SURFACE FIBRATIONS OF NON-POSITIVE CURVATURE

F.E.A. JOHNSON

Department of Mathematics
University College London
Gower Street
London WCIE 6BT
U.K.

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
5300 Bonn 3
Germany

MPI/91-15

SURFACE FIBRATIONS OF NON-POSITIVE CURVATURE

F.E.A. Johnson

§ 0: Introduction:

It is widely believed that the Whitehead group Wh(G) of a torsion free group is zero. In [16], Waldhausen set up a purely algebraic machine which showed this is true in some cases. However, Waldhausen's programme runs into difficulties as soon as non-Noetherian amalgamations are encountered, with the consequence that some very obvious geometrically defined groups cannot be dealt with by this method; in particular, Waldhausen's method breaks down on an arbitrary product of surface groups

$$G = K_1 \times \cdots\cdots \times K_m$$

$$K_i = \pi_1(\text{Surface of genus } g_i \geq 2).$$

More recently, in a remarkable series of papers which make extensive and highly ingenious use of dynamics and differential geometry, F.T. Farrell and L.E. Jones have circumvented many of the apparent difficulties which arise from a purely algebraic approach. Their theorem is

Theorem (Farrell — Jones [7]): *Let M be a closed connected Riemannian manifold having nonpositive sectional curvature. Then*

$$Wh(\pi_1(M)) = 0.$$

In particular, the Farrell-Jones Theorem shows immediately that $Wh(G) = 0$ when G is a product of Surface groups, by using its geometrical representation as a group of motions of a product of hyperbolic planes. However, if instead of taking direct products one takes iterated extensions one obtains the class of so called "poly Surface" groups (see §4) which are geometrically interesting but algebraically recalcitrant.

In this paper we single out a subclass of "strongly poly Surface groups" and show.

Theorem A: *If G is a strongly poly Surface group then $Wh(G) = 0$.*

Our proof makes direct use of the Farrell-Jones Theorem. We first prove

Theorem B: *If G is a strongly poly Surface group then there is a smooth closed Riemannian manifold X_G of nonpositive (sectional) curvature such that $\pi_1(X_G) = G$.*

Clearly the Farrell-Jones Theorem and Theorem B immediately imply Theorem A.

It is easy to show that in general a poly Surface group is not a discrete cocompact lattice in a semisimple Lie group. Thus the universal covers $\widetilde{X}_G$ which arise in Theorem B are not symmetric spaces. Theorem B is therefore of some purely geometric interest in that it provides a new class of simply connected complete Riemannian manifolds of nonpositive curvature which admit a discrete cocompact group of isometries (see, e.g [4] [6]).

Poly-surface groups have arisen in other contexts. The examples of Atiyah [1] and Kodaira [13] which established non multiplicativity of the signature are also poly Surface. In §4 we show directly that Wh(G)=0 when G is the fundamental group of an Atiyah-Kodaira manifold.

The paper is arranged as follows: in §1 we review the Earle-Eells theory of Teichmüller space. This not only makes the task of bundle classification easier, but is essential in the later construction of nonpositive curvature metrics. Our construction of nonpositive curvature metrics is in §2, and the applications to Whitehead torsion in §3.

It is a pleasure to be able to thank Professor Hirzebruch and all the staff at the Max-Planck-Institut for their hospitality and help.

§ 1: Classification of Surface bundles:

Throughout this paper we shall be dealing with smooth closed orientable surfaces. Although our primary concern is with geometry, it is technically convenient to parametrize things by means of the fundamental group. Thus let K denote the Surface group with presentation

$$K = \left\langle X_1, \ldots, X_g, Y_1, \ldots, Y_g : \prod_{i=1}^{g} X_i Y_i X_i^{-1} Y_i^{-1} \right\rangle$$

and let Σ_K be a fixed smooth closed orientable surface with $\pi_1(\Sigma_K) = K$; the genus g will be taken invariably to be at least two. $\mathrm{Diff}(\Sigma_K)$ will denote the group of diffeomorphisms of Σ_K, topologized with the C^∞ topology, and $\varphi : \mathrm{Diff}(\Sigma_K) \to \mathrm{Out}(K)$ the natural homomorphism to the group, $\mathrm{Out}(K)$, of outer automorphisms of K. A theorem of Baer [2] asserts that φ is surjective and that $\mathrm{Ker}(\varphi)$ is precisely $\mathrm{Diff}_o(\Sigma_K)$, the identity component in $\mathrm{Diff}(\Sigma_K)$. We record this fact thus:

__Proposition 1.1 (Baer [2]):__ *There is a natural isomorphism*

$$\mathrm{Diff}(\Sigma_K)/_{\mathrm{Diff}_o(\Sigma_K)} \xrightarrow{\simeq} \mathrm{Out}(K).$$

We now recall the salient features of the theory of Earle and Eells [5], which, for our purposes, be summarized conveniently as follows:

Let $\mathcal{C}(\Sigma_K)$ denote the set of all Riemannian metrics on Σ_K which have constant curvature equal to –1; with the C^∞ topology, $\mathcal{C}(\Sigma_K)$ has the structure of a smooth *contractible* Frechet manifold. There an action of $\mathrm{Diff}(\Sigma_K)$ on $\mathcal{C}(\Sigma_K)$ given by

$$\mathcal{C}(\Sigma_K) \times \mathrm{Diff}(\Sigma_K) \to \mathcal{C}(\Sigma_K)$$
$$(\nu, \varphi) \mapsto \varphi^*(\nu).$$

This action is effective and proper, and its restriction to $\mathrm{Diff}_o(\Sigma_K)$ is free; moreover, the quotient $\mathcal{C}(\Sigma_K)/_{\mathrm{Diff}_o(\Sigma_K)}$ may be identified with the classical Teichmüller space $\mathcal{T}(\Sigma_K)$. In particular, $\mathcal{C}(\Sigma_K)/_{\mathrm{Diff}_o(\Sigma_K)}$ is contractible, since we are insisting that $g \geq 2$. We now have a principal fibre bundle

$$\begin{array}{ccc} \mathrm{Diff}_o(\Sigma_K) & \to & \mathcal{C}(\Sigma_K) \\ & & \downarrow \\ & & \mathcal{T}(\Sigma_K) \end{array}$$

with connected group, in which base $\mathcal{T}(\Sigma_K)$ and total space $\mathcal{C}(\Sigma_K)$ are contractible; from this follows immediately the analogue for diffeomorphisms of the theorem of Hamstrom [8].

Theorem 1.2 (Earle-Eells [5]): $\mathrm{Diff}_o(\Sigma_K)$ *is contractible.*

Now, on making the identification

$$\mathrm{Out}(K) \cong \mathrm{Diff}(\Sigma_K)/_{\mathrm{Diff}_o(\Sigma_K)}$$

there is an induced action

$$\mathcal{T}(\Sigma_K) \times \mathrm{Out}(K) \to \mathcal{T}(\Sigma_K)$$

which is effective and properly discontinuous, and whose quotient $\mathcal{T}(\Sigma_K)/_{\mathrm{Out}(\mathrm{K})}$ is the classical Riemann moduli space. This last action coincides with that of Kravetz. [14].

By the Uniformization Theorem, the space $\mathcal{C}(\Sigma_K)$ is diffeomorphic to the space $\mathcal{M}(\Sigma_K)$ of smooth complex structures on Σ_K. The reader will observe that we have departed slightly from [5] in employing $\mathcal{C}(\Sigma_K)$ in preference to its diffeomorph $\mathcal{M}(\Sigma_K)$.

The above theory gives us an algebraic description of smooth bundles with fibre Σ_K. Observe that the classifying space functor $G \mapsto BG$ preserves homotopy equivalences [3]. Thus $B\mathrm{Diff}_0(\Sigma_K)$ is also contractible, and from the fibration

$$\begin{array}{ccc} B\mathrm{Diff}_0(\Sigma_K) & \to & B\mathrm{Diff}(\Sigma_K) \\ & & \downarrow \\ & & B\mathrm{Out}(\mathrm{K}) \end{array}$$

it follows that

<u>**Proposition 1.3:**</u> *The induced map*

$$B\varphi : B\mathrm{Diff}(\Sigma_{\mathrm{K}}) \to B\mathrm{Out}(\mathrm{K})$$

is a homotopy equivalence.

Let $\mathcal{B}_{\Sigma_K}(X)$ denote the set of smooth equivalence classes of smooth bundles with fibre Σ_K over a smooth, connected manifold X. Standard approximation arguments show that $\mathcal{B}_{\Sigma_K}(X)$ is naturally equivalent to the set of based homotopy classes $[X, B\mathrm{Diff}(\Sigma_{\mathrm{K}})]$, which, by (1.3), is isomorphic to $[X, B\mathrm{Out}(\mathrm{K})]$. However, $\mathrm{Out}(\mathrm{K})$ is discrete so that $[X, B\mathrm{Out}(\mathrm{K})] \cong \mathrm{Hom}_{Groups}(\pi_1(X), \mathrm{Out}(K))$. Thus we have

<u>**Theorem 1.4:**</u> *There is a natural bijection*

$$\mathcal{B}_{\Sigma_K}(X) \xrightarrow{\simeq} \mathrm{Hom}_{Groups}(\pi_1(X), \mathrm{Out}(K))$$

for any smooth connected manifold X.

We now compare this classification with that of congruence classes of group extensions. For any discrete group Q, let $\mathfrak{C}(K,Q)$ denote the set of congruence classes of extensions of the form

$$1 \to \mathrm{K} \to ? \to \mathrm{Q} \to 1.$$

Since $g \geq 2$, K has trivial centre, so that $\mathfrak{C}(K,Q)$ is naturally equivalent to $\mathrm{Hom}_{\mathrm{Groups}}(Q, \mathrm{Out}(\mathrm{K}))$. (See, for example, [15] Chapter IV). We may express our final result as follows; (Compare [10]).

Theorem 1.5: *Let X_Q be a smooth connected manifold with $\pi_1(X_Q) = Q$. Then there are natural equivalences*

$$\mathcal{B}_{\Sigma_K}(X_Q) \cong \mathfrak{C}(K,Q) \cong \mathrm{Hom}_{Groups}(Q, \mathrm{Out}(\mathrm{K}))$$

where $\mathcal{B}_{\Sigma_K}(X_Q)$ is the set of smooth equivalence classes of smooth Σ_K bundles over X_Q, and $\mathfrak{C}(K,Q)$ is the set of congruence classes of extensions of K by Q.

§2: A construction for metrics of nonpositive curvature:

The construction assumes its simplest form for a direct product. Let μ be a Riemannian metric on a smooth manifold X. For any smooth function $f : X \to \mathcal{C}(\Sigma_K)$ we define a Riemannian metric $[f,\mu]$ on $\Sigma_K \times X$; let $\pi : \Sigma_K \times X \to X$ denote the projection map, giving rise to the following exact sequence of vector bundles over $\Sigma_K \times X$:

$$\mathcal{E} = (0 \to \mathrm{Ker}\,(T\pi) \to T\,(\Sigma_K \times X) \to \pi^*(TX) \to 0).$$

If $\nu : \pi^*(TX) \to TX$ is the bundle map over π which is the identity on fibres, and if, for some base point $* \in \Sigma_k$, $Ti : TX \to T(\Sigma_K \times X)$ is the induced tangent map of the inclusion $i : X \to \Sigma_K \times X$, $i(x) = (*, x)$, then $\sigma = (Ti) \circ \nu : \pi^*(TX) \to T\,(\Sigma_K \times X)$ is a right splitting of $\mathcal{E}$, yielding a total splitting.

$$h\sigma : T\,(\Sigma_K \times X) \stackrel{\cong}{\to} \mathrm{Ker}\,(T\pi) \oplus \pi^*(TX). \tag{2.1}$$

To define a Riemannian metric on $T(\Sigma_K \times X)$ it suffices to define Riemannian metrics on $\mathrm{Ker}(T\pi)$, $\pi^*(TX)$ separately and take the orthogonal direct sum. First we define a Riemannian metric [f] on $\mathrm{Ker}(T\pi)$; for this, choose a smooth function

$$f : X \to \mathcal{C}(\Sigma_K).$$

To simplify notation, we temporarily suppress K and write $\Sigma = \Sigma_K$. For each $x \in X$, $f(x)$ gives a Riemannian metric

$$f(x) : T\Sigma \underset{\Sigma}{\times} T\Sigma \to \mathbf{R};$$

that is,

$$f(x)_s : T\Sigma_s \times T\Sigma_s \to \mathbf{R}$$

for each $s \in \Sigma$. However, $\mathrm{Ker}(T\pi)_{(s,x)} = T\Sigma_s$, so we have a Riemannian metric

$$[f] : \mathrm{Ker}(T\pi) \underset{\Sigma \times X}{\times} \mathrm{Ker}(\mathrm{T}\pi) \to \mathbf{R}$$

by

$$[f]_{(s,x)} = f(x)_s : T\Sigma_s \times T\Sigma_s \to \mathbf{R}.$$

As a metric on $\pi^*(TX)$ we take the pullback $\pi^*(\mu)$ of the given metric μ, and define $[f, \mu]$ to be the orthogonal direct sum

$$[f, \mu] = [f] \perp \pi^*(\mu((T\pi)))$$

under the identification

$$T(\Sigma_K \times X) \cong \text{Ker}\,(T\pi) \oplus \pi^*(TX)$$

given by (2.1). The following is now easy to check.

Theorem 2.2: *If (X, μ) is a Riemannian manifold of nonpositive curvature then for any smooth function $f : X \to \mathcal{C}(\Sigma_K)$, $[f, \mu]$ is a Riemannian metric of nonpositive curvature on $\Sigma_K \times X$.*

We now investigate how to refine this construction to apply to fibre bundles rather than direct products. We begin with a smooth fibre bundle with fibre Σ_K.

$$\xi = \left\{ \begin{array}{ccc} \Sigma_K & \to & E \\ & & \downarrow \pi \\ & & X \end{array} \right\}$$

We first pass to the principal $\text{Diff}(\Sigma_K)$;

$$\check{\xi} = \left\{ \begin{array}{ccc} \text{Diff}(\Sigma_K) & \to & \check{E} \\ & & \downarrow \\ & & X \end{array} \right\},$$

and then to associated $\mathcal{C}(\Sigma_K)$ bundle

$$\bar{\xi} = \left\{ \begin{array}{ccc} \mathcal{C}(\Sigma_K) & \to & \bar{E} \\ & & \downarrow \\ & & X \end{array} \right\}.$$

The fibre of $\bar{\xi}$ is contractible, so that $\bar{\xi}$ admits a section, f, say.

However, a section f of $\bar{\xi}$ defines exactly a Riemannian metric $[f]$ on $\mathrm{Ker}\,(T\pi)$

$$\mathcal{E} = (0 \to \mathrm{Ker}\,(T\pi) \to TE \to \pi^*(TXI \to 0)).$$

As in the case of a direct product, ξ splits. If μ is a metric on TX then $[f,\mu] = [f]\perp\pi^*(\mu)$ defines a Riemannian metric on TE. Moreover $[f,\mu]$ has nonpositive curvature provided μ does. We have proved;

Theorem 2.3: *Let*

$$\xi = \left\{ \begin{array}{ccc} \Sigma_K & \to & E \\ & & \downarrow \\ & & X \end{array} \right\}$$

be a smooth bundle with fibre Σ_K in which the base space X is a closed manifold admitting a Riemannian metric of nonpositive curvature; then E also admits a Riemannian metric of nonpositive curvature.

§3: Vanishing of the Whitehead group for strongly poly Surface groups:

If $\mathcal{C}$ is a class of abstract groups, a group G is said to be a $poly - \mathcal{C} - group$ when there exists a filtration $\mathcal{G} = (G_r)_{0 \leq r \leq n}$ on G by subgroups G_r such that

(i) $\{1\} = G_0 \subset G_1 \subset \dots \subset G_n = G$.
(ii) $G_r \lhd G_r + 1$ and $G_{r+1}/G_r \in \xi$ of each r, $0 \leq r \leq n-1$. G is said to be *strongly poly* $- \mathcal{C}$ when, in addition ,
(iii) $G_r \lhd G$ for each r.

It follows easily that a strongly poly-$\mathcal{C}$ group $G(n)$ length n is constructed as an extension.

$$1 \to K \to G(n) \to G(n-1) \to 1$$

where $G(n-1)$ is a strongly poly-$\mathcal{C}$ group of length $n-1$.

In certain cases the restriction of being strongly poly C is not excessive; in particular, this is true when $C = \text{SURFACE} = \{\pi_1(\Sigma) : \Sigma \textit{ closed orientable surface of genus} \geq 2\}$. As we have shown elsewhere [9] [10].

<u>**Proposition 3.1**</u> *Every poly-Surface group contains a strongly poly Surface subgroup of finite index.*

<u>**Theorem 3.2:**</u> *If G is a strongly poly Surface group then there is a smooth closed Riemannian manifold X_G of nonpositive (sectional) curvature such that $\pi_1(X_G) = G$.*

<u>**Proof:**</u> To any strongly poly Surface group G we can associate a canonical smooth model X_G of homotopy type $K(G,1)$. The procedure is, briefly, as follows (see also [9],[10]: if G is a Surface group, we let X_G be the surface such that $\pi_1(X_G) = G$. If G is a strongly poly Surface group of length n, defined by an extension.

$$1 \to K \to G \to H \to 1.$$

in which K is a Surface group and H is strongly poly Surface of length $n-1$, then by (1.5), we may realize G as the fundamental group of a smooth fibre bundle

$$\begin{array}{ccc} \Sigma_K & \to & X_G \\ & & \downarrow \\ & & X_H \end{array}$$

where X_H is the canonical model, previously constructed, for H. However, it is clear from (2.3) that at each stage of the construction, X_H admits a metric of nonpositive curvature, so that, by (2.3), X_G also admits a metric of nonpositive curvature. □

As a Corollary we get

<u>**Theorem 3.3:**</u> *If G is a strongly poly Surface group then $Wh(G) = 0$.*

Since every poly-Surface group contains a strongly poly Surface subgroup of finite index we obtain

Corollary 3.4: If G is a poly-Surface group then G contains a subgroup G_o of finite index such that $Wh(G_o) = 0$.

The class of poly Surface groups is a subclass of Waldhausen's generalized free products. However, except for the very lowest dimensional examples, Waldhausen's vanishing criterion does not apply to them because of difficulties arising from "non-Noetherian amalgamations". In particular, (3.3) does not follow from Waldhausen's results.

§4: The examples of Atiyah and Kodaira:

By a *Kodaira fibration* we mean a holomorphic mapping $p : E \to X$ where E is a nonsingular connected projective algebraic surface X is a nonsingular projective algebraic curve, and p is C^∞ locally trivial but *not* holomorphically locally trivial; that is, if Σ denotes a typical fibre, E describes a variation of complex structure on Σ, parametrized by X. It is known ([11]) that the genus of the algebraic curve Σ must be at least three.

In [1], [13] Atiyah and Kodaira separately described examples of such objects having the additional property that $\operatorname{sign}(E) \neq 0$.

We will show

Theorem 4.1:

Let E be the total space of a Kodaira fibration. Then E admits a Riemannian metric of nonpositive curvature.

Proof: Choose $* \in X$ and put $\Sigma = p^{-1}(*)$. Let $\hat{E}$ be the covering of E with

$$\pi_1(\hat{E}) = \operatorname{Ker}(\mathrm{p} : \pi_1(\mathrm{E}) \to \pi_1(\mathrm{X})).$$

Observe that there is a diffeomorphism $h : \hat{E} \to \Sigma \times \tilde{X}$. The complex structure on E lifts to one on $\hat{E}$ which, by transport of structure via h, we may interpret as a complex structure on $\Sigma \times \tilde{X}_1$ or, alternatively, as a smooth mapping $\tilde{f} : \tilde{X} \to \mathcal{M}(\Sigma)$, where $\mathcal{M}(\Sigma)$ is the space of smooth complex structures on Σ.

Making the previously observed identification between $\mathcal{M}(\Sigma)$ and $\mathcal{C}(\Sigma)$, we now have a smooth map $\tilde{f} : \tilde{X} \to \mathcal{M}(\Sigma)$. Let $\tilde{\mu}$ be a lifting to $\tilde{X}$ of a Riemannian metric μ of constant negative curvature on X. Thus we obtain a metric $\left[\tilde{f}, \tilde{\mu}\right]$ of nonpositive curvature on $\Sigma \times \tilde{X}$. By transport of structure from h we see that $\hat{E}$ also admits a metric of nonpositive curvature. Finally, it is easy to see that the construction is $\pi_1(X)$ equivariant, so that $E = \hat{E}/_{\pi_1(X)}$ also admits a metric of nonpositive curvature $\Box$.

Corollary 4.2: *Let E be the total space of a Kodaira fibration. Then $Wh(\pi_1(E)) = 0$.*

REFERENCES

[1] M.F. Atiyah: The signature of fibre bundles, in "Global Analysis: Papers in honor of K. Kodaira" pp. 75–84. Princeton University Press 1969.

[2] R. Baer: Isotopie von Kurven auf orientierbaren Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen. J. Reine Angew. Math. 159 (1928) 101–111.

[3] J.M. Boardman and R. Vogt: Homotopy invariant algebraic structures: Lecture Notes in Mathematics vol 347 Springer-Verlag 1973.

[4] K. Burns and R. Spatzier: Manifolds of nonpositive curvature and their buildings. Publ. Math. de l' I.H.E.S no 65 (1987) 35–39.

[5] C.J. Earle and J. Eells: A fibre bundle description of Teichmüller theory. J. Diff. Geom. 3 (1969) 19–43.

[6] P. Eberlein: Isometry groups of simply connected manifolds of nonpositive curvature II. Acta Math. 149 (1982) 41–69.

[7] F.T. Farrell and L.E. Jones: Rigidity and other topological aspects of compact nonpositively curve manifolds Bull. A.M.S. vol 22 (1990) pp.59–64.

[8] M.E. Hamstrom: Homotopy groups of the space of homeomorphisms on a 2-manifold. Illinois J. Math. 10 (1966) 563–573.

[9] F.E.A. Johnson: On the releasability of poly-Surface groups. Jour. Pure Appl. Alg. 15 (1979) 235–241.

[10] F.E.A. Johnson: The theory of Surface fibrations. Preprint. University College London 1990.

[11] A. Kas: On deformations of a certain type of irregular algebraic surface. Amer. J. Math. 90 (1968) 789–804.

[12] S.P. Kerckhoff: The Nielsen realization problem. Ann of Math. 117 (1983) 235–265.

[13] K. Kodaira: A certain type of irregular algebraic surface. Jour. Anal Math. 19 (1967) 207–215.

[14] S. Kravetz: On the geometry of Teichmüller spaces and the structure of their modular groups. Ann. Acad. Sci. Fenn. 278 (1959) 1–35.

[15] S. MacLane: Homology Springer-Verlag 1963.

[16] F. Waldhausen: Algebraic K-theory of generalized free products I, II. Ann. of Math. 108 (1978) 135–256.