C \subset \partial M$ is an oriented copy $(-\infty_C, \infty_C)$ of $\mathbf{R}$. We assume that ∂M has at least two components. Otherwise M is homeomorphic to the half-space $\mathbf{R}^2_+$, and $\overline{\overline{C}} = \{*\}$.

Take a sequence $M_1 \subset M_2 \dots$ as in (3.5). Let $\mathcal{C}_i$ denote the family of all the components of ∂M which meet M_i. It is obvious that $\mathcal{C}_i \subset \mathcal{C}_{i+1}$. On the other hand, let $\mathcal{B}_i$ denote the family of arcs whose union is the clousure

$$\overline{\partial M_i - \bigcup\{C; C \in \mathcal{C}_i\}} = M_i \cap \overline{M_{i+1} - M_i}$$

We fix a component $C_0 \subset \partial M$ with $C_0 \cap M_1 \neq \emptyset$. Since each M_i is a 2-disk, the orientation of M and the component C_0 define compatible "clockwise" orderings on $\mathcal{C}_i$ which give a total ordering on $\bigcup \mathcal{C}_i - \{C_0\}$. Moreover, this ordering satisfies the following condition (A).

(A) "Given $C \in \mathcal{C}_i$ there exist exactly two components $C_1, C_2 \in \mathcal{C}_{i-1}$ such that C lies between C_1 and C_2 in the clockwise ordering of $\mathcal{C}_i$"

The orderings we have already defined yield a total ordering $\leq$ on the sets of ends $\{\pm\infty_C\}$ of the components of ∂M. Namely, we define $\leq$ as follows

a) $\infty_{C_0} < \pm\infty_C < -\infty_{C_0}$ for all $C \neq C_0$

b) $-\infty_C < \infty_C$ for all $C \neq C_0$

c) $\infty_C < -\infty_{C'}$ if C precedes C' in the above ordering on $\bigcup \mathcal{C}_i - \{C_0\}$

We now embed C_0 in the 1-sphere $S^1 = \partial D^2$ by an orientation preserving embedding $\psi_0 : C_0 \to S^1$. We can assume $S^1 - \psi_0(C_0) = [0,1]$ with $\psi_0(-\infty_{C_0}) = 1$ and $\psi_0(\infty_{C_0}) = 0$. We extend ψ_0 to an orientation preserving embedding $\psi_1 : \bigcup\{C; C \in \mathcal{C}_1\} \to S^1$ by embedding each C in $[0,1]$ according to the ordering in $\mathcal{C}_1 - \{C_0\}$. In addition, we define ψ_1 satisfying the following extra condition

(B) "If $C, C' \in \mathcal{C}_1$, and $[\infty_C, -\infty_{C'}]$ is a small interval in the sense of (1.1) then $\psi_1(\infty_C) = \psi_1(-\infty_{C'})$."

Assume we have already defined an orientation preserving embedding

$$\psi_i : \bigcup\{C; C \in \mathcal{C}_i\} \to S^1$$

which follows the "clockwise" ordering of $\mathcal{C}_i$ and such that ψ_i verifies condition (B) for $\mathcal{C}_i$. By using (A) we can now extend ψ_i to an embedding ψ_{i+1} with the same properties. In this way we can inductively define an embedding $\psi : \partial M \to S^1$ verifying (B) for all $\mathcal{C}_i$. By construction $S^1 - \psi(\partial M)$ is a totally disconnected compact subspace os S^1, and so it is a Cantor set.

We extend ψ to an embedding $\xi_1 : \partial M \cup \{\Gamma; \Gamma \in \mathcal{B}_1\} \to D^2$ with the condition that $\xi_1(\Gamma) \in \overset{o}{D}{}^2$ for each arc Γ. These arcs together the components in $\mathcal{C}_1$ defines a 2-disk in D^2, and we homeomorphically map M_1 to that disk by extending ξ_1. So, we have defined an embedding $h_1 : M_1 \cup \partial M \to D^2$.

Let $D_{r_1} \subset D_{r_2} \subset \ldots$ be a sequence of 2-disks in $\overset{o}{D}{}^2$ with radii $r_n = \frac{n}{n+1}$. Assume we have constructed a embedding

$$h_k : M_k \cup \partial M \to D^2$$

with the following two properties: (i) h_k extends h_{k-1}, and (ii) $D_{r_{k-1}} \subset h_k(\overset{o}{M}_k) \subset \overset{o}{D}{}^2$.

We extend h_k to h_{k+1} as follows. It is not hard to find an extension of h_k

$$\xi_{k+1} : M_k \cup \partial M \cup \{\Gamma; \Gamma \in \mathcal{B}_{k+1}\} \to D^2$$

such that $\xi_{k+1}(int\Gamma) \subset \overset{o}{D}{}^2$, and $\xi_{k+1}(\Gamma) \cap (h_k(M_k) \cap D^2_{r_k}) = \emptyset$. Since $\overline{M_{k+1} - M_k}$ is a finite set of disjoint closed 2-disks whose boundaries are disjointly embedded by ξ_{k+1}, one can easily define an extension h_{k+1} of ξ_{k+1}

disjointly embedded by ξ_{k+1}, one can easily define an extension h_{k+1} of ξ_{k+1} which verifies conditions (i) and (ii) above. Therefore the union of all the embeddings h_k yields an embedding $h : M \to D^2$. extending the embedding $\psi : \partial M \to D^2$ Moreover, by condition (ii) above we have $h(M) = D^2 - K$ where $K = S^1 - \psi(\partial M)$ is a Cantor set. Hence the correspondence in (1.2) is surjective.

q.e.d.

§4 Contractible planar polyhedra

In this section we discuss some properties of contractible planar polyhedra and we prove theorem (2.3) and then theorem (2.2). For the convenience of the reader we first describe notations and some basic facts on polyhedra and planar polyhedra.

We recall that for any planar polyhedron $X = |K|$ it is always possible to choose an embedding $h : X \to \mathbf{R}^2$ which is linear on each simplex of the triangulation K (see [Mo; 10.13]). A polyhedron $X = |K|$ is said to be purely n-dimensional if each point $x \in X$ belongs to some n-simplex of K. We also recall that an n-dimensional polyhedron $X = |K|$ is said to be strongly connected if given two n-simplices σ, τ in K there exists a finite sequence $\sigma = \sigma_0, \sigma_1, \dots \sigma_k = \tau$ of n-simplices such that $\sigma_i \cap \sigma_{i-1}$ is a common $n-1$-face $(1 \leq i \leq k)$. In general, the notion of strongly connected component, M_σ, of an n-simplex $\sigma \in K$ can be easily given for any n-dimensional polyhedron. Moreover, it is straightforwardly checked that M_σ is a purely n-dimensional subcomplex of K. In addition we have (see [M; 5.3.3] for a proof):

(4.1) Proposition: *Any n-dimensional polyhedron $X = |K|$ can be decomposed as a union of two subpolyhedra $X = R(X) \cup L(X)$ where $R(X)$ is the union of all strongly connected components $M_\sigma \subseteq X$. Furthermore $R(X)$ is purely n-dimensional, and $dim L(X) \leq n-1$ with $dim(M_\sigma \cap M_{\sigma'}) \leq n-2$ and $dim(M_\sigma \cap L(X)) \leq n-2$ for every pair of n-simplices $\sigma, \sigma' \in K$.*

The singular part of X is the union $S(X) = L(X) \cup O(X)$ where $O(X) = \bigcup\{M_\sigma \cap M_{\sigma'}; \sigma, \sigma' \in K\}$. Moreover the above decomposition only depends on the homeomorphism type of X.

With the notation of (4.1) we have the following properties for planar polyhedra.

(4.2) Proposition: *Let $X = |K|$ be a connected planar polyhedron. Then the following statements hold for each strongly connected component $M_\sigma \subseteq X$*

(1) M_σ is a 2-pseudomanifold whose interior $\overset{o}{M}_\sigma$ is a 2-manifold.

(2) For each M_σ, $M_\sigma \cap S(X) \subseteq \partial M_\sigma$.

In addition, if X is simply connected then X is contractible and the three further statements below hold

(3) Each M_σ is a 2-manifold

(4) Two points $x, y \in M_\sigma \cap S(X)$ can not be joined outside M_σ.

(5) Each component of the graph $S(X)$ is a tree.

We recall that an n-pseudomanifold is a strongly connected purely n-dimensional polyhedron $X = |K|$ such that any $(n-1)$-simplex of K is the face of at most two n-simplices of K. The boundary of X, ∂X, is the union of all the $(n-1)$-simplices which are contained in exactly one n-simplex of K. The difference $\overset{o}{X} = X - \partial X$ is called the interior of X. It is a well known fact that any 1-pseudomanifold is a 1-manifold.

Given $x \in |K|$, the star of x in K, $star(x; K)$, is the subcomplex of K generated by the set $\{\sigma \in K;\ x \in K\}$. And the link of x in K is the subcomplex of $star(x; K)$ denoted by

$$link(x; K) = \{\tau \in K;\ \tau < \sigma \text{ with } x \in \sigma - \tau\}$$

It is a basic fact that $star(x; K)$ is a cone over $link(x; K)$. Moreover, if $X = |K|$ is a n-pseudomanifold, $link(x; K)$ is an $(n-1)$-pseudomanifold (with boundary if $x \in \partial X$).

(4.3) Proof of (4.2): As a simple consequence of the Jordan Curve Theorem ([Mo; §8]) it follows that $link(x; M_\sigma) = S^1$ when $x \in \overset{o}{M}_\sigma$. This yields (1) and (2). Since X is planar the homology groups $H_i(X)$ are trivial for $i \geq 2$. So, it follows from the Whitehead Theorem ([M; 8.3.10]) that X is contractible if X is simply connected. Moreover, as X is 2-dimensional the inclusions $M_\sigma \subseteq X$ induce injections $\pi_1(M_\sigma, *) \to \pi_1(X, *)$, and so each component M_σ is simply connected and then contractible. By using the Jordan Curve Theorem it is not hard to check that the contractibility of M_σ implies that ∂M_σ is a 1-manifold. This yields (3) since $\overset{o}{M}_\sigma$ is already a 2-manifold by (1). Finally (4) follows from arguments similar to Van Kampen's Theorem; and (5) is obvious. **q.e.d.**

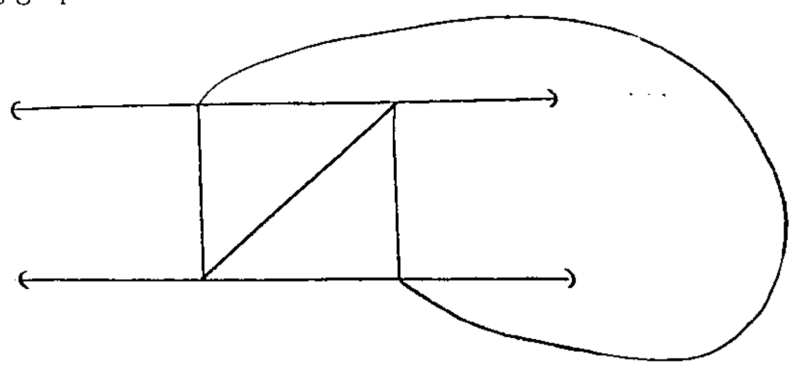
(4.4) Remark: For a contractible planar polyhedron X such that $M_\sigma = \mathbf{R}^2$ for some 2-simplex σ, we necessarily have $X = M_\sigma = \mathbf{R}^2$. Therefore if $X \neq \mathbf{R}^2$ each M_σ is a 2-manifold with boundary.

Starting with a planar polyhedron $X \subseteq \mathbf{R}^2$, by "thickening" the singular set $S(X) \subsetneq X$ it is possible to define a planar 2-pseudomanifold $M(X) \subseteq \mathbf{R}^2$ such that $X \subseteq M(X)$ is a proper strong deformation retract. We recall that a continuous map $f : X \to Y$ is proper when $f^{-1}(K)$ is a compact for each compact subset $K \subseteq Y$. Moreover when all the components M_σ are 2-manifolds, $M(X)$ turns to be a 2-manifold. In particular, according to 4.2(3) if X is contractible $M(X)$ is a contractible planar 2-manifold. And by (3.5) it is not hard to find a proper embedding $M(X) \subseteq \mathbf{R}^2$. We can also use (3.5) to define a tree $T \subsetneq M(X)$ such that $M(X)$ is in fact a regular neighbourhood of T. As consequences of these observations we can now state

(4.5) Proposition: *Any planar contractible polyhedron can be properly embedded in* $\mathbf{R}^2$.

(4.6) Proposition: *Any planar contractible polyhedron has the proper homotopy type of a tree.*

(4.7) Remark: Obviously (4.5) does not hold for any planar polyhedron, as the following graph shows



We now finish this section with the proofs of (2.3) and (2.2)

(4.8) On the definition in (2.2): The polyhedron $X(T)$ defined in §2 is in fact a contractible planar polyhedron. The contractibility follows from the push out construction of $X(\mathcal{T})$ since all the 2-manifolds $D^2 - C_t$ involved are contractible.

We now define an embedding $X(\mathcal{T})$ in $\mathbf{R}^2$. We can assume that $P \neq \emptyset$ since otherwise $A = \emptyset$ and $X(\mathcal{T}) = T$ is a locally finite tree. Let $t_0 \in T^0$. We take t_0 as the root vertex of T, and this induces a partial ordering in T^0 by taking $v \leq w$ when v appears in the unique path γ_w going from w to t_0. In addition, the paths γ_v induce a height function $h : T^0 \to \mathbf{N}$ where $h(v)$ is the number of vertices in γ_v. Using the function h we can define inductively an embedding $\xi : X(\mathcal{T}) \to \mathbf{R}^2$ as follows. Let $\mathcal{T}_n$ be the finite Cantorian subtree generated by the set of vertices $\{v \in T^0; h(v) \leq n\}$. Assume we have already defined an embedding $\xi_n : X(\mathcal{T}_n) \to \mathbf{R}^2$.

In order to extend ξ_n to an embedding $\xi_{n+1} : X(\mathcal{T}_{n+1}) \to \mathbf{R}^2$ we consider all vertices $v \in T^0$ with $h(v) = n + 1$. Let w_v be the unique vertex with $w_v \leq v$ and $h(w_v) = n$. If $v \in T^0 - P$ we can easily define an extension ξ_{n+1} of ξ_n to the edge (w_v, v) in such a way that when $v \in A$ and $w_v \in P$ then $\xi_n = \xi_{n+1}$, since in this case $(w_v, v) \subseteq D^2 - C_{w_v}$ and ξ_n is already defined on $D^2 - C_{w_v}$. If $v \in P$ then w_v necessarily belongs to A, and we can extend ξ_n to an embedding ξ_{n+1} of $X(\mathcal{T}_{n+1}) \cup D^2 - C_v$. Therefore the union $\xi = \cup \xi_n$ defines a planar embedding of $X(\mathcal{T})$. By the push out construction of $X(\mathcal{T})$ it clear that $X(\mathcal{T}) \cong X(\mathcal{T}')$ when $\mathcal{T}$ and $\mathcal{T}'$ are two equivalent Cantorian trees.

q.e.d.

In this way the function $\psi : \mathcal{T} \mapsto X(\mathcal{T})$ is a well defined function which carries equivalence classes of Cantorian trees to homeomorphism types of contractible planar polyhedra $\neq D^2, \mathbf{R}^2$. We now proceed to show that ψ is a 1-1 correspondence.

(4.9) Proof of (2.3) (surjectivity in (2.2)): Let X be any contractible planar polyhedron $\neq D^2, \mathbf{R}^2$. According to (4.1) and (4.2) we can write $X = R(X) \cup S(X)$ where $R(X)$ is a union of contractible 2-manifolds $\{M_\alpha\}$. Moreover, by (4.4) we know that $\partial M_\alpha \neq \emptyset$ for each α. By using (1.2) we can identify M_α with $D^2 - C_\alpha$ for some cyclic Cantor set $\bar{\bar{C}}_\alpha$. Furthermore, by (3.6) each component of ∂M_α is identified with a small interval of C_α (including the "outside interval").

For a small interval $C_\alpha[a, b]$, let $H^\alpha_{a,b}$ be the corresponding component of ∂M_α. We consider the intersections $\overset{o}{S}{}^\alpha_{a,b} = H^\alpha_{a,b} \cap S(X)$ and for each α the union $\overset{o}{\mathcal{S}}_\alpha = \bigcup\{ \overset{o}{\mathcal{S}}{}^\alpha_{a,b} \}$. Now we take $\mathcal{S}^\alpha_{a,b} = \overset{o}{\mathcal{S}}{}^\alpha_{a,b} \cup \{a, b\}$, and $\mathcal{S}^\alpha = \bigcup \mathcal{S}^\alpha_{a,b}$. Then we define the Cantorian tree

$$\mathcal{T} = (T, P, A, \sigma, \theta)$$

where T is the tree consisting of the union of the cones $C(\overset{o}{\mathcal{S}}{}^\alpha)$ and the singular set $S(X)$. The set of vertices A consists of all points in $\bigcup \overset{o}{\mathcal{S}}{}^\alpha$, and the set P consists of all the cone points $t_\alpha \in C(\overset{o}{\mathcal{S}}{}^\alpha)$. The function σ is given by $\sigma(t_\alpha) = (C_\alpha, \mathcal{S}^\alpha)$. There is now an obvious way of defining λ_α and it is easily checked from the definition of $X(\mathcal{T})$ that $X(\mathcal{T}) \cong X$.

q.e.d.

(4.10) <u>Proof of the injectivity in (2.2)</u>: Assume we have a homeomorphism $f : X(\mathcal{T}_1) \to X(\mathcal{T}_2)$ between two Cantorian trees $\mathcal{T}_i = (T_i, P_i, A_i, \sigma_i, \lambda_i)$, $(i = 1,\ 2)$. For each vertex $t_i \in P_i$ we have an ordered Cantor set $(C_{t_i}, \leq)$ with $\sigma_i(t_i) = (C_{t_i}, \mathcal{S}_{t_i})$. If $C'_{t_i} = C_{t_i} - \overset{o}{\mathcal{S}}_{t_i}$, then the definition of $X(\mathcal{T}_i)$ shows that the strong connected components of the polyhedron $X(\mathcal{T}_i)$ are the 2-manifolds $D^2 - C'_{t_i}$. By definition of $D^2 - C_{t_i}$ (see (1.2)), the ordering on C'_{t_i} defines an orientation on $\partial(D^2 - C'_{t_i})$. Moreover, the small intervals of C'_{t_i} can be identified with the connected components $H \subseteq \partial(D^2 - C'_{t_i})$. In addition, each intersection $S(X(\mathcal{T}_i)) \cap H$ is determined by the sequences in $\sigma_i(t_i)$.

On the other hand, the topological invariance of the decomposition in (4.1) implies that the given homeomorphism f induces a homeomorphism of pairs

$$f_{t_1} : (D^2 - C'_{t_1}, S(X(\mathcal{T}_1)) \cap D^2 - C'_{t_1}) \longrightarrow (D^2 - C'_{s(t_1)}, S(X(\mathcal{T}_2)) \cap D^2 - C'_{s(t_1)})$$

for each vertex $t_1 \in P_1$. Furthermore, the function $t_1 \mapsto s(t_1)$ defines a 1-1 correspondence $s : P_1 \longrightarrow P_2$. If f_{t_1} is an orientation preserving (reversing) homeomorphism, then f_{t_1} induces an order preserving (reversing, respectively) homeomorphism

$$\tilde{f}_{t_1} : (C_{t_1}, \mathcal{S}_{t_1}) \longrightarrow (C_{s(t_1)}, \mathcal{S}_{s(t_1)}).$$

We now proceed to define an homeomorphism

$$\tau : \mathcal{T}_1 = (T_1, P_1, A_1) \longrightarrow \mathcal{T}_2 = (T_2, P_2, A_2)$$

as follows. We define $\tau|P_1 = s$ *on* P_1. Next we shall define τ in the set

$$A_1 = \bigcup\{\ link(t_1;T_1);\ t_1 \in T_1\ \}$$

Each point $a \in link(t_1;T_1)$ is identified by θ_1 to a point p_a in certain sequence $\overset{o}{S} \subseteq \overset{o}{\mathcal{S}}_1$. As it was remarked above we have $\overset{o}{S} = H \cap S(X(\mathcal{T}_1))$ for a unique component $H \subseteq \partial(D^2 - C'_{t_1})$, and since the given homeomorphism f verifies $f(S(X(\mathcal{T}_1))) = S(X(\mathcal{T}_2))$ we can define $\tau(a) = \theta_2(f(p_a)), a \in A_1$. The extension of $\tau|P_1 \cup A_1$ to $\cup\{star(t_1;T_1); t_1 \in P_1\}$ is the canonical linear extension. Finally, we define $\tau = f$ between the singular sets

$$S(X(\mathcal{T}_i)) = T_i - \bigcup\{\ \overset{o}{star}\ (t_i;T_i); t_i \in P_i\ \} \quad (i = 1,\ 2)$$

Hence τ actually defines a homeomorphism between the Cantorian trees $\mathcal{T}_i$.

q.e.d.

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