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Abstract

The famous H. Schubert theorem (1949) states that any nontrivial knot in S^3 admits a decomposition into connected sum of prime factors, which are unique up to order. We prove a similar result for knots in $T \times I$, where T is a two-dimensional torus. However, we only consider knots of geometric degree one, use a different type of connected summation, and take into account the order of prime factors.

Keywords: Knot, prime decomposition, thick torus
MSC 57M25, 57M15

1 Introduction

Let T be a two-dimensional torus and $I = [0, 1]$. By a *thick torus* we mean a 3-manifold homeomorphic to the product $T \times I$ equipped with a fixed orientation.

Definition 1. *A knot in $T \times I$ is an oriented simple closed curve $K \subset Int\,(T \times I)$. Two knots $K_i \subset T_i \times I, i = 1, 2$, are* equivalent *if there is a homeomorphism of pairs $h\colon (T_1 \times I, K_1) \to (T_2 \times I, K_2)$ which takes $T_1 \times \{0\}$ to $T_2 \times \{0\}$ and preserves orientations of the thick tori and knots.*

Definition 2. *Let $K \subset T \times I$ be a knot. A proper annulus $A \subset T \times I$ is called* vertical *if it is isotopic to an annulus of the type $c \times I$, where c is a nontrivial simple closed curve in T. A vertical annulus $A \subset T \times I$ is* admissible *(with respect to K) if K intersects A transversally at one point. By a vertical* multi-annulus *in $T \times I$ we mean the disjoint union $\mathbb{A} = A_0 \cup A_1 \cup \cdots \cup A_{n-1} \subset T \times I$ of vertical annuli. $\mathbb{A}$ is admissible so are all A_i.*

Definition 3. *We shall say that a knot $K \subset T \times I$ is* of degree one *if $T \times I$ contains an admissible annulus.*

Let $K_i \subset T_i \times I, 0 \leq i \leq n - 1$, be a collection of $n \geq 2$ degree one knots in thick tori. Choose admissible annuli $A_i \subset T_i \times I$. For each i we cut $T_i \times I$ along A_i and get a *thick annulus* $M_i \approx A_i \times [0, 1]$ with two copies $A_i' = A_i \times \{0\}, A_i'' = A_i \times \{1\}$ of A_i in ∂M_i. The annuli are joined by the

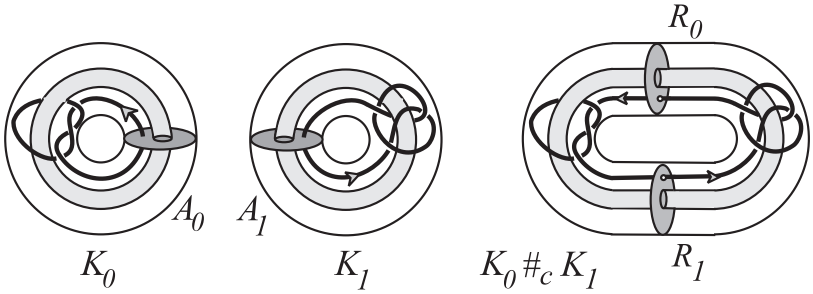

Figure 1: Circular connected sum

oriented arc $l_i = K \cap M_i$. We assume that the initial and terminal points of l_i lie in A'_i and A''_i respectively. For each i choose a homeomorphism $h_i \colon A''_i \to A'_{i+1}$ which reverses the induced orientations of the annuli, takes $A''_i \cap (T \times \{0\})$ to $A'_i \cap (T \times \{0\})$, and takes the terminal point of l_i to the initial point of l_{i+1} (indices are taken modulo n).

Definition 4. *The knot $K = K_0 \#_c K_1 \#_c \ldots \#_c K_{n-1} \subset T \times I$ obtained by gluing together the pairs (M_i, l_i) along h_i is called a* circular connected sum *of K_i. Admissible annuli in $T \times I$ obtained by identifying A''_i with A'_{i+1} are denoted $R_i, 0 \leq i \leq n-1$. See Fig. 1 for $n = 2$.*

The circular connected sum of degree one knots may depend on the choice of the annuli $A_i \subset T_i \times I$ used for the construction. However, if A_i are fixed, then K and the admissible multi-annulus $\mathbb{R} = R_0 \cup R_1 \cup \cdots \cup R_{n-1}$ are uniquely determined. In turn, K and $\mathbb{R}$ determine K_i and A_i. Suppose we are considering a circular connected sum $K_0 \#_c K_1$ of two knots such that one of them is *horizontal*, (i.e., isotopic to a simple closed curve in a middle torus $T_i \times \{*\}$). Then the sum $K_0 \#_c K_1$ is equivalent to the second knot. Such a summation is called *trivial*.

Definition 5. *A nonhorizontal degree one knot $K \subset T \times I$ is called* prime *if it cannot be represented as a nontrivial circular connected sum of two other knots.*

Let K be a degree one knot in $T \times I$. Suppose there is a 3-ball $B \subset T \times I$ such that $l = K \cap B$ is a knotted arc in B. Replacing l by an unknotted arc $l_1 \subset B$ with the same endpoints, we get a new degree one knot $K_1 \subset T \times I$.

Definition 6. *We shall say that K_1 is obtained from K by* cutting off a local knot *and that K is obtained from K_1 by* inserting a local knot.

Note that the exact place for the insertion, i.e., a ball $B \subset T \times I$ such that $l_1 = K_1 \cap B$ is an unknotted arc, is not important. Indeed, B can be moved by an isotopy of pairs $h_t\colon (T \times I, K_1) \to (T \times I, K_1)$ to any other position along K_1. This fact is well-known in the classical knot theory, where it is used for proving commutativity of connected sum operation.

Definition 7. *A degree one nonhorizontal knot K in $T \times I$ is called* essential *if it does not contain local knots. K is called* almost horizontal *if it can be obtained from a horizontal knot in $T \times I$ by inserting local knots.*

One can easily see that inserting local knots is equivalent to taking circular connected sums with the corresponding almost horizontal knots. It follows that almost horizontal summands of a circular connected sum can be shifted to any position, for example, one may write them at the end of the sum.

Theorem 1. *Any nonhorizontal degree one knot K can be represented as a circular connected sum*

$$K = K_0 \#_c K_1 \#_c \dots \#_c K_{n-1} \#_c L_0 \#_c L_1 \#_c \dots \#_c L_{m-1},$$

where K_i are essential and L_j are almost horizontal prime knots. The summands K_i are uniquely determined up to cyclic permutation while the summands L_j are uniquely determined up to any permutation.

For knotted theta-curves in S^3 and in arbitrary 3-manifolds similar prime decomposition theorems, which take into account the order of prime factors, can be found in [4, 2]

2 Properties of admissible annuli

Definition 8. *Let $K \subset T \times I$ be a degree one knot and $\mathbb{R} = R_0 \cup R_1 \cup \dots \cup R_{n-1} \subset T \times I, n \geq 2$, an admissible multi-annulus. Then a vertical multi-annulus $\mathbb{C} \subset T \times I$ is called* tight *(with respect to $\mathbb{R}$) if either $\mathbb{C} \cap \mathbb{R} = \emptyset$ or the following holds:*

1. *$\mathbb{C} \cap \mathbb{R}$ consists of radial arcs of the annuli.*
2. *These arcs decompose $\mathbb{C}$ into strips (embedded rectangles) such that the lateral sides of each strip lie in different annuli of $\mathbb{R}$.*

Definition 9. *Let $K \subset T \times I$ be a degree one knot and $\mathbb{C} \subset T \times I$ a vertical multi-annulus such that K intersects $\mathbb{C}$ transversally. Then the* weight $w(\mathbb{C})$ *of $\mathbb{C}$ is the number of points in $K \cap \mathbb{C}$.*

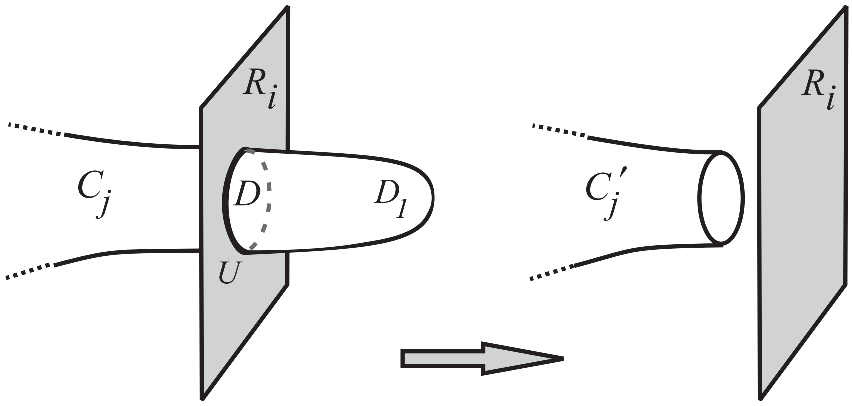

Figure 2: Removing trivial circles

Lemma 1. *Let $K \subset T \times I$ be a degree one knot and $\mathbb{R} = R_0 \cup R_1 \cup \dots \cup R_{n-1}, n \geq 2$, an admissible multi-annulus. Then for any vertical multi-annulus $\mathbb{C} \subset T \times I$ there is an isotopy $h_t \colon T \times I \to T \times I, 0 \leq t \leq 1$, such that $h_0(\mathbb{C}) = \mathbb{C}$, the multi-annulus $\mathbb{C}' = h_1(\mathbb{C})$ is tight, and $w(\mathbb{C}) \geq w(\mathbb{C}')$. Moreover, if K is essential and $\mathbb{C}$ is admissible, h_t may be chosen so as to be invariant on K, i.e., $h_t(K) = K$ for all t.*

Proof. We may assume that $\mathbb{C}$ and $\mathbb{R}$ are in general position. Then any connected component of $\mathbb{C} \cap \mathbb{R}$ is one of the following curves: a trivial circle, a trivial arc, a nontrivial circle, or a radial arc. Our goal is to remove all curves of the first three types and some curves of the last type.

Step 1. Suppose $\mathbb{C} \cap \mathbb{R}$ contains a trivial circle $U \subset R_i$. Using an innermost disc argument, we may assume that U bounds a disc $D \subset R_i \subset \mathbb{R}$ such that $D \cap \mathbb{C} = U$. Denote by D_1 the disc bounded by U in $C_j \subset \mathbb{C}$. Then $D \cup D_1$ is a sphere bounding a ball $B \subset T \times I$. We use B for constructing an isotopy h_t which moves D_1 to the other side of R_i and $\mathbb{C}$ to a new multi-annulus $\mathbb{C}'$, thus annihilating U and maybe some other circles in $\mathbb{C} \cap \mathbb{R}$. See Fig. 2. Since R_i is admissible, $K \cap D$ is either empty or consists of one point. In the latter case $K \cap D_1 \neq \emptyset$. It follows that in both cases $w(\mathbb{C}') \leq w(\mathbb{C})$.

Suppose $\mathbb{C}$ is admissible. Then either $l = K \cap B$ is empty or l is an arc. If K is essential, then l is unknotted. Therefore h_t may be chosen so as to be invariant on K. Further on we will assume that K contains no trivial circles.

Step 2. All trivial arcs in $\mathbb{C} \cap \mathbb{R}$ can be removed just in the same way as above, using an outermost arc argument and half-discs bounded by trivial arcs and arcs in $\partial(T \times I)$ instead of discs. Further we assume that $\mathbb{C} \cap \mathbb{R}$ contains no trivial arc.

Step 3. Suppose that $\mathbb{C}$ intersects an annulus R_i of $\mathbb{R}$ along nontrivial circles, which decompose it into smaller annuli. Then R_i contains two out-

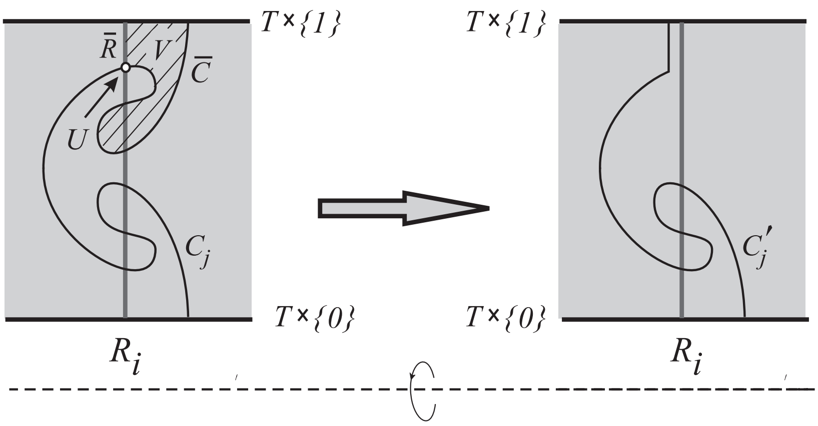

Figure 3: Removing nontrivial circles

ermost annuli, each bounded by a circle in $\mathbb{C} \cap R_i$ and a circle in $\partial(T \times I)$. Since R_i is admissible, at least one of them (denote it $\bar{R}$) has with K no common points. The circle $U = \partial\bar{R} \cap \mathbb{C}$ cuts off an annulus $\bar{C} \subset C_j \subset \mathbb{C}$ having a boundary circle in the same torus of $\partial(T \times I)$ as $\bar{R}$. Then $\bar{R} \cup \bar{C}$ together with an annulus in $\partial(T \times I)$ bound in $T \times I$ a solid torus V. We use V for constructing an isotopy $h_t \colon T \times I \to T \times I$ which moves $\bar{C}$ to the other side of $\bar{R}$ and $\mathbb{C}$ to a new multi-annulus $\mathbb{C}'$, thus annihilating U and maybe some other circles in $\mathbb{C} \cap \mathbb{R}$. Clearly $w(\mathbb{C}') \leq w(\mathbb{C})$. If $\mathbb{C}$ is admissible, then $V \cap K = \emptyset$, since $\bar{R} \cap K = \emptyset$. Therefore we may construct h_t such that it keeps K fixed. See Fig. 3. In order to get a 3-dimensional illustration, rotate it around the axis shown at the bottom of the figure).

Step 4. Suppose that $\mathbb{C} \cap \mathbb{R}$ consists of radial arcs. They decompose $\mathbb{C}$ and $\mathbb{R}$ into strips. If $\mathbb{C}$ is not tight, then there are strips $P \subset C_j \subset \mathbb{C}$ and $Q \subset R_i \subset \mathbb{R}$ such that they have common lateral sides and $P \cup Q$ cuts off a 3-ball B from $T \times I$. We use B for constructing an isotopy of $T \times I$ which moves P to the other side of R_i and C_j to a new annulus C'_j, thus annihilating two or more radial arcs of $\mathbb{C} \cap \mathbb{R}$. Clearly $w(\mathbb{C}') \leq w(\mathbb{C})$. If K is essential and $\mathbb{C}$ is admissible, we use the same argument as in Step 1 for constructing an isotopy which is invariant on K. ∎

Let $K \subset T \times I$ be a degree one knot and $R, R' \subset T \times I$ be disjoint admissible annuli. They decompose $T \times I$ into two parts $M_i \approx R \times [0,1], i = 1, 2$. We shall say that R, R' are *parallel in* $(T \times I, K)$ if for at least one i the arc $K \cap M_i$ is trivial in M_i, i.e., has the form $\{*\} \times [0,1]$.

Lemma 2. *Let $K \subset T \times I$ be a degree one essential knot and $\mathbb{R} = R_0 \cup R_1 \cup \cdots \cup R_{n-1}, n \geq 2$, an admissible multi-annulus such that at least two annuli of $\mathbb{R}$ are not parallel in $(T \times I, K)$. Then for any admissible multi-*

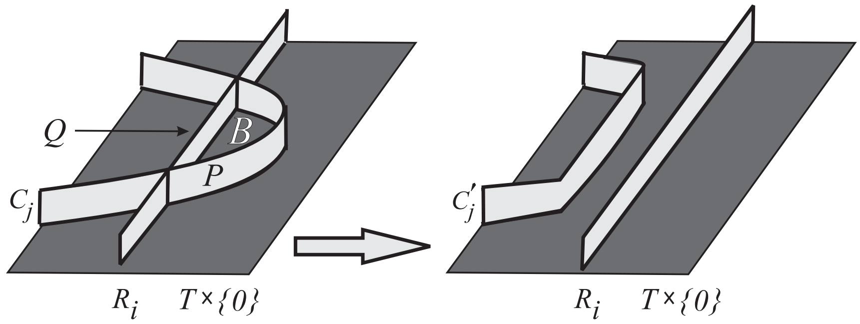

Figure 4: Removing radial arcs

annulus $\mathbb{C} \subset T \times I$ *there is an isotopy* $h_t \colon T \times I \to T \times I, 0 \le t \le 1$, *such that* $h_0(\mathbb{C}) = \mathbb{C}$, h_t *is invariant on* K, *and the multi-annulus* $\mathbb{C}' = h_1(\mathbb{C})$ *is disjoint with* $\mathbb{R}$.

Proof. By Lemma 1 we may assume that $\mathbb{C}$ is tight. We claim that $\mathbb{C} \cap \mathbb{R} = \emptyset$. On the contrary, assume that $\mathbb{C}$ intersects $\mathbb{R}$. Then $\mathbb{C}$ consists of strips such that each strip P joins two neighboring annuli $R_i, R_{i+1} \subset \mathbb{R}$ and lies in the thick annulus M_i between them. Note that P cuts M_i into a ball. If $K \cap P = \emptyset$, then the arc $l_i = K \cap M_i$ is contained in this ball. Since K is essential, l_i is unknotted. Thus R_i, R_{i+1} are parallel and the pair (M_i, l_i) is trivial, i.e., homeomorphic to $(R_i \times [0,1], \{*\} \times [0,1])$.

Recall that $\mathbb{C}$ is admissible. It follows that K intersects only one strip. Therefore, only one thick annulus between neighboring annuli may be nontrivial, but then its complement in $T \times I$ consists of trivial regions and thus is also trivial. This contradicts our assumption that $\mathbb{R}$ contains nonparallel annuli. ∎

Remark 1. *Suppose that an essential knot* $K \subset T \times I$ *is nonprime. Then any two admissible annuli in* $T \times I$ *are isotopic in* $T \times I$. *Indeed, since* K *is nonprime,* $T \times I$ *contains a pair of disjoint admissible annuli* R', R'' *which are not parallel in* $(T \times I, K)$. *In fact we can take any pair of annuli decomposing* K *into a nontrivial circular connected sum. Let* $C \subset T \times I$ *be another admissible annulus. By Lemma 2* C *isotopic in* $(T \times I, K)$ *to an annulus which is disjoint with* R' *and thus is isotopic to* R'. *Note that the assumption that* K *be nonprime is essential. See Fig. 5 for a knot having two nonisotopic admissible annuli* R_0, R_1.

Lemma 3. *Let* $K \subset T \times I$ *be an essential knot and* $\mathbb{R} = R_0 \cup R_1 \cup \cdots \cup R_{n-1}, n \ge 2$, *an admissible multi-annulus in* $T \times I$ *such that no two annuli of* $\mathbb{R}$ *are parallel in* $(T \times I, K)$. *Suppose that an annulus* $C \subset T \times I$ *intersects*

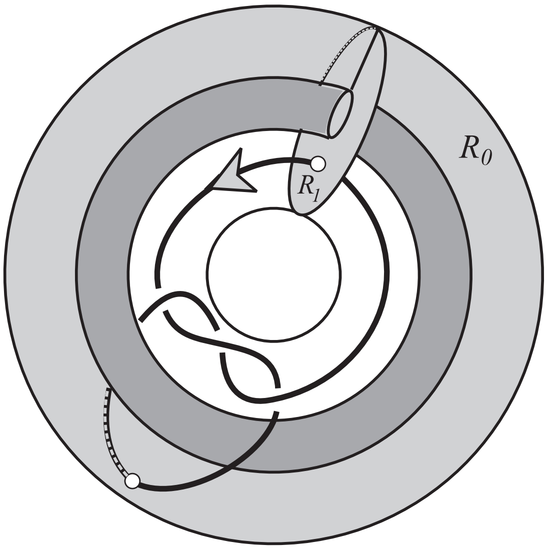

Figure 5: Two nonisotopic admissible annuli

K transversally and the base circles $C \cap (T \times \{0\})$ and $\mathbb{R} \cap (T \times \{0\})$ of C and $\mathbb{R}$ are not homotopic in $T \times \{0\}$. Then $n \leq w(C)$.

Proof. By Lemma 1 we may transform C into a tight position without increasing its weight. Since the base circles of C and $\mathbb{R}$ are not homotopic, $C \cap \mathbb{R}$ is a nonempty collection of radial arc, which decompose C and $\mathbb{R}$ into strips. Note that K must intersect each strip of C in any region of $T \times I$ between two neighboring annuli. Otherwise the region would be trivial and the annuli parallel. Since any region contains at least one strip intersecting K, we may conclude that $n \leq w(\mathbb{C})$. ∎

3 Proof of the main theorem

Let a nonhorizontal degree one knot K be given. First we cut off all local knots. By [3] and Theorem 7 of [1], any knot K in a 3-manifold without nonseparating 2-spheres contains only finitely many local knots, which are uniquely determined by K. Therefore the set of almost horizontal summands L_j of K is finite and these summands are unique up to order. Further on we shall assume that K does not contain local knots, i.e., is essential.

Let us prove that a prime decomposition of K does exist. If K is prime, we are done. Suppose K is not prime. Then among all decompositions of K into circular connected sums we take a decomposition $K = K_0 \#_c K_1 \#_c \ldots \#_c K_{n-1}$ having the maximal number n of summands. Clearly $n \geq 2$.

We claim that all K_i are prime. Let $\mathbb{R} = R_0 \cup R_1 \cup \dots \cup R_{n-1} \subset T \times I$ be the admissible multi-annulus corresponding to that decomposition. The annuli of $\mathbb{R}$ split $T \times I$ into thick annuli $M_i \approx R_i \times [0, 1]$. On the contrary, suppose that for some i the knot $K_i \subset T_i \times I$ is not prime. Then $T_i \times I$ contains a pair of disjoint admissible annuli R', R'' such that they are not parallel in $(T_i \times I, K_i)$. Consider the admissible multi-annulus $R' \cup R'' \subset T_i \times I$ and the annulus $A_i \subset T_i \times I$ used for constructing the circular connected sum. According to Lemma 2 we may assume that R', R'' are disjoint with A_i and thus can be considered as annuli in the thick annulus M_i between R_i and R_{i+1}. Since R', R'' are not parallel in $(T_i \times I, K_i)$, at least one of them is not parallel to R_i or R_{i+1}. This contradicts our assumption that n is maximal.

Let us prove that the summands of a prime decomposition of K into a circular connected sum are unique up to cyclic permutation. Let $K = K_0 \#_c K_1 \#_c \dots \#_c K_{n-1}$, $K = K'_0 \#_c K'_1 \#_c \dots \#_c K'_{m-1}$ be two representations and $\mathbb{R} = R_0 \cup R_1 \cup \dots \cup R_{n-1}$, $\mathbb{R}' = R'_0 \cup R'_1 \cup \dots \cup R'_{m-1}$ the corresponding admissible multi-annuli in $T \times I$. By Lemma 2 we may assume that $\mathbb{R} \cap \mathbb{R}' = \emptyset$. It follows that any annulus R'_j lies in a thick annulus M_i between two neighboring annuli R_i and R_{i+1}. Since K_i is prime, R'_j must be parallel to exactly one of them. Similarly, any annulus of $\mathbb{R}$ is parallel to exactly one annulus of $\mathbb{R}'$. We may conclude that $m = n$ and that after an appropriate isotopic deformation of $\mathbb{R}$ we get $\mathbb{R} = \mathbb{R}'$. Therefore both decompositions have the same set of prime summands. Their orderings are determined by K, so may differ only by a cyclic permutation.

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