COURANT-DORFMAN ALGEBRAS AND THEIR COHOMOLOGY

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To the memory of I.Ya. Dorfman (1948 - 1994)

Abstract. We introduce a new type of algebra, the Courant-Dorfman algebra. These are to Courant algebroids what Lie-Rinehart algebras are to Lie algebroids, or Poisson algebras to Poisson manifolds. We work with arbitrary rings and modules, without any regularity, finiteness or non-degeneracy assumptions. To each Courant-Dorfman algebra $(\mathcal{R}, \mathcal{E})$ we associate a differential graded algebra $\mathcal{C}(\mathcal{E}, \mathcal{R})$ in a functorial way by means of explicit formulas. We describe two canonical filtrations on $\mathcal{C}(\mathcal{E}, \mathcal{R})$, and derive an analogue of Cartan relations for derivations of $\mathcal{C}(\mathcal{E}, \mathcal{R})$; we classify central extensions of $\mathcal{E}$ in terms of $H^2(\mathcal{E}, \mathcal{R})$ and study the canonical cocycle $\Theta \in \mathcal{C}^3(\mathcal{E}, \mathcal{R})$ whose class $[\Theta]$ obstructs re-scalings of the Courant-Dorfman structure. In the non-degenerate case, we also explicitly describe the Poisson bracket on $\mathcal{C}(\mathcal{E}, \mathcal{R})$; for Courant-Dorfman algebras associated to Courant algebroids over finite-dimensional smooth manifolds, we prove that the Poisson dg algebra $\mathcal{C}(\mathcal{E}, \mathcal{R})$ is isomorphic to the one constructed in [17] using graded manifolds.

1. Introduction

1.1. **Historical background.** This is the first in a series of papers devoted to the study of Courant-Dorfman algebras. These algebraic structures first arose in the work of Dorfman [8] and Courant [4] on reduction in classical mechanics and field theory (with [5] a precursor to both, ultimately leading back to [7]). Courant considered sections of the vector bundle $\mathbf{T}M = TM \oplus T^*M$ over a finite-dimensional C^∞ manifold M, endowed with the canonical pseudo-metric

$$\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle = \iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1$$

and a new bracket he introduced:

$$[\![(v_1, \alpha_1), (v_2, \alpha_2)]\!] = (\{v_1, v_2\}, L_{v_1}\alpha_2 - L_{v_2}\alpha_1 - \frac{1}{2}d_0(\iota_{v_1}\alpha_2 - \iota_{v_2}\alpha_1)),$$

while Dorfman was working in a more general abstract setting involving a Lie algebra $\mathfrak{X}^1$ and a complex Ω acted upon by the differential graded Lie algebra $T[1]\mathfrak{X}^1 = \mathfrak{X}^1[1] \oplus \mathfrak{X}^1$ (i.e. to each $v \in \mathfrak{X}^1$ there are associated operators ι_v and L_v on Ω satisfying the usual Cartan relations). She considered the space $\mathcal{Q} = \mathfrak{X}^1 \oplus \Omega^1$ equipped with the above pseudo-metric and a bracket given by

$$[(v_1, \alpha_1), (v_2, \alpha_2)] = (\{v_1, v_2\}, L_{v_1}\alpha_2 - \iota_{v_2}d_0\alpha_1)$$

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Here $\{\cdot,\cdot\}$ denotes the commutator of vector fields (resp. the bracket on $\mathfrak{X}^1$), while d_0 denotes the exterior derivative (resp. the differential on Ω). In both cases, a *Dirac structure* was defined to be a subbundle $D \subset \mathbf{T}M$ (resp. a subspace $\mathcal{D} \subset \mathcal{Q}$) which is maximally isotropic with respect to $\langle\cdot,\cdot\rangle$ and closed under the Courant (resp. Dorfman) bracket; each Dirac structure defines a Poisson bracket on a subalgebra of $C^\infty(M)$ (resp. a subspace of Ω^0), thus explaining the role of this formalism in the theory of constrained dynamical systems, both in mechanics and field theory.

The notion of a *Courant algebroid* was introduced in [13] where it was used to generalize the theory of Manin triples to Lie bialgebroids; it involved a vector bundle equipped with a pseudo-metric, a Courant bracket and an anchor map to the tangent bundle, satisfying a set of compatibility conditions. The notion has since turned up in other contexts. Ševera [21] discovered that a Courant (or Dorfman) bracket could be twisted by a closed 3-form, as a result of which the Courant algebroid $\mathbf{T}M$ took the place of TM in Hitchin's "generalized differential geometry" (= differential geometry in the presence of an abelian gerbe [10]); he also noted that transitive Courant algebroids could be used to give an obstruction-theoretic interpretation of the first Pontryagin class (this theory was fully worked out by Bressler [3], who also elucidated the relation with vertex operator algebras). In general, there is mounting evidence that Courant algebroids play the same rôle in string theory as Poisson structures do in particle mechanics [21, 3, 1].

1.2. **The aim and content of this paper.** In our earlier work [16, 17] we made an attempt to explain Courant algebroids in terms of graded differential geometry by constructing, for each vector bundle E with a non-degenerate pseudo-metric, a graded symplectic (super)manifold $M(E)$. We proved that Courant algebroid structures on E correspond to functions $\Theta \in \mathcal{C}^3(M(E))$ obeying the Maurer-Cartan equation

$$\{\Theta,\Theta\} = 0$$

The advantage of this approach is geometric clarity: after all, graded manifolds are just manifolds with a few bells and whistles, and our construction uses nothing more than a cotangent bundle. As a by-product, it yielded new examples of topological sigma-models [18]. Moreover, the graded manifold approach enabled Ševera [22] to envision an infinite hierarchy of graded symplectic structures similar to the hierarchy of higher categories (our construction in [17] is equivalent to a special case of his).

Nevertheless, the formulation in terms of graded manifolds has certain drawbacks. In particular, we were unable to describe the algebra of functions $\mathcal{C}(M(E))$ explicitly in terms of E, which made it somewhat difficult to work with: general considerations (such as grading) would carry one a certain distance, but to go beyond that one had to either resort to local coordinates, or introduce unnatural extra structure, such as a connection, which rather spoiled the otherwise beautiful picture.

The aim of this paper is to obtain a completely explicit description of the algebra $\mathcal{C}(M(E))$. We work from the outset with a commutative algebra $\mathcal{R}$ and an $\mathcal{R}$-module $\mathcal{E}$ equipped with a pseudo-metric $\langle\cdot,\cdot\rangle$; these can be completely arbitrary: no regularity or finiteness conditions are imposed on $\mathcal{R}$ or $\mathcal{E}$, nor is $\langle\cdot,\cdot\rangle$ required to be non-degenerate[1]. A *Courant-Dorfman algebra* consists of this underlying structure, plus an $\mathcal{E}$-valued derivation ∂ and a (Dorfman) bracket $[\cdot,\cdot]$, satisfying compatibility conditions generalizing those defining a Courant algebroid.

Given a metric $\mathcal{R}$-module $(\mathcal{E},\langle\cdot,\cdot\rangle)$, we construct a graded commutative $\mathcal{R}$-algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ whose degree-q component consists of (finite) sequences $\omega=(\omega_0,\omega_1,\ldots)$, where each ω_k is an $\mathcal{R}$-valued function of $q-2k$ arguments from $\mathcal{E}$ and k arguments from $\mathcal{R}$. With respect to the $\mathcal{R}$-arguments, ω_k is a symmetric k-derivation; the behavior of ω_k under permutations of the $\mathcal{E}$-arguments and the $\mathcal{R}$-module structure is controlled by ω_{k+1}. The algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ is actually a subalgebra of the convolution algebra $\mathrm{Hom}(U(L),\mathcal{R})$ where L is a certain graded Lie algebra.

Furthermore, every Courant-Dorfman structure on the metric module $\mathcal{E}$ gives rise to a differential on the algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ for which we give an explicit formula (4.1). The construction is functorial with respect to (strict) morphisms of Courant-Dorfman algebras; this is the content of our first main Theorem 4.10. The resulting cochain complex, which we call the *standard complex*, is related to the Loday-Pirashvili complex [14] for the Leibniz algebra $(\mathcal{E},[\cdot,\cdot])$ in a way analogous to how the de Rham complex of a manifold is related to the Chevalley-Eilenberg complex of its Lie algebra of vector fields.

We then conduct further investigation of the differential graded algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$. In particular, we describe natural filtrations and subcomplexes, related to those considered in [20] and [9], which we expect to be an important tool in cohomology computations; derive commutation relations among certain derivations of $\mathcal{C}(\mathcal{E},\mathcal{R})$, similar to the well-known Cartan relations among contractions and Lie derivatives by vector fields; classify central extensions of the Courant-Dorfman algebra $\mathcal{E}$ in terms of $H^2(\mathcal{E},\mathcal{R})$. We also consider the canonical cocycle $\Theta=(\Theta_0,\Theta_1)\in\mathcal{C}^3(\mathcal{E},\mathcal{R})$ given by the formula:

$$\begin{aligned}\Theta_0(e_1,e_2,e_3) &= \langle[e_1,e_2],e_3\rangle\\ \Theta_1(e;f) &= -\rho(e)f\end{aligned}$$

generalizing the Cartan 3-form on a quadratic Lie algebra appearing in the Chern-Simons theory.

When the pseudo-metric $\langle\cdot,\cdot\rangle$ is non-degenerate, the algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ has a Poisson bracket for which we also give an explicit formula ((6.1), (6.2) and (6.3)); the differential then becomes Hamiltonian for the canonical cocycle Θ (Theorem 6.3). Finally, for Courant-Dorfman algebras coming from finite-dimensional vector bundles, we prove (Theorem 6.7) that the differential graded Poisson algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ is isomorphic to the algebra $\mathcal{C}(M(E))$ constructed in [17]. The isomorphism associates to every $\omega\in\mathcal{C}^p(M(E))$ the sequence $\Phi\omega=((\Phi\omega)_0,(\Phi\omega_1),\ldots)\in\mathcal{C}^p(\mathcal{E},\mathcal{R})$ where

$$(\Phi\omega)_k(e_1,\ldots,e_{p-2k};f_1,\ldots,f_k)=$$
$$=(-1)^{\frac{(p-2k)(p-2k-1)}{2}}\{\cdots\{\omega,e_1^\flat\},\cdots\},e_{p-2k}^\flat\},f_1\},\cdots\},f_k\}$$

where $e^\flat=\langle e,\cdot\rangle$. Under this isomorphism, our canonical cocycle Θ corresponds to the one constructed in *loc. cit.* In fact, this formula is the main creative input for

[1]This is still more than Dorfman [8] required: what she was dealing with is an example of a structure we called *hemi-strict Lie 2-algebra* in [19].

this work: all the other formulas were "reverse-engineered" from this one and then shown to be valid in the general case.

We would like to emphasize that, apart from overcoming the drawbacks of the graded manifold formulation mentioned above and being completely explicit, our constructions apply in a much more general setting where extra structures, such as local coordinates or connections, may not be available. One point worth mentioning is that the algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ is generally *not* freely generated over $\mathcal{R}$ (in a sense which we hope to eventually make precise, it is as free as possible in the presence of $\langle\cdot,\cdot\rangle$); rather, it has a filtration such that the associated graded algebra is free graded commutative over $\mathcal{R}$. The situation is, more or less, the following: when $\mathcal{R}$ and $\mathcal{E}$ satisfy some finiteness conditions and $\langle\cdot,\cdot\rangle$ is non-degenerate, the set of isomorphisms of $\mathcal{C}(\mathcal{E},\mathcal{R})$ with the free algebra $\mathrm{gr}\mathcal{C}(\mathcal{E},\mathcal{R}) = S_{\mathcal{R}}(\mathfrak{X}^1[-2]\oplus\mathcal{E}^\vee[-1])$ is in 1-1 correspondence with the set of splittings of the extension of $\mathcal{R}$-modules

$$\Lambda^2_{\mathcal{R}}\mathcal{E}^\vee \rightarrowtail \mathcal{C}^2(\mathcal{E},\mathcal{R}) \twoheadrightarrow \mathrm{Der}(\mathcal{R},\mathcal{R}) = \mathfrak{X}^1_{\mathcal{R}}$$

This is known as an *Atiyah sequence*; its splitting is nothing but a metric connection on $\mathcal{E}$. The set of splittings may be empty, but when $\mathcal{R}$ is "smooth" (in the sense that the module $\mathrm{Der}(\mathcal{R},\mathcal{R})$ is projective), splittings do exist and form a torsor under $\Omega^1\otimes_{\mathcal{R}}\Lambda^2_{\mathcal{R}}\mathcal{E}^\vee$; nevertheless, it is important to keep in mind that, when working with a smooth scheme or a complex manifold, such splittings generally exist only locally, while a global splitting is obstructed by the Atiyah class. In [17] we wrote down the Poisson bracket on $S_{\mathcal{R}}(\mathfrak{X}^1[-2]\oplus\mathcal{E}^\vee[-1])$ corresponding to the canonical one on $\mathcal{C}(M(E))$ under a given metric connection ∇; we have since been informed that this bracket had been known to physicists under the name of *Rothstein bracket*. For an approach using this formulation we refer to [12]; our work here was motivated by the desire to avoid any unnatural choices.

1.3. **The sequel(s).** We plan to write (at least) two sequels to this paper, in which we address several issues not covered here. In the first one, we introduce a closed 2-form Ξ on the algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$; it corresponds to the one we constructed on $\mathcal{C}(M(E))$ in [17] (even for degenerate $\langle\cdot,\cdot\rangle$). We use this extra structure to, on the one hand, restrict the class of morphisms of differential graded algebras to those which also preserve this structure, and on the other hand, to expand the class of morphisms of Courant-Dorfman algebras to include lax morphisms, so as to make the functor from Theorem 4.10 fully faithful. We will also consider morphisms of Courant-Dorfman algebras over different base rings. Furthermore, the 2-form Ξ gives rise to a Poisson bracket on a certain subalgebra of $\mathcal{C}(\mathcal{E},\mathcal{R})$ by a graded version of Dirac's formalism [7].

In the second sequel, we consider the general notion of a module over a Courant-Dorfman algebra, based on the notion of a dg module over the dg algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ (possibly with some extra conditions involving Ξ), and study the (derived) category of these modules. One such module is the *adjoint module* $\mathcal{C}(\mathcal{E},\mathcal{E})$ consisting of derivations of $\mathcal{C}(\mathcal{E},\mathcal{R})$ preserving Ξ. It forms a differential graded Lie algebra under the commutator bracket; this dg Lie algebra controls the deformation theory of Courant-Dorfman structures on a fixed underlying metric module, and is analogous to the dg Lie algebra controlling deformations of Lie-Rinehart structures on a fixed underlying module, described in [6].

Eventually, we hope to be able to re-write the whole story using an approach involving nested operads.

1.4. **On relation with other work and choice of terminology.** Our definition of a Courant-Dorfman algebra is very similar to Weinstein's "$(R,\mathcal{A})C$-algebras" [23], except for his non-degeneracy assumption and use of Courant, rather than Dorfman, brackets. Keller and Waldmann [12] gave an "algebraic" definition of a Courant algebroid, while still retaining the finiteness, regularity and non-degeneracy assumptions, and obtained formulas similar to some of those derived here. Our description of the algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ has the same spirit as the formulas describing exterior powers of adjoint and co-adjoint representations of a Lie algebroid in ([2], Example 3.26 and subsection 4.2).

We feel justified in our choice of the term "Courant-Dorfman algebra": not only is it natural and easy to remember, but it also recognizes the contributions of two mathematicians (one of whom has long since left active research while the other is, sadly, no longer with us) to the subject that has since grown in scope far beyond what they had envisioned.

1.5. **Organization of the paper.** The paper is organized as follows. In Section 2 we define Courant-Dorfman algebras, derive some of their basic properties and give a number of examples of these structures, emphasizing connection with the various areas of mathematics where they arise; Section 3 is devoted to the preliminary construction of a convolution algebra associated to a graded Lie algebra; Section 4 is the heart of the paper, where we construct the differential graded algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ and study its properties; Section 5 is devoted to classifying central extensions and studying the canonical class of a Courant-Dorfman algebra; in Section 6 we consider the non-degenerate case and derive formulas for the Poisson bracket; here we also elucidate the relation of our constructions with earlier work on Courant algebroids. Finally, Section 7 is devoted to concluding remarks and speculations. For the convenience of the reader we have also included several appendices where we have collected the necessary facts about derivations, Kähler differentials, Lie-Rinehart algebras and Leibniz algebras.

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2. Definition and basic properties

2.1. **Conventions and notation.** We fix once and for all a commutative ring $\mathbb{K} \ni \frac{1}{2}$ as our ground ring (the condition ensures that $\mathbb{K}$-linear derivations annihilate constants and polarization identities hold). All tensor products and Hom's are assumed to be over $\mathbb{K}$; tensor products and Hom's over other rings will be explicitly indicated by appropriate subscripts.

By a graded module we shall always mean a collection $\mathcal{M} = \{\mathcal{M}_i\}_{i\in\mathbb{Z}}$ of modules indexed by $\mathbb{Z}$. The dual module $\mathcal{M}^\vee$ is defined by setting $\mathcal{M}_i^\vee = (\mathcal{M}_{-i})^\vee$. For a $k \in \mathbb{Z}$, the shifted module $\mathcal{M}[k]$ is defined by $\mathcal{M}[k]_i = \mathcal{M}_{k+i}$, so that $(\mathcal{M}[k])^\vee = (\mathcal{M}^\vee)[-k]$.

The (graded) commutator of operators will always be denoted by $\{\cdot,\cdot\}$.

2.2. Courant-Dorfman algebras and related categories.

Definition 2.1. A *Courant-Dorfman algebra* consists of the following data:

- a commutative $\mathbb{K}$-algebra $\mathcal{R}$;
- an $\mathcal{R}$-module $\mathcal{E}$;
- a symmetric bilinear form (*pseudometric*) $\langle\cdot,\cdot\rangle : \mathcal{E}\otimes_{\mathcal{R}}\mathcal{E}\longrightarrow\mathcal{R}$;
- a derivation $\partial:\mathcal{R}\longrightarrow\mathcal{E}$;
- a *Dorfman bracket* $[\cdot,\cdot]:\mathcal{E}\otimes\mathcal{E}\longrightarrow\mathcal{E}$.

These data are required to satisfy the following conditions:

(1) $[e_1, fe_2] = f[e_1,e_2] + \langle e_1, \partial f\rangle e_2$;
(2) $\langle e_1, \partial\langle e_2,e_3\rangle\rangle = \langle [e_1,e_2],e_3\rangle + \langle e_2,[e_1,e_3]\rangle$;
(3) $[e_1,e_2]+[e_2,e_1] = \partial\langle e_1,e_2\rangle$;
(4) $[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] + [e_2,[e_1,e_3]]$;
(5) $[\partial f, e] = 0$;
(6) $\langle\partial f,\partial g\rangle = 0$

for all $e, e_1, e_2, e_3 \in \mathcal{E}$, $f, g\in\mathcal{R}$.

When only conditions (1), (2) and (3) are satisfied, we shall speak of an *almost Courant-Dorfman algebra* and treat (4), (5) and (6) as integrability conditions.

Remark 2.2. A $\mathbb{K}$-module $\mathcal{E}$ equipped with a bracket $[\cdot,\cdot]$ satisfying condition (4) above is called a ($\mathbb{K}$-) *Leibniz algebra*. For basic facts about these algebras we refer to Appendix D.

Given a Courant-Dorfman algebra, the *Courant bracket* $[\![\cdot,\cdot]\!]$ is defined by the formula

$$[\![e_1,e_2]\!] = \frac{1}{2}([e_1,e_2]-[e_2,e_1])$$

Conversely, the Dorfman bracket can be recovered from the Courant bracket:

$$[e_1,e_2] = [\![e_1,e_2]\!] + \frac{1}{2}\partial\langle e_1,e_2\rangle$$

If $\frac{1}{3}\in\mathcal{K}$, the definition of a Courant-Dorfman algebra can be rewritten in terms of the Courant bracket, as was done originally in [13].

Definition 2.3. The bilinear form $\langle\cdot,\cdot\rangle$ gives rise to a map

$$(\cdot)^\flat : \mathcal{E}\longrightarrow\mathcal{E}^\vee = \mathrm{Hom}_{\mathcal{R}}(\mathcal{E},\mathcal{R})$$

defined by

$$e^\flat(e') = \langle e, e'\rangle$$

We say $\langle\cdot,\cdot\rangle$ is *strongly non-degenerate* if $(\cdot)^\flat$ is an isomorphism, and call a Courant-Dorfman algebra *non-degenerate* if its $\langle\cdot,\cdot\rangle$ is strongly non-degenerate. In this case the inverse map is denoted by

$$(\cdot)^\sharp : \mathcal{E}^\vee\longrightarrow\mathcal{E}$$

and there is a symmetric bilinear form

$$\{\cdot,\cdot\}:\mathcal{E}^\vee\otimes_{\mathcal{R}}\mathcal{E}^\vee\longrightarrow\mathcal{R}$$

defined by

$$\{\lambda,\mu\} = \langle\lambda^\sharp,\mu^\sharp\rangle \qquad (2.1)$$

for $\lambda,\mu\in\mathcal{E}^\vee$.

Remark 2.4. For non-degenerate Courant-Dorfman algebras, it can be shown that conditions (1), (5) and (6) of Def. 2.1 are redundant.

Definition 2.5. A *strict morphism* between Courant-Dorfman algebras $\mathcal{E}$ and $\mathcal{E}'$ is a map of $\mathcal{R}$-modules $f : \mathcal{E} \to \mathcal{E}'$ respecting all the operations.

Remark 2.6. It is possible to define a morphism of Courant-Dorfman algebras over different base rings, as well as weak morphisms which preserve the operations up to coherent homotopies. For the purposes of this paper, strict morphisms over a fixed base suffice; we shall refer to them as simply morphisms from now on.

Courant-Dorfman algebras over a fixed $\mathcal{R}$ form a category, which we denote by $\mathbf{CD}_{\mathcal{R}}$.

Remark 2.7. The Courant-Dorfman structure consists of several layers of underlying structure: the $\mathcal{R}$-module $\mathcal{E}$, the *metric $\mathcal{R}$-module* $(\mathcal{E}, \langle\cdot,\cdot\rangle)$, the *differential metric $\mathcal{R}$-module* $(\mathcal{E}, \langle\cdot,\cdot\rangle, \partial)$ and the $\mathbb{K}$-Leibniz algebra $(\mathcal{E}, [\cdot,\cdot])$. Correspondingly, there are obvious forgetful functors from $\mathbf{CD}_{\mathcal{R}}$ to the categories $\mathbf{Mod}_{\mathcal{R}}$, $\mathbf{Met}_{\mathcal{R}}$, $\mathbf{dMet}_{\mathcal{R}}$ and $\mathbf{Leib}_{\mathbb{K}}$. We shall refer to the respective fiber categories $\mathbf{CD}_{\mathcal{E}}$, $\mathbf{CD}_{(\mathcal{E},\langle\cdot,\cdot\rangle)}$ and $\mathbf{CD}_{(\mathcal{E},\langle\cdot,\cdot\rangle,\partial)}$ when we wish to consider Courant-Dorfman algebra with the indicated underlying structure fixed. We shall frequently speak of just a Courant bracket or a Dorfman bracket, with the rest of the data implicitly understood.

Definition 2.8. Given a locally ringed space $(X, \mathcal{O}_X)$ over $\mathbb{K}$, a *Courant algebroid* over X is an $\mathcal{O}_X$-module $\mathcal{E}$ equipped with a compatible Courant-Dorfman algebra structure.

Remark 2.9. Definition 2.8 differs somewhat from the earlier versions. Traditionally [13], X was required to be a C^∞ manifold, $\mathcal{E}$ locally free of finite rank (i.e. sections of a vector bundle), and $\langle\cdot,\cdot\rangle$ strongly non-degenerate; Bressler [3] drops the finite-rank and non-degeneracy assumptions while still requiring that X be a smooth manifold. Our definition is equivalent to those of loc. cit. under the aforementioned additional assumptions.

2.3. **The anchor, coanchor and tangent complex.** Let $\Omega^1 = \Omega^1_{\mathcal{R}}$ be the $\mathcal{R}$-module of Kähler differentials, with the universal derivation

$$d_0 : \mathcal{R} \longrightarrow \Omega^1.$$

Furthermore, let

$$\mathfrak{X}^1 = \mathfrak{X}^1_{\mathcal{R}} = \mathrm{Der}(\mathcal{R}, \mathcal{R}) \simeq \mathrm{Hom}_{\mathcal{R}}(\Omega^1, \mathcal{R})$$

Now, let $(\mathcal{R}, \mathcal{E})$ be a Courant-Dorfman algebra. By the universal property of Ω^1, there is a unique map of $\mathcal{R}$-modules

$$\delta : \Omega^1 \longrightarrow \mathcal{E}$$

such that $\delta(d_0 f) = \partial f$ (see Appendix B). This map will be referred to as the *coanchor*. Define further the *anchor map*

$$\rho : \mathcal{E} \longrightarrow \mathfrak{X}^1$$

by setting

$$\rho(e) \cdot f = \langle e, \partial f\rangle \tag{2.2}$$

for all $e \in \mathcal{E}$, $f \in \mathcal{R}$.

Remark 2.10. In a non-degenerate almost Courant-Dorfman algebra, ∂ can be recovered from ρ, and condition (1) of Def. 2.1 follows from (2) and (3).

The condition (6) of Definition 2.1 can now be restated as

$$\rho \circ \delta = 0 \tag{2.3}$$

In other words, the following is a cochain complex of $\mathcal{R}$-modules:

$$\Omega^1[2] \xrightarrow{\delta} \mathcal{E}[1] \xrightarrow{\rho} \mathfrak{X}^1 \tag{2.4}$$

This complex will be denoted by $\mathbb{T} = \mathbb{T}_{\mathcal{E}}$ and referred to as the *tangent complex* of the Courant-Dorfman algebra $\mathcal{E}$; the differential on $\mathbb{T}$ will also be denoted by δ (that is, $\delta_{-2} = \delta$, $\delta_{-1} = \rho$).

Definition 2.11. A Courant-Dorfman algebra is *exact* if its tangent complex is acyclic.

The complex $\mathbb{T}$ has an extra structure: namely, the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{E}$ extends to a graded *skew*-symmetric bilinear map of graded $\mathcal{R}$-modules

$$\Xi : \mathbb{T} \otimes_{\mathcal{R}} \mathbb{T} \longrightarrow \mathcal{R}[2]$$

if we define

$$\Xi(v, \alpha) = \iota_v \alpha = -\Xi(\alpha, v) \tag{2.5}$$
$$\Xi(e_1, e_2) = \langle e_1, e_2 \rangle$$

for $v \in \mathfrak{X}^1$, $\alpha \in \Omega^1$, $e_1, e_2 \in \mathcal{E}$.

Proposition 2.12. Ξ *is* δ*-invariant, i.e.*

$$\Xi(\delta a, b) + (-1)^{\deg(a)} \Xi(a, \delta b) = 0 \tag{2.6}$$

for all homogeneous $a, b \in \mathbb{T}$.

Proof. This amounts to saying that, for all $e \in \mathcal{E}$ and $\alpha \in \Omega^1$, one has

$$\langle e, \delta\alpha \rangle = \iota_{\rho(e)} \alpha, \tag{2.7}$$

which is just a restatement of the definitions. □

Proposition 2.13. *The anchor* ρ *is a homomorphism of Leibniz algebras.*

Proof. First, observe that conditions (3) and (5) of Def. 2.1 imply

$$[e, \partial f] = \partial \langle e, \partial f \rangle \tag{2.8}$$

Furthermore, by (2),

$$\langle e_1, \partial \langle \partial f, e_2 \rangle \rangle = \langle [e_1, \partial f], e_2 \rangle + \langle \partial f, [e_1, e_2] \rangle$$

Combining these and using the definition of ρ, we immediately get

$$\rho([e_1, e_2]) \cdot f = \rho(e_1) \cdot (\rho(e_2) \cdot f) - \rho(e_2) \cdot (\rho(e_1) \cdot f),$$

as claimed. □

Corollary 2.14. *Let* $(\mathcal{R}, \mathcal{E})$ *be a Courant-Dorfman algebra, and let* $\mathcal{K} = \ker \rho$. *Then* $(\mathcal{R}, \mathcal{K})$ *is a Courant-Dorfman subalgebra (with zero anchor).*

Proof. By Proposition 2.13, $\mathcal{K}$ is closed under $[\cdot, \cdot]$; by (2.3), the image of ∂ is contained in $\mathcal{K}$. □

Proposition 2.15. *The image $\delta\Omega^1$ is a two-sided ideal with respect to the Dorfman bracket $[\cdot,\cdot]$. More precisely, the following identities hold:*

$$[e, \delta\alpha] = \delta L_{\rho(e)}\alpha$$
$$[\delta\alpha, e] = \delta(-\iota_{\rho(e)} d_0 \alpha)$$

In particular,

$$[\delta\alpha, \delta\beta] = 0$$

for all $\alpha, \beta \in \Omega^1$, $e \in \mathcal{E}$.

Proof. For the first identity, it suffices to consider α of the form $f d_0 g$. The identity then follows by applying condition (1) and the formula (2.8). The second identity then follows immediately from condition (3) and the Cartan identity. The last identity is then a consequence of (2.3). □

Corollary 2.16. *Let $\bar{\mathcal{E}} = \mathcal{E}/\delta\Omega^1$. Then $(\mathcal{R}, \bar{\mathcal{E}})$ is a Lie-Rinehart algebra under the induced bracket and anchor; furthermore, the pseudometric $\langle\cdot,\cdot\rangle$ induces one on $\bar{\mathcal{K}} = \ker \bar{\rho}$ which is, moreover, $\bar{\mathcal{E}}$-invariant (with respect to the natural action of $\bar{\mathcal{E}}$ on $\bar{\mathcal{K}}$, see Appendix C).*

Proof. By Proposition 2.15, the bracket on $\mathcal{E}$ descends to $\bar{\mathcal{E}}$; the induced bracket is skew-symmetric by condition (3). Similarly, by (2.3), one gets the induced anchor $\bar{\rho} : \bar{\mathcal{E}} \longrightarrow \mathfrak{X}^1$. The axioms for a Lie-Rinehart algebra follow immediately from those for Courant-Dorfman algebra.

To prove the last statement, observe that $\bar{\mathcal{K}} = \mathcal{K}/\delta\Omega^1$. Now, for all $e \in \mathcal{K}$, $\alpha \in \Omega^1$,

$$\langle e, \delta\alpha\rangle = \iota_{\rho(e)}\alpha = 0 \tag{2.9}$$

by (2.7), hence $\langle\cdot,\cdot\rangle$ descends to $\bar{\mathcal{K}}$. The $\bar{\mathcal{E}}$-invariance follows from axiom (2). The equation (2.9) implies, in particular, that $\delta\Omega^1$ is isotropic. □

Remark 2.17. Of course, $\mathcal{E}/\partial\mathcal{R}$ is always a Lie algebra (over $\mathbb{K}$).

Definition 2.18. Suppose $\mathcal{E}$ is a Courant-Dorfman algebra. An $\mathcal{R}$-submodule $\mathcal{D} \subset \mathcal{E}$ is said to be a *Dirac submodule* if $\mathcal{D}$ is isotropic with respect to $\langle\cdot,\cdot\rangle$ and is closed under $[\cdot,\cdot]$ (equivalently, under $[\![\cdot,\cdot]\!]$).

Proposition 2.19. *If $\mathcal{D}$ is a Dirac submodule, $(\mathcal{R}, \mathcal{D})$ is a Lie-Rinehart algebra under the restriction of the anchor and bracket.*

Proof. Clear. □

Even though $\langle\cdot,\cdot\rangle$ is allowed to be degenerate, even zero, it is *not* true that a Lie-Rinehart algebra is a special case of a Courant-Dorfman algebra, because of the relation (2.2) between the anchor and $\langle\cdot,\cdot\rangle$. Nevertheless, the notion of a morphism between a Courant-Dorfman algebra and a Lie-Rinehart algebra does make sense.

Definition 2.20. A *strict morphism* from a Lie-Rinehart algebra $\mathcal{L}$ to a Courant-Dorfman algebra $\mathcal{E}$ is a map of $\mathcal{R}$-modules $p : \mathcal{L} \longrightarrow \mathcal{E}$ satisfying the following conditions:

(1) p commutes with anchors and brackets;
(2) $\langle\cdot,\cdot\rangle \circ (p \otimes p) = 0$

Definition 2.21. A *strict morphism* from a Courant-Dorfman algebra $\mathcal{E}$ to a Lie-Rinehart algebra $\mathcal{L}$ is an $\mathcal{R}$-module map $r : \mathcal{E} \longrightarrow \mathcal{L}$ satisfying the following conditions:

(1) r commutes with anchors and brackets;
(2) $r \circ \delta = 0$

Proposition 2.22. *The following are morphisms in the sense of the above definitions:*

- *the anchor* $\rho : \mathcal{E} \longrightarrow \mathfrak{X}^1$*;*
- *the canonical projection* $\pi : \mathcal{E} \longrightarrow \bar{\mathcal{E}}$ *from Corollary 2.16;*
- *the inclusion* $i : \mathcal{D} \longrightarrow \mathcal{E}$ *of a Dirac submodule.*

Proof. Obvious, in view of the already established facts. □

2.4. **Twists.** Given a Courant-Dorfman algebra $\mathcal{E}$ and a 3-form $\psi \in \Omega^3$, we can define a new bracket

$$[e_1, e_2]_\psi = [e_1, e_2] + \delta\iota_{\rho(e_2)}\iota_{\rho(e_1)}\psi \tag{2.10}$$

This twisted bracket $[\cdot,\cdot]_\psi$ will be again a Dorfman bracket (with the same $\langle\cdot,\cdot\rangle$ and ∂) if and only if $d_0\psi = 0$. It is clear that this defines an invertible endofunctor $\mathrm{Tw}(\psi)$ on the category $\mathbf{CD}_\mathcal{R}$, restricting to each $\mathbf{CD}_{(\mathcal{E},\langle\cdot,\cdot\rangle,\partial)}$, and that

$$\mathrm{Tw}(\psi_1 + \psi_2) = \mathrm{Tw}(\psi_1) \circ \mathrm{Tw}(\psi_2)$$

Furthermore, each $\beta \in \Omega^2$ defines a natural transformation $\exp(-\beta)$ from $\mathrm{Tw}(\psi)$ to $\mathrm{Tw}(\psi + d_0\beta)$ via

$$\exp(-\beta)(e) = e - \delta\iota_{\rho(e)}\beta$$

which is also additive. In fact, this yields an action of the group crossed module $\Omega^2 \xrightarrow{d_0} \Omega^{3,\mathrm{cl}}$ on the category $\mathbf{CD}_\mathcal{R}$, restricting to each $\mathbf{CD}_{(\mathcal{E},\langle\cdot,\cdot\rangle,\partial)}$. In particular, the group $\Omega^{2,\mathrm{cl}}$ acts on every Courant-Dorfman algebra by automorphisms.

We refer to [3] for the relevant calculations.

2.5. **Some examples.**

Example 2.23. Let $(\mathcal{R}, \mathcal{E})$ be a Courant-Dorfman algebra with $\langle\cdot,\cdot\rangle = 0$. A quick glance at the axioms then shows that $\mathcal{E}$ is a Lie algebra over $\mathcal{R}$, while ∂ is a derivation with values in the center of $\mathcal{E}$. There are no further restrictions.

As a special case of this, let $\mathcal{E} = \mathcal{R}$. Then the bracket must vanish, while the derivation ∂ can be arbitrary.

More fundamentally, consider $\mathcal{E} = \Omega^1$ with $\partial = d_0$. This is the initial object in $\mathbf{CD}_\mathcal{R}$.

Example 2.24. At the opposite extreme, let $\partial = 0$. Then the definition reduces to that of a quadratic Lie algebra over $\mathcal{R}$ (i.e. a Lie algebra equipped with an ad-invariant quadratic form).

Example 2.25. Given an $\mathcal{R}$, let $\mathcal{Q}_0 = \mathfrak{X}^1 \oplus \Omega^1$. It becomes a Courant-Dorfman algebra with respect to

$$\begin{aligned}
\langle(v_1, \alpha_1), (v_2, \alpha_2)\rangle &= \iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1 \\
\partial f &= (0, d_0 f) \\
[(v_1, \alpha_1), (v_2, \alpha_2)] &= (\{v_1, v_2\}, L_{v_1}\alpha_2 - \iota_{v_2}d_0\alpha_1)
\end{aligned}$$

The bracket here is *the* original Dorfman bracket [8], while the corresponding Courant bracket is

$$[\![(v_1,\alpha_1),(v_2,\alpha_2)]\!] = (\{v_1,v_2\}, L_{v_1}\alpha_2 - L_{v_2}\alpha_1 - \frac{1}{2}d_0(\iota_{v_1}\alpha_2 - \iota_{v_2}\alpha_1))$$

which is *the* original Courant bracket [4].

For any $\psi \in \Omega^{3,\mathrm{cl}}$, the Courant-Dorfman algebra $\mathcal{Q}_\psi = \mathrm{Tw}(\psi)(\mathcal{Q}_0)$ is exact. Conversely, it can be shown [21] that, if $\mathcal{Q}$ is exact and its tangent complex $\mathbb{T}_{\mathcal{Q}}$ (2.4) admits an isotropic splitting, $\mathcal{Q}$ is isomorphic to $\mathcal{Q}_\psi$ for some ψ; since isotropic splittings form an Ω^2-torsor, such exact Courant-Dorfman algebras are classified by $H^3_{\mathrm{dR}}(\mathcal{R})$.

Example 2.26. As a variant of the previous example, we can replace $\mathfrak{X}^1$ by another Lie-Rinehart algebra $(\mathcal{R},\mathcal{L})$, and let $\mathcal{E} = \mathcal{L}\oplus\Omega^1$. Given any $\psi\in\Omega^{3,\mathrm{cl}}$, define the structure maps as follows:

$$\begin{aligned}
\langle (a_1,\alpha_1),(a_2,\alpha_2)\rangle &= \iota_{\rho(a_1)}\alpha_2 + \iota_{\rho(a_2)}\alpha_1\\
\partial f &= (0, d_0 f)\\
[(a_1,\alpha_1),(a_2,\alpha_2)] &= ([a_1,a_2], L_{\rho(a_1)}\alpha_2 - \iota_{\rho(a_2)}d_0\alpha_1 + \iota_{\rho(a_1)}\iota_{\rho(a_2)}\psi),
\end{aligned}$$

where ρ is the anchor of $\mathcal{L}$.

More generally, we can consider a dual pair of compatible Lie-Rinehart algebras (a Lie bialgebroid) [13].

Example 2.27. Consider a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ equipped with an ad-invariant pseudometric $\langle\cdot,\cdot\rangle$. Given a $\mathbb{K}$-algebra $\mathcal{R}$, let $\underline{\mathfrak{g}} = \mathcal{R}\otimes\mathfrak{g}$; extend $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$ to $\underline{\mathfrak{g}}$ by $\mathcal{R}$-linearity. Finally, let $\mathcal{E} = \underline{\mathfrak{g}}\oplus\Omega^1$ and define the structure maps as follows:

$$\begin{aligned}
\langle (F_1,\alpha_1),(F_2,\alpha_2)\rangle &= \langle F_1,F_2\rangle\\
\partial f &= (0,d_0 f)\\
[(F_1,\alpha_1),(F_2,\alpha_2)] &= ([F_1,F_2],\langle F_1, d_0F_2\rangle),
\end{aligned}$$

where, for $F_i = f_i\otimes x_i$ $(i=1,2)$, $\langle F_1,d_0F_2\rangle$ means $\langle x_1,x_2\rangle f_1 d_0 f_2$. Again, it can be easily checked that this defines a Courant-Dorfman structure (with zero anchor map). This algebra goes back to the work of Spencer Bloch on algebraic K-theory (see [3] and references therein).

Example 2.28. As a special case of the previous example, assume that $\mathbb{Q}\subset\mathbb{K}$ and consider $\mathcal{R} = \mathbb{K}[z,z^{-1}]$, the ring of Laurent polynomials. In this case, $\underline{\mathfrak{g}}$ is better known as $L\mathfrak{g}$, the loop Lie algebbra of $\mathfrak{g}$, and the Lie algebra structure on $\mathcal{E}/\partial\mathcal{R} = L\mathfrak{g}\oplus(\Omega^1/d_0\mathcal{R})\simeq L\mathfrak{g}\oplus\mathbb{K}$ is very well-known. The latter isomorphism is induced by the residue map

$$\oint : \Omega^1 \longrightarrow \mathbb{K},$$

and so the Lie bracket is given by the famous Kac-Moody formula

$$[F_1,F_2] + \oint\langle F_1, d_0F_2\rangle$$

Example 2.29. It is possible to combine Examples 2.26 and 2.27. Let $\underline{\mathfrak{g}}$ be a Lie algebra over $\mathcal{R}$. Assume there is an connection ∇ on $\underline{\mathfrak{g}}$ which acts by derivations of the the Lie bracket. Let $\omega\in\Omega^2\otimes\underline{\mathfrak{g}}$ be the curvature. Then $\mathcal{L} = \mathfrak{X}^1\oplus\underline{\mathfrak{g}}$ becomes a Lie-Rinehart algebra with the bracket given by

$$[(v_1,\xi_1),(v_2,\xi_2)] = ([v_1,v_2],[\xi_1,\xi_2] + \nabla_{v_1}\xi_2 - \nabla_{v_2}\xi_1 + \iota_{v_1}\iota_{v_2}\omega)$$

and the anchor given by the projection onto the first factor.

Suppose now that $\underline{\mathfrak{g}}$ is equipped with an ad-invariant ∇-invariant (i.e. $\mathcal{L}$-invariant) pseudometric $\langle\cdot,\cdot\rangle$ and that, moreover, there exists a 3-form $\psi \in \Omega^3$ such that

$$d_0\psi = \frac{1}{2}\langle\omega,\omega\rangle$$

(the vanishing first Pontryagin class condition). Then the Lie-Rinehart structure on $\mathcal{A}$ extends to a Courant-Dorfman structure on $\mathcal{E} = \mathcal{L} \oplus \Omega^1$ as follows:

$$\begin{aligned}
\partial f &= (0, 0, d_0 f)\\
\langle(v_1,\xi_1,\alpha_1),(v_2,\xi_2,\alpha_2)\rangle &= \iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1 + \langle\xi_1,\xi_2\rangle\\
[(v_1,\xi_1,\alpha_1),(v_2,\xi_2,\alpha_2)] &= ([v_1,v_2],[\xi_1,\xi_2] + \nabla_{v_1}\xi_2 - \nabla_{v_2}\xi_1 + \iota_{v_1}\iota_{v_2}\omega,\\
&\quad \langle\xi_1,\nabla\xi_2\rangle + L_{v_1}\alpha_2 - \iota_{v_2}d_0\alpha_1 + \iota_{v_1}\iota_{v_2}\psi)
\end{aligned}$$

We refer to [3] for the relevant calculations.

3. A preliminary construction: universal enveloping and convolution algebras.

Let V and W $\mathbb{K}$-modules, $(\cdot,\cdot) : V\otimes V \to W$ a symmetric bilinear form. Consider the graded $\mathbb{K}$-module $L = W[2] \oplus V[1]$; it becomes a graded Lie algebra over $\mathbb{K}$ with the only nontrivial brackets given by $-(\cdot,\cdot)$. Consider its universal enveloping algebra $U(L)$. As an algebra, it is a quotient of the tensor algebra $T(L)$ (with grading induced by that of L) by the homogeneous ideal generated by elements of the form $v_1 \otimes v_2 + v_2 \otimes v_1 + (v_1, v_2)$, $v \otimes w + w \otimes v$ and $w_1 \otimes w_2 + w_2 \otimes w_1$. Consequently, for $p \geq 0$, we have

$$U(L)_{-p} = \bigoplus_{k=0}^{\left[\frac{p}{2}\right]} (V^{\otimes(p-2k)} \otimes S^k W)/R$$

where R is the submodule generated by elements of the form

$$\begin{gathered}
v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_{p-2k} \otimes w_1 \cdots w_k +\\
+ v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_{p-2k} \otimes w_1 \cdots w_k +\\
+ v_1 \otimes \cdots \otimes \hat{v}_i \otimes \hat{v}_{i+1} \otimes \cdots \otimes v_{p-2k} \otimes (v_i, v_{i+1}) w_1 \cdots w_k
\end{gathered}$$

for $i = 1, \ldots, p-2k-1$, $k = 0, \ldots, \left[\frac{p}{2}\right]$.

Recall that $U(L)$ is also a graded cocommutative coalgebra with comultiplication

$$\Delta : U(L) \longrightarrow U(L) \otimes U(L)$$

uniquely determined by the requirement that the elements of L be primitive and that Δ be an algebra homomorphism. Explicitly,

$$\Delta(v_1 \cdots v_{p-2k} w_1 \cdots w_k) =$$

$$= \sum_{i=0}^{k} \sum_{j=0}^{p-2k} \sum_{\sigma,\tau} (-1)^\sigma v_{\sigma(1)} \cdots v_{\sigma(j)} w_{\tau(1)} \cdots w_{\tau_i} \otimes v_{\sigma(j+1)} \cdots v_{\sigma(p-2k-j)} w_{\tau(i+1)} \cdots w_{\tau(k)}$$

where σ runs over $(j, p-2k-j)$-shuffles, τ – over $(i, k-i)$-shuffles.

Now, recall that, whenever U is a graded $\mathbb{K}$-coalgebra and $\mathcal{R}$ is a graded $\mathbb{K}$-algebra, the graded $\mathcal{R}$-module $\mathrm{Hom}(U, \mathcal{R})$ is naturally an $\mathcal{R}$-algebra, called the *convolution algebra* (this is a general fact about a pair (comonoid, monoid) in any monoidal category).

Let us apply this construction to $U = U(L)$ and an arbitrary $\mathbb{K}$-algebra $\mathcal{R}$ (concentrated in degree 0). Denote the corresponding convolution algebra by $\mathcal{A} = \mathcal{A}(V, W; \mathcal{R}) = \mathrm{Hom}(U(L), \mathcal{R})$. Since $U(L)$ is non-positively graded and $\mathcal{R}$ sits in degree 0, $\mathcal{A}$ is non-negatively graded. Explicitly, for $p \geq 0$, $\mathcal{A}^p$ consists of $\left[\frac{p}{2}\right] + 1$-tuples

$$\omega = (\omega_0, \omega_1, \ldots, \omega_{[\frac{p}{2}]})$$

where

$$\omega_k : V^{\otimes^{p-2k}} \otimes W^{\otimes^k} \longrightarrow \mathcal{R}$$

is symmetric in the W-arguments and satisfying

$$\omega_k(\ldots, v_i, v_{i+1}, \ldots; \ldots) + \omega_k(\ldots, v_{i+1}, v_i, \ldots; \ldots) = \\ = -\omega_{k+1}(\ldots, \hat{v}_i, \hat{v}_{i+1}, \ldots; (v_i, v_{i+1}), \ldots) \tag{3.1}$$

for all $i = 1, \ldots, p - 2k - 1$. By adjunction, ω_k can be viewed as a map

$$V^{\otimes^{p-2k}} \longrightarrow \mathrm{Hom}(S^k W, \mathcal{R})$$

Again, since $S(W[2])$ is a coalgebra (concentrated in even non-positive degrees), $\mathrm{Hom}(S(W[2]), \mathcal{R})$ is an algebra with multiplication given by

$$HK(w_1, \ldots, w_{i+j}) = \\ \sum_{\tau \in \mathrm{sh}(i,j)} H(w_{\tau(1)}, \ldots, w_{\tau(i)}) K(w_{\tau(i+1)}, \ldots, w_{\tau(i+j)}) \tag{3.2}$$

This leads to the following formula for the multiplication in $\mathcal{A}$:

$$(\omega\eta)_k(v_1, \ldots, v_{p+q-2k}) = \\ = \sum_{i+j=k} \sum_{\sigma \in \mathrm{sh}(p-2i, q-2j)} (-1)^\sigma \omega_i(v_{\sigma(1)}, \ldots, v_{\sigma(p-2i)}) \eta_j(v_{\sigma(p-2i+1)}, \ldots, v_{\sigma(p+q-2k)}) \tag{3.3}$$

where the multiplication in each summand takes place in $\mathrm{Hom}(S(W[2]), \mathcal{R})$ according to formula (3.2). In particular,

$$(\omega\eta)_0(e_1, \ldots, e_{p+q}) = \\ = \sum_{\sigma \in \mathrm{sh}(p,q)} (-1)^\sigma \omega_0(e_{\sigma(1)}, \ldots, e_{\sigma(p)}) \eta_0(e_{\sigma(p+1)}, \ldots, e_{\sigma(p+q)}) \tag{3.4}$$

where the multiplication in each summand takes place in $\mathcal{R}$.

Recall further that, as any universal enveloping algebra, $U(L)$ has a canonical increasing filtration

$$\mathbb{K} = U^0 \subset U^1 \subset \cdots \subset U^n \subset U^{n+1} \subset \cdots \subset U(L)$$

where U^n is the submodule spanned by products of no more than n elements of L. This induces a decreasing filtration of $\mathcal{A} = \mathcal{A}(V, W; \mathcal{R})$ by $\mathcal{R}$-submodules

$$\mathcal{A} \supset \mathrm{Ann}(U^1) \supset \cdots \supset \mathrm{Ann}(U^n) \supset \mathrm{Ann}(U^{n+1}) \supset \cdots \supset 0$$

where $\mathrm{Ann}(U^n)$ denotes the annihilator of U^n in $\mathcal{A}$. Now, given $q, i \geq 0$, define

$$\mathcal{A}_i^q = \mathrm{Ann}(U^{q-i-1}) \cap \mathcal{A}^q$$

and

$$\mathcal{A}_i = \bigoplus_{q \geq 0} \mathcal{A}_i^q$$

(set $\mathcal{A}_i = 0$ for $i < 0$). It is easy to see that this defines an *increasing* filtration on $\mathcal{A}$ which is finite in each (superscript) degree, and that, furthermore, $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$ (in particular, $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$ with respect to the multiplication (3.4)). Explicitly,

$$\mathcal{A}_i = \{\omega \in \mathcal{A} | \omega_k = 0 \forall k > i\}$$

(finite sum). Define, as usual, $\mathrm{gr}_i\mathcal{A}^q := \mathcal{A}_i^q/\mathcal{A}_{i-1}^q$, and let

$$\mathrm{gr}\mathcal{A}^q = \bigoplus_i \mathrm{gr}_i\mathcal{A}^q$$

The following is then immediate:

Proposition 3.1. *There is a canonical isomorphism of graded $\mathcal{R}$-modules*

$$\mathrm{gr}\mathcal{A} \simeq \mathrm{Hom}(S(L), \mathcal{R})$$

where the grading on the left hand side is with respect to the superscript degree. In particular,

$$\mathcal{A}_0 = \mathrm{gr}_0\mathcal{A} = \mathrm{Hom}(S(V[1]), \mathcal{R})$$

Remark 3.2. If $\mathbb{K} \supset \mathbb{Q}$, the symmetrization map

$$\Psi : S(L) \longrightarrow U(L)$$

is a coalgebra isomorphism by Poincaré-Birkhoff-Witt theorem. Hence, for any $\mathcal{R}$, the dual map

$$\Psi^* : \mathrm{Hom}(U(L), \mathcal{R}) \longrightarrow \mathrm{Hom}(S(L), \mathcal{R}) = \mathrm{Hom}(S(V[1]), \mathrm{Hom}(S(W[2]), \mathcal{R}))$$

is an isomorphism of algebras. Explicitly,

$$(\Psi^*\omega)_k(v_1, \ldots, v_{p-2k}) = \frac{1}{(p-2k)!} \sum_{\sigma \in S_{p-2k}} (-1)^\sigma \omega_k(v_{\sigma(1)}, \ldots, v_{\sigma(p-2k)}) \tag{3.5}$$

4. The standard complex.

4.1. **The algebra $\mathcal{C}(\mathcal{E}, \mathcal{R})$.** Let $(\mathcal{E}, \langle\cdot,\cdot\rangle)$ be a metric $\mathcal{R}$-module; consider the convolution algebra $\mathcal{A} = \mathcal{A}(\mathcal{E}, \Omega^1; \mathcal{R})$ as in the previous section, with $(\cdot,\cdot) = d_0\langle\cdot,\cdot\rangle$. Let $\mathcal{C}^0 = \mathcal{R}$ and for each $p > 0$, define the submodule $\mathcal{C}^p \subset \mathcal{A}^p$ as consisting of those $\omega = (\omega_0, \omega_1, \ldots)$ which satisfy the following two extra conditions:

(1) Each ω_k takes values in $\mathfrak{X}^k = \mathrm{Hom}_{\mathcal{R}}(S^k_{\mathcal{R}}\Omega^1, \mathcal{R}) \subset \mathrm{Hom}(S^k\Omega^1, \mathcal{R})$;
(2) Each ω_k is $\mathcal{R}$-linear in the last ($p-2k$-th) argument.

For $e_1, \ldots, e_{p-2k} \in \mathcal{E}$, $\omega_k(e_1, \ldots, e_{p-2k})$ can be viewed as either a symmetric k-derivation of $\mathcal{R}$ whose value on $f_1, \ldots, f_k \in \mathcal{R}$ will be denoted by

$$\omega_k(e_1, \ldots, e_{p-2k}; f_1, \ldots, f_k)$$

or as a symmetric $\mathcal{R}$-multilinear function on Ω^1 whose value on a k-tuple $\alpha_1, \ldots, \alpha_k$ will be similarly denoted by

$$\bar{\omega}_k(e_1, \ldots, e_{p-2k}; \alpha_1, \ldots, \alpha_k)$$

so that

$$\bar{\omega}_k(\ldots; d_0f_1, \ldots, d_0f_k) = \omega_k(\ldots; f_1, \ldots, f_k)$$

Evidently $\bar{\omega}_0 = \omega_0$. Often we shall drop the bar from the notation altogether when it is not likely to cause confusion.

Remark 4.1. If there exist $e, e' \in \mathcal{E}$ such that $\langle e, e' \rangle = 1$, then an $\omega = (\omega_0, \omega_1, \ldots)$ is uniquely determined by ω_0, for then

$$-\omega_1(e_1, \ldots; f) = \omega_0(fe, e', e_1, \ldots) + \omega_0(e', fe, e_1, \ldots)$$

and so on by induction. This condition is very often satisfied and is a great help when one needs to prove, for instance, that some cochain vanishes.

Proposition 4.2. *For all $1 \leq i < p - 2k$ the following holds:*

$$\bar{\omega}_k(\ldots, fe_i, \ldots) = f\bar{\omega}_k(\ldots, e_i, \ldots) + \\ + \sum_{j=1}^{p-2k-i} (-1)^j \langle e_i, e_{i+j} \rangle \iota_{d_0 f} \bar{\omega}_{k+1}(\ldots, \hat{e}_i, \ldots, \hat{e}_{i+j}, \ldots)$$

where ι_α denotes contraction with $\alpha \in \Omega^1$.

Proof. By induction from $i = p - 2k$ downward, using (3.1) at each step. □

Define $\mathcal{C} = \mathcal{C}(\mathcal{E}, \mathcal{R}) = \{\mathcal{C}^p\}_{p \geq 0}$.

Proposition 4.3. *$\mathcal{C}(\mathcal{E}, \mathcal{R}) \subset \mathcal{A}(\mathcal{E}, \Omega^1; \mathcal{R})$ is a graded subalgebra.*

Proof. Given $\omega \in \mathcal{C}^p$, $\eta \in \mathcal{C}^q$ we must show that $\omega\eta$ satisfied conditions (1) and (2) defining $\mathcal{C}$. The first one is clear, while the second one follows from the observation that, since the expression (3.3) for $(\omega\eta)_k$ is a sum over *shuffle* permutations, the last argument of $(\omega\eta)_k$ occurs either as the last argument of ω_i or the last argument of η_j. □

Let $s : (\mathcal{E}, \langle \cdot, \cdot \rangle) \longrightarrow (\mathcal{E}', \langle \cdot, \cdot \rangle')$ be a map of metric $\mathcal{R}$-modules. It induces a map $s^\vee : \mathcal{C}(\mathcal{E}', \mathcal{R}) \longrightarrow \mathcal{C}(\mathcal{E}, \mathcal{R})$ given by

$$(s^\vee \omega)_k(e_1, \ldots, e_{q-2k}) = \omega_k(s(e_1), \ldots, s(e_{q-2k}))$$

This map is obviously a morphism of graded $\mathcal{R}$-algebras. In other words,

Proposition 4.4. *The assignment $(\mathcal{E}, \langle \cdot, \cdot \rangle) \mapsto \mathcal{C}(\mathcal{E}, \mathcal{R})$, $s \mapsto s^\vee$ is a contravariant functor from the category $\mathbf{Met}_\mathcal{R}$ of metric $\mathcal{R}$-modules to the category $\mathbf{gra}_\mathcal{R}$ of graded commutative $\mathcal{R}$-algebras.*

4.2. **The filtration $\{\mathcal{C}_i\}_{i \geq 0}$.** The filtration $\{\mathcal{A}_i\}$ on $\mathcal{A}$ induces one on $\mathcal{C}$ by $\mathcal{C}_i = \mathcal{A}_i \cap \mathcal{C}$.

Proposition 4.5. *There is a canonical isomorphism of graded $\mathcal{R}$-modules*

$$\mathrm{gr}\mathcal{C} \simeq \mathrm{Hom}_\mathcal{R}(S_\mathcal{R}(\mathcal{E}[1] \oplus \Omega^1[2]), \mathcal{R})$$

In particular,

$$\mathcal{C}_0 = \mathrm{Hom}_\mathcal{R}(S_\mathcal{R}(\mathcal{E}[1]), \mathcal{R})$$

is a subalgebra of $\mathcal{C}$.

Proof. Observe that, if $\omega = (\omega_0, \ldots, \omega_i, 0, \ldots) \in \mathcal{C}_i^q$, then ω_i is completely skew-symmetric in the first $q - 2i$ variables and hence $\mathcal{R}$-linear in each of them by Proposition 4.2. Clearly, ω_i only depends on the class of ω in $\mathrm{gr}_i\mathcal{C}^q$, and vanishes if and only if $\omega \in \mathcal{C}_{i-1}^q$. □

Remark 4.6. Observe that, in particular, $\mathcal{C}^0 = \mathcal{C}_0^0 = \mathcal{R}$ and $\mathcal{C}^1 = \mathcal{C}_0^1 = \mathrm{Hom}_\mathcal{R}(\mathcal{E}, \mathcal{R}) = \mathcal{E}^\vee$. One always has the natural inclusion $\Lambda_\mathcal{R}\mathcal{E}^\vee \hookrightarrow \mathcal{C}_0$. If $\mathcal{E}$ is sufficiently nice (e.g. locally free of finite rank), this inclusion is an isomorphism, so that $\mathcal{C}_0$ is generated as an algebra by $\mathcal{C}^{\leq 1}$. Moreover, in that case $\mathcal{C}^{\leq 2}$ generates *all* of $\mathcal{C}$.

Remark 4.7. If $\mathbb{K} \supset \mathbb{Q}$, the image of $\mathcal{C}(\mathcal{E}, \mathcal{R})$ under the symmetrization map Ψ^* (3.5) is the subalgebra $\hat{\mathcal{C}}(\mathcal{E}, \mathcal{R})$ of $\mathrm{Hom}(S(\mathcal{E}[1]), \mathrm{Hom}(S(\Omega^1[2]), \mathcal{R})$ consisting of those $\hat{\omega} = (\hat{\omega}_0, \hat{\omega}_1, \ldots)$ which satisfy the following two conditions:

(1) Each $\hat{\omega}_k$ takes values in $\mathfrak{X}^k = \mathrm{Hom}_{\mathcal{R}}(S^k_{\mathcal{R}}\Omega^1, \mathcal{R}) \subset \mathrm{Hom}(S^k\Omega^1, \mathcal{R})$;

(2) For any $i = 1, \ldots, \deg\hat{\omega} - 2k$ and $f \in \mathcal{R}$,

$$\hat{\omega}_k(\ldots, fe_i, \ldots) = f\hat{\omega}_k(\ldots, e_i, \ldots)+$$
$$+\frac{1}{2}\sum_{j\neq i}(-1)^{i-j+\theta(i-j)}\langle e_i, e_j\rangle \iota_{d_0 f}\hat{\omega}_{k+1}(\ldots, \hat{e}_i, \ldots, \hat{e}_j, \ldots)$$

where θ is the Heaviside function (so that $(-1)^{\theta(i-j)} = \frac{j-i}{|j-i|}$).

This algebra is relevant for the Courant bracket-based formulation, which some researchers may prefer.

4.3. **The differential.** Suppose now that $(\mathcal{R}, \mathcal{E})$ is equipped with an almost Courant-Dorfman structure. For $\eta \in \mathcal{C}^q(\mathcal{E}, \mathcal{R})$, define $d\eta = ((d\eta)_0, (d\eta)_1, \ldots)$ by setting

$$(d\eta)_k(e_1, \ldots, e_{q-2k+1}; f_1, \ldots, f_k) = \tag{4.1}$$
$$= \sum_{\mu=1}^{k}\eta_{k-1}(\partial f_\mu, e_1, \ldots, e_{q-2k+1}; f_1, \ldots, \widehat{f_\mu}, \ldots, f_k)+$$
$$+\sum_{i=1}^{q-2k+1}(-1)^{i-1}\langle e_i, \partial(\eta_k(e_1, \ldots, \widehat{e_i}, \ldots, e_{q-2k+1}; f_1, \ldots, f_k))\rangle+$$
$$+\sum_{i<j}(-1)^i\eta_k(e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, [e_i, e_j], e_{j+1}, \ldots, e_{q-2k+1}; f_1, \ldots, f_k)$$

Proposition 4.8. *The operator d is a derivation of the algebra $\mathcal{C}(\mathcal{E}, \mathcal{R})$ of degree +1; if the almost Courant-Dorfman structure is a Courant-Dorfman structure, it squares to zero.*

In this generality, the only proof we have is a verification of all the claims (that $d\mathcal{C}^q \subset \mathcal{C}^{q+1}$, d is a derivation and $d^2 = 0$) by a direct calculation. It is completely straightforward but extremely tedious; to save space and time, we omit it. However, it is worth noting that, under the conditions of Remark 4.6, it suffices to do the calculations in low degrees. We display these calculations here as it is certainly instructive to see how the conditions (4), (5) and (6) of Def 2.1 imply that $d^2 = 0$. Thus, for $f \in \mathcal{C}^0 = \mathcal{R}$ we have $df = (df)_0 \in \mathcal{C}^1 = \mathcal{E}^\vee$ with

$$(df)_0(e) = \langle e, \partial f\rangle = \rho(e)f$$

whereas for $\lambda \in \mathcal{C}^1$ we have $d\lambda = ((d\lambda)_0, (d\lambda)_1)$ with

$$\begin{aligned}(d\lambda)_0(e_1, e_2) &= \rho(e_1)(\lambda(e_2)) - \rho(e_2)(\lambda(e_1)) - \lambda([e_1, e_2])\\ (d\lambda)_1(g) &= \lambda(\partial g)\end{aligned}$$

Therefore,

$$(d(df))_0(e_1, e_2) = \rho(e_1)(\rho(e_2)f) - \rho(e_2)(\rho(e_1)f) - \rho([e_1, e_2])f = 0$$

by Proposition 2.13, while

$$(d(df))_1(g) = df(\partial g) = \langle \partial g, \partial f\rangle = 0$$

by condition (6) of Def. 2.1.

Now, if $\omega = (\omega_0, \omega_1) \in \mathcal{C}^2$, $d\omega = ((d\omega)_0, (d\omega)_1)$ where

$$\begin{aligned}(d\omega)_0(e_1, e_2, e_3) &= \rho(e_1)\omega_0(e_2, e_3) - \rho(e_2)\omega_0(e_1, e_3) + \rho(e_3)\omega_0(e_1, e_2) - \\ &\quad -\omega_0([e_1, e_2], e_3) - \omega_0(e_2, [e_1, e_3]) + \omega_0(e_1, [e_2, e_3]) \\ (d\omega)_1(e, f) &= \omega_0(\partial f, e) + \rho(e)\omega_1(f)\end{aligned}$$

from which we obtain, using Proposition 2.13:

$$(d(d\lambda))_0(e_1, e_2, e_3) = \lambda([[e_1, e_2], e_3] + [e_2, [e_1, e_3]] - [e_1, [e_2, e_3]]) = 0$$

by condition (4) of Def 2.1, and

$$(d(d\lambda))_1(e, f) = \rho(\partial f)\lambda(e) - \lambda([\partial f, e]) = 0$$

by conditions (6) and (5) of Def. 2.1.

Corollary 4.9. *Given $\eta \in \mathcal{C}^q$, $\overline{d\eta} = ((\overline{d\eta})_0, (\overline{d\eta})_1, \ldots)$ is given by*

$$(\overline{d\eta})_k(e_1, \ldots, e_{q-2k+1}; \alpha_1, \ldots, \alpha_k) = \tag{4.2}$$

$$\begin{aligned}&= \sum_{\mu=1}^{k} \bar\eta_{k-1}(\delta\alpha_\mu, e_1, \ldots; \alpha_1, \ldots, \widehat{\alpha_\mu}, \ldots, \alpha_k) + \\ &+ \sum_{i=1}^{q-2k+1} (-1)^{i-1}\rho(e_i)\bar\eta_k(e_1, \ldots, \widehat{e_i}, \ldots; \alpha_1, \ldots, \alpha_k) + \\ &+ \sum_{i=1}^{q-2k+1} \sum_{\mu=1}^{k} (-1)^i \bar\eta_k(e_1, \ldots, \widehat{e_i}, \ldots; \alpha_1, \ldots, \iota_{\rho(e_i)} d_0\alpha_\mu, \ldots, \alpha_k) + \\ &+ \sum_{i<j} (-1)^i \bar\eta_k(e_1, \ldots, \widehat{e_i}, \ldots, \widehat{e_j}, [e_i, e_j], e_{j+1}, \ldots; \alpha_1, \ldots, \alpha_k)\end{aligned}$$

Proof. It is obvious that

$$(\overline{d\eta})_k(\ldots; d_0 f_1, \ldots, d_0 f_k) = (d\eta)_k(\ldots; f_1, \ldots, f_k)$$

so we only have to prove $\mathcal{R}$-linearity in the α's. This is done with the help of Proposition 4.2. □

If $s : \mathcal{E} \longrightarrow \mathcal{E}'$ is a strict morphism of Courant-Dorfman algebras, a quick inspection of the formulas reveals that $s^\vee$ commutes with differentials. We summarize the preceding discussion in our main theorem, extending Proposition 4.4:

Theorem 4.10. *The assignment $(\mathcal{R}, \mathcal{E}) \mapsto (\mathcal{C}(\mathcal{E}, \mathcal{R}), d)$, $s \mapsto s^\vee$ is a contravariant functor from the category $\mathbf{CD}_\mathcal{R}$ of Courant-Dorfman algebras over $\mathcal{R}$ and strict morphisms to the category $\mathbf{dga}_\mathcal{R}$ of differential graded algebras with zero-degree component equal to $\mathcal{R}$ and $\mathcal{R}$-linear dg morphisms.*

Remark 4.11. The tangent complex $\mathbb{T}_\mathcal{E}$ we have constructed (2.4) is indeed the tangent complex of the dg algebra $\mathcal{C}(\mathcal{E}, \mathcal{R})$ in the sense of algebraic geometry.

Corollary 4.12. *Given a locally ringed space $(X, \mathcal{O}_X)$, there is a (covariant) functor from the category $\mathbf{CA}_X$ of Courant algebroids over X to the category $\mathbf{dgS}_X$ of differential graded spaces over X.*

The complex $(\mathcal{C}(\mathcal{E},\mathcal{R}),d)$ will be referred to as the *standard complex* of $(\mathcal{R},\mathcal{E})$, and its q-th cohomology module will be denoted by $H^q(\mathcal{E},\mathcal{R})$. It is an analogue, for Courant-Dorfman algebras, of the de Rham complex of a Lie-Rinehart algebra $(\mathcal{R},\mathcal{L})$ (see Appendix C). In the latter case there is an evident chain map from the de Rham complex to the Chevalley-Eilenberg complex of the Lie algebra $\mathcal{L}$ with coefficients in the module $\mathcal{R}$. There is an analogous statement in our case:

Proposition 4.13. *The assignment $\eta \mapsto \eta_0$ is a chain map from the standard complex $\mathcal{C}(\mathcal{E},\mathcal{R})$ to the Loday-Pirashvili complex $\mathcal{C}_{\mathrm{LP}}(\mathcal{E},\mathcal{R})$ of the Leibniz algebra $\mathcal{E}$ with coefficients in the symmetric $\mathcal{E}$-module $\mathcal{R}$.*

Proof. We have

$$(d\eta)_0(e_1,\ldots,e_{q+1}) = \tag{4.3}$$

$$= \sum_{i=1}^{q+1}(-1)^{i-1}\rho(e_i)\eta_0(e_1,\ldots,\widehat{e_i},\ldots,e_{q+1}) +$$

$$+ \sum_{i<j}(-1)^i\eta_0(e_1,\ldots,\widehat{e_i},\ldots,\widehat{e_j},[e_i,e_j],e_{j+1},\ldots,e_{q+1})$$

which coincides with the expression (D.2) for $d_{LP}(\eta_0)$, where one defines

$$[e,f] = \rho(e)f = -[f,e].$$

□

4.4. **On morphisms between Lie-Rinehart and Courant-Dorfman algebras.** Let $\mathcal{L}$ be a Lie-Rinehart algebra and $\mathcal{E}$ Courant-Dorfman algebra. Suppose $p : \mathcal{L} \longrightarrow \mathcal{E}$ is a morphism in the sense of Def. 2.20. We define the induced map

$$p^\vee : \mathcal{C}(\mathcal{E};\mathcal{R}) \longrightarrow \widetilde{\Omega}(\mathcal{L},\mathcal{R})$$

(see Appendix C) by setting

$$(p^\vee\omega)(x_1,\ldots,x_q) = \omega_0(p(x_1),\ldots,p(x_q))$$

Similarly, given a morphism $r : \mathcal{E} \longrightarrow \mathcal{L}$ in the sense of Def. 2.21, we define

$$r^\vee : \widetilde{\Omega}(\mathcal{L},\mathcal{R}) \longrightarrow \mathcal{C}(\mathcal{E},\mathcal{R})$$

by

$$\begin{aligned}(r^\vee\omega)_0(e_1,\ldots,e_q) &= \omega(r(e_1),\ldots,r(e_q))\\ (r^\vee\omega)_{i>0} &= 0\end{aligned}$$

(so the image of $r^\vee$ is contained in $\mathcal{C}_0(\mathcal{E},\mathcal{R})$).

Proposition 4.14. *The maps $p^\vee$ and $r^\vee$ are morphisms of differential graded algebras.*

Proof. Given $\omega \in \mathcal{C}^q(\mathcal{L},\mathcal{R})$, $(dr^\vee\omega)_0 = (r^\vee d\omega)_0$ by condition (1) of Def. 2.21 and because for alternating cochains, the Loday-Pirashvili formula (4.3) coincides with the Cartan-Chevalley-Eilenberg formula (C.1), whereas $(dr^\vee\omega)_1 = 0$ by condition (2) of Def. 2.21.

On the other hand, $dp^\vee - p^\vee d = 0$ by formula (3.1) and conditions (1) and (2) of Def. 2.20. Details are left to the reader. □

In particular, the morphisms $\rho : \mathcal{E} \longrightarrow \mathfrak{X}^1$, $\pi : \mathcal{E} \longrightarrow \bar{\mathcal{E}}$ and $i : \mathcal{D} \longrightarrow \mathcal{E}$ (see Prop. 2.22) give rise to the corresponding dg maps $\rho^\vee : \widetilde{\Omega}_\mathcal{R} \longrightarrow \mathcal{C}(\mathcal{E},\mathcal{R})$, $\pi^\vee : \widetilde{\Omega}(\bar{\mathcal{E}},\mathcal{R}) \longrightarrow \mathcal{C}(\mathcal{E},\mathcal{R})$ and $i^\vee : \mathcal{C}(\mathcal{E},\mathcal{R}) \longrightarrow \widetilde{\Omega}(\mathcal{D},\mathcal{R})$.

Finally, since the de Rham algebra $(\Omega_\mathcal{R}, d_0)$ is initial in the category $\mathbf{dga}_\mathcal{R}$, there is an evident dg map from $\Omega_\mathcal{R}$ to each of these dg algebras, commuting with the above maps. We shall denote this map by ρ^*, where ρ is the anchor. Explicitly,

$$(\rho^*\omega)_0(e_1,\ldots,e_q) = \iota_{\rho(e_q)}\cdots\iota_{\rho(e_1)}\omega \tag{4.4}$$

$$(\rho^*\omega)_{>0} = 0 \tag{4.5}$$

4.5. **Filtration $\{\mathcal{F}_i\}_{i\geq 0}$ and ideal $\mathcal{I}$.** Observe that the differential d does not preserve the filtration $\{\mathcal{C}_i\}$. In fact, for $\omega \in \mathcal{C}_k$, $\omega_{k+1} = 0$ but

$$(d\omega)_{k+1}(e_1,\ldots;\alpha_1,\ldots,\alpha_{k+1}) = \sum_{\mu=1}^{k+1} \omega_k(\delta\alpha_\mu, e_1,\ldots;\alpha_1,\ldots,\widehat{\alpha}_\mu,\ldots,\alpha_{k+1})$$

does not vanish in general. Nevertheless, this formula suggests a fix. Let us define $\mathcal{F}_k \subset \mathcal{C}_k$ as consisting of such $\omega = (\omega_0,\omega_1,\ldots)$ that, for each $i = 1,\ldots,k$, ω_i vanishes if any $k-i+1$ of its arguments are of the form $\delta\alpha$ for some $\alpha \in \Omega^1$. Notice that, because of (3.1), it does not matter *which* of the arguments those are. Obviously, $\mathcal{F}_{k+1} \subset \mathcal{F}_k$.

Proposition 4.15. *$d\mathcal{F}_k \subset \mathcal{F}_k$ and $\mathcal{F}_{k_1}\mathcal{F}_{k_2} \subset \mathcal{F}_{k_1+k_2}$. In particular, $\mathcal{F}_0$ is a differential graded subalgebra of $\mathcal{C}(\mathcal{E},\mathcal{R})$ equal to $\pi^\vee\mathcal{C}(\bar{\mathcal{E}},\mathcal{R})$.*

Proof. The first statement follows by inspection of formula (4.2), using Proposition 2.15. For the second one, suppose $\omega \in \mathcal{F}_{k_1}$, $\eta \in \mathcal{F}_{k_2}$, and consider the expression (3.3) for $(\omega\eta)_k$:

$$(\omega\eta)_k(e_1,\ldots) =$$
$$= \sum_{i\geq 1}\sum_\sigma (-1)^\sigma \omega_i(e_{\sigma(1)},\ldots)\eta_{k-i}(e_{\sigma(\deg\omega-2i+1)},\ldots)$$

Suppose that $n = k_1+k_2-k+1$ of the arguments are $\delta\alpha$'s. By our assumption, ω_i vanishes if at least $r = k_1-i+1$ of its arguments are $\delta\alpha$'s, while η_{k-i} vanishes if at least $s = k_2-k+i+1$ of its arguments are $\delta\alpha$'s. Now, in each term on the right hand side, some m of the arguments of ω_i are $\delta\alpha$'s, while the $n-m$ remaining $\delta\alpha$'s are arguments of η_{k-i}. Since $r+s = n+1$, either $m \geq r$ or $n-m \geq s$. Therefore, either ω_i or η_{k-i} vanishes; hence, so does $(\omega\eta)_k$.

The last statement is obvious. □

Let us also consider, for each $q > 0$, the submodule $\mathcal{I}^q \subset \mathcal{C}^q$ consisting of those $\omega = (\omega_0,\omega_1,\ldots)$ such that for each k and all $\alpha_1,\ldots,\alpha_{q-2k} \in \Omega^1$,

$$\omega_k(\delta\alpha_1,\ldots,\delta\alpha_{q-2k}) = 0$$

Let $\mathcal{I} = \{I^q\}_{q>0}$.

Proposition 4.16. *$\mathcal{I}$ is a differential graded ideal of $\mathcal{C}(\mathcal{E},\mathcal{R})$.*

Proof. Follows immediately upon inspecting formulas (3.3) and (4.2), in view of Proposition 2.15. □

We expect that the filtrations $\{\mathcal{F}_i\}$ and $\{\mathcal{I}^{(i)}\}$ (powers of the ideal $\mathcal{I}$) will be useful in computing the cohomology of $\mathcal{E}$ (see subsection 6.5 below).

4.6. **Some Cartan-like formulas.** Given an $\alpha \in \Omega^1$, consider the operator $\iota_\alpha : \mathcal{C} \longrightarrow \mathcal{C}[-2]$ defined by $(\iota_\alpha \bar{\omega})_k = \iota_\alpha \bar{\omega}_{k+1}$, i.e

$$(\iota_\alpha \bar{\omega})_k(e_1, \ldots; \alpha_1, \ldots, \alpha_k) = \bar{\omega}_{k+1}(e_1, \ldots; \alpha, \alpha_1, \ldots, \alpha_k) \tag{4.6}$$

It is easy to check that this defines a derivation of the algebra $\mathcal{C}(\mathcal{E}, \mathcal{R})$. For $f \in \mathcal{R}$, define the operator ι_f so that

$$\overline{\iota_f \omega} = \iota_{d_0 f} \bar{\omega} \tag{4.7}$$

Similarly, for any $e \in \mathcal{E}$, the operator $\iota_e : \mathcal{C} \longrightarrow \mathcal{C}[-1]$ given by

$$(\iota_e \omega)_k(e_2, \ldots) = \omega_k(e, e_2, \ldots) \tag{4.8}$$

defines a derivation of $\mathcal{C}(\mathcal{E}, \mathcal{R})$ of degree -1.

Recall that the $\mathbb{K}$-module $L = \mathcal{R}[2] \oplus \mathcal{E}[1]$ forms a graded Lie algebra with respect to the brackets $-\langle \cdot, \cdot \rangle$.

Proposition 4.17. *The assignments $f \mapsto \iota_f$ and $e \mapsto \iota_e$ define an action of the graded Lie algebra L on $\mathcal{C}(\mathcal{E}, \mathcal{R})$ by derivations.*

Proof. The only non-trivial commutation relation is

$$\{\iota_{e_1}, \iota_{e_2}\} = \iota_{-\langle e_1, e_2 \rangle} \tag{4.9}$$

which follows immediately from (3.1). □

Let us now define

$$L_e = \{\iota_e, d\} \quad \text{and} \quad L_f = \{\iota_f, d\} \tag{4.10}$$

Then the following analogues of the well-known Cartan commutation relations hold:

$$L_f = \iota_{\partial f} \tag{4.11}$$

$$\{L_f, \iota_e\} = \iota_{-\langle \partial f, e \rangle} = \iota_{-\rho(e) f} = \{L_e, \iota_f\} \tag{4.12}$$

$$\{L_{e_1}, \iota_{e_2}\} = \iota_{[e_1, e_2]} \tag{4.13}$$

$$\{L_f, L_g\} = 0 \tag{4.14}$$

$$\{L_e, L_f\} = L_{\langle e, \partial f \rangle} = L_{\rho(e) f} = -\{L_f, L_e\} \tag{4.15}$$

$$\{L_{e_1}, L_{e_2}\} = L_{[e_1, e_2]} \tag{4.16}$$

We leave the derivation of these identities as an easy exercise for the reader.

Remark 4.18. The assignment $\alpha \mapsto \iota_\alpha$ is $\mathcal{R}$-linear while $f \mapsto \iota_f$ and $e \mapsto \iota_e$ are not. If $d_0 \alpha = 0$, one has

$$L_\alpha = \{\iota_\alpha, d\} = \iota_{\delta \alpha}$$

but otherwise the algebra does not close. This is because there are more derivations of $\mathcal{C}(\mathcal{E}, \mathcal{R})$ of negative degree than we have accounted for here: there are also derivations coming from maps $\phi \in \mathrm{Hom}_{\mathcal{R}}(\mathcal{E}, \Omega^1)$, of the form

$$(\iota_\phi \omega)_k(e_1, \ldots) = \sum_{i \geq 0} (-1)^{i-1} \iota_{\phi(e_i)} \omega_{k+1}(e_1, \ldots, \widehat{e}_i, \ldots)$$

A description of the full algebra of derivations will be done in the sequel.

5. Some applications

5.1. H^2 **and central extensions.** Let us consider extensions of $\mathcal{R}$-modules of the form

$$\mathcal{R} \overset{i}{\rightarrowtail} \widehat{\mathcal{E}} \overset{p}{\twoheadrightarrow} \mathcal{E} \tag{5.1}$$

Definition 5.1. Suppose $(\mathcal{E}, \langle\cdot,\cdot\rangle, \partial, [\cdot,\cdot])$ and $(\widehat{\mathcal{E}}, \langle\cdot,\cdot\rangle', \partial', [\cdot,\cdot]')$ are Courant-Dorfman algebras and $p : \widehat{\mathcal{E}} \longrightarrow \mathcal{E}$ is a strict morphism fitting into (5.1). We say that (5.1) is a *central extension* of Courant-Dorfman algebras if the following conditions hold:

(1) $(i(f))^\flat = 0$ for all $f \in \mathcal{R}$;
(2) $[\hat{e}, i(f)] = \rho'(\hat{e})f$ for all $\hat{e} \in \widehat{\mathcal{E}}$ and $f \in \mathcal{R}$, where ρ' is the anchor of $\widehat{\mathcal{E}}$.

A (necessarily iso) *morphism* of central extensions is a morphism of extensions (5.1) which is also a Courant-Dorfman morphism.

Proposition 5.2. *The* $\mathbb{K}$*-module of isomorphism classes of central extensions* (5.1) *which are split as metric* $\mathcal{R}$*-modules is isomorphic to* $H^2(\mathcal{E}, \mathcal{R})$.

Proof. The extension being split as metric $\mathcal{R}$-modules means that $\widehat{\mathcal{E}}$ is isomorphic to $\mathcal{E} \oplus \mathcal{R}$ as an $\mathcal{R}$-module in such a way that

$$\langle (e_1, f_1), (e_2, f_2)\rangle' = \langle e_1, e_2\rangle \tag{5.2}$$

The argument follows the well-known pattern: ∂' necessarily has the form

$$\partial' f = (\partial f, -\omega_1(f)) \tag{5.3}$$

for some $\omega_1 \in \mathfrak{X}^1$, while the bracket must have the form

$$[(e_1, f_1), (e_2, f_2)]' = ([e_1, e_2], \rho(e_1)f_2 - \rho(e_2)f_1 + \omega_0(e_1, e_2)) \tag{5.4}$$

for some ω_0 such that $\omega = (\omega_0, \omega_1) \in \mathcal{C}^2(\mathcal{E}, \mathcal{R})$; these define a Courant-Dorfman structure if and only if $d\omega = 0$. Conversely, any 2-cocycle ω defines a Courant-Dorfman structure on $\mathcal{E} \oplus \mathcal{R}$ by the formulas (5.2), (5.3) and (5.4). Furthermore, the central extensions given by cocycles ω and ω' are isomorphic if and only if $\omega - \omega' = d\lambda$ for a $\lambda \in \mathcal{C}^1(\mathcal{E}, \mathcal{R}) = \mathcal{E}^\vee$, the isomorphism given by $\hat{e} \mapsto \hat{e} + i(\lambda(p(\hat{e})))$, and conversely, every such λ gives an isomorphism of extensions. We leave it to the reader to check the details. □

Example 5.3. Every closed $\omega \in \Omega^{2,\mathrm{cl}}$ gives rise to a central extension of *any* Courant-Dorfman algebra by cocycle $\rho^*\omega$ ((4.4),(4.5)).

5.2. H^3 **and the canonical class.** Given an almost Courant-Dorfman algebra $\mathcal{E}$, consider the cochain $\Theta = (\Theta_0, \Theta_1) \in \mathcal{C}^3(\mathcal{E}, \mathcal{R})$ defined as follows:

$$\Theta_0(e_1, e_2, e_3) = \langle [e_1, e_2], e_3\rangle \tag{5.5}$$

$$\Theta_1(e; f) = -\rho(e)f \tag{5.6}$$

To see that $\Theta \in \mathcal{C}^3$, we need to verify relations (3.1):

$$\Theta_0(e_1, e_2, e_3) + \Theta_0(e_2, e_1, e_3) =$$
$$= \langle [e_1, e_2], e_3\rangle + \langle [e_2, e_1], e_3\rangle = \rho(e_3)\langle e_1, e_2\rangle = -\Theta_1(e_3; \langle e_1, e_2\rangle)$$

and

$$\Theta_0(e_1, e_2, e_3) + \Theta_0(e_1, e_3, e_2) =$$
$$= \langle [e_1, e_2], e_3\rangle + \langle [e_1, e_3], e_2\rangle = \rho(e_1)\langle e_2, e_3\rangle = -\Theta_1(e_1; \langle e_2, e_3\rangle)$$

are consequences of conditions (3) and (2) of Def. (2.1), respectively.

Proposition 5.4. *If $\mathcal{E}$ is a Courant-Dorfman algebra, $d\Theta = 0$; for any $\psi \in \Omega^{3,\mathrm{cl}}$, the Courant-Dorfman algebra* $\mathrm{Tw}(\psi)(\mathcal{E})$ *has*

$$\Theta_\psi = \Theta + \rho^*\psi$$

Proof. In fact, a computation using conditions (2) and (3) of Def. 2.1 yields

$$(d\Theta)_0(e_1, e_2, e_3, e_4) = 2\langle [e_1, [e_2, e_3]] - [[e_1, e_2], e_3] - [e_2, [e_1, e_3]], e_4\rangle \tag{5.7}$$

$$(d\Theta)_1(e_1, e_2; f) = 2\langle [\partial f, e_1], e_2\rangle \tag{5.8}$$

$$-(d\Theta)_2(f_1, f_2) = 2\langle \partial f_1, \partial f_2\rangle \tag{5.9}$$

Therefore, $d\Theta = 0$ by conditions (4), (5) and (6) of Def. 2.1. The second statement follows immediately from the formulas (2.10), (4.4) and (4.5). □

We shall call Θ the *canonical cocycle* of $\mathcal{E}$ and its class $[\Theta] \in H^3(\mathcal{E}, \mathcal{R})$ – the *canonical class* of $\mathcal{E}$.

Remark 5.5. If $i : \mathcal{D} \longrightarrow \mathcal{E}$ is an isotropic submodule, $i^\vee\Theta$ is $\mathcal{R}$-trilinear and alternating; if $\mathcal{D}$ is Dirac, $i^\vee\Theta = 0$. When $\langle\cdot,\cdot\rangle$ is strongly non-degenerate and $\mathcal{D}$ is *maximally* isotropic, we can say "and only if". This is the criterion originally used by Courant and Weinstein [5] to define Dirac structures in $\mathcal{Q}_0$.

Example 5.6. For the "original" Courant-Dorfman algebra $\mathcal{Q}_0 = \mathfrak{X}^1 \oplus \Omega^1$ (Example 2.25), we have $\Theta = d\omega$ where

$$\omega_0((v_1, \alpha_1), (v_2, \alpha_2)) = \iota_{v_1}\alpha_2 - \iota_{v_2}\alpha_1$$
$$\omega_1 = 0$$

(verify!). Hence, the canonical class of $\mathcal{Q}_0$ is zero. It follows that for any $\psi \in \Omega^{3,\mathrm{cl}}$ the canonical class of $\mathcal{Q}_\psi$ is the image of $[\psi] \in H^3_{\mathrm{dR}}$.

6. The non-degenerate case

Let us now restrict attention to the special case of Courant-Dorfman algebras which are non-degenerate in the sense of Def. 2.3.

6.1. **The Poisson bracket.**

Recall that a strongly non-degenerate $\langle\cdot,\cdot\rangle$ has an inverse

$$\{\cdot,\cdot\} : \mathcal{E}^\vee \otimes_\mathcal{R} \mathcal{E}^\vee \longrightarrow \mathcal{R}$$

defined by formula (2.1). This operation can be extended to a Poisson bracket on $\mathcal{C} = \mathcal{C}(\mathcal{E}, \mathcal{R})$:

$$\{\cdot,\cdot\} : \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C}[-2]$$

which we shall now define. Recall that, for an $\omega = (\omega_0, \omega_1, \ldots) \in \mathcal{C}^p$, each ω_k is a $\mathbb{K}$-linear map

$$\omega_k : \mathcal{E}^{\otimes^{p-2k}} \longrightarrow \mathfrak{X}^k$$

which is $\mathcal{R}$-linear in the $(p-2k)$-th argument. Hence, by adjunction, it gives rise to an $\mathbb{K}$-linear map

$$\tilde{\omega}_k : \mathcal{E}^{\otimes^{p-2k-1}} \longrightarrow \mathrm{Hom}_\mathcal{R}(S^k\Omega^1, \mathcal{E}^\vee)$$

defined as follows:

$$\tilde{\omega}_k(e_1, \ldots, e_{p-2k-1})(f_1, \ldots, f_k)(e) = \omega_k(e_1, \ldots, e_{p-2k-1}, e; f_1, \ldots, f_k)$$

(the sign is inserted for later convenience). Define

$$\omega_k^\sharp : \mathcal{E}^{\otimes^{p-2k-1}} \longrightarrow \mathrm{Hom}_\mathcal{R}(S^k\Omega^1, \mathcal{E})$$

by $\omega_k^\sharp = (\tilde{\omega}_k)^\sharp$. Denote the inverse of $(\cdot)^\sharp$ by $(\cdot)^\flat$.

Remark 6.1. These maps define an isomorphism (extending that of Def. 2.3) of graded $\mathcal{R}$-modules between $\mathcal{C}(\mathcal{E},\mathcal{R})$ and $\mathcal{C}(\mathcal{E},\mathcal{E})$ whose elements are tuples $T = (T_0, T_1, \ldots)$ where

$$T_k : \mathcal{E}^{\otimes^{p-2k-1}} \longrightarrow \mathrm{Hom}_{\mathcal{R}}(S^k\Omega^1, \mathcal{E})$$

satisfies conditions obtained by applying $(\cdot)^\sharp$ to equations (3.1); these make sense even when $\langle\cdot,\cdot\rangle$ is degenerate.

Given $H \in \mathrm{Hom}_{\mathcal{R}}(S^i\Omega^1, \mathcal{E})$, $K \in \mathrm{Hom}_{\mathcal{R}}(S^j\Omega^1, \mathcal{E})$, we can obtain $\langle H \cdot K\rangle \in \mathrm{Hom}_{\mathcal{R}}(S^{i+j}\Omega^1, \mathcal{R})$ by composing the product (B.1) in $\mathfrak{X}$ with $\langle\cdot,\cdot\rangle$, i.e.

$$\langle H \cdot K\rangle(f_1, \ldots, f_{i+j}) = \sum_{\tau \in \mathrm{sh}(i,j)} \langle H(f_{\tau(1)}, \ldots, f_{\tau(i)}), K(f_{\tau(i+1)}, \ldots, f_{\tau(i+j)})\rangle$$

Now let $\omega = (\omega_0, \omega_1, \ldots) \in \mathcal{C}^p$, $\eta = (\eta_0, \eta_1, \ldots) \in \mathcal{C}^q$. Let us define operations

$$\langle \omega \bullet \eta\rangle = (\langle \omega \bullet \eta\rangle_0, \langle \omega \bullet \eta\rangle_1, \ldots)$$

and

$$\omega \diamond \eta = ((\omega \diamond \eta)_0, (\omega \diamond \eta)_1, \ldots)$$

with

$$\langle \omega \bullet \eta\rangle_k, (\omega \diamond \eta)_k : \mathcal{E}^{\otimes^{p+q-2k-2}} \longrightarrow \mathfrak{X}^k$$

given by the formulas

$$\langle \omega \bullet \eta\rangle_k(e_1, \ldots, e_{p+q-2k-2}) = \tag{6.1}$$

$$= (-1)^{q-1} \sum_{i+j=k} \sum_{\sigma} (-1)^\sigma \langle \omega_i^\sharp(e_{\sigma(1)}, \ldots, e_{\sigma(p-2i-1)}) \cdot \eta_j^\sharp(e_{\sigma(p-2i)}, \ldots, e_{\sigma(p+q-2k-2)})\rangle$$

where σ runs over $\mathrm{sh}(p-2i-1, q-2j-1)$, and

$$(\omega \diamond \eta)_k(e_1, \ldots, e_{p+q-2k-2}) = \tag{6.2}$$

$$= \sum_{i+j=k} \sum_{\sigma} (-1)^\sigma \omega_{i+1}(e_{\sigma(1)}, \ldots, e_{\sigma(p-2i-2)}) \circ \eta_j(e_{\sigma(p-2i-1)}, \ldots, e_{\sigma(p+q-2k-2)})$$

where σ runs over $\mathrm{sh}(p-2i-2, q-2j)$, and $\circ$ in each summand is defined as in (B.3).

And finally, define

$$\{\omega, \eta\} = \omega \diamond \eta + \langle \omega \bullet \eta\rangle - (-1)^{pq}\eta \diamond \omega \tag{6.3}$$

Remark 6.2. The subalgebra $\mathcal{C}_0 = \mathrm{Hom}_{\mathcal{R}}(S(\mathcal{E}[1]), \mathcal{R})$ is also closed under $\{\cdot,\cdot\}$; the restriction is given by

$$\{\omega_0, \eta_0\} = \langle \omega \bullet \eta\rangle_0$$

and in particular, for $\lambda, \mu \in \mathcal{C}^1$, the bracket reduces to the formula (2.1). At the other extreme, $\mathcal{E} = 0$ ("vacuously non-degenerate"), we get $\mathcal{C} = \mathrm{Hom}_{\mathcal{R}}(S_{\mathcal{R}}(\Omega^1[2]), \mathcal{R})$, and the formulas (3.3) and (6.3) reduce, respectively, to the classical formulas (B.1) and (B.2).

Theorem 6.3. *Let $\mathcal{E}$ be a metric $\mathcal{R}$-module with a strongly non-degenerate $\langle\cdot,\cdot\rangle$.*

(i) *The formula (6.3) defines a non-degenerate Poisson bracket on the algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ of degree -2;*

(ii) *For any almost Courant-Dorfman structure on $\mathcal{E}$, the canonical cochain Θ, defined by formulas* (5.5) *and* (5.6)*, and the derivation d, defined by formula* (4.1)*, are related by*

$$d = -\{\Theta, \cdot\}$$

(iii) *The almost Courant-Dorfman structure is a Courant-Dorfman structure if and only if*

$$\{\Theta, \Theta\} = 0 \tag{6.4}$$

Proof. The first two statements are proved by a direct verification. The "if" part of (iii) follows from (ii) and Proposition 5.4, the "only if" – by formulas (5.7), (5.8), (5.9), the nondegeneracy of $\langle\cdot,\cdot\rangle$ and the assumption that $\frac{1}{2} \in \mathbb{K}$. □

Remark 6.4. Observe that $[\cdot,\cdot]$ and ∂ can be recovered from Θ via

$$\begin{aligned} [e_1, e_2] &= \Theta_0^\sharp(e_1, e_2) \\ -\partial f &= \Theta_1^\sharp(f) \end{aligned}$$

So the Poisson bracket $\{\cdot,\cdot\}$ defines the differential graded Lie algebra controlling the deformation theory of Courant-Dorfman algebras with fixed underlying metric module with non-degenerate $\langle\cdot,\cdot\rangle$. In fact, we can use $(\cdot)^\sharp$ to lift $\{\cdot,\cdot\}$ to $\mathcal{C}(\mathcal{E},\mathcal{E})$ and obtain an explicit description of this bracket which makes sense even if $\langle\cdot,\cdot\rangle$ is degenerate. This is similar to the description of the deformation complex of a Lie algebroid by Crainic and Moerdijk [6]. We shall postpone writing down these formulas until the sequel to this paper, dealing with modules and deformation theory.

6.2. **The canonical class as obstruction to re-scaling.** The canonical class $[\Theta]$ has a familiar deformation-theoretic interpretation. Let t be a formal variable, and extend everything $\mathbb{K}[[t]]$-linearly to $\mathcal{R}[[t]]$, $\mathcal{E}[[t]]$. If Θ satisfies the Maurer-Cartan equation (6.4) and thus defines a Courant-Dorfman structure on $(\mathcal{R},\mathcal{E})$, so does $\Theta_t = e^t\Theta$ on $(\mathcal{R}[[t]],\mathcal{E}[[t]])$. The question is, when is Θ_t isomorphic to Θ? "Isomorphic" here means that there exists an automorphism $\phi(t)$ of the Poisson algebra $\mathcal{C}[[t]]$ with $\phi(0) = \mathrm{id}$, and whose infinitesimal generator is Hamiltonian with respect to an $\omega(t) \in \mathcal{C}^2[[t]]$, such that

$$\phi(t)\Theta = \Theta_t$$

Differentiating at $t = 0$ immediately yields

$$d\omega(0) = \Theta$$

so in particular $[\Theta] = 0$. Conversely, if this is the case, $\phi(t) = \exp(t\{\omega(0),\cdot\})$ does the trick.

If $\mathbb{K} \supset \mathbb{R}$, we can ask the same question for t a real number, rather than a formal variable. In this case, the condition $[\Theta] = 0$ is still necessary but not sufficient unless there exists an $\omega(t)$ that integrates to a flow.

6.3. **Cartan relations and iterated brackets.** The following is easily verified:

Proposition 6.5. *Given $f \in \mathcal{R}$, $e \in \mathcal{E}$,*

$$\begin{aligned} -\iota_f &= \{f, \cdot\} \\ \iota_e &= \{e^\flat, \cdot\} \end{aligned}$$

where ι_f and ι_e are given by (4.7) *and* (4.8)*. Thus, the equations* (4.9) – (4.16) *express commutation relations among Hamiltonian derivations of $\mathcal{C}(\mathcal{E},\mathcal{R})$, analogous to the well-known Cartan relations among derivations of $\Omega_{\mathcal{R}}$.*

Corollary 6.6. *For any $\omega = (\omega_0, \omega_1, \ldots) \in \mathcal{C}^p(\mathcal{E},\mathcal{R})$, the following relation holds:*

$$\omega_k(e_1, \ldots, e_{p-2k}; f_1, \ldots, f_k) = \tag{6.5}$$

$$= (-1)^{\frac{(p-2k)(p-2k-1)}{2}} \{\cdots\{\omega, e_1^\flat\}, \cdots\}, e_{p-2k}^\flat\}, f_1\}, \cdots\}, f_k\}$$

6.4. **Relation with graded symplectic manifolds.** In this subsection we follow the notation and terminology of [17]. Let M_0 be a finite-dimensional C^∞ manifold, $E \longrightarrow M_0$ a vector bundle of finite rank equipped with a pseudometric $\langle\cdot,\cdot\rangle$. Consider the isometric embedding

$$\begin{array}{rcl} j : E & \longrightarrow & E \oplus E^* \\ e & \longmapsto & (e, \frac{1}{2}e^\flat) \end{array}$$

with respect to the canonical pseudometric on $E \oplus E^*$, inducing an embedding of graded manifolds

$$j[1] : E[1] \longrightarrow (E \oplus E^*)[1]$$

Define $M = M(E)$ to be the pullback of

$$T^*[2]E[1] \xrightarrow{p} (E \oplus E^*)$$

along $j[1]$, and let $\Xi \in \Omega^2(M)$ be the pullback of the canonical symplectic form on $T^*[2]E[1]$. This Ξ is closed, has degree +2 with respect to the induced grading, and is non-degenerate if and only if $\langle\cdot,\cdot\rangle$ is, in which case its inverse gives a Poisson bracket on the algebra $\mathcal{C}(M)$ of polynomial functions on M, of degree -2. Conversely, we proved in [17] that every degree-two graded symplectic manifold is isomorphic to $M(E)$ for some E.

Theorem 6.7. *Let $\mathcal{R} = C^\infty(M_0)$, $\mathcal{E} = \Gamma(E)$. The map*

$$\Phi : \mathcal{C}(M(E)) \longrightarrow \mathcal{C}(\mathcal{E},\mathcal{R})$$

given, for $\omega \in \mathcal{C}^p(M(E))$, by

$$(\Phi\omega)_k(e_1, \ldots, e_{p-2k}; f_1, \ldots, f_k) =$$

$$= (-1)^{\frac{(p-2k)(p-2k-1)}{2}} \{\cdots\{\omega, e_1^\flat\}, \cdots\}, e_{p-2k}^\flat\}, f_1\}, \cdots\}, f_k\}$$

is an isomorphism of graded Poisson algebras.

Proof. That Φ takes values in $\mathcal{C}(\mathcal{E},\mathcal{R})$ (i.e. the relations (3.1) hold) is a consequence of the Jacobi identity for $\{\cdot,\cdot\}$ and (2.1). That Φ is a map of Poisson algebras follows by applying Lemma A.1 to $\star$ being first the product and then the Poisson bracket on $\mathcal{C}(M(E))$. The injectivity of Φ amounts to the statement that ω is uniquely determined by the functions $(\Phi\omega)_k$, $k = 0, 1, \ldots, [\deg\omega/2]$; this is most easily seen in local coordinates where these functions are just the Taylor coefficients of ω. Surjectivity is a consequence of Corollary 6.6. ∎

6.5. **Relation with "naive cohomology".** Let $\mathcal{E}$ be a Courant-Dorfman algebra, $\mathcal{K} = \ker\rho$, $\bar{\mathcal{E}} = \mathcal{E}/\delta\Omega^1$. The map $(\cdot)^\flat : \mathcal{E} \to \mathcal{E}^\vee$ extends to

$$(\cdot)^\flat : \Lambda_{\mathcal{R}}\mathcal{K} \longrightarrow \mathcal{C}(\mathcal{E},\mathcal{R})$$

whose image is actually contained in $\mathcal{F}_0$, in view of (2.7). For $\mathcal{R} = C^\infty(M_0)$, $\mathcal{E} = \Gamma(E)$ and $\langle\cdot,\cdot\rangle$ non-degenerate, this map is an isomorphism onto $\mathcal{F}_0$, which in turn is isomorphic to $\mathcal{C}(\bar{\mathcal{E}},\mathcal{R})$ (see Subsection 4.5). Stiénon and Xu [20] defined a differential on the algebra $\Lambda_{\mathcal{R}}\mathcal{K}$ (in view of this isomorphism, it is just the standard differential for the Lie-Rinehart algebra $\bar{\mathcal{E}}$) and called its cohomology the "naive cohomology" of the Courant algebroid E. They conjectured that, if ρ is surjective, the inclusion

$$\Phi^{-1}\circ(\cdot)^\flat : \Lambda_{\mathcal{R}}\mathcal{K} \longrightarrow \mathcal{C}(M(E))$$

is a quasi-isomorphism. This was proved by Ginot and Grutzmann [9] who also obtained further results by considering the spectral sequence associated to the filtration of $\mathcal{C}(M(E))$ by the powers of what they called "the naive ideal". This ideal corresponds under Φ to the ideal $\mathcal{I}$ we defined in Subsection 4.5.

Remark 6.8. For general $(\mathcal{R},\mathcal{E},\langle\cdot,\cdot\rangle)$ it is not known (and probably false) that the image of $\Lambda_{\mathcal{R}}\mathcal{K}$ in $\mathcal{C}(\mathcal{E},\mathcal{R})$ is closed under d.

7. Concluding remarks, speculations and open ends

In conclusion, let us mention a few important issues we have not touched upon here, which we plan to address in a sequel (or sequels) to this paper.

7.1. **The pre-symplectic structure.** The algebra $\mathcal{C}(\mathcal{E},\mathcal{R})$ has an extra structure: a closed 2-form $\Xi \in \Omega^2_{\mathcal{C}}$ which has degree 2 with respect to induced grading, and is d-invariant in the sense that

$$L_d\Xi = 0 \tag{7.1}$$

where L is the Lie derivative operator on $\Omega_{\mathcal{C}}$. This two-form exists on general principles: for strongly non-degenerate $\langle\cdot,\cdot\rangle$ it is just the inverse of the Poisson tensor (6.3), while for $\mathcal{R} = C^\infty(M_0)$ and $\mathcal{E} = \Gamma(E)$ the construction from [17] yields Ξ for an arbitrary $\langle\cdot,\cdot\rangle$ (see subsection 6.4 for a review). The formulas (2.5) define the induced bilinear form on the tangent complex $\mathbb{T}_{\mathcal{E}}$; its δ-invariance (2.6) is just the linearization of (7.1). By Dirac's formalism [7] adapted to the graded setting, the closed 2-form Ξ induces a Poisson bracket on a certain subalgebra $\mathcal{C}_\flat(\mathcal{E},\mathcal{R})$ of $\mathcal{C}(\mathcal{E},\mathcal{R})$.

7.2. **Morphisms.** The functors we have constructed,

$$\begin{array}{ccc} \mathbf{Met}_{\mathcal{R}} & \longrightarrow & \mathbf{gra}^{\mathrm{op}}_{\mathcal{R}} \\ (\mathcal{E},\langle\cdot,\cdot\rangle) & \longmapsto & \mathcal{C}(\mathcal{E},\mathcal{R}) \end{array}$$

and

$$\begin{array}{ccc} \mathbf{CD}_{\mathcal{R}} & \longrightarrow & \mathbf{dga}^{\mathrm{op}}_{\mathcal{R}} \\ (\mathcal{E},\langle\cdot,\cdot\rangle,\partial,[\cdot,\cdot]) & \longmapsto & (\mathcal{C}(\mathcal{E},\mathcal{R}),d) \end{array}$$

are not fully faithful for two reasons. The first has to do with infinite dimensionality issues: not all maps $\mathcal{F}^\vee \to \mathcal{E}^\vee$ come from maps $\mathcal{E} \to \mathcal{F}$, duals of tensor products are not tensor products of duals, and so on. These issues can be dealt with by calling those maps of duals which are duals of maps *admissible* and restricting attention only to such maps; one can similarly define admissible derivations, and so on. Of course, this only make sense for objects in the image of the above functors.

However, even if we restrict attention to finite-dimensional and locally free case, the functors above are still not full. This is because we have only defined *strict* maps of Courant-Dorfman algebras; the more general notion of a *lax* map can be obtained as admissible dg map preserving Ξ in an evident way; this way we can also describe maps of Courant-Dorfman algebras over different base rings.

Finally, we have defined (strict) morphisms from Lie-Rinehart to Courant-Dorfman algebras and back, but no category containing both kinds of algebras as objects. This problem can be solved by introducing "Lie-Rinehart 2-algebras" (algebraic analogues of Lie 2-algebroids) and their weak (and maybe also higher) morphisms, which can again be reduced to studying dg algebras of a certain kind and admissible dg morphisms between them.

7.3. **Modules.** We have not defined the notion of a module over a Courant-Dorfman algebra and cohomology with coefficients, except in the trivial module $\mathcal{R}$. Again, this can be done by analyzing (the derived category of) dg modules over the dga $\mathcal{C}(\mathcal{E},\mathcal{R})$ and trying to describe them explicitly in terms of $\mathcal{E}$. It is not clear though what, if any, compatibility with Ξ we should require.

7.4. **The Courant-Dorfman operad.** The infinite-dimensionality problems mentioned above arise because our construction of the algebra $\mathcal{C}(\mathcal{E};\mathcal{R})$ involves dualization. It seems more natural to try to construct some sort of coalgebra instead. In operad theory, Koszul duality provides a systematic way of obtaining such a differential graded coalgebra from an algebra over a given *quadratic* operad. There is a an operad, $\mathcal{CD}$, on the set of two colors, whose algebras are Courant-Dorfman algerbras; as operads go, this is a pretty nasty one: inhomogeneous cubic, so Koszul duality does not apply. However, if $\langle\cdot,\cdot\rangle$ is non-degenerate, one can replace ∂ by an action of $\mathcal{E}$ on $\mathcal{R}$ via the anchor ρ and get rid of the offending relations, ending up with an algebra over a nice homogeneous quadratic operad (this is actually the formulation given in [17]). Of course, non-degeneracy is not a condition that can be expressed in operadic terms; more importantly, even if we ignore this and try to apply Koszul duality to the resulting quadratic operad, we will get a wrong answer, because we are really interested in Courant-Dorfman structures over a *fixed* underlying metric module (which we can assume to be non-degenerate if we want to). What is relevant in this situation (which also arises in several other contexts we know of) is a kind of *relative* deformation theory for algebras over a *pair* of operads $P \subset Q$, where we want to vary the Q-algebra structure while keeping the underlying P-structure fixed. As far as we are aware, such a theory is not yet available, but it would be interesting and useful to try to develop it.

Appendix A. Derivations and shuffles

Let A be a $\mathbb{K}$-module equipped with a bilinear operation

$$\star : A \otimes A \longrightarrow A$$

A derivation of $\star$ is a $\mathbb{K}$-linear map $D : A \longrightarrow A$ satisfying the Leibniz rule:

$$D(a \star b) = Da \star b + a \star Db$$

Lemma A.1. *Let $D_1, \dots, D_k : A \longrightarrow A$ be derivations, and let $D = D_1 \cdots D_n$. Then*

$$D(a \star b) = \sum_{i+j=k} \sum_{\sigma \in \mathrm{sh}(i,j)} (D_{\sigma(1)} \cdots D_{\sigma(i)} a) \star (D_{\sigma(i+1)} \cdots D_{\sigma(k)} b)$$

Proof. Induction. □

If A and the D's are graded, the lemma holds with appropriate Koszul signs put in place.

Appendix B. Kähler differential forms and multiderivations.

Let $\mathcal{R}$ be a commutative $\mathbb{K}$-algebra, $\mathcal{M}$ an $\mathcal{R}$-module. A $\mathbb{K}$-linear map

$$D : \mathcal{R} \longrightarrow \mathcal{M}$$

is called a *derivation* if it satisfies the Leibniz rule:

$$D(fg) = (Df)g + f(Dg) \quad \forall f, g \in \mathcal{R}$$

$\mathcal{M}$-valued derivations form an $\mathcal{R}$-module denoted $\mathrm{Der}(\mathcal{R}, \mathcal{M})$; the assignment is functorial in $\mathcal{M}$.

The functor $\mathcal{M} \mapsto \mathrm{Der}(\mathcal{R}, \mathcal{M})$ is (co)representable: there exists an $\mathcal{R}$-module $\Omega^1 = \Omega^1_{\mathcal{R}}$, unique up to a unique isomorphism, together with a natural (in $\mathcal{M}$) isomorphism of $\mathcal{R}$-modules

$$\mathrm{Der}(\mathcal{R}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathcal{R}}(\Omega^1, \mathcal{M})$$

In particular, putting $\mathcal{M} = \Omega^1$, the identity map on the right hand side corresponds to the universal derivation

$$d_0 : \mathcal{R} \longrightarrow \Omega^1$$

Ω^1 is referred to as the *module of Kähler differentials*; it can be described explicitly as consisting of formal finite sums of terms of the form $f d_0 g$ with $f, g \in \mathcal{R}$, subject to the Leibniz relation

$$d_0(fg) = (d_0 f)g + f d_0 g$$

The algebra of Kähler differential forms is obtained by taking $\Omega = \{\Omega^k\}_{k \geq 0}$ with $\Omega^k = \Lambda^k_{\mathcal{R}} \Omega^1$. It is associative and graded-commutative with respect to exterior multiplication. The universal derivation d_0 extends to an odd derivation of Ω satisfying $d_0^2 = 0$, called the *de Rham differential*, or the *exterior derivative*. The algebra of Kähler differential forms is the universal differential algebra containing $\mathcal{R}$.

The module $\mathfrak{X}^1 = \mathrm{Der}(\mathcal{R}, \mathcal{R})$ forms a Lie algebra under the commutator bracket $\{\cdot, \cdot\}$. By the universal property of Ω^1 one has

$$\mathfrak{X}^1 \simeq \mathrm{Hom}_{\mathcal{R}}(\Omega^1, \mathcal{R}) = (\Omega^1)^\vee$$

Given $v \in \mathfrak{X}^1$, we denote the corresponding operator on the right hand side by ι_v. It extends to a unique odd derivation of Ω, denoted by the same symbol. The Lie derivative operator is defined by the Cartan formula

$$L_v = \{d_0, \iota_v\}$$

The operators ι_v, L_v and d_0 are subject to the usual Cartan commutation relations

$$\{\iota_v, \iota_w\} = 0; \quad \{L_v, \iota_w\} = \iota_{\{v,w\}}; \quad \{L_v, L_w\} = L_{\{v,w\}},$$

describing an action of the differential graded Lie algebra $T[1]\mathfrak{X}^1 = \mathfrak{X}^1[1] \oplus \mathfrak{X}^1$ on Ω.

Kähler differential forms should be distinguished from the usual differential forms $\widetilde{\Omega}_{\mathcal{R}} = \{\widetilde{\Omega}^k\}_{k\geq 0}$ on $\mathcal{R}$, where $\widetilde{\Omega}^k$ is defined as the module of alternating k-multilinear functions on $\mathfrak{X}^1$:

$$\widetilde{\Omega}^k = \mathrm{Hom}_{\mathcal{R}}(\Lambda_{\mathcal{R}}\mathfrak{X}^1, \mathcal{R})$$

Of course, one has the canonical inclusion

$$\Omega = \Lambda_{\mathcal{R}}\Omega^1 \hookrightarrow (\Lambda_{\mathcal{R}}(\Omega^1)^{\vee})^{\vee} = \widetilde{\Omega}$$

which generally fails to be an isomorphism unless $\mathcal{R}$ satisfies certain finiteness conditions. Nevertheless, the exterior multiplication and differential d_0 extend to $\widetilde{\Omega}$ and are defined by the usual Cartan formulas.

Let $\mathfrak{X}^0 = \mathcal{R}$ and, for $k > 0$, let $\mathfrak{X}^k$ denote the $\mathcal{R}$-module of symmetric k-derivations of $\mathcal{R}$, that is, symmetric k-linear forms (over $\mathbb{K}$) on $\mathcal{R}$ with values in $\mathcal{R}$ which are derivations in each argument. Again, by abstract nonsense we have

$$\mathfrak{X}^k \simeq \mathrm{Hom}_{\mathcal{R}}(S^k_{\mathcal{R}}\Omega^1, \mathcal{R})$$

The function on the right hand side corresponding to a k-derivation H on the left will be denoted by $\bar{H}$, so that

$$H(f_1, \ldots, f_k) = \bar{H}(d_0 f_1, \ldots, d_0 f_k).$$

The graded module of symmetric multi-derivations, $\mathfrak{X} = \{\mathfrak{X}^k\}_{k\geq 0}$, forms a graded commutative algebra over $\mathcal{R}$ (if we assign to elements of $\mathfrak{X}^k$ degree $2k$); the multiplication is given by the following explicit formula:

$$HK(f_1, \ldots, f_{i+j}) = \sum_{\tau\in \mathrm{sh}(i,j)} H(f_{\tau(1)}, \ldots, f_{\tau(i)})K(f_{\tau(i+1)}, \ldots, f_{\tau(i+j)}) \tag{B.1}$$

Furthermore, $\mathfrak{X}$ has a natural Poisson bracket, extending the commutator of derivations and the natural action of $\mathfrak{X}^1$ on $\mathcal{R}$; it is given by the formula:

$$\{H, K\} = H \circ K - K \circ H \tag{B.2}$$

where

$$H \circ K(f_1, \ldots, f_{i+j-1}) = \sum_{\tau\in \mathrm{sh}(i,j-1))} H(K(f_{\tau(1)}, \ldots, f_{\tau(i)}), f_{\tau(i+1)}, \ldots, f_{\tau(i+j-1)}) \tag{B.3}$$

for $H \in \mathfrak{X}^i$, $K \in \mathfrak{X}^j$. This Poisson bracket has degree -2 with respect to the grading just introduced.

Given an $\alpha \in \Omega^1$, denote by ι_α the evident contraction operator on $\mathfrak{X}$. It is a derivation of the multiplication, but not of the Poisson bracket, unless $d_0\alpha = 0$.

Appendix C. Lie-Rinehart algebras.

Definition C.1. A Lie-Rinehart algebra consists of the following data:

- a commutative $\mathbb{K}$-algebra $\mathcal{R}$;
- an $\mathcal{R}$-module $\mathcal{L}$;
- an $\mathcal{R}$-module map $\rho : \mathcal{L} \longrightarrow \mathfrak{X}^1$, called the *anchor*;
- a $\mathbb{K}$-bilinear Lie bracket $[\cdot,\cdot] : \mathcal{L} \otimes \mathcal{L} \longrightarrow \mathcal{L}$.

These data are required to satisfy the following additional conditions:

(1) $[x_1, fx_2] = f[x_1, x_2] + (\rho(x_1)f)x_2$;

(2) $\rho([x_1, x_2]) = \{\rho(x_1), \rho(x_2)\}$

for all $x_1, x_2 \in \mathcal{L}$, $f \in \mathcal{R}$.

A *morphism* of Lie-Rinehart algebras over $\mathcal{R}$ is a map of the underlying $\mathcal{R}$-modules commuting with anchors and brackets in an obvious way. Lie-Rinehart algebras over $\mathcal{R}$ form a category denoted by $\mathbf{LR}_{\mathcal{R}}$.

Example C.2. $\mathfrak{X}^1 = \mathrm{Der}(\mathcal{R}, \mathcal{R})$ becomes a Lie-Rinehart algebra with respect to the commutator bracket $\{\cdot, \cdot\}$ and the identity map $\mathfrak{X}^1 \longrightarrow \mathfrak{X}^1$ as the anchor. This is the terminal object in $\mathbf{LR}_{\mathcal{R}}$: the anchor of each Lie-Rinehart algebra gives the unique map.

Example C.3. Let $\mathcal{M}$ be an $\mathcal{R}$-module; a *derivation of* $\mathcal{M}$ is a pair (D, σ), where $D : \mathcal{M} \longrightarrow \mathcal{M}$ is a $\mathbb{K}$-linear map and $\sigma = \sigma_D \in \mathrm{Der}(\mathcal{R}, \mathcal{M})$, satisfying the following compatibility condition:

$$D(fm) = fD(m) + \sigma(f)m$$

Derivations of $\mathcal{M}$ form an $\mathcal{R}$-module which we denote $\mathrm{Der}(\mathcal{M})$; moreover, $\mathrm{Der}(\mathcal{M})$ is a Lie-Rinehart algebra with respect to the commutator bracket and the anchor π given by the assignment $(D, \sigma) \mapsto \sigma$.

Definition C.4. A *representation* of a Lie-Rinehart algebra $\mathcal{L}$ on an $\mathcal{R}$-module $\mathcal{M}$ is a map of Lie-Rinehart algebras $\nabla : \mathcal{L} \longrightarrow \mathrm{Der}(\mathcal{M})$. In other words, ∇ assigns, in an $\mathcal{R}$-linear way, to each $x \in \mathcal{L}$ a derivation $(\nabla_x, \rho(x))$ such that

$$\nabla_{[x,y]} = \{\nabla_x, \nabla_y\}$$

An $\mathcal{R}$-module $\mathcal{M}$ equipped with a representation of $\mathcal{L}$ is said to be an $\mathcal{L}$-module.

Example C.5. For every Lie-Rinehart algebra $\mathcal{L}$, $\mathcal{R}$ is an $\mathcal{L}$-module with $\nabla = \rho$.

Example C.6. Let $\mathcal{L}$ be a Lie-Rinehart algebra and let $\mathcal{K} = \ker(\rho)$. Then $\mathcal{K}$ becomes an $\mathcal{L}$-module with

$$\nabla_x(y) = [x, y]$$

for $x \in \mathcal{L}$, $y \in \mathcal{K}$. Moreover, $\mathcal{K}$ is a Lie algebra over $\mathcal{R}$ with respect to the restricted bracket, and ∇ acts by derivations of this bracket.

Given an $\mathcal{L}$-module $\mathcal{M}$, one defines for each $q \geq 0$ the module of q-cochains on $\mathcal{L}$ with coefficients in $\mathcal{M}$ to be

$$\widetilde{\Omega}^q(\mathcal{L}, \mathcal{M}) = \mathrm{Hom}_{\mathcal{R}}(\Lambda_{\mathcal{R}}\mathcal{L}, \mathcal{M}).$$

The differential $d : \widetilde{\Omega}^q(\mathcal{L}, \mathcal{M}) \longrightarrow \widetilde{\Omega}^{q+1}(\mathcal{L}, \mathcal{M})$ is given by the standard (Chevalley-Eilenberg-Cartan-de Rham) formula

$$d\eta(x_1, \ldots, x_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i-1} \nabla_{x_i} \eta(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1}) + \\ + \sum_{i<j} (-1)^{i+j} \eta([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}) \tag{C.1}$$

Remark C.7. Notice that, for $\mathcal{L} = \mathfrak{X}^1$ and $\mathcal{M} = \mathcal{R}$, this yields $\widetilde{\Omega}_{\mathcal{R}}$, rather than $\Omega_{\mathcal{R}}$. It is possible (and probably more correct in general) to consider the complex $\Omega(\mathcal{L}, \mathcal{M})$ with differential given by the universal property of the Kähler forms.

Remark C.8. The term "Lie-Rinehart algebra" is due to J. Huebschmann [11], and is based on the work of G.S. Rinehart who studied these structures in a seminal paper [15] (although Rinehart himself referred to earlier work of Herz and Palais).

Appendix D. Leibniz algebras, modules and cohomology.

This section follows Loday and Pirashvili [14] closely. A *Leibniz algebra* over $\mathbb{K}$ is a $\mathbb{K}$-module $\mathcal{E}$ equipped with a bilinear operation

$$[\cdot,\cdot] : \mathcal{E}\otimes\mathcal{E} \longrightarrow \mathcal{E}$$

satisfying the following version of the Jacobi identity:

$$[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] + [e_2,[e_1,e_3]]$$

(i.e., $[e,\cdot]$ is a derivation[2] of $[\cdot,\cdot]$ for each $e\in\mathcal{E}$).

Given a Leibniz algebra $\mathcal{E}$, an $\mathcal{E}$*-module* is a $\mathbb{K}$-module $\mathcal{M}$ equipped with two structure maps: a left action

$$\begin{array}{ccc} \mathcal{E}\otimes\mathcal{M} & \longrightarrow & \mathcal{M} \\ (e,m) & \mapsto & [e,m] \end{array}$$

and a right action

$$\begin{array}{ccc} \mathcal{M}\otimes\mathcal{E} & \longrightarrow & \mathcal{M} \\ (m,e) & \mapsto & [m,e] \end{array}$$

satisfying the following equations:

$$\begin{array}{rcl} [e_1,[e_2,m]] & = & [[e_1,e_2],m] + [e_2,[e_1,m]] \\ [e_1,[m,e_2]] & = & [[e_1,m],e_2] + [m,[e_1,e_2]] \\ [m,[e_1,e_2]] & = & [[m,e_1],e_2] + [e_1,[m,e_2]] \end{array}$$

Maps of $\mathcal{E}$-modules are defined in an obvious way.

Given any Leibniz algebra $\mathcal{E}$, a left $\mathcal{E}$-action on $\mathcal{M}$ satisfying the first of the above three equations can be extended to an $\mathcal{E}$-module structure in two standard ways, by defining the right action either by

$$[m,e] := -[e,m]$$

or by

$$[m,e] = 0$$

Following Loday and Pirashvili, we call the first one *symmetric*, the second – *anti-symmetric*.

Given a Leibniz algebra $\mathcal{E}$ and an $\mathcal{E}$-module $\mathcal{M}$, define the complex of cochains on $\mathcal{E}$ with values in $\mathcal{M}$ by setting, for $q\geq 0$,

$$\mathcal{C}^q_{\mathrm{LP}}(\mathcal{E},\mathcal{M}) = \mathrm{Hom}(\mathcal{E}^{\otimes^q},\mathcal{M})$$

with the differential

$$d_{\mathrm{LP}} : \mathcal{C}^q_{\mathrm{LP}}(\mathcal{E},\mathcal{M}) \longrightarrow \mathcal{C}^{q+1}_{\mathrm{LP}}(\mathcal{E},\mathcal{M})$$

given by

$$d_{\mathrm{LP}}\eta(e_1,\ldots,e_{q+1}) = \sum_{i=1}^{q}(-1)^{i-1}[e_i,\eta(\ldots,\hat{e}_i,\ldots)]+ \tag{D.1}$$

---

[2]In fact, this defines a *left* Leibniz algebra, whereas Loday and Pirashvili considered *right* Leibniz algebras, in which $[\cdot,e]$ is a right derivation of $[\cdot,\cdot]$. The assignment $[\cdot,\cdot]\longrightarrow[\cdot,\cdot]^{\mathrm{op}}$ where

$$[x,y]^{\mathrm{op}} = -[y,x]$$

establishes an isomorphism of the categories of these two kinds of Leibniz algebras; the formulas for modules and differentials have to be modified accordingly.

$$+(-1)^{q+1}[\eta(e_1,\dots,e_q),e_{q+1}]+\sum_{i<j}(-1)^i\eta(e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,[e_i,e_j],e_{j+1},\dots,e_{q+1})$$

If the module $\mathcal{M}$ is symmetric, this reduces to

$$d_{\mathrm{LP}}\eta(e_1,\dots,e_{q+1})=\sum_{i=1}^{q+1}(-1)^{i-1}[e_i,\eta(\dots,\hat{e}_i,\dots)]+ \tag{D.2}$$

$$+\sum_{i<j}(-1)^i\eta(e_1,\dots,\hat{e}_i,\dots,\hat{e}_j,[e_i,e_j],e_{j+1},\dots,e_{q+1})$$

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