MODULI OF HYPER–KÄHLERIAN MANIFOLDS I.

("Filling in" problem and the construction of moduli space)

by

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MPI/90-41

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#0. INTRODUCTION.

It is a well known fact that if X is a compact complex simply connected Kähler manifold with

$$c_1(X)=O$$

then

$$X=\sqcap X_j x \sqcap Y_j$$

where

a) for each j

$$\dim_{\mathbb{C}} H^o(X_j,\Omega^2)=1$$

and if ϕ_j is a non-zero holomorphic two form on X_j, then at each point $x \in X_j$ it is a non-degenerate, i.e. if

$$\phi_{j|U}=\sum(\phi_j)_{\alpha,\beta}dz^\alpha \wedge dz^\beta$$

then

$$\det(\phi_j|_U)\in\Gamma(U,O_U^*)$$

Such manifolds we will call Hyper-Kählerian.

b) For each i and

$$O<p<n=\dim_{\mathbb{C}} Y_i \quad \dim_{\mathbb{C}} H^o(Y_i,\Omega^p)=O$$

and $H^o(Y_i,\Omega^n)$ is spanned by a holomorphic n-form $\omega_{Y_i}(n,0)$ which has no zeroes.

This fact is due to Calabi and Bogomolov. See [3]. An elegant proof based on Yau's solution of Calabi conjecture was given by M. L. Michelson. See [16].

The purpose of this article is to study the moduli space of the so called marked algebraic Hyper-Kählerian manifolds.

Definition. A triple

$$(X,\gamma_1,\dots,\gamma_{b_2};L)$$

will be called a marked algebraic Hyper-Kählerian manifold if X is a Hyper-Kählerian manifold

$$\gamma_1,\ldots,\gamma_{b_2}$$

is a basis of $H_2(X,\mathbb{Z})$ and L is the imaginary part of Hodge metric on X as a class of cohomology.

The aim of this article is to prove that the moduli space of marked polarized Hyper-Kähler manifolds exists and up to a component is isomorphic to

$$SO(2,b_2-3)/SO(2)xSO(b_2-2)$$

where

$$b_2=\dim_{\mathbb{R}}H^2(X,\mathbb{R}).$$

The content of this article is the following:

In #1 we introduce the basic definitions and notations

In #2 we prove the following Theorem:

THEOREM 1.

Suppose that:

$$\pi^*:\mathfrak{X}^*\to D^*$$

is a family of non-singular Hyper-Kählerian manifolds such that:

a) $\pi^*:\mathfrak{X}^*\to D^*$ has a trivial monodromy on $H_2(X_t,\mathbb{Z})$

b) $$\begin{array}{ccc} \mathfrak{X} & \subset & \mathbb{P}^N x D^* \\ \downarrow & & \downarrow \\ D^* & = & D^* \end{array}$$

Then there exists a family

$$\pi:\mathfrak{X}\to D$$

such that all its fibres are non-singular Hyper-Kählerian manifolds and we have

$$\begin{array}{ccc} \mathfrak{X}^* & \subset & \mathfrak{X} \\ \downarrow & & \downarrow \\ D^* & \subset & D \end{array}$$

(here $D=\{t|\ t\in\mathbb{C}$ and $|t|<1\}$)

The idea of the proof of THEOREM 1.

First step.

We need to prove that the family $\mathfrak{S}^* \to D^*$ can be embedded into a family $\mathcal{Y} \to U^o$, where $U^o = U \backslash \mathcal{A}$, U is a polycylinder and $\mathcal{A}$ is a cmplex analytic subspace in U. Moreover U has dimension equal to $\dim_{\mathbb{C}} H^1(X_t, \Omega^1_t) - 1$ and $\mathcal{Y} \to U^o$ is the maximal subfamily in tyhe Kuranishi family for which the polarization class L is of type (1,1).

Second step.

For any $t \in U^o$ we can define the isometric deformations with respect to Yau's metric corresponding to L and take the union of all these deformations. It is easy to see that they form an open set in the the Kuranishi space. From the definition of an isometric deformation it follows that the group SO(3) acts on them. Now if we change the complex structures on

$$\mathfrak{S}^* \to D^*$$

simultaniously with an element

$$A \in SO(3)$$

we will get another family

$$\mathfrak{S}^*_A \to D^*_A$$

which is not in "general" complex analytic one. The main point is that we can find $A \in SO(3)$ such that the family

$$\mathfrak{S}^*_A \to D^*_A$$

can be prolonged to a smooth family of Hyper-Kählerian manifolds $\mathfrak{S}_A \to D_A$, i.e. all the fibres of $\mathfrak{S}_A \to D_A$ are smooth Hyper-Kählerian manifolds. From this result it is not so difficult to get THEOREM 1.

In #3 we prove the following Theorem:

THEOREM 2.

There exists a universal family of marked polarized algebraic Hyper-Kählerian manifolds:

$$\mathfrak{X}_L \to \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

The construction follows Burns and Rapoport. See [6].

We have the so called period map:

$$p: \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \to \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$$

where

$$p(t) := (\dots, \int_{\gamma_i} \omega_t(2,0), \dots)$$

where $\omega_t(2,O)$ is the unique up to a constant holomorphic two-form on $X_t=\pi^{-1}(t)$. From Bogomolov's result, that there are no obstructions to deformations and Local Torelli theorem we get that the irreducible component $\mathfrak{M}(L;\gamma_1,\dots,\gamma_{b_2})$ is a non-singular manifold and

$$\dim_{\mathbb{C}}\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}=b_2-2, \text{ where } b_2=\dim_{\mathbb{C}}H^2(X,\mathbb{C})$$

From Griffith's theory of variation of Hodge structures we get that:

$$p:\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}\to SO(2,b_2-2)/SO(2)xSO(b_2-2)\subset\mathbb{P}(H^2(X,\mathbb{Z})\otimes\mathbb{C})$$

is a local isomorphism. See [2].

The second part of this article

"MODULI OF HYPER-KÄHLERIAN MANIFOLDS II".

contains #4.

In #4 we prove THEOREM 3.

THEOREM 3. The period map

$$p:\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}\to\mathbb{P}(H^2(X,\mathbb{Z})\otimes\mathbb{C})$$

is an embedding up to a component of

$$\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

Theorem 3 is a positive answer to the so called global Torelli problem, and is in some aspects a generalization of the theorem of Piatetski-Shapiro and Shafarevich about K3 surfaces. See [20].

Ideas and methods of the proof of THEOREM 3.

In order to prove Theorem 3 we need to compatify partially the family

$$\mathfrak{X}_L \to \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

to a family

$$\overline{\mathfrak{X}}_L \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

by adding singular Hyper-Kählerian algebraic manifolds for which L is a very ample line bundle.

Next we prove that

$$\overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

is a Hausdorff space and p can be extended to a proper étale map $\overline{p}$.

$$\overline{p}: \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \to SO(2,b_2-2)/SO(2)xSO(b_2\text{-}2)$$

But

$$SO(2,b_2-2)/SO(2)xSO(b_2\text{-}2)$$

is a Siegel domain of IV type and so it is simply connected domain of holomorphy. From this fact and since $\overline{p}$ is a proper and étale map it follows that $\overline{p}$ is a surjective and one to one map up to a component of

$$\overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

So this proves that the period map is both injective and surjective up to a component of the moduli space of marked polarized Hyper-Kählerian manifolds. This generalizes a theorem proved in [21].

The main step of the proof of Theorem 3 is the partial compactification of the moduli space (one of its components) and it is based on Theorem 1.

The proof of Theorem 1 is based on the proof of Calabi's conjecture given by Yau. See [22]. More precisely we are using the existence of Ricci flat metrics on Hyper-Kählerian manifolds and the so called isometric deformations which existence is based on the solution of the Calabi's conjecture.

Theorem 1 gives an affirmative answer to the so called "filling in problem" posed by Ph. Griffiths. See [11] and [18] for counterexamples in case of surfaces of general type.

Theorem 1 is a generalization of some results of Kulikov's . See [15]. Our proof is entirely different from the proof of Kulikov's theorem for K3 surfaces and in my opinion his proof can not be generalized for higher dimensions.

The first examples of Hyper-Kählerian manifolds of

$$\dim_{\mathbb{C}} X > 3$$

were constructed by Fujiki. See [12]. These examples were generalized by Beauville and Miyaoka. See [1].

It is not very difficult to prove the surjectivity of the period map for all Hyper-Kähler manifolds. This will be done in a future paper.

Recently O. Debarre constructed using the so-called elementary transformations introduce by Mukai in [17] constructed two bimeromorphic but not biholomorphic non-algebraic Hyper-Kählerian manifolds. See [7].

CONJECTURE. Let X and X' be two marked Hyper-Kählerian manifolds which have the same periods, then X is bimeromorphic to X'.

Part of this work was done during my stay in IAS in Princeton and was supported by an NSF grant. It was finished in Max-Plank Institute fur Mathematik in Bonn. The auther expresses his gratitude to both institutes for the hospitality and excellent conditions for work.

CONTENT.

#1. SOME DEFINITIONS AND NOTATIONS.

Definition 1.1. Let X be a Kähler compact manifold such that:

a) $\pi_1(X)=O$

b) $\dim_C X=2n$, $n>3$

c) $\dim_C H^o(X,\Omega^2)=1$ and let $\omega_X(2,O)$ is a non-zero holomorphic two form on X, then $\omega_X(2,O)$ is a non-degenerate form on X, which means that $\wedge^n\omega_X(2,O):=\omega_X(2n,O)$ is a holomorphic 2n form which has no zeroes.

Then X will be called a Hyper-Kählerian manifold.

Some notations:

$\omega_X(k,O)$ will be a holomorphic k-form on X.

$\omega_X(O,k)=\overline{\omega_X(k,O)}$, i.e. the anti-holomorphic k-forms on X.

D-will be the unit disk, i.e. $D=\left\{t\in C|\ |t|<1\right\}$

$D^*=D\setminus\{O\}$

If $\pi:\mathfrak{X}\to D$ is a family of manifolds, then $X_t=\pi^{-1}(t)$.

If g is a Riemannian metric on X by ∇ we will denote the Levi-Chevita connection on T^*X, where TX is the tangent bundle on X and T^*X is the cotangent bundle. By $T^*X\otimes C$, we will denote the complexified cotangent bundle. ∇ induces a covariant derivative on $\wedge^pT^*X$ for any $p\in\mathbf{Z}$. This covariant derivative we will denote it again ∇.

$\Gamma(X,\mathfrak{F})$ will denote the global sections of any sheaf $\mathfrak{F}$ on X.

If $\phi\in\Gamma(X,\wedge^pT^*X)$, then locally:

$$\phi=\sum_{p+q=m}\phi_{A_p,\overline{B}_q}dz^{A_p}\wedge d\overline{z}^{\overline{B}_q}$$

where

$$A_p=(\alpha_1,\ldots,\alpha_p)\ \&\ B_q=(\beta_1,\ldots,\beta_q)$$

are multi-indices.

$dz^{A_p}=dz^{\alpha_1}\wedge\ldots\wedge dz^{\alpha_p}$ and $z^1,\ldots,z^{2n}$ are local coordinates.

If $\phi\in\Gamma(X,\wedge^pT^*X)$ and $d\phi=O$, then $[\phi]$ we will denote the class of cohomology that ϕdefines in $H^p(X,C)$.

#2. PROOF OF THEOREM 1:

THEOREM 1.

Suppose that:

$$\pi^*:\mathfrak{X}^* \to D^*$$

is a family of non-singular Hyper-Kählerian manifolds such that:

a) $\pi^*:\mathfrak{X}^* \to D^*$ has a trivial monodromy on $H_2(X_t,\mathbf{Z})$

b)
$$\begin{array}{ccc} \mathfrak{X} & \subset & \mathbf{P}^N \times D^* \\ \downarrow & & \downarrow \\ D^* & = & D^* \end{array}$$

Then there exists a family $\pi:\mathfrak{X}\to D$ such that all its fibres are non-singular Hyper-Kählerian manifolds and we have

$$\begin{array}{ccc} \mathfrak{X}^* & \subset & \mathfrak{X} \\ \downarrow & & \downarrow \\ D^* & \subset & D \end{array}$$

(here $D=\{t|\ t\in C \text{ and } |t|<1\}$)

This problem was first posed by Ph. A. Griffiths.

For the proof of Theorem 1 we will need some preliminary matirial.

#2.1. HODGE STRUCTURES OF WEIGHT TWO ON HYPER-KÄHLERIAN MANIFOLDS.

Definition 2.1.1. The triple

$$(X;\gamma_1,\dots,\gamma_{b_2};L)$$

we will call a marked, polarized Hyper-Kählerian manifold if

a) X is a Hyper-Kählerian manifold;

b) $\gamma_1,\dots,\gamma_{b_2}$ is a basis of $H_2(X,\mathbf{Z})$ and

c) L is the cohomology class of the imaginary part of a Kähler metric on X, i.e.

$$L=[\mathrm{Im}(g_{\alpha,\overline{\beta}})]\in H^2(X,\mathbf{Z})$$

Remark. Notice that two marked, polarized Hyper-Kählerian manifolds

$$(X;\gamma_1,\dots,\gamma_{b_2};L)$$

and

$$(X';\beta_1,\ldots,\beta_{b_2};L')$$

are isomorphic iff there exists a biholomorphic map

$$\phi:X\simeq X'$$

such that

a) $\phi^*(L')=L \;;\phi^*:H^2(X',\mathbf{Z})\rightarrow:H^2(X,\mathbf{Z})$

b) $\phi_*(\gamma_i)=\beta_i \;;\phi_*:H_2(X,\mathbf{Z})\rightarrow:H_2(X',\mathbf{Z})$

<u>Definition 2.1.2.</u>

Suppose that

$$\pi:\mathfrak{X}\rightarrow S$$

is a family of a non-singular Hyper-Kählerian manifolds and suppose that the monodromy operator T induced by the action of

$$\pi_1(S) \text{ on } H_2(X_t,\mathbf{Z})$$

is the identity operator. It is clear that if we fix a basis

$$\gamma_1,\ldots,\gamma_{b_2}$$

of $H_2(X_t,\mathbf{Z})$, then since the monodromy operator

$$T=id \text{ for every } s\in S$$

$$\gamma_1,\ldots,\gamma_{b_2}$$

will be a basis in $H_2(X_s,\mathbf{Z})$ for every $s\in S$. So we can define the period map:

$$p:S\rightarrow\mathbb{P}(H^2(X,\mathbf{C}))$$

in the following manner:

$$p(s):=(\ldots,\int_{\gamma_i}\omega_{X_s}(2,O),\ldots)$$

Now we want to see where the image of S lies in $\mathbf{P}(H^2(X,\mathbf{C}))$.

For this reason we will define a scalar product in $H^2(X,\mathbf{R})$, where X is a marked Hyper-Kählerian manifold.

<u>Definition 2.1.3.</u> The scalar product < , > in $H^2(X,\mathbf{R})$ is defined as follows

$$<w_1,w_2>=\int_X w_1\wedge w_2\wedge L^{n-2}$$

and L is the polarization class.

<u>Proposition 2.1.3.4.</u> The scalar product < , > has signature $(3,b_2-3)$, where

$$b_2=\dim_{\mathbf{R}}H^2(X,\mathbf{R}).$$

Proof:

It is easy to see that

$$<L,L>=\int_X L^{2n} = \mathrm{Vol}(X)>0$$

where Vol(X) is the volume of X with tespect to the metric $(g_{\alpha,\overline{\beta}})$ and $[\mathrm{Im}(g_{\alpha,\overline{\beta}})]=L$.

Next we will prove the following relations:

(2.1.4.) $\quad <\omega_X(2,0),\omega_X(2,0)>=0$

(2.1.5) $\quad <\omega_X(2,0),\omega_X(0,2)>>0$

(2.1.6) $\quad <\omega_X(2,0),L>=0$

(2.1.4.) and (2.1.6.) follow from the definition of < , > and comparing the types of forms.

In order to prove (2.1.5.) we need the following lemma:

LEMMA. If η is a primitive form of type (p,q), then

$$*\eta=\frac{(\sqrt{-1}\ )^{p-q}}{(2n-p-q)}(-1)^{\frac{(p+q)(p+q+1)}{2}}L^{2n-p-q}$$

where $*$ is the Hodge star operator.

Proof: See [8].

Q.E.D.

From this lemma it follows that

$$<\omega_X(2,0),\overline{\omega_X(2,0)}>=\int_X \omega_X(2,0)\wedge*\overline{\omega_X(2,0)}=\|\omega_X(2,0)\|^2>0$$

So (2.1.5.) is proved.

Let

$$\omega_X(2,0) = \mathrm{Re}\omega_X(2,0)+i\ \mathrm{Im}\omega_X(2,0)$$

then from 2.1.4. and 2.1.5. it follows that:

$$<\mathrm{Re}\omega_X(2,0),\ \mathrm{Re}\omega_X(2,0)>=<\mathrm{Im}\omega_X(2,0),\mathrm{Im}\omega_X(2,0)>=\tfrac{1}{2}\|\omega_X\|^2>0$$

and

$$<\mathrm{Re}\omega_X(2,0),\mathrm{Im}\omega_X(2,0)>=0$$

So we see that L, $\mathrm{Re}\omega_X(2,0)$ & $\mathrm{Im}\omega_X(2,0)$ are three orthonormal vectors in $H^2(X,\mathbb{R})$ and they have positive self intersection number. So from here it follows that < , > has at least signature $(3,b_2-3)$. Since we have

$$H^2(X,\mathbf{R})=\mathbf{R}\mathrm{Re}\omega_X(2,0)+\mathbf{R}\mathrm{Im}\omega_X(2,0)+\mathbf{R}L+H^{1,1}(X,\mathbf{R})_o$$

where

$$H^{1,1}(X,\mathbf{R})_o=\left\{\omega\in H^{1,1}(X,\mathbf{R})|<\omega,L>=0\right\}$$

i.e. $H^{1,1}(X,\mathbf{R})_o$ are the primitive cohomology classes of type (1,1). From the LEMMA it follows that if $\omega\in H^{1,1}(X,\mathbf{R})_o$, then

$$<\omega,\omega><0$$

It is easy to see that if $\omega\in H^{1,1}(X,\mathbf{R})$, then

$$<\omega,\omega_X(2,0)>=<\omega,\omega_X(0,2)=0$$

So Proposition 2.1.3.4. is proved.

Q.E.D

The scalar product < , > defines a non-singular quadric

$$Q\subset \mathbf{P}(H^2(X,\mathbf{C}))$$

in the following way:

(2.1.7.) $$Q:=\left\{u\in\mathbf{P}(H^2(X,\mathbf{C}))|\ <u,u>=0\right\}$$

Let Ω be

$$\Omega:=\left\{u\in Q|\ <u,\overline{u}>>0\right\}$$

Ω is an open subset in Q.

(2.1.8.) Let $\Omega(L)=\left\{u\in\Omega|\ <u,L>=0\right\}$

From Griffith's theory [13] we obtain that if

$$\mathfrak{X}\rightarrow S$$

is a family of marked polarized Hyper-Kählerian manifolds, then

$$p(S)\subseteq\Omega(L)$$

where p is the period map.

Definition 2.1.10.

$\Omega(L)$ we will call the period domain of the polarized Hodge structures of weight two on Hyper-Kählerian manifolds.

Remark 2.1.11.

a) If $L\in H^2(X,\mathbf{Z})$, then < , > is defined over $\mathbf{Z}$.

b) It is not difficult to see that:

$$\Omega(L)\doteq SO_o(2,b_2-3)/U(1)xSO(b_2-3)$$

#2.2. GEOMETRY OF Ω.

Proposition 2.2.1.

There exists a one-to-one map ϕ between points of Ω and all two dimensional oriented vector subspaces $E \subset H^2(X,\mathbf{R})$ such that $<\ ,\ >$ (defines in #2.1.) when restricted to E is positive, i.e. $<u,u>>O$ for $\forall u \in E$.

Proof: The map ϕ is constructed in the following way:

Let

$$x \in \Omega \subset \mathbf{P}(H^2(X,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C}),$$

then x defines a line

$$l_x \subset H^2(X,\mathbf{Z}) \otimes \mathbf{C}$$

Let

$$\omega_x = \mathrm{Re}\omega_x + i\mathrm{Im}\omega_x \neq 0 \quad \omega_x \in l_x$$

From the definition of Ω it follows that

$$<x,x>=O \ \& \ <x,\overline{x}>>O \Rightarrow x \neq \overline{x}$$

So

$$\mathrm{Re}\omega_x \neq O \quad \mathrm{Im}\omega_x \neq O$$

Now we can define ϕ in the following way:

$$\phi(x) = E_x$$

where E_x is an oriented two dimensional subspace in $H^2(X,\mathbf{R})$ spanned by

$$\mathrm{Re}\omega_x \text{ and } \mathrm{Im}\omega_x$$

The orientation of E_x is given by $\{\mathrm{Re}\omega_x , \mathrm{Im}\omega_x\}$.

Since from

$$<x,x>=O \text{ and } <x,\overline{x}>>O \Rightarrow x \neq \overline{x} \quad \text{if } x \in \Omega,$$

then it follows that to the point $\overline{x}$ coresponds $E_{\overline{x}}$, i.e

$$\phi(\overline{x}) = E_{\overline{x}}$$

where

$$E_{\overline{x}} \equiv E_x \text{ (as subspaces without orientation)}$$

but

E_x has a different orientation then $E_{\overline{x}}$.

Now it is very easy to show that ϕ is a one-to-one map. Indeed let E be a positive two dimensional subspace in $H^2(X,\mathbf{Z})\otimes\mathbf{R}$.

Let e_1 and e_2 be two orthonormal vectors in E_X and $x=e_1+ie_2$.

Clearly

$$<x,x>=O \text{ and } <x,\bar{x}>>O$$

So the vector $x\neq 0$ defines a line l_X in $H^2(X,\mathbf{R})\otimes C$ and the line l_X defines a point $u\in\Omega$.

Q.E.D.

Corollary 2.2.2 Let

$$\pi:\mathfrak{S}\rightarrow S$$

be a family of marked polarized Hyper-Kählerian manifolds, then the period map

$$p:S\rightarrow\Omega$$

can be defined in the following way:

$$p(s) = E_s^+ \overset{\text{def}}{=} \{Re\omega_s(2,O),Im\omega_s(2,O)\}$$

where E_s^+ means E_s with an orientation

$$\{Re\ \omega_s(2,O),Im\omega_s(2,O).$$

Corollary 2.2.3. $\Omega\doteq SO_o(2,b_2-3)/U(1)xSO(b_2-3)$.

#2.3. GEOMETRY OF PLANE QUADRICS ON Ω.

Proposition 2.3.1.

Let E be a three dimensional subspace in $H^2(X,\mathbb{R})$ such that the restriction of $<\ ,\ >$ on E is srtrictly positive, i.e $<\ ,\ >_{|E}>O$.

Then

$$\mathbf{P}(E\otimes\mathbf{C})\cap\Omega$$

will be a non-singular projective plane quadric.

Proof: From the definition of Ω it follows that

$$\Omega \text{ is an open subset in } Q,$$

where Q is a non-singular hypersurface of degree 2 in $\mathbf{P}(H^2(X,\mathbf{C}))$. Clearly

$$\mathbf{P}(E\otimes\mathbf{C})\cap Q$$

is a plane quadric. We will prove first that $\mathbf{P}(E\otimes\mathbf{C})\cap Q=\mathbf{P}(E\otimes\mathbf{C})\cap\Omega$.

Since

$$E\subset H^2(X,\mathbf{R}) \ \&\ \dim_{\mathbf{C}}E=3$$

and the restriction of $<\ ,\ >$ on E is srtrictly positive it follows that

$$\mathbf{P}(E\otimes\mathbf{C})\cap Q\subset\Omega$$

Indeed if

$$u\in\mathbf{P}(E\otimes\mathbf{C})\cap Q$$

then any vector $w\in l_u$ defined by u in $H^2(X,\mathbb{R})\otimes\mathbf{C})$, (where l_u is the one dimensional subspace in $H^2(X,\mathbb{R})\otimes\mathbf{C})$, that corresponds to u) has the property that

$$<w,\overline{w}>>O \ \&\ <w,w>=0$$

So we get that

$$<u,\overline{u}>>O \ \&\ <u,u>=0 \text{ in } \mathbf{P}(H^2(X,\mathbb{R})\otimes\mathbf{C}))$$

Since this inequality is valid for any

$$u\in \mathbf{P}(E\otimes\mathbf{C})\cap Q$$

we get that

$$\mathbf{P}(E\otimes\mathbf{C})\cap Q\subset\Omega$$

Q.E.D.

Next we will prove that $\mathbf{P}(E\otimes\mathbf{C})\cap\Omega$ is nonsingular projective curve of deg=2.

Proof: Suppose that $\mathbf{P}(E\otimes\mathbf{C})\cap Q$ is a singular plane quadric, then

$$\mathbf{P}(E\otimes\mathbf{C})\cap Q$$

should have a unique singular point q.

From the definition of Ω we know that

$$\forall u \in \Omega \Rightarrow u \neq \overline{u}$$

So we get that $q \neq \overline{q}$. Remember q was a singular point on the plane quadric

$$\mathbb{P}(E \otimes \mathbb{C}) \bigcap Q$$

From here and the fact that

$$E \subset H^2(X,\mathbb{R}) \Rightarrow \overline{E \otimes \mathbb{C}} = E \otimes \mathbb{C}$$

we get that the plane quadric

$$\mathbb{P}(E \otimes \mathbb{C}) \bigcap Q \subset \Omega$$

has two different singular point q & $\overline{q}$.

This is so since

$$\mathbb{P}(E \otimes \mathbb{C}) \bigcap \Omega \equiv \overline{\mathbb{P}(E \otimes \mathbb{C}) \bigcap \Omega}\ ,\ q\ \&\ \overline{q} \in \Omega \Rightarrow q \neq \overline{q}$$

This is clearly a contradiction with the fact that deg $\mathbb{P}(E \otimes \mathbb{C}) \bigcap \Omega = 2$.

Q.E.D.

Definition 2.3.2.

$$\text{Grass}(3,b_2;\mathbb{R}) \overset{\text{def}}{=} \left\{ \text{all oriented 3-dim subspaces } E \subset H^2(X,\mathbb{R}) \mid <\ ,\ >_{|E} > O \right\}$$

Corollary 2.3.2.1.

There is a one two-one map

$$\nu : \mathbb{Q}(\mathbb{R}) \rightarrow \text{Grass}(3,b_2;\mathbb{R})$$

where

$$\mathbb{Q}(\mathbb{R}) \overset{\text{def}}{=} \left\{ \text{all projective plane quadrics } F \subset \Omega \mid F = \overline{F} \right\}$$

Definition 2.3.3.

If $E \subset H^2(X,\mathbb{R})$ & $<u,u> > O\ \forall u \in E$

then we will denote by $\mathbb{P}^1(E)(\mathbb{R}) \subset \Omega$ the plane quadric

$$\Omega \bigcap \mathbb{P}(E \otimes \mathbb{C}) \equiv Q \bigcap \mathbb{P}(E \otimes \mathbb{C}) \text{ (See Prop. 2.3.1.)}$$

Proposition 2.3.4. Let $L \in H^2(X,\mathbb{Z})$ & $<L,L> > O$, $\Omega(L) := \{u \in \Omega \mid <u,L> = O\}$ and $V \subset \Omega(L)$ be a complex analytic submanifold. Let $z \in \Omega$ be any fixed point such that $z \notin \Omega(L)$, then the set

$$\mathcal{A}_z(V)(\mathbb{R}) \overset{\text{def}}{=} \{\mathbb{P}^1(E)(\mathbb{R}) \mid z \in \mathbb{P}^1(E)(\mathbb{R})\ \&\ \mathbb{P}^1(E)(\mathbb{R}) \bigcap V \neq \emptyset\}$$

is a real analytic subset in $\text{Grass}(3,b_2;\mathbb{R})$.

Proof:

This is a standard fact from the theory of the grassmanian manifolds. See [13].

Q.E.D.

Definition 2.3.5.

Let $z\in\Omega$ & z be a fixed point then we will denote by $\mathcal{A}_z(\mathbb{R})$ the following set:

$$\mathcal{A}_z(\mathbb{R}) \overset{\text{def}}{=} \{\mathbf{P}^1(E)(\mathbb{R}) \mid z\in\mathbf{P}^1(E)(\mathbf{R})\}$$

Remark 2.3.5.1.

It is a standart fact that $\mathcal{A}_z(\mathbb{R})$ is a real analytic subset in Grass(3,b_2;**R**) and $\dim_{\mathbf{R}} \mathcal{A}_z(\mathbb{R})=b_2-3$.(See [13].)

Proof of the fact that $\dim_{\mathbf{R}} \mathcal{A}_z(\mathbb{R})=b_2-3$:

We know from 2.2.1. that to the point $z\in\Omega$ corresponds to a two-dimensional space

$$E_z\subset H^2(X,\mathbf{R})$$

Clearly that there is one-to-one correspondence between the following three sets

$$\left\{E\subset H^2(X,\mathbf{R}) \mid <\ ,\ >_{|E}>O\ ,\ \dim_{\mathbf{R}}E=3\ \&\ E_z\subset E\right\}$$
$$\left\{\text{the points of } \mathcal{A}_z(\mathbb{R})\subset \text{Grass}(3,b_2;\mathbf{R})\right\}$$

and the lines in in the convex cone

$$\Upsilon_z(\mathbf{R}) \overset{\text{def}}{=} \{u\in H^2(X,\mathbb{R}) \mid u\perp E_z\ <u,u>>O\}$$

i.e.

$$\dim_{\mathbf{R}} \mathcal{A}_z(\mathbb{R})=\dim_{\mathbb{R}} \mathbb{P}(\Upsilon_z(\mathbf{R}))$$

where $\mathbb{P}(\Upsilon_z(\mathbf{R}))$ means the projectivization of $\Upsilon_z(\mathbb{R})$.

So

$$\dim_{\mathbf{R}} \mathcal{A}_z(\mathbb{R})=\dim_{\mathbb{R}}\Upsilon_z(\mathbf{R})-1=b_2-3$$

This follows directly from the definition of $\Upsilon_z(\mathbf{R})$.

Q.E.D.

Definition 2.3.6. Let

$$\text{Grass}(3,b_2;\mathbf{C}) \overset{\text{def}}{=} \{\text{all oriented } E\subset H^2(X,\mathbf{C}) \mid \dim_{\mathbf{C}}E=3\ \&\ <\ ,\ >_{|E}>O, \text{ i.e. } <u,\overline{u}>>O\ \ \forall u\in E\}$$

Corollary 2.3.6.1.

There is a one two-one map $\nu:\mathbb{Q}(\mathbf{C})\rightarrow\text{Grass}(3,b_2;\mathbf{C})$ where

$$\mathbb{Q}(\mathbf{C}) \overset{\text{def}}{=} \left\{\text{all projective plane quadrics } F\subset\Omega\right\}$$

Definition 2.3.7.

If $E\subset H^2(X,\mathbf{C})$, $\dim_{\mathbf{C}}E=3$ & $<u,\overline{u}>>O$ for all $u\in E$

then we will denote by $\mathbf{P}^1(E)(\mathbf{C})\subset\Omega$ the plane quadric $\Omega\cap\mathbb{P}(E)\equiv Q\cap\mathbb{P}(E)$ (See Prop. 2.3.1.)

Proposition 2.3.8.

Let $L\in H^2(X,\mathbb{Z})$ & $<L,L>>0$, $\Omega(L):=\{u\in\Omega|\ <u,L>=0\}$, $V\subset\Omega(L)$ be a complex analytic submanifold. Let $z\in\Omega$ be any fixed point such that $z\notin\Omega(L)$, then the set

2.3.8.1 $$\mathcal{A}_z(V)(\mathbf{C}) \stackrel{def}{=} \{\mathbb{P}^1(E)|\ z\in\mathbb{P}^1(E)\ \&\ \mathbb{P}^1(E)\cap V\neq\emptyset\}$$

is a complex analytic subset in $Grass(3,b_2;\mathbf{C})$.

Proof:

This is a standard fact from the theory of the Grassmanian manifolds. See [13].

Q.E.D.

Remark 2.3.9. Let τ be the complex analytic conjugation in $H^2(X,\mathbf{R})\otimes\mathbf{C}$, i.e.

$$\tau(u)=\overline{u} \text{ for } u\in H^2(X,\mathbf{R})\otimes\mathbf{C}$$

then τ acts on $Grass(3,b_2;\mathbf{C})$ in the following manner:

$$\tau(E)=\overline{E}$$

Clearly that

$$Grass(3,b_2;\mathbf{C})^{\tau}=Grass(3,b_2;\mathbf{R})$$

where

$$Grass(3,b_2;\mathbf{C})^{\tau}=\{E\subset Grass(3,b_2;\mathbf{C})|E^{\tau}=E\}$$

Definition 2.3.10. Let

$$\Omega(L) \stackrel{def}{=} \{u\in\Omega|\ <u,L>=0, \text{ where } L\in H^2(X,\mathbf{R})\ \&\ <L,L>>0\}$$

Remark 2.3.11.

Let $z\in\Omega(L)$ and let E_z be the two dimensional subspace in $H^2(X,\mathbf{R})$ that corresponds to the point z, i.e. $E_z=\phi(z)$(See (2.2.1.))

then

$$<\ ,\ >_{|E_z(L)}>0,$$

E(L) is the three dimensional subspace in $H^2(X,\mathbf{R})$ spanned by E_z and L.

Proof: We know from 2.2.1. that

$$<\ ,\ >_{|E_z}>0$$

From the definition of $\Omega(L)$ it follows that $L\perp E_z$ and $<L,L>>0$

So Remark 2.3.11. is proved if we use 2.2.1.

Q.E.D.

Main Lemma 2.3.12.

Let V be a complex analytic submanifold in $\Omega(L)$, where $\Omega(L)$ is defined as in 2.3.10.

Let $z \in V$ and $E_z(L)$ be defined as in Remark 2.3.11. Let $U \subset \Omega(L)$ be any open neighborhood of the point $z \in V$.

Then there exists a point

$$y \in U \ \& \ y \notin V$$

such that

$$\mathbf{P}^1(E_y(L))(\mathbb{R}) \bigcap \mathbf{P}^1(E_z(L))(\mathbb{R}) \neq \emptyset$$

i.e.

$$\mathbf{P}^1(E_y(L))(\mathbb{R}) \bigcap \mathbf{P}^1(E_z(L))(\mathbb{R}) = t \bigcup \bar{t}$$

and

$$\bar{t} \ \& \ t \notin \Omega(L)$$

Proof:

Let $x \in \mathbf{P}^1(E_z(L))(\mathbb{R})$ and $x \notin \Omega(L)$

Sublemma 1. $\mathcal{A}_x(\mathbb{C}) \bigcap \Omega(L)$ contains an open subset $U' \subset \Omega(L)$ and $V' = V \bigcap \mathcal{A}_x(\mathbb{C}) \subset U'$, where

$$\mathcal{A}_x(\mathbf{C}) \stackrel{\text{def}}{=} \{\mathbf{P}^1(E) | \ x \in \mathbf{P}^1(E) \ \}$$

Proof:

Step1. $\dim_{\mathbb{C}} \mathcal{A}_x(\mathbf{C}) = b_2 - 2$.

Proof of step1:

Since $x \in \Omega \subset \mathbb{P}(H^2(X,\mathbf{C}))$ and from the definition of $\mathbf{P}(H^2(X,\mathbf{C}))$ it follows that x corresponds to a line

$$l_x \subset H^2(X,\mathbf{C})$$

Clearly from the definition of $\mathcal{A}_x(\mathbb{C})$ it follows that $\mathcal{A}_x(\mathbf{C})$ is parametrized by all lines in:

$\Upsilon_x(\mathbf{C}) \stackrel{\text{def}}{=} \{$all l in $H^2(X,\mathbf{C}) |$ l is one dim subspace, $u \in l$, $u \neq 0$ $\langle u,\bar{u} \rangle > 0$ & $\langle l_x,\bar{u} \rangle = 0\}$

It is not difficult to see, using the fact that $\langle \ , \ \rangle$ has a signature $(3, b_2 - 3)$ that

$$\Upsilon_x(\mathbf{C}) \text{ is an open cone in } \mathbf{C}^{b_2 - 1}$$

So $\dim_{\mathbb{C}} \Upsilon_X(\mathbb{C}) = b_2 - 1 \Rightarrow \dim_{\mathbb{C}} \mathbb{P}(\Upsilon_X(\mathbb{C})) = b_2 - 2 \Rightarrow \dim_{\mathbb{C}} \mathcal{A}_X(\mathbb{C}) = b_2 - 2 = \dim_{\mathbb{C}} \Omega$

Q.E.D.

<u>Step 2</u>. $\mathcal{A}_X(\mathbb{C}) \bigcap \Omega(L)$ contains an open subset.

<u>Proof:</u> Since $\mathcal{A}_X(\mathbb{C})$ contains $\mathbb{P}^1(E_X \otimes \mathbb{C})$ where $E_X \subset H^2(X,\mathbb{R})$ we have

$$\mathbb{P}^1(E_X \otimes \mathbb{C}) = \overline{\mathbb{P}^1(E_X \otimes \mathbb{C})} = \mathbb{P}(E_X \otimes \mathbb{C}) \bigcap \Omega = \mathbb{P}(E_X \otimes \mathbb{C}) \bigcap Q (\text{ See 2.2.1.})$$

So we get that

$\mathbb{P}^1(E_X \otimes \mathbb{C})$ **intersects** $\Omega(L)$ **transversally**

This is so since

a) $\mathbf{P}^1(E_X \otimes \mathbb{C})$ is a plane quadric, i.e. plane curve of degree 2

b) $\mathbf{P}^1(E_X \otimes \mathbb{C})$ contains z and $\overline{z}$, where both $z \neq \overline{z} \in \Omega(L)$ since $E_X \otimes \mathbb{C} = \overline{E_X \otimes \mathbb{C}}$.

So from here and the fact that transversality is an open condition we get what we need from the fact that $\dim_{\mathbb{C}} \Upsilon_X(\mathbb{C}) = b_2 - 1$. See [13].

Q.E.D.

So the Sublemma is proved

Q.E.D.

<u>Step 3</u>. $\mathcal{A}_X(\mathbf{R}) \bigcap \Omega(L)$ is not contained in V, where

$\mathcal{A}_X(\mathbf{R}) \stackrel{\text{def}}{=} \{\mathbf{P}^1(E)(\mathbb{R}) |\ x \in \mathbb{P}^1(E)(\mathbb{R})\}$ where x is fixed and V is the submanifold in $\Omega(L)$ defined in 2.3.12.

where $\dim_{\mathbf{R}} E = 3$ and $<,>_{|E} > O$.

<u>Proof of step 3:</u>

Suppose that Step 3 is not true. This means that we have the following inclusion:

$$\mathcal{A}_X(V)(\mathbf{R}) \bigcap \Omega(L) \subset V$$

where

$$\mathcal{A}_X(V)(\mathbf{R}) \stackrel{\text{def}}{=} \{\mathbb{P}^1(E)(\mathbb{R}) | x \in \mathbb{P}^1(E)(\mathbf{R}) \ \&\ \mathbf{P}^1(E)(\mathbb{R}) \bigcap V \neq \emptyset\} \subset \text{Grass}(3, b_2; \mathbf{R})$$

where $\dim_{\mathbf{R}} E = 3$ and $<,>_{|E} > O$.

We will show that this inclusion is absurd.

It was proved that $\mathcal{A}_X(V)(\mathbf{R})$ is a real anlytic subset in $\text{Grass}(3, b_2; \mathbf{R})$. More over we have

$$\mathcal{A}_X(V)(\mathbf{R}) = \mathcal{A}_X(V)(\mathbf{C})^{\tau}$$

On the other hand we have the follwing inclusions

(*) $\quad \mathcal{A}_X(\mathbf{C})^{\tau} = \mathcal{A}_X(\mathbb{R}) \subset \mathcal{A}_X(V)(\mathbf{C}) \subset \mathcal{A}_X(\mathbb{C})$

From $(*)$ we obtain that the complex analytic submanifold $\mathcal{A}_X(V)(\mathbf{C})$ in $\mathcal{A}_X(\mathbf{C})$ is locally defined by

$$(**) \qquad f_1(z^1,\dots,z^m)=o\ ,f_2(z^1,\dots,z^m)=O,\dots,f_k(z^1,\dots,z^m)=O$$

where

$$f_1(z^1,\dots,z^m)\ ,\ f_2(z^1,\dots,z^m),\dots,f_k(z^1,\dots,z^m)$$

are complex-analytic functions in

$$\mathcal{A}_X(\mathbf{C})$$

From

$$(*) \qquad \mathcal{A}_X(\mathbf{C})^\tau=\mathcal{A}_X(\mathbf{R})\subset\mathcal{A}_X(V)(\mathbf{C})\subset\mathcal{A}_X(\mathbf{C})$$

We obtain that

$$f_1(\mathrm{Re}z^1,\dots,\mathrm{Re}z^m)=O\ ,f_2(\mathrm{Re}z^1,\dots,\mathrm{Re}z^m)=O,\dots,f_k(\mathrm{Re}z^1,\dots,\mathrm{Re}z^m)=O$$

on $\mathcal{A}_X(\mathbf{C})^\tau=\mathcal{A}_X(\mathbb{R})$ and so on $\mathcal{A}_X(\mathbf{C})$. From here it follows that

$$f_1(z^1,\dots,z^m)\equiv O\ ,\ f_2(z^1,\dots,z^m)\equiv O,\dots,f_k(z^1,\dots,z^m)\equiv O$$

on $\mathcal{A}_X(\mathbf{C})$.

This is so since the following trivial fact is valid:

Trivial fact.

If $f(z^1,\dots,z^m)$ is a complex analytic function on $\mathbf{C}^m$ and

$$f(\mathrm{Re}z^1,\dots,\mathrm{Re}z^m)\equiv O$$

then

$$f(z^1,\dots,z^m)\equiv O \text{ on } \mathbf{C}^m.$$

See [13].

But this is a contrudiction since $\mathcal{A}_X(V)(\mathbf{C})$ is a proper analytic subset in $\mathcal{A}_X(\mathbf{C})$ defined locally by

$$f_1(z^1,\dots,z^m)\equiv O\ ,\ f_2(z^1,\dots,z^m)\equiv O,\dots,f_k(z^1,\dots,z^m)\equiv O$$

Step 3 is proved.

Q.E.D.

The end of the proof of Lemma 2.3.12.:

From Step 3 it follows that there exists a plane quadric

$$\mathbf{P}^1(E_y(w))(\mathbf{R}) \text{ in } \mathcal{A}_X(V)(\mathbf{R})$$

such that

$$\mathbf{P}^1(E_y(w))(\mathbf{R})\bigcap\Omega(L)=y\bigcup\bar{y}\notin V$$

where

$$E_y \text{ is the two dimensional subspace in } H^2(X,\mathbf{R})$$

that corresponds to the point $y \in \Omega(L)$ by 2.2.1. and $E_y(w)$ is a three dimensional subspace spanned by E_y and a vector $w \in H^2(X,\mathbf{R})$ is such that

$$<w,w> > O \text{ and } <w,E_y> = O$$

If w is proportional to L then our Lemma is proved.

Suppose that $w \neq L$.

Let us consider the four dimensional subspace in $H^2(X,\mathbf{R})$ spanned by E_y and L. Let us denote this four dimensional subspace by $\mathcal{S}$. Clearly

$$E_y(L) \subset \mathcal{S} \text{ and } E_z(L) \subset \mathcal{S}$$

and

(**) $$< , >_{|E_y(L)} > O \text{ and } < , >_{|E_z(L)} > O$$

So

$$E_y(L) \cap E_z(L) = E_t \ \& \ \dim_{\mathbf{R}} E_t = 2$$

From (**) and 2.2.1. it follows that

$$< , >_{|E_t} > O$$

So again using 2.2.1. we get that

$$\mathbb{P}^1(E_z(L))(\mathbf{R}) \cap \mathbb{P}^1(E_y(L))(\mathbf{R}) = t \cup \bar{t}$$

Q.E.D.

#2.4. CALABI-YAU METRICS AND ISOMETRIC DEFORMATIONS.

Definition 2.4.1.

A Kähler metric $g_{\alpha,\overline{\beta}}$ on a Hyper-Kählerian manifold X will be called Calabi-Yau metric if

$$\text{Ricci } (g_{\alpha,\overline{\beta}})=\overline{\partial}\partial \log \det (g_{\alpha,\overline{\beta}})\equiv O$$

The existence of a Calabi-Yau metric follows from the deep work of Yau [22]. In the polarization class L, there exists a unique Calabi-Yau metric $g_{\alpha,\overline{\beta}}$ such that

$$[g_{\alpha,\overline{\beta}}]\equiv L$$

Let us fix the Calabi-Yau $(g_{\alpha,\overline{\beta}})$ metric in L. This metric induces a covariant differentiation on

$$\wedge^2(T^*X\otimes\mathbf{C})$$

We will denote it by ∇.

Lemma 2.4.2. $\nabla\omega_X(2,O)=\overline{\nabla\omega_X(2,O)}\equiv O$

Proof: See [1].

Q.E.D.

Corollary 2.4.2.1. If $\omega_X(2,O)=\text{Re}\omega_X(2,O)+i\text{Im}\omega_X(2,O)$, then

$$\nabla\text{Re}\omega_X(2,O)= \nabla\text{Im}\omega_X(2,O)=O$$

(2.4.3.) From the definition of a Kähler metric, it follows that

$$\nabla(\sqrt{-1}\sum g_{\alpha,\overline{\beta}}dz^\alpha\wedge\overline{dz^\beta})=\nabla(\text{Im } g_{\alpha,\overline{\beta}})=O\ .$$

(2.4.4.) $\text{Re}\omega_X(2,O)$, $\text{Im}\omega_X(2,O)$ and $\text{Im}(g_{\alpha,\overline{\beta}})$ define a three-dimensional subspace

$$E_X(L)\subset\Gamma(X,\wedge^2T^*X\otimes\mathbf{C})$$

$E_X(L)$ is spanned by three forms parallel with the respect to the connection induced by the Calabi-Yau metric $(g_{\alpha,\overline{\beta}})$.

Since

$$\text{Re}\omega_X(2,O),\ \text{Im}\omega_X(2,O)\ \&\ \text{Im } g_{\alpha,\overline{\beta}}$$

are harmonic forms, then

(2.4.4.1.) $$E_X(L)\subset H^2(X,\mathbf{R})$$

Proposition 2.4.4.2. $\mathrm{Re}\omega_X(2,O)$, $\mathrm{Im}\omega_X(2,O)$ & $\mathrm{Im}\ g_{\alpha,\overline{\beta}}$ is an orthonormal basis in

$$E_X(L)\subset\Gamma(X,\wedge^2T^*X\otimes\mathbb{C})$$

Proof: Since

$$H^o(X,\Omega^2)\simeq\mathbb{C}\omega_X(2,O)$$

and the definition of < , > we may suppose that

$$<\mathrm{Re}\omega_X(2,O),\mathrm{Re}\omega_X(2,O>=<\mathrm{Im}\omega_X(2,O),\mathrm{Im}\omega_X(2,O)>=<\mathrm{Im}\ g_{\alpha,\overline{\beta}},\mathrm{Im}\ g_{\alpha,\overline{\beta}}>=1$$

From the definition of < , > and comparing the types of the forms it follows that

$$<\mathrm{Re}\omega_X(2,O),\mathrm{Im}\omega_X(2,O>=<\mathrm{Im}\omega_X(2,O),\mathrm{Im}g_{\alpha,\overline{\beta}}>=<\mathrm{Im}\omega_X(2,O),\mathrm{Im}\ g_{\alpha,\overline{\beta}}>=O$$

This proves (2.4.4.2)

Q.E.D.

(2.4.5.) Isometric deformations.

Let us define γ in the following way:

$$\gamma \overset{\mathrm{def}}{=} a\mathrm{Re}\omega_X(2,O)+b\mathrm{Im}\omega_X(2,O)+c\mathrm{Im}\ g_{\alpha,\overline{\beta}}$$

where

$$a,\ b\ \&\ c\in\mathbb{R} \text{ and } a^2+b^2+c^2=1$$

Since

$$\gamma\in E_X(L)\subset\Gamma(X,\wedge^2T^*X\otimes\mathbb{C})$$

then

(*) $$\nabla\gamma=O$$

Locally γ can be written in the following way:

$$\gamma=\sum\gamma_{\mu,\nu}dx^\mu\wedge dx^\nu$$

If

$$\sum g_{\tau,\nu}dx^\tau\wedge dx^\nu$$

is the Riemannian Ricci flat metric on X defined by the Calabi-Yau metric $(g_{\alpha,\overline{\beta}})$ on X, then we will define the complex structure operator $J(\gamma)$ in the following manner:

(2.4.5.1.) $$\left(J(\gamma)^\alpha_\beta\right)=\left(\sum_\tau g^{\alpha\tau}\gamma_{\tau\beta}\right)\in\Gamma(X,T^*\otimes T)$$

Clearly

$$\nabla(J(\gamma))=O$$

LEMMA 2.4.5.2.

a) $J(\gamma)$ defines a new integrable complex structure on X.

b) γ is an imaginary part of a Calabi-Yau metric with respect to the new complex structure $J(\gamma)$ and this metric defined by γ is equivelent as a Riemannian metric to the Calabi-Yau metric $g_{\alpha,\overline{\beta}}$, that we started with.

Proof: Since

$$\nabla J(\gamma)=O$$

if we prove that in each point $x \in X$ we have

$$J(\gamma) o J(\gamma) = -id$$

then $J(\gamma)$ will define an almost complex structure globally on X. Then we will need to show that $J(\gamma)$ is an integrable one.

(2.4.5.2.1.) $J(\gamma) o J(\gamma) = -id$ at $\forall x \in X$.

Proof:

Since $\omega_X(2,O)$ is a parallel with respect to the connection ∇ of the Ricci flat metric, it follows that the holonomy group of the Calabi-Yau metric is Sp(n). This means that globally there exists

$$j \in \Gamma(X, T^* \otimes T)$$

such that

$$\nabla j = O \ \& \ joj = \text{-id} \text{ (j defines a quaternionic structure on X)}$$

and we have at each point x

$$T^{*\,1,O}_{x,X} \simeq \mathbf{H}^n = \mathbf{C}^n + \mathbf{C}^n j$$

This splitting is global.

On the other hand the Calabi-Yau metric on

$$T^{*\,1,O}_{x,X} = \mathbb{H}^n = \mathbb{R}^n + \mathbf{R}^n i + \mathbf{R}^n j + \mathbf{R}^n k, \text{ where } k = ioj$$

is induced by the standart scalar product on $\mathbb{H}^n$, so from here it follows that we can find an orthonormal quaternionic basis in

$$T^{*\,1,O}_{x,X} \simeq \mathbf{H}^n = \mathbf{C}^n + \mathbf{C}^n j$$

$$h^1=e^1+e^{1+n}j,\ldots\ldots,h^n=e^n+e^{2n}j$$

Then at a point $x\in X$ we have:

(*) $$\mathrm{Im}(g_{\alpha,\overline{\beta}})|_{T^{*\,1,O}_{x,X}}=\sqrt{-1}\sum_{i=1}^{n} e^i\wedge\overline{e}^i$$

(**) $$\omega_X(2,O)|_{T^{*\,1,O}_{x,X}}=e^1\wedge e^{1+n}+\ldots+e^n\wedge e^{2n}=\sum_{i=n}^{n} e^i\wedge e^{i+n}$$

Let us denote by I the original complex structure on X, then

$$J(\mathrm{Im}(g_{\alpha,\overline{\beta}}))=I$$

Let

$$J=J(\mathrm{Re}\omega_X(2,O))\ \&\ K=J(\mathrm{Im}\omega_X(2,O))$$

From (*) and (**) we get:

(***) $$I^2=J^2=K^2=-\mathrm{id},\ IJ+JI=IK+KI=JK+KJ=O$$

Let me remind You that

$$\gamma \overset{\mathrm{def}}{=} a\mathrm{Re}\omega_X(2,O)+b\mathrm{Im}\omega_X(2,O)+c\mathrm{Im}\ g_{\alpha,\overline{\beta}}$$

and

$$a^2+b^2+c^2=1;\ a,\ b\ \&\ c\in\mathbf{R}$$

From (***) we get

$$J(\gamma)oJ(\gamma)=a^2IoI+b^2JoJ+c^2KoK=(a^2+b^2+c^2)(-\mathrm{id})=-\mathrm{id}$$

So we have proved that the $J(\gamma)$ is an almost complex structure on X.

Proof of the fact that $J(\gamma)$ is an integrable complex structure.

Proof: The proof is based on the following fact:

ANDREOTTI-WEIL REMARK.

Let ω be a n-complex-valued C^∞ form in a neighbothood of a point $x\in X$, where

$$\dim_{\mathbf{R}} X=2n$$

Let ω satisfy:

a) $P(\omega)=O$, where P are the Plücker relations. This means that at each point $x\in X$

$$\omega|_{x\in X}=\zeta^1\wedge\ldots\wedge\zeta^n \quad \zeta^i\in T^*_{x,X}\otimes\mathbf{C}$$

so ω defines a subspace

$$T^{1,O}_X\subset T^*_{x,X}\otimes\mathbf{C}$$

at $\forall x\in X$.

b) $\omega\wedge\overline{\omega}=f(x^1,...x^{2n})dx^1\wedge..\wedge dx^{2n}$, where $f(x^1,...x^{2n})dx^1\wedge..\wedge dx^{2n}>O$ in U. This means that

$$T_x^{1,O}\dot{+}\overline{T_x^{1,O}}=T^*_{x,X}\otimes\mathbf{C}$$

in U

c) $d\omega=O$

a) and b) means that ω defines an almost complex-structure in U. c) means that this almost complex structure is an integrable one.

So in order to use the Andreotti-Weil remark we need to construct the form ω, that satisfies a), b) and c). So first we will construct a globally defined form $\omega_{J(\gamma)}(2,O)$ of type (2,O) with respect to $J(\gamma)$ and then we will prove that

$$\omega_{J(\gamma)}(2n,O)=\wedge^n\ \omega_{J(\gamma)}(2,O)$$

fulfills the conditions of Andreotti-Weil remark.

Construction of $\omega_{J(\gamma)}(2,0)$.

Let

$$(\alpha,\beta,\gamma)$$

be an orthonormal basis of

$$E_X(L)\subset\Gamma(X,\wedge^2T^*X)$$

with respect to the scalar product induced by Calabi-Yau metric on $\Gamma(X,\wedge^2T^*X)$. We suppose that

$$(\alpha,\beta,\gamma)$$

define the same orientation of $E_X(L)$ as

$$\{Re\omega_X(2,O),Im\omega_X(2,O),Im(g_{\alpha,\overline{\beta}})\}$$

(2.4.5.2.1.) $$\omega_{J(\gamma)}(2,O)=\alpha+i\beta$$

Proposition 2.4.5.2.2.

$\omega_{J(\gamma)}(2,O)=\alpha+i\beta$ is a form of type (2,O) with respect to the almost complex structure on X defined by $J(\gamma)$.

Proof:

Since both $\omega_{J(\gamma)}(2,O)=\alpha+i\beta$ & $J(\gamma)$

are parallel with respect to the connection ∇. We need to check that

$$\omega_{J(\gamma)}(2,O)=\alpha+i\beta$$

is a form of type (2,O) at one point $x\in X$ with respect to $J(\gamma)$. We will define an action of Sp(1) on T^*X. Remember that the holonomy group of the Calabi-Yau metric $(g_{\alpha,\overline{\beta}})$ is Sp(n), so we can introduce on $T^*_{x,X}$ a quaternionic structure, i.e.

$$T^*_{x,X}=\mathbf{C}^n+\mathbf{C}^n j=\mathbf{H}^n \text{ (}\mathbf{H}\text{ is the quatenionic field)}$$

The Calabi-Yau metric $(g_{\alpha,\overline{\beta}})$ induces the standard quaternionic scalar product.

Let

$$h^1=e^1+e^{n+1}j,\ldots,h^n=e^n+e^{2n}j$$

be a quaternionic orthonormal basis in $\mathbf{H}^n$, then the restriction of Calabi-Yau metric on $T^*_{x,X}$ is obtained from the following quaternionic product in $\mathbf{H}^n$. Let

$$u=\sum h^i u_i \;\&\; v=\sum h^i v_i$$

then

$$<u,v>=\sum u_i\overline{v}_i$$

We can identify

$$Sp(1)=\{A\in\mathbf{H}|\ A\overline{A}=1\}$$

Then Sp(1) acts on $\mathbf{H}^n$ in the following way:

Let $A\in Sp(1)$ and let

$$u=\sum h^i u_i$$

then

$$Au=\sum h^i u_i A$$

Clearly $Sp(1)\subset Sp(n)$; i.e. this action of Sp(1) preserves the quaternionic scalar product

$$<u,v>=\sum u_i\overline{v}_i$$

The following remark is an easy exercise.

Remark. Sp(1) induces an action on $\wedge^2 T^*_{x,X}$ and

$$E_X(L)\subset\Gamma(X,\wedge^2 T^*_{x,X})$$

is invariant under the induced action of Sp(1). Moreover Sp(1) induces the standard SO(3) action on $E_X(L)$ with respect to the Euclidean metric on $E_X(L)$ induced by the orthonormal basis

$$\{\mathrm{Re}\omega_X(2,O),\mathrm{Im}\omega_X(2,O),\mathrm{Im}(g_{\alpha,\overline{\beta}})\}$$

From this remark it follows immediately that there exists

$$A\in Sp(1)\subset Sp(n)$$

such that:

(*) $$A(\mathrm{Re}\omega_X(2,O))=\alpha,\ A(\mathrm{Im}\omega_X(2,O))=\beta\ \&\ A(\mathrm{Im}(g_{\alpha,\overline{\beta}}))=\gamma$$

So

$$A(\omega_X(2,O))=\omega_{J(\gamma)}(2,O)$$

On the other hand from the definition of $J(\gamma)$ we see immediately that

(**) $$J(\gamma)=AIA^t$$

So from (*) and (**) we get that $\omega_{J(\gamma)}(2,O)$ is a form of type (2,O) with respect to the almost complex structure $J(\gamma)$. This is so since $\wedge^{2,O}$ is a subspace of vectors of type (2,O) in

$$\wedge^2(T^*_{x,X}\otimes C)$$

with respect to the complex structure defined by I and if

$$J(\gamma)=AIA^t,$$

then

$$A(\wedge^{2,O})\subset\wedge^2(T^*_{x,X}\otimes C)$$

and if

$$\omega\in\wedge^2(T^*_{x,X}\otimes C),$$

is of type (2,O) with respect to I, then $A(\omega)$ is of type (2,0) with respect to $J(\gamma)=AIA^t$.

Q.E.D.

Proof of 2.4.5.2.b): If

$$\gamma=\sum\gamma_{\mu,\nu}dx^{\mu}\wedge dx^{\nu}$$

then γ defines a scalar product in the following way on $T^*_{x,X}$:

Let

$$u=\sum u_{\alpha}dx^{\alpha}\ \text{ and }\ v=\sum v_{\beta}dx^{\beta}$$

then

$$<u,v>_{\gamma}=\sum u_{\alpha}\gamma_{\alpha\beta}v_{\beta}$$

If we prove that for

$$\forall u \in T^*_{x,X}$$

we have:

$$<J(\gamma)u,u>_\gamma>O$$

then it will follow that γ is an imaginary part of a Kähler metric on X with respect to $J(\gamma)$, this follows from the definition of a Kähler metric and since

$$d\gamma=O$$

We may suppose that at $\forall x \in X$

$$g_{\alpha,\overline{\beta}}=\delta_{\alpha\overline{\beta}}$$

then

$$J(\gamma)^{\alpha}_{\beta}=\gamma_{\alpha\beta},\ \gamma_{\alpha\beta}=-\gamma_{\beta\alpha}\ \&\ J(\gamma)oJ(\gamma)=\text{-id} \Rightarrow \sum_{\beta}\gamma_{\alpha\beta}\gamma_{\beta\nu}=-\delta_{\alpha\nu}$$

If

$$u=\sum u_\alpha dx^\alpha$$

then

$$<J(\gamma)u,u>_\gamma=\sum\gamma_{\mu\alpha}u_\alpha\gamma_{\mu\beta}u_\beta=\sum u_\alpha(-\gamma_{\alpha\mu})\gamma_{\mu\beta}u_\beta=$$

$$\sum u_\alpha(\delta_{\alpha\beta})u_\beta=\sum u^2>O$$

The last calculation shows that γ is an imagunary part of a Kähler metric on X with respect to the complex structure $J(\gamma)$ and this new Kähler metric is equivelent as Riemann metric to the Calabi-Yau metric we started with.

Q.E.D.

Definition 2.4.5.3.

From Lemma 2.4.5. it follows that every oriented two dimensional submanifold $E \subset E_X(L) \subset \Gamma(X,\wedge^2 T^*X)$ defines a new complex structure on X. Since all oriented planes in three dimensional space is parametrized by the two dimensional sphere S^2 we obtain a family of Hyper-Kählerian manifolds

$$\pi: \mathfrak{H} \rightarrow S^2$$

Such family we will call a family of isometric deformations with respect to the Calabi-Yau metric $g_{\alpha,\overline{\beta}}$.

Proposition 2.4.5.4. Let

$$\pi: \mathfrak{S} \to S^2$$

be a family of marked isometric deformations with respect to the Calabi-Yau metric

$$g_{\alpha,\overline{\beta}} \text{ such that } [\mathrm{Im}\ g_{\alpha,\overline{\beta}}]=L,$$

then

$$p(S^2)=\mathbf{P}^1(E_X(L))(\mathbb{R})\subset\Omega$$

where $E_X(L)$ is the three dimensional space spanned by

$$\mathrm{Re}\omega_X(2,O),\ \mathrm{Im}\omega_X(2,O),\ \mathrm{Im}g_{\alpha,\overline{\beta}}$$

p is the period map

Proof:

Every point $t\in S^2$ defines an oriented two plane $E_t\subset E_X(L)$ in the following manner

$$E_t\equiv\{\mathrm{Re}\omega_t(2,O),\mathrm{Im}\omega_t(2,0)\}$$

where

$$\{\mathrm{Re}\omega_t(2,O),\mathrm{Im}\omega_t(2,O)\}$$

is an orthonormal basis in E_t and

$$\omega_t(2,O)=\mathrm{Re}\omega_t(2,O)+i\mathrm{Im}\omega_t(2,O)$$

Now our proposition follows from 2.2.1.

Q.E.D.

#2.5. HILBERT SHEME OF HYPER-KAHLERIAN MANIFOLDS.

Let X be a projective Hyper-Kählerian manifold embedded in $\mathbf{P}^N$. The Fubbini-Schtudy metric on $\mathbf{P}^N$ in a natural way defines a class of a polarazation L.

Definition 2.5.1.

Let $\overline{\mathrm{Hilb}}_{X/\mathbf{P}^N}$ be the irreducible component of the Hilbert scheme that contains X.

Let $\mathrm{Hilb}_{X/\mathbf{P}^N}$ be the the subscheme of $\overline{\mathrm{Hilb}}_{X/\mathbf{P}^N}$ that parametrizes all non-singular Hyper-Kählerian manifolds in the flat family:

$$\bar{\mathfrak{Y}} \to \overline{\mathrm{Hilb}}_{X/\mathbf{P}^N}$$

Remark.

Grothendieck proved in [SGA] that $\mathrm{Hilb}_{X/\mathbf{P}^N}$ is a quasi-projective algebraic space.

Proposition 2.5.2. $\mathrm{Hilb}_{X/\mathbf{P}^N}$ is a non-singular manifold.

Proof: Bogomolov proved in [4] that the Kuranishi family $\pi:\mathfrak{X}\to\mathcal{K}$ has a non-singular base $\mathcal{K}$ and

$$\dim_{\mathbf{C}}\mathcal{K}=\dim_{\mathbf{C}}H^1(X,\Theta_X)$$

From the local Torelli theorem (See [3]) it follows that we may suppose that $\mathcal{K}\subset\Omega\subset\mathbf{P}(H^2(X,\mathbf{C}))$. Let L be a fixed class in $H^2(X,\mathbf{Z})$ and let

$$\mathcal{K}_L=\{t\in\mathcal{K} \mid L \text{ is of type } (1,1) \text{ on } X_t=\pi^{-1}(t)\}$$

It is an easy exercise to see that $\mathcal{K}_L$ can be defined also in the following manner:

$$\mathcal{K}_L=\Omega\cap H_L,\ \text{where } H_L=\{u\in\Omega \mid \langle u,L\rangle=0\}$$

i.e.

$$\dim_{\mathbf{C}}\mathcal{K}=h^{1,1}-1$$

On the other hand we may consider $\mathcal{K}_L$ to be a maximal local slice to the orbits of the action of subgroup $G\subset\mathbf{PGL}(N)$ on $\tilde{H}_{X/\mathbf{P}^N}$. where $\tilde{H}_{X/\mathbf{P}^N}$ is the universal covering of $\mathrm{Hilb}_{X/\mathbf{P}^N}$.

REMARK. 1)$\mathbf{PGL}(N)$ that preserve the fixed marking of the family $\tilde{\mathfrak{Y}}\to\tilde{H}_{X/\mathbf{P}^N}$, where this family is just the pullback of the standart family

$$\mathfrak{Y}\to\mathrm{Hilb}_{X/\mathbf{P}^N}$$

2) The action of G on $\tilde{H}_{X/\mathbf{P}^N}$ is defined correctly, since $\mathbf{PGL}(N)$ acts on $\mathrm{Hilb}_{X/\mathbf{P}^N}$ in a natural manner and so G acts on $\tilde{H}_{X/\mathbf{P}^N}$.

3)Notice that since $\pi_1(\tilde{H}_{X/\mathbf{P}^N})=0$ it is enouph to fix the marking of one of the fibres of $\tilde{\mathfrak{Y}} \to \tilde{H}_{X/\mathbf{P}^N}$

then the marking of all the fibres will be fixed.

From Lemma 3.1. it follows that if G_o is the group of biholomorphic automprphisms of a fixed Hyper-Kählerian manifold that preserve the marking of a fixed Hyper-Kähler manifold then G_o is the same group for all Hyper-Kähler manifold. It is clear that G_o is a normal subgroup in G and G/G_o acts freely on $\tilde{H}_{X/\mathbf{P}^N}$.

From here it follows that locally $\tilde{H}_{X/\mathbf{P}^N}$ is a product of $\mathfrak{K}_L \mathrm{x} \mathrm{Orb}(G/G_o)$ So $\tilde{H}_{X/\mathbf{P}^N}$ is a non-singular manifold. From here it follows that

$$\mathrm{Hilb}_{X/\mathbf{P}^N}$$

is a nonsingular quasi-projective manifold.

Q.E.D.

Definition 2.5.3.

$$\Gamma_L \overset{\mathrm{def}}{=} \{\gamma \in \mathrm{Aut}\ H^2(X,\mathbf{Z}) \mid <\gamma(u),\gamma(u)>=<u,u> \ \& \ \gamma(L)=L\}$$

Remark 2.5.4.

a) We can define correctly the period map, $p:\mathrm{Hilb}_{X/\mathbf{P}^N} \to \Omega(L)/\Gamma_L$

b) From general Baily-Borel compactification theory, it follows that $\Omega(L)/\Gamma_L$ is a quasi-projective manifold.

LEMMA 2.5.5. There exists an open Zariski set $\mathrm{Hilb}'_{X/\mathbf{P}^N} \subset \mathrm{Hilb}_{X/\mathbf{P}^N}$ such that

$$W \overset{\mathrm{def}}{=} p(\ \mathrm{Hilb}'_{X/\mathbf{P}^N}) = p(\mathrm{Hilb}_{X/\mathbf{P}^N})$$

is an open Zariski subset in $\Omega(L)/\Gamma_L$ and every point of W corresponds to an algebraic Hyper-Kähler maifold. (p is the period map)

Proof: From the famous Hironaka's "resolution of singularıries" Theorem it follows that we can find

$$\tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N}$$

such that

1) $\mathrm{Hilb}_{X/\mathbf{P}^N} \subset \tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N}$

2) $\tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N}$ is a projective manifold obtained from $\overline{\mathrm{Hilb}_{X/\mathbf{P}^N}}$ by successive blows up on non-singular submanifolds.

3) $\tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N} \setminus \mathrm{Hilb}_{X/\mathbf{P}^N}$ is a divisor with normal crossings.

4) $\tilde{\mathfrak{Y}} \rightarrow \tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N}$ is a flat family obtained by the pull back of the family

$$\mathfrak{Y} \rightarrow \mathrm{Hilb}_{X/\mathbf{P}^N} \text{ on } \tilde{\mathrm{Hilb}}_{X/P^N}$$

Borel proved in [5] that the period map: $p:\mathrm{Hilb}_{X/\mathbf{P}^n} \rightarrow \Omega(L)/\Gamma_L$ can be prolonged to a holomorphic map:

$$\bar{p}:\tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N} \rightarrow \overline{\Omega(L)/\Gamma_L}$$

Proposition 2.5.5.1. The map $\bar{p}:\tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N} \rightarrow \overline{\Omega(L)/\Gamma_L}$ is a surjective map.

Proof of 2.5.5.1.: In Proposition 2.5.3. we proved that locally $\mathrm{Hilb}_{X/\mathbf{P}^N}$ is a product of

$$\mathfrak{K}_L x G/G_o$$

where over $\mathfrak{K}_L$ we have a family of marked polarized Hyper-Kählerian manifolds:

$$\pi: \mathfrak{S} \rightarrow \mathfrak{K}_L \subset \mathfrak{K}\text{(the base of the Kuranishi family)}$$

and from local Torelli Theorem we know that

(*) $$\mathfrak{K}_L \subset \Omega(L) \ \&\ \dim_{\mathbf{C}} \mathfrak{K}_L = \dim_{\mathbf{C}} \Omega(L)$$

From (*) and the fact that the morphism between two projective varieties is proper it follows that $p(\tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N})$ is a proper algebraic subsvariety in the projective algebraic variety

$$\overline{\Omega(L)/\Gamma_L}$$

with the same dimension, so $\bar{p}(\tilde{\mathrm{Hilb}}_{X/\mathbf{P}^N}) \equiv \overline{\Omega(L)/\Gamma_L}$

Q.E.D.

Since the map

$$\tilde{p}:\tilde{\text{Hilb}}_{X/\mathbf{P}^N}\to \overline{\Omega(L)/\Gamma_L}$$

is a proper surjective map, then

$$\tilde{p}(\tilde{\text{Hilb}}_{X/\mathbf{P}^N}\setminus \text{Hilb}_{X/\mathbf{P}^N})=\overline{V}$$

is a proper algebraic submanifold in

$$\overline{\Omega(L)/\Gamma_L}.$$

Let

(2.5.5.2.) $$V \overset{\text{def}}{=} \overline{V}\setminus\left(\overline{V}\cap(\overline{\Omega(L)/\Gamma_L}\setminus\Omega(L)/\Gamma_L)\right)$$

Since

$$(\overline{\Omega(L)/\Gamma_L})\setminus(\Omega(L)/\Gamma_L)$$

is a proper algebraic submanifold in

$$\overline{\Omega(L)/\Gamma_L}$$

it follows that V is a proper algebraic submanifold in

$$\Omega(L)/\Gamma_L.$$

Let

(2.5.5.3.) $$W \overset{\text{def}}{=} (\Omega(L)/\Gamma_L)\setminus V$$

Let

$$\text{Hilb'}_{X/\mathbf{P}^N} \overset{\text{def}}{=} \text{Hilb}_{X/\mathbf{P}^N}\setminus(\text{Hilb}_{X/\mathbf{P}^N}\cap\tilde{p}^{-1}(V))$$

Then we will have p($\text{Hilb'}_{X/\mathbf{P}^N}$)=W. So $\text{Hilb'}_{X/\mathbf{P}^N}$ is what we need. On the other hand

from the definition of V' it follows immediately that p($\text{Hilb}_{X/\mathbf{P}^N}$)=W.

Q.E.D.

<u>Corollary 2.5.5.4.</u>

In $\Omega(L)$ there exists countable unions of complex analytic submanifolds V' such that every point

$$v\in\Omega(L)\setminus V' \overset{\text{def}}{=} W \ \&\ W \text{ is an open subset in } \Omega(L)$$

corresponds to a marked algebraic polarazid Hyper-Kählerian manifold X_v.

<u>Proof of 2.5.5.4.:</u>

Let $\tau:\Omega(L)\to\Omega(L)/\Gamma_L$ and $V'=\tau^{-1}(V)$

where we shall remind that V is a proper subspace in $\Omega(L)/\Gamma_L$ defined as follows:

(2.5.5.2.) $$V \overset{\text{def}}{=} \overline{V}\setminus\left(\overline{V}\cap(\overline{\Omega(L)/\Gamma_L}\setminus\Omega(L)/\Gamma_L)\right)$$

Since Γ_L consists of countable elements, then from the definition of V' and τ we get that V' consists of countable number of proper subspaces in $\Omega(L)$.

Q.E.D.

Corollary 2.5.5.5. Let $\tilde{H}_{X/\mathbf{P}^N}$ be the universal covering of $Hilb_{X/\mathbf{P}^N}$ and let $\pi:\tilde{\mathfrak{Y}}\to\tilde{H}_{X/\mathbf{P}^N}$ be the pullback of the family

$$\mathfrak{Y}\to Hilb_{X/\mathbf{P}^N}$$

then $p(\tilde{H}_{X/\mathbf{P}^N})=W=\Omega(L)\setminus V'$ where p is the period map and it is well defined since

$$\pi_1(\tilde{H}_{X/\mathbf{P}^N})=0$$

and if we mark one of the fibres of

$$\pi:\tilde{\mathfrak{Y}}\to\tilde{H}_{X/\mathbf{P}^N}$$

then we can assume that the whole family

$$\pi:\tilde{\mathfrak{Y}}\to\tilde{H}_{X/\mathbf{P}^N}$$

is a marked family of polarrized Hyper-Kählerian manifolds.

Proof: This follows immediately from the way we define W in $\Omega(L)$.

Q.E.D.

#2.6. THE PROOF OF THEOREM 2.

PROOF: The proof is based on several lemmas and on the THEOREM 2 which will be proved in #3. Let me remind the statement of THEOREM 2:

THEOREM 2.

There exists a universal family of marked polarized algebraic Hyper-Kählerian manifolds:

$$\mathfrak{X}_L \rightarrow \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

From THEOREM 2 it follows that we may consider the family of marked algebraic polarized Hyper-Kählerian manifolds

$$\pi^*:\mathfrak{S}^* \rightarrow D^*$$

that fulfills the conditions a) and b) of THEOREM 1 as a subfamily of

$$\mathfrak{X}_L \rightarrow \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

i.e.

$$\begin{array}{ccc} \mathfrak{S}^* & \subset & \mathfrak{X}_L \\ \downarrow & & \downarrow \\ D^* & \subset & \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})} \end{array}$$

LEMMA 2.6.1. There exists an open set U^o in $\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ such that

a) $D^* \subset U^o$

b) $p(U^o)=U\setminus\mathcal{A}$ in $\Omega(L)$, where $\mathcal{A}=U\bigcap V'$ is a complex analytic subspace in U & U is a policylinder, which contains $p(D^*)\subset\Omega(L)$. (V' was defined in #2.5.5.4. & p is the period map)

Proof: From a Theorem 9 proved by Ph. A. Griffiths in [13] it follows that we can prolong the period map

$$p^*:D^* \rightarrow \Omega(L)$$

to a map

$$p:D \rightarrow \Omega(L)$$

since the monodromy of the family

$$\pi^*:\mathfrak{S}^* \rightarrow D^*$$

is trivial.

2.6.1.1.

Let us denote by z the point $p(o)\in\Omega(L)$, where $o=D\setminus D^*$. We may suppose that $p(D^*)$ is a punctured disc in $\Omega(L)$. Let U be a policylinder containing $p(D)\subset\Omega(L)$. Let $\{U_i\}$ be a covering of $U\bigcap W$ by polycylineders. Remember that $W=\Omega(L)\backslash V'$, V' is an union of complex analytic subspaces in $\Omega(L)$. and every point of W corresponds to a marked algebraic Hyper-Kählerian manifolds. (See #2.5.5.4.) Even more for the period map

we have

$$p:\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}\rightarrow\Omega(L)$$

$$p^{-1}(W)=\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}.$$

(See 2.5.5.4. & #3.)

We may suppose that over each component of $p^{-1}U_i$ we have a family of Hyper-Kählerian manifolds.

Clearly $\{p^{-1}U_i\}$ is a covering of

$$D^*\subset\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

It is an obvious fact that if we glue all

$$\{p^{-1}U_i\}$$

along isomorphic marked polarized Hyper-Kählerian manifolds then we will get what we need, i.e. we will construct

$$U^o\subset\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

such that we have a family of marked Hyper-Kählerina manifolds over U^o

$$\pi:\mathfrak{X}^o\rightarrow U^o$$

and

$$p(U^o)=U\backslash\mathcal{A}$$

where U is a policylinder in $\Omega(L)$ and $\mathcal{A}=U\bigcap V'$ is a complex-analytic subset in U.

Remark 2.6.1.2. $\mathcal{A}$ defined as in Lemma 2.6.2. contains the point $z=p(o)\in D$.

(See Definition 2.6.1.1.)

Remark 2.6.1.3.

Over U^o we have a family of marked polarized Hyper-Kählerian manifolds $\mathfrak{X}^o\rightarrow U^o$ with a fixed class of polarization L.

Definition 2.6.2. Let $\mathfrak{Y}\rightarrow U^o xS^2$ be C^∞ family of isometric deformations with respect to the Ricci falat metric that corresponds to the class $L\in H^{1,1}(X_t,\mathbf{Z})$ for each $t\in U^o\subset\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ in the family $\mathfrak{S}^o \to U^o$.

Remark 2.6.2.1. The family $\mathcal{Y} \to U^o x S^2$ of isometric deformations with respect to the Ricci-flat metric that corresponds to the class L that corresponds to a very ample line bundle,is correctly defined, since the family $\mathfrak{S}^o \to U^o$ from which we obtained $\mathcal{Y} \to U^o x S^2$ is just the restriction of the universal family $\mathfrak{X}_L \to \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ which existence is proved in THEOREM 2.

Proposition 2.6.3. Let $\mathcal{U}=p(U^o x S^2)$ be the image of $U^o x S^2$ under the period map p, then every point $u \in \mathcal{U}$ is contained in an open set $U_u \subset \Omega$ such that $u \in U_u \subset \mathcal{U} \subset \Omega$ i.e. $\mathcal{U}$ is an open subset in Ω.

Proof: We will use the following Proposition:

Proposition 2.2.1.

There exists a one-to-one map ϕ between points of Ω and all two dimensional oriented vector subspaces $E \subset H^2(X,\mathbb{R})$ such that $< , >$ (defines in #2.1.) when restricted to E is positive, i.e. $<u,u> > O$ for $u \in E$.

Sublemma 2.6.3.1. A point $u \in \mathcal{U}=p(U^o x S^2) \subset \Omega$, where $U^o \subset \Omega(L)$ iff $E_u=\phi(u)$ and L spanned a three dimensional subspace $E_u(L)$ such that:

a) $< , >_{|E_u(L)} > O$ and b) $E_u(L)$ contains $E_x=\phi(x)$, where $x \in U^o \subset \Omega(L)$

Proof of the Sublemma:

From the definition of isometric deformations with respect to a Calabi-Yau metric with a fixed imaginary class L, Proposition 2.2.1. and the way we define the family $\mathcal{Y} \to U^o x S^2$. Sublemma 2.6.3.1. follows directly.

Q.E.D.

Now Proposition 2.6.3. follows immediately from Proposition 2.2.1., Sublemma 4.6.3.1. & the following fact:

Fact.

The condition that the restriction of $< , >$ on a two-dimensional subbspace in $H^2(X,\mathbb{Z})$ is strictly positive is an open condition in the Grassmanian of all two dimensional subspaces in $H^2(X,\mathbb{Z})$. The same is true for the three dimensinal subspaces in $H^2(X,\mathbb{Z})$.

The end of the proof of Proposition 2.6.3.

Indeed if $u \in \mathcal{U}$ then from Sulemma 2.6.3.1.$\Rightarrow$that E_u and L spanned a three dimensional subspace $E_u(L)$ in $H^2(X,\mathbb{Z})$ on which $< , >$ is strictly positive. From here and continuity arguments it follows that if u' is a point which is nearly enouph to the point $u\in\mathfrak{U}$, then $E_{u'}$ and L will span a three dimensional subapace $E_{u'}(L)$ in $H^2(X,\mathbb{Z})$ on which

$$< , >_{|E_{u'}(L)}>O$$

and $E_{u'}(L)$ will contain a two dimensional subspace

$E_x \perp$ to L, where $x\in U^o$ & $\phi(x)=E_x$.

Q.E.D.

Proposition 2.6.4. Let $\pi:\mathfrak{X}^*\to D^*$ be a family of marked Hyper-Kählerian manifolds that fulfills the conditions a) and b) of THEOREM 1, then

A) $\mathfrak{X}^*$ as a C^∞ manifold is diffeomorphic to XxD^*, where X is a Hyper-Kählerian manifold.

B) $\lim\limits_{\substack{u\to o\\ u\in D^*}} \omega_u(2,O)=\omega_z(2,O)$ exists and $\omega_z(2,O)$ is a complex non-degenerate C^∞ form on X.

Proof of Proposition 2.6.4.:

Let me remind You the following Definition:

Definition 2.6.1.1.

Let us denote by z the point $p(o)\in\Omega(L)$ where $o=D\backslash D^*$. From the following Lemma

(LEMMA 2.6.1.

There exists an open set U^o in $\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ such that

a) $D^*\subset U^o$

b) $p(U^o)=U\backslash\mathcal{A}$ in $\Omega(L)$, where $\mathcal{A}$ is a complex analytic subspace in U & U is a policyclinder, which contains $p(D^*)\subset\Omega(L)$. (remember p is the period map))

it follows that we may suppose that

$z\in U$ & $z\in V'\subset\Omega(L)$(for the defintion of V' see #2.5.5.4.)

From the definition of $\Omega(L)$ it follows that

$$< , >_{|E_z(L)}>O$$

where $\phi(z)=E_z$ and $E_z(L)$ is the 3-dimensional space in $H^2(X,\mathbb{Z})$ spanned by E_z and L

So we have a plane quadric

$\mathbf{P}^1(E_z(L))(\mathbf{R})\subset\Omega$

We can use now Lemma 2.3.12. Let me remind it:

Main LEMMA 2.3.12.

Let V be a complex analytic submanifold in $\Omega(L)$, where $\Omega(L)$ is defined as in 2.3.10.

Let $z \in V$.

Let $E_z(L)$ be defined as in Remark 2.3.11.

Let U be any open neighborhood of the point $z \in V$.

Then there exists a point

$$y \in U \ \& \ y \notin V$$

such that

$$\mathbb{P}^1(E_y(L))(\mathbb{R}) \bigcap \mathbb{P}^1(E_z(L))(\mathbb{R}) \neq \emptyset$$

i.e.

$$\mathbf{P}^1(E_y(L))(\mathbf{R}) \bigcap \mathbf{P}^1(E_z(L))(\mathbf{R}) = t \bigcup \bar{t}$$

and

$$\bar{t} \ \& \ t \notin \Omega(L).$$

from 2.3.12. it follows that there exists a point

$$y \in U^o$$

such that

$$\mathbf{P}^1(E_y(L))(\mathbf{R}) \bigcap \mathbb{P}^1(E_z(L))(\mathbb{R}) = t \bigcup \bar{t}$$

and

$$\bar{t} \ \& \ t \notin \Omega(L).$$

Definition 2.5.4.1. Let U_i be a policylinder in U with the following properties:

a) The closure $\overline{U}_i \subset U$ and $z \in \overline{U}_i$

b) $U_i \bigcap p(D^*) = D_i \neq \emptyset$ & D_i is a disk in $p(D^*)$.

c) $y \in U_i$, where y is defined by Lemma 2.3.12.

b) The closure of D_i is contained in p(D).

It is an obvious fact that such U_i exists. Even more from local Torelli Theorem we may suppose that $p^{-1}(U_i)$ is a disjoint union of policylinders in $\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$

(Remark 2.6.4.1.

From now on we will denote one of the components of $p^{-1}(U_i)$ again by U_i, where $p: U \subset \Omega(L))$ then we have a family of marked Hyper-Kählerian manifolds $\mathfrak{X}_i \to U_i$

Definition 2.6.4.2. Let $\mathfrak{Y}_i \to \mathfrak{U}_i$ be a family of marked polarized Hyper-Kählerian manifolds that corresponds to all isometric deformation with respect to the Ricci-flat metric that corresponds to the polarization class L of all fibers of the family

$$\mathfrak{S}_i \to U_i$$

which is subfamily of the universal family of marked polarized Hyper-Kählerian manifolds

$$\mathfrak{X}_L \to \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

Proposition 2.6.4.2. The period map p restricted to $\mathfrak{U}_i$(may be after shrinking U_i) is an embedding, i.e.

$$p:\mathfrak{U}_i \subset \Omega$$

Proof: By assumption we have:

$$U_i \subset \Omega(L) \subset \Omega$$

On the other hand from the definition of isometric deformation and Proposition 2.4.5.4. it follows dirctly that p restricted to $\mathfrak{U}_i$ is an embedding.

Q.E.D.

Remark 2.6.4.2.1. From now on we will suppose that $\mathfrak{U}_i$ is contained in Ω.

2.6.4.3. From the proof of Proposition 2.6.3. it follows that every point x of $\mathfrak{U}_i$ is contained in $\mathfrak{U}_i$ with on open neighborhood in Ω, i.e. $\mathfrak{U}_i$ is an open set in Ω.

2.6.4.4. Since

$y \in U_i$, where y is defined as in Lemma 2.3.12.

it folows from the defintion of $\mathfrak{U}_i$ and the isometric deformations that

$$\mathbf{P}^1(E_y(L))(\mathbf{R}) \subset \mathfrak{U}_i$$

where $\phi(y)=E_y$ & $E_y(L)$ is the subspace in $H^2(X,\mathbf{Z})$ spanned by E_y and L, ϕ is defined in 2.2.1.

2.6.4.5. From Lemma 2.3.12. and the Definition of $\mathfrak{U}_i$ it follows that

$$t \in \mathfrak{U}_i$$

where

$$\mathbf{P}^1(E_y(L))(\mathbf{R}) \cap \mathbf{P}^1(E_z(L))(\mathbf{R}) = t \cup \bar{t} \text{ (See Lemma 2.3.12.)}$$

2.6.4.6. Since

$y \in U_i$ & the Definition of U_i

we get that the point y corresponds to a marked Hyper-Kählerian manifold X_y. So every point $t \in \mathbf{P}^1(E_y(L))(\mathbf{R})$ corresponds to a marked Hyper-Kählerian manifold X_t.

2.6.4.7.a. Since $< , >_{|E_Z(L)} > 0$ then the group SO(3) acts on $E_Z(L)$ and this action is defined in the followin way:

First we fix an orthonormal basis, namely let $\{e_1, e_2\}$ be an orthonormal basis in E_Z and $e_3 = L$. Then if $A \in SO(3)$ and

$$v = \sum_{i=1}^{3} a_i e_i \in E_Z(L)$$

then

$$A(v) \stackrel{\text{def}}{=} \sum_{i=1}^{3} a_i A(e_i) \in E_Z(L)$$

2.6.4.7.b. We know from Lemma 2.3.12. that

$$E_Z(L) \cap E_Y(L) = E_t \Rightarrow E_t \subset E_Z(L)$$

Let

$$A \in SO(3)$$

and such that

$$A(E_Z) = E_t$$

2.6.4.8. For each

$$u \in U_i \cap D^* = D_i^* \subset \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

we will define on X_u a new comp,ex structure X_u^A in the following way:

Let

$$E_u(L) = \{\mathrm{Re}\,\omega_u(2,0),\ \mathrm{Im}\,\omega_u(2,0), g_{\alpha,\overline{\beta}}(u)\} \subset \Gamma(X, \wedge^2 T^*)$$

where $g_{\alpha,\overline{\beta}}(u)$ is the Calabi-Yau metric on X_u that corresponds to the class L.

From #2.1. & #2.4. we know that we mau suppose that

$$\{\mathrm{Re}\,\omega_u(2,0),\ \mathrm{Im}\,\omega_u(2,0), g_{\alpha,\overline{\beta}}(u)\}$$

is an orthonormal basis in $E_u(L)$, which is defined by $\omega_u(2,0)$ depending holomorphically on u.

From #2.4. we know that

$$A(E_u) = \{A(\mathrm{Re}\,\omega_u(2,0)), A(\mathrm{Im}\,\omega_u(2,0))\} \subset \Gamma(X, \wedge^2 T^*)$$

defines a new complex structure on X_u, which we will denote by X_u^A. So we get a new family:

$$\mathfrak{X}_i^A \to D_{i,A}, \text{ where } D_i \overset{\text{def}}{=} U_i \cap D$$

In the same way we can get a new family

$$\mathfrak{X}^{*A} \to D_A^*$$

from the family

$$\mathfrak{X}^* \to D^*$$

in the way described above.

Remark. The family $\mathfrak{X}_i^A \to D_{i,A}$ is not a holomorphic family but only a C^∞ family of complex structures over the disc $D_{i,A}$.

2.6.4.9. From the way we defined $\mathfrak{U}_i$ it follows that $D_{i,A} \subset \mathfrak{U}_i$ even more

2.6.4.9.a. Proposition.

If $u \to z$(convrging), where $u \in D_i$(remember that the closere of D_i contains z) then $A(u) \to A(z) = t$(converging), where

$$\mathbb{P}^1(E_y(L))(\mathbb{R}) \cap \mathbb{P}^1(E_z(L))(\mathbb{R}) = t \cup \bar{t}$$

and $A(u)$ corresponds in $\mathfrak{U}_i$ to the comlex structure X_u^A on X. Clearly $A(u) \in D_{i,A} \subset \mathfrak{U}_i$.

Proof: 2.6.4.9.a. follows from the way we define the family $\mathfrak{X}_i^A \to D_{i,A}$

Q.E.D.

Sublemma 2.6.4.10.

Let X_t be the the marked Hyper-Kählerian manifold that corresponds to the point

$$t \in \mathbb{P}^1(E_y(L))(\mathbb{R}) \cap \mathbb{P}^1(E_y(L))(\mathbb{R}) \subset \mathfrak{U}_i \subset \Omega$$

let $u \to z$, where $u \in D_i$, let $\omega_u^A(2,0)$ be the holomorphic two-form on X_u^A (where $A \in SO(3)$ and $A(E_z) = E_t$.) normalized in the following way $<\omega_u^A(2,0), \omega_u^A(2,0)> = 1$ then

$$\lim_{u \to z} \omega_u^A(2,O) = \omega_t(2,0)$$

where $A(z) = t$ & $u \in D_i$ and $\omega_t(2,0)$ is the holomorphic two form on X_t.

Proof: From 2.6.4.3. we know that every point $t \in \mathfrak{U}_i$ is contained in $\mathfrak{U}_i$ together with an open neighborhood in Ω. From the fact that we have a holomorphic family of marked Hyper-Kählerian manifolds over $\mathfrak{U}_i$, i.e.

$$\mathfrak{X}_i \to \mathfrak{U}_i$$

the fact that

$$\mathfrak{U}_i \subset \Omega \subset \mathbb{P}(H^2(X,\mathbb{C}))$$

and the normalization condition, i.e.

$$<\omega_u^A(2,0),\omega_u^A(2,0)>=1$$

we get that as cohomology classes .

$$\lim_{u\to z}[\omega_u^A(2,O)]=[\omega_t(2,0)]$$

where A(z)=t & $u\in D_i$ & and $\omega_t(2,0)$ is the normalized holomorphic two form on X_t.

From

$$\lim_{u\to z}[\omega_u^A(2,O)]=[\omega_t(2,0)]$$

we obtain that

$$\lim_{u\to z}\omega_u^A(2,O)=\omega_t(2,0)$$

This is so since $\dim_{\mathbb{C}} H^o(X_t,\Omega_t^2)=1$ for all $t\in\mathfrak{U}_i$ and we have a holomorphic family

$$\mathfrak{X}_i \to \mathfrak{U}_i$$

of marked Hyper-Kählerian manifolds and $u\to z$ in $\mathfrak{U}_i$. This follows from 2.6.4.3.

Q.E.D.

<u>Cor. 2.6.4.10.1.</u> The family

$$\mathfrak{X}^{*A} \to D_A^*$$

defined in 2.6.4.8. can be embedded in C^∞ family of non-singular marked Hyper-Kählerian manifolds over the disk D_A, where D_A is the closure of D_A^*, i.e. in $\mathfrak{X}^A \to D_A$.

<u>Proof of 2.6.4.10.1.:</u> Since

a) $D_{i,A}\subset D_A^*\subset\mathfrak{U}_i\subset\mathfrak{U}$

b) The closure of $D_{i,A}$ contains t=A(z) and is contained in D_A

c) Every point of $\mathfrak{U}_i$ is contained together with an open set in Ω

d) the closure $\overline{D}_A$ of the punctured disc D_A^* is contained in $\mathfrak{U}$

e) Over $\mathfrak{U}$ we have a holomorphic family $\mathfrak{X}\to\mathfrak{U}$ of marked Hyper-Kählerian manifolds and from 2.6.4.10. we get immediately that 2.6.4.10.1. is proved.

Q.E.D.

From 2.6.4.10.1. $\Rightarrow$that the family $\mathfrak{X}^A \to D_A$ as C^∞ manifold is diffeomorphic to DxX, where X is Hyper-Kählerian manifold . From here we obtain that $\mathfrak{X}^{*A} \to D_A^*$ is topologocally the same as $\mathfrak{X}^* \to D^*$. This follows directly from the Definition of Isometric deformations. So 2.6.4.A) is proved.

Q.E.D.

Proof of 2.6.4.B):

From Lemma 2.3.12. it follows that there exists a point $t \in \mathbb{P}^1(E_z(L))(\mathbb{R})$ such that

$$t \bigcup \bar{t} = \mathbb{P}^1(E_z(L))(\mathbb{R}) \bigcap \mathbb{P}^1(E_y(L))(\mathbb{R})$$

where

$$y \in U \setminus V'$$

and so y is the image under the period map of marked Hyper-Kählerian manifold with a class of polarization L. See Lemma 2.6.1.

Let

$$S_L \overset{\text{def}}{=} \{u \in \mathbb{P}^1(E_z(L))(\mathbb{R}) | E_u = \phi(u) \ \& \ E_u \text{ contains } L\}$$

(ϕ is defined in 2.2.1.)

It is easy to prove that as C^∞ manifold $S_L \doteq \{t \in \mathbb{C} | \ |t|=1\}$.

Sublemma 2.6.4.B.1. $t \bigcup \bar{t} \in S_L$, where $t \bigcup \bar{t} = \mathbb{P}^1(E_y(L))(\mathbb{R}) \bigcap \mathbb{P}^1(E_z(L))(\mathbb{R})$ for $\forall y \in U_i \subset \Omega(L)$.

Proof of Sublemma 2.6.4.B.1.: From the definition of

$$\mathbb{P}^1(E_y(L))(\mathbb{R}) \ \& \ \mathbb{P}^1(E_z(L))(\mathbb{R})$$

it follows that a point u

$$u \in \mathbb{P}^1(E_y(L))(\mathbb{R}) \bigcap \mathbb{P}^1(E_z(L))(\mathbb{R})$$

iff

$$\phi(u) = E_u = E_y(L) \bigcap E_z(L) (\text{ See 2.2.1.})$$

so $L \in E_u$ and Sublemma 2.6.4.B.1. follows from the definition of S_L.

Q.E.D.

Sublemma 2.6.4.B.2. There exist three points t_1, t_2 & t_3 on $S_L \subset \mathbb{P}^1(E_z(L))(\mathbb{R})$ such that:

a) t_1, t_2 & $t_3 \in \mathfrak{U}_i$ and t_1, t_2 & t_3 are three diiferent points.

b) t_1, t_2 & t_3 define three classes of cohomologies $[\omega_1(2,0)]$, $[\omega_2(2,0)]$ & $[\omega_3(2,0)]$ that are linearly independent in $E_z(L) \subset H^2(X, \mathbb{C})$.

Proof of Sublemma 2.6.4.B.2.:

From Lemma 2.3.12. and the definition of $\mathfrak{U}_i$ it follows that $\mathfrak{U}_i \bigcap S_L \neq \emptyset$, i.e. $t \in \mathfrak{U}_i \bigcap S_L$. From

2.6.4.3. we get that t is contained in $\mathfrak{U}_i$ together with an open set. From here 2.6.4.B.2.a. follows immediately.

Q.E.D.

In order to prove 2.6.4.B.2.b. we need to notice that if $t_1 \neq t_2$ in S_L then the classes of cohomologies $[\omega_1(2,0)]$ & $[\omega_2(2,0)]$ that are defined by t_1 & t_2 are linearly independent in $H^2(X,C)$. If $[\omega_3(2,0)]$ is a linear combination of $[\omega_1(2,0)]$ & $[\omega_2(2,0)]$, then

$$t_3 \in \mathbf{P}(E_{1,2}) \cap \mathbf{P}^1(E_Z(L))(\mathbf{R})$$

where $E_{1,2}$ is the plane in $H^2(X,C)$ spanned by $[\omega_1(2,0)]$ & $[\omega_2(2,0)]$(This is an easy exersice.) but

$$\mathbf{P}(E_{1,2}) \cap \mathbf{P}^1(E_Z(L))(\mathbf{R})$$

consists of at most of two points, since $\mathbf{P}^1(E_Z(L))(\mathbf{R})$ is plane quadric and so have deg 2. Now 2.6.4.B.2.b. follows from 2.6.4.B.2.a. and the fact that $S_L \cap \mathfrak{U}_i$ is an open set in S_L.(See 2.6.4.3.)

Q.E.D.

Remark. Since t_1, t_2 & $t_3 \in S_L \cap \mathfrak{U}_i$ so they corresponds to three marked Hyper-Kählerian manifolds Z_1,Z_2 & Z_3 that are in isometric families of three Hyper-Kählerina manifolds X_1, X_2 & X_3 with respect to the Calabi-Yau metric that correponds to L. X_1, X_2 & X_3 are fibres in

$$\mathfrak{X}_i \to U_i \subset \Omega(L)$$

over the points u_1, u_2 & $u_3 \in U_i$. This follows from the definition of $\mathfrak{U}_i$. See 2.6.4.

Definition. Let A, B & C∈SO(3) such that $A(E_Z)=E_{t_1}$, $B(E_Z)=E_{t_2}$ & $C(E_Z)=E_{t_3}$. From 2.6.4.7. we know that SO(3) acts on $E_Z(L)$.

Sublemma 2.6.4.B.3.

a) For each $u \in D^*$ the forms $\omega_u^A(2,0)$, $\omega_u^B(2,0)$ & $\omega_u^C(2,0)$ defined three linearly independent classes of cohomologies in $E_u(L) \subset H^2(X,C)$ where $E_u(L)$ is the three dimensional space spanned by $[\mathrm{Re}\,\omega_u(2,0)]$, $[\mathrm{Im}\,\omega_u(2,0)]$ & L. and this is an orthonormal basis for each $u \in D^*$ in $E_u(L)$.

b) There exists three constants a, b & c∈**C** such that $\omega_u(2,0)=a\omega_u^A(2,0)+b\omega_u^B(2,0)+c\omega_u^C(2,0)$ as a form for each $u \in D^*$. where A, B & C are fixed elements in SO(3) and $A(z)=t_1$, $B(z)=t_2$ & $C(z)=t_3$ and t_1, t_2 & t_3 are defined as in 2.6.4.B.2.

Proof of a): 2.6.4.B.3.

a) follows immediately from 2.6.4.B.2. and continuity arguments.

Q.E.D.

Proof of b):

From 2.6.4.B.3.a) it follows that there exists three constants a, b & c∈C such that

$$[\omega_Z(2,0)]=a[\omega_Z^A(2,0)]+b[\omega_Z^B(2,0)]+c[\omega_Z^C(2,0)]$$

Now we must prove that:

(*) $$\omega_u(a,b,c,) \overset{\text{def}}{=} a\omega_u^A(2,0)+b\omega_u^B(2,0)+c\omega_u^C(2,0$$

is a form on X_u for $\forall u\in D^*$.

Proof of(*): (*) follows from the way we define the action of SO(3) on

$$E_u(L)\subset\Gamma(X,\Lambda^2T^*X)$$

Let me remind You how we define this action. First we fixed an orthonormal basis that depends holomorphically on $u\in D^*$.

$$e_1(u)=\mathrm{Re}\,\omega_u(2,0),\ e_2=\mathrm{Im}\,\omega_u(2,0)\ \&\ e_3(u)=\mathrm{Im}(g_{\alpha,\overline{\beta}})$$

where

$$[e_3(u)]=L \text{ in } H^2(X,\mathbb{Z})$$

if A∈SO(3) and

$$v(u)=\sum_{i=1}^{3} a_i e_i(u)$$

then

$$A(v(u)) \overset{\text{def}}{=} \sum_{i=1}^{3} a_i A(e_i(u))$$

From the Definition of $\omega_u(a,b,c)$ it follows that

(I) $$\omega_u(a,b,c)\in E_u(L)\subset\Gamma(X,\Lambda^2T^*X\otimes\mathbb{C})$$

From the defintion of the isometric deformations we know that

(II) $\omega_u(a,b,c)$ is a holomorphic two-form on $X_u \Leftrightarrow <\omega_u(a,b,c),e_3(u)>=0$

So if we prove that

$$<\omega_u(a,b,c),e_3(u)>=0$$

then (*) will be proved. So we need to *prove (II).*

Proof of (II):

From the defintion of the isometric deformations it follows that we need to prove (II) on the level of cohomology classes, since $e_i(u)$ are parallel forms with respect to the metric $(g_{\alpha,\overline{\beta}})$. From the definition of $\omega_u(a,b,c)$ we get that

(F) $$\omega_u(a,b,c)=a\sum_{i=1}^{3} a_{1i}e_i(u)+b\sum_{i=1}^{3} b_{2i}e_i(u)+c\sum_{i=1}^{3} c_{3i}e_i(u)$$

From (F) follows that

(F1)

$$<\omega_u(a,b,c),e_3(u)>=a\sum_{i=1}^{3}a_{1i}<e_i(u),e_3(u)>+b\sum_{i=1}^{3}b_{2i}<e_i(u),e_3(u)>+c\sum_{i=1}^{3}c_{3i}<e_i(u),e_3(u)>$$

From the definition of the orthonormal basis we obtain that the formula (F1) does not depend on $u\in D$. From definition of the constants a, b, & c, i.e.

$$[\omega_z(2,0)]=a[\omega_z^A(2,0)]+b[\omega_z^B(2,0)]+c[\omega_z^C(2,0)]$$

and since

$$z\in\Omega(L)\Rightarrow<[\omega_z(2,0)],[e_3(z)]=L>=0$$

we obtain what we need, i.e.

$$<\omega_u(a,b,c),e_3(u)>=0$$

So (*) is proved and with this 2.6.4.B.3.b).

Q.E.D.

From 2.6.4.10. it follows that all the limits as C^∞ forms of the following forms exist

$$\lim_{u\to z}\omega_u^A(2,O)=\omega_{t_1}(2,0)$$

$$\lim_{u\to z}\omega_u^B(2,O)=\omega_{t_2}(2,0)$$

$$\lim_{u\to z}\omega_u^C(2,O)=\omega_{t_3}(2,0)$$

where

$A(z)=t_1$ & $u\in D_i$ & and $\omega_{t_1}(2,0)$ is the holomorphic two form on X_{t_1}.

$B(z)=t_2$ & $u\in D_i$ & and $\omega_{t_2}(2,0)$ is the holomorphic two form on X_{t_2}.

$C(z)=t_3$ & $u\in D_i$ & and $\omega_{t_3}(2,0)$ is the holomorphic two form on X_{t_3}.

From here and the fact that:

There exists three constants a, b & $c\in C$ such that

$$\omega_u(2,0)=a\omega_u^A(2,0)+b\omega_u^B(2,0)+c\omega_u^C(2,0)$$

as forms on X.

So we get that

$$\lim_{\substack{u \to o \\ u \in D^*}} \omega_u(2,0) = \omega_z(2,0)$$

exists as a C^∞ form on X.

In order to finish the proof of 2.6.4.B. we need to show that $\omega_z(2,0)$ is a non degenerate two-form on X. Clearly

$$d\omega_z(2,0)=0$$

From the Definiton of isometric deormations we get that for each $A \in SO(3)$ we have:

$$\wedge^n \omega_u(2,0) \wedge \overline{(\wedge^n \omega_u(2,0))} = vol(g_{\alpha,\overline{\beta}}) = \wedge^n \omega_u^A(2,0) \wedge \overline{(\wedge^n \omega_u^A(2,0))}$$

(III) $$\lim_{u \to z} \wedge^n \omega_u(2,0) \wedge \overline{(\wedge^n \omega_u(2,0))} = \lim_{u \to z} \wedge^n \omega_u^A(2,0) \wedge \overline{(\wedge^n \omega_u^A(2,0))}$$

Since

(IV) $$\lim_{u \to z} \omega_u^A(2,0) = \omega_z^A(2,0)$$

and $\omega_z^A(2,0)$ is a non-degenerate form defined by the Hyper-Kählerian manifold X_t, where $t=A(z)$.

From (III) & (IV) 2.6.4.B. follows directly.

Q.E.D.

In order to finish the proof of THEOREM 1. we need to use first the fact that the family $\mathfrak{X}^* \to D^*$ as C^∞ manifold is diffeomorpfic to D^*xX, where X is a Hyper-Kählerian manifold. So we can compactify topologocally the family $\mathfrak{X}^* \to D^*$ to DxX.

From the fact that

$$\lim_{u \to z} \omega_u(2,0) = \omega_z(2,0) \text{ exists}$$

and $\omega_z(2,0)$ is a non-degenerate form, we need to chek that the 2n-form $\wedge^n \omega_z(2,0)$ fulfills conditions a), b) & c) of the Andreotti-Weil remark. Clearly

$$d(\wedge^n \omega_z(2,0)) = O$$

$$\wedge^n \omega_z(2,0) \wedge \overline{(\wedge^n \omega_z(2,0))} > O$$

So b) & c) are filfilled.

Let P be the Plücker relations. Since they are polynomial relations, it follows that these are closed relations, i.e.

$$\lim_{u\to z} P(\wedge^n \omega_u(2,0)) = P(\lim_{u\to z} \omega_u(2,0)) = 0$$

So THEOREM 1. is proved.

Q.E.D.

#3. CONSTRUCTION OF THE MODULI SPACE.

The construction is based on the following Lemma:

LEMMA 3.1.

Let g be a holomorphic automorphism of X such that g^*=id, where

$$g^*:H^2(X,\mathbb{Z})\to :H^2(X,\mathbb{Z})$$

then g induces the identity map on the Kuranishi space of X, i.e on

$$\begin{array}{ccc} X & \subset & \mathfrak{X} \\ \downarrow & & \downarrow \\ O & \in & \mathcal{K} \end{array}$$

Proof: See [12].

Q.E.D.

LEMMA 3.2. Let

$$\begin{array}{ccc} X & \subset & \mathfrak{X} \\ \downarrow & & \downarrow \\ O & \in & \mathcal{K} \end{array}$$

be the

Kuranishi family of marked Hyper-Kähلerian manifolds,

$$(X,\gamma_1,\ldots,\gamma_{b_2})$$

then $\mathfrak{X}\to\mathcal{K}$ is the local universal family of marked Hyper-Kählerian manifolds,

$$(X,\gamma_1,\ldots,\gamma_{b_2})$$

Proof: We need to prove that if

$$\begin{array}{ccc} X_o & \to & Y \\ \downarrow & & \downarrow \\ x_o \in & & W \end{array}$$

is a family of marked Hyper-Kählerian manifolds, where W is a "small" policylinder, then there exists a unique map f of families:

$$\begin{array}{ccc} Y & \to & \mathfrak{X} \\ \downarrow & & \downarrow \\ W & \to & \mathcal{K} \end{array}$$

such that:

a) $f(x_o)=0$ and $f:X_o\to X_o$ is an isomorphism of marked Hyper-Kählerian manifolds.

b) the family $Y \to W$ is the pull back of the

Kuranishi family.

We know that the Kuranishi family is complete. See [14]. This means that there exists a holomorphic map f of families:

$$\begin{array}{ccc} Y & \to & \mathfrak{X} \\ \downarrow & & \downarrow \\ W & \to & \mathcal{K} \end{array}$$

such that:

a) $f(x_0)=0$ and $f:X_0 \to X_0$ is an isomorphism of marked Hyper-Kähₗerian manifolds.

b) the family $Y \to W$ is the pull back of the Kuranishi family.

Let g be a map between the families

$$Y \to W \text{ and } \mathfrak{X} \to \mathcal{K}$$

which fulfills the conditions a) and b) as for the map f, then from [14] it follows that we must have:

$$f(x)=\sigma(g(x)) \text{ for } x \in W$$

where σ is an isomorphism of the Kuranishi family such that

$$\sigma:X_0 \to X_0$$

preserve the marking, i.e.

$$\sigma^*=\mathrm{id} \text{ on } H^2(X,\mathbb{Z})$$

From 2.1. it follows that $\sigma=\mathrm{id}$ on $\mathcal{K}$, so

$$f \equiv g$$

Q.E.D.

#3.3. The construction of the moduli space.

Let

$$\begin{array}{ccc} X_0 & \to & \mathfrak{X} \\ \downarrow & & \downarrow \\ x_0 & \in & \mathcal{K} \end{array}$$

be the Kuranishi family of marked polarized algebraic Hyper-Kählerian manifolds,

$$(X_0,\gamma_1,\ldots,\gamma_{b_2};L)$$

where $\gamma_1,\ldots,\gamma_{b_2}$ is a fixed basis in $H_2(X,\mathbb{Z})$ and L is a fixed class of cohomology in $H^2(X,\mathbb{Z})$

corresponding to an imaginary part of a Hodge metric on X_0. From the local Torelli theorem it follows that we may consider $\mathfrak{K}$ as an open subset in

$$\Omega\subset\mathbb{P}(H^2(X,\mathbb{Z})\otimes\mathbb{C})$$

Let

$$H_L=\{x\in\mathbb{P}(H^2(X,\mathbb{Z})\otimes\mathbb{C})\mid <x,L>=0\}$$

From the local Torelli Theorem it follows that if we restrict the Kuranishi family

$$\mathfrak{S}\rightarrow\mathfrak{K}$$

to the family

$$\mathfrak{S}_L\rightarrow\mathfrak{K}_L,\text{ where }\mathfrak{K}_L=\mathfrak{K}\cap H_L$$

we will get the local universal family of all Hyper-Kählerian manifolds for which L is the imaginary part of a Hodge metric on X_t, for every $t\in\mathfrak{K}_L$.

From 3.1. it follows that we can glue all families

$$\left\{\mathfrak{S}_L\rightarrow\mathfrak{K}_L\right\}$$

by identifying isomorphic marked algebraic Hyper-Kählerian manifolds with fixed class of polarization L. In such a way we get an universal family

$$\mathfrak{X}_L\rightarrow\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

of marked polarized Hyper-Kählerian manifolds. This is so since if

$$\phi:X\rightarrow X$$

is a biholomorphic map of X such that

$$\phi^*(L)=L$$

then ϕ must be an isometry with respect to Calabi-Yau metric that corresponds to L and so for generic X $\phi^*=\mathrm{id}$ on $H^2(X,\mathbb{Z})$. See [6] and [11].

So we have proved the following THEOREM:

THEOREM 2.

There exists a universal family of marked polarized algebraic Hyper-Kählerian manifolds:

$$\mathfrak{X}_L\rightarrow\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

REMARK.

There is another way of constructing the universal family of marked polarized algebraic Hyper-Kählerian manifolds:

Namely let $\tilde{H}_{X/\mathbf{P}^N}$ be the universal covering of $\mathrm{Hilb}_{X/\mathbf{P}^N}$ and let

$$\pi:\tilde{\mathfrak{Y}}\to\tilde{H}_{X/\mathbf{P}^N}$$

be the pullback of the family

$$\pi:\tilde{\mathfrak{Y}}\to\tilde{H}_{X/\mathbf{P}^N}$$

Then it is easy to see that

G/G_0 acts on $\tilde{H}_{X/\mathbf{P}^N}$, where G and G_0 are defined in #2.5.

It is not very difficult to prove that this action is a free and proper using a Theorem by Mumford and Mutsusaka. See [25]. So by a general Theorem due to Palais we get that

$$\tilde{H}_{X/\mathbf{P}^N}/(G/G_0)\cong\mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$. See [26].

From this we get the following fact:

Fact

$$p(\mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})})=\Omega(L)\backslash V'=W$$

for the Definitions of V' and W see #2.5.5.4.

REFERENCES

1. A. Beauville, "Varriétés Kählericnnes dont la primiére class de Chern ext nulle," J. of Diff. Geometry 18(1983).

2. A. Beauville, "Some remarks of Kähler manifolds with $c_1=0$," Progress in Mathematics vol. 39, Birkhauser Inc., 1-26.

3. F. Bogomolov, "On decomposition of Kähler manifolds with trivial canonical class," Math. USSR Sbornik 22(1974), 580-583.

4. F. Bogomolov, "Kählerian varieties with trivial canonical class," Preprint, I.H.E.S. February 1981.

5. A. Borel, "Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem," J. of Diff. Geometry (1972), 543-560.

6. D. Burns and M. Rapoport, "On the Torelli problem for Kählerian K3 surfaces," Ann. Sci. ENS 8 (1975), 235-274.

7. E. Calabi, "Métrique Kähleriennes et fibres holomorphes," Ann. Sci. E.N.S. 12 (1979), 269-294.

8. S. S. Chern, "Complex Manifolds," Chicago University, 1957.

9. S. S. Chern, "Seminar on Partial Differential Equations," Math. Sciences Research Institute Publ. vol. 2, Springer Verlag, New York, 1984.

10. O. Debarre, in preparation.

11. R. Friedman, "A degeneration family of quintic surfaces with trivial monodromy," Duke Math. J. vol. 50 N1 (1981) 203-214.

12. A. Fujiki, "On primitive symplectic compact Kähler V-manifolds," Progress in Math. vol. 39, Birkhäuser, Boston, 1983, 435-442.

13. Ph. A. Griffiths, "Periods of integrals on algebraic manifolds III," I.H.E.S. Publ. Math. vol. 38 (1970), 125-180.

14. K. Kodaira, J. Morrow, "Complex Manifolds," Holt, Rinehart and Winston, Inc., New-York, 1971.

15. V. S. Kulikov, "Degeneration of K3 and Enriques surfaces," Math. USSR Izvestia 11 (1977), 957-989.

16. M. L. Michelson, "Clifford and spinor cohomology of Kähler manifolds," Amer. J. of Math. 102 (1980), 1083-1196.

17. S. Mukai, "Symplectic structure of the moduli of sheaves on an abelian or K3 surface," Inv. Math. 71 (1984) 101-116.

18. D. Morrison, personal communication.

19. U. Persson & H. Pinkham, "Degeneration of surfaces with trivial canonical bundle," Ann. of Math. 113 (1981), 45-66.

20. I. I. Piateckii-Shapiro & I. R. Shafarevich, "A Torelli Theorem for algebraic surfaces of type K3," Math. USSR Izvestia 5 (1971), 547-588.

21. A. Todorov, "Applications of Kähler-Einstein-Calabi-Yau metric to moduli of K3 surfaces," Inv. Math. 61 (1980), 547-588.

22. S. T. Yau, "On Ricci curvature of a compact Kähler manifold and complex Monge-Amper equation I," Comm. Pure and Applied Math. 31 (1978), 229-441.

23. D. Mumford and J. Fogarty, "Geometric Invariant Theory", Second Enlarged Edition, Springer-Verlag Berlin, Heidelberg, New-York 1982.

24. D. Morrison, "Some remarks on moduli of K3 surfaces" Progress in Math. vol. 39, Birkhäuser, Boston, 1983, 303-333.

25. Matsusaka T. and Mumford D, "Two fundamental theorems on deformations of polarized varieties", Amer. Journal of Math. 86(1964), 668-684.

26. Palais R., "On the existence of slices for actions of non-compact Lie groups", Ann. of Math. (2) 73(1961), 295-323.