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Abstract

We construct explicit quantization of semisimple conjugacy classes of the complex orthogonal group $SO(N)$ with non-Levi isotropy subgroups through an operator realization on highest weight modules of the quantum group $U_q\big(\mathfrak{so}(N)\big)$.

Mathematics Subject Classifications: 81R50, 81R60, 17B37.

Key words: Quantum groups, deformation quantization, conjugacy classes, representation theory.

1 Introduction

This is a continuation of our recent works [1, 2] on quantization of closed conjugacy classes of simple complex algebraic groups with non-Levi stabilizers. There, we extend the methods developed for classes with Levi isotropy subgroups in [3], to the non-Levi case, with the focus on symplectic groups. In this paper, we extend the ideas of [1, 2] to orthogonal groups. This solves the quantization problem for all non-Levi conjugacy classes of simple complex matrix groups. Along with [1, 2, 3, 4], this result yields quantization of all semisimple classes of symplectic and special linear groups and "almost all" classes of orthogonal groups. A special "thin" family of orthogonal classes with Levi stabilizer is left beyond our scope. This family can be called "boundary" as it shares some properties of non-Levi classes. We will give a special consideration to this case in a separate publication. Then the quantization problem will be closed for all semisimple conjugacy classes of the four classical series of simple groups.

We recall that semisimple conjugacy classes of simple complex groups fall into two families distinguished by the type of their isotropy subgroup: whether it is Levi or not. With regard to the classical matrix series, the second type appears only in the symplectic and orthogonal cases. The isotropy subgroup of a given class is not Levi if and only if the eigenvalues ± 1 are both present in the spectrum, the multiplicity of -1 is greater or equal to 4 and, for even N, the multiplicity of 1 is greater or equal to 4. This is what we assume in what follows.

If both eigenvalues ± 1 are present but the multiplicity of -1 is 2, the stabilizer is of Levi type: the block $SO(2)$ rotating this eigenspace is isomorphic to $GL(1)$. Quantization of such classes methodologically lies in between of [3] and the present work. We postpone this case to a separate study, in order to simplify the current presentation.

Recall that closed conjugacy classes are affine subvarieties of the algebraic group G of complex orthogonal $N \times N$-matrices, [5]. We consider them as Poisson homogeneous spaces over the Poisson group G equipped with the Drinfeld-Sklyanin bracket. Their Poisson structure restricts from a Poisson structure on G, which is compatible with the adjoint action of the Poisson group G. We are searching for quantization of the affine coordinate ring of a class as a quotient of the quantized algebra, $\mathbb{C}_\hbar[G]$, of polynomial functions on G.

The algebra $\mathbb{C}_\hbar[G]$ can be realized as a subalgebra in $U_q(\mathfrak{g})$ and therefore represented on all $U_q(\mathfrak{g})$-modules. We are looking for a $U_q(\mathfrak{g})$-module to represent the quantized polynomial ring of the class by linear operators. Then it will be realized as a quotient of $\mathbb{C}_\hbar[G]$ by the annihilator. We have managed to find such a module among those of highest weight. Contrary to the Levi case, it is not a parabolic Verma module.

The key step of our approach is finding an appropriate submodule in an auxiliary parabolic Verma module $\hat{M}_\lambda$ and pass to the quotient module M_λ, where the quantized coordinate ring of the class $K\backslash G$ is realized by linear operators. The module $\hat{M}_\lambda$ is associated with the quantum universal enveloping subalgebra $U_q(\mathfrak{l}) \subset U_q(\mathfrak{g})$ of a Levi subgroup $L \subset G$. This Levi subgroup is maximal among those contained in the stabilizer K. There are two such subgroups, and we choose one by reducing to $GL(m)$ the orthogonal block $SO(2m) \subset K$, which rotates the eigenspace of -1.

On construction of M_λ we proceed to the study of the $U_q(\mathfrak{g})$-module $\mathbb{C}^N \otimes M_\lambda$. We find the spectrum and the minimal polynomial of the image of an invariant matrix $\mathcal{Q} \in \mathbb{C}^N \otimes U_q(\mathfrak{g})$ in $\mathrm{End}(\mathbb{C}^N \otimes M_\lambda)$ whose entries generate the algebra $\mathbb{C}_\hbar[K\backslash G]$. We start from the minimal polynomial on $\mathbb{C}^N \otimes \hat{M}_\lambda$, which is known from [3]. Further we analyze the structure of $\mathbb{C}^N \otimes \hat{M}_\lambda$ and show that a submodule responsible for a simple divisor of the minimal polynomial disappears under the projection $\mathbb{C}^N \otimes \hat{M}_\lambda \to \mathbb{C}^N \otimes M_\lambda$. This reduction yields a polynomial equation on $\mathcal{Q}$ with correct classical limit.

As a result, we obtain an explicit expression of the annihilator of M_λ in $\mathbb{C}_\hbar[G]$ in terms of the "quantum coordinate matrix" $\mathcal{Q}$. This annihilator is the quantized defining ideal of the class $K\backslash G$. This way we obtain an explicit description of $\mathbb{C}_\hbar[K\backslash G]$ as a quotient of $\mathbb{C}_\hbar[G]$, in terms of generators and relations.

An important special case of orthogonal non-Levi conjugacy classes comprises symmetric spaces $SO(2m) \times SO(N-2m)\backslash SO(N)$. There is an extended literature on their quantum counterparts, which find their applications in integrable models, [6], and representation theory, [7, 8, 9]. They were basically viewed as subalgebras in the Hopf dual to $U_\hbar(\mathfrak{g})$ annihilated by certain coideal subalgebras playing the role of quantum stabilizers. An advanced theory of quantum symmetric pairs was developed in [10, 11, 12, 13]. In the present paper, we adopt a different approach to quantization realizing it by endomorphisms in a $U_\hbar(\mathfrak{g})$-module. The two approaches are complementary, as in the classical geometry a closed conjugacy class can be alternatively presented as a subalgebra and a quotient algebra of $\mathbb{C}[G]$. It is an interesting problem to match these approaches, as that could facilitate further advances, for instance, in description of quantum associated vector bundles.

2 Classical conjugacy classes

Throughout the paper, G designates the algebraic group $SO(N)$, $N \geqslant 7$ or $N = 5$, of orthogonal matrices preserving a non-degenerate symmetric bilinear form $||C_{ij}||_{i,j=1}^N$ on the complex vector space $\mathbb{C}^N$; the Lie algebra of G will be denoted by $\mathfrak{g}$. We choose the realization of $\mathfrak{g}$ corresponding to $C_{ij} = \delta_{ij'}$, where δ_{ij} is the Kronecker symbol, and $i' = N + 1 - i$ for $i = 1, \ldots, N$.

The polynomial ring $\mathbb{C}[G]$ is generated by the matrix coordinate functions $||A_{ij}||_{i,j=1}^N$, modulo the set of N^2 relations written in the matrix form as

$$ACA^t = C. \tag{2.1}$$

The right conjugacy action of G on itself induces a left action on $\mathbb{C}[G]$ by duality; the matrix A is invariant as an element of $\mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}[G]$.

The group G is equipped with the Drinfeld-Sklyanin bivector field

$$\{A_1, A_2\} = \frac{1}{2}(A_2A_1r - rA_1A_2), \tag{2.2}$$

where $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a solution of the classical Yang-Baxter equation, [14]. This equation is understood in $\mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}[G]$, and the subscripts indicate the natural embeddings of $\mathrm{End}(\mathbb{C}^N)$ in $\mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathbb{C}^N)$, as usual.

The bivector field (2.2) is skew-symmetric when restricted to G and defines a Poisson bracket, which makes G a Poisson group. Of all possible solutions of the classical Yang-

Baxter equation we fix

$$r = \sum_{i=1}^{N}(e_{ii}\otimes e_{ii} - e_{ii}\otimes e_{i'i'}) + 2\sum_{\substack{i,j=1\\ i>j}}^{N}(e_{ij}\otimes e_{ji} - e_{ij}\otimes e_{i'j'}), \tag{2.3}$$

which is the simplest factorizable r-matrix, [16]. At the end of the article, we lift this restriction.

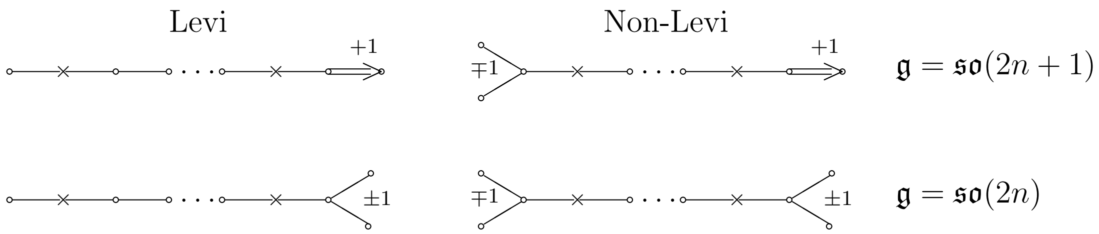
We regard the group G as a G-space under the conjugation action. The object of our study is another Poisson structure on G,

$$\{A_1, A_2\} = \frac{1}{2}(A_2 r_{21} A_1 - A_1 r A_2 + A_2 A_1 r - r_{21} A_1 A_2), \tag{2.4}$$

see [15]. It is compatible with the conjugation action and makes G a Poisson space over the Poisson group G with the Drinfeld-Sklyanin bracket (2.2).

We reserve n to denote the rank of the Lie algebra $\mathfrak{so}(N)$, so N is either $2n$ or $2n+1$. A closed conjugacy class $O \subset G$ consists of diagonalizable matrices and is determined by the multi-set of eigenvalues $S_O = \{\mu_i, \mu_i^{-1}\}_{i=1}^n \cup \{1\}$, where $\{1\}$ is present when N is odd. Every eigenvalue μ enters S_O with its reciprocal μ^{-1} and, in particular, may degenerate to $\mu = \mu^{-1} = \pm 1$. For a class to be non-Levi, both $+1$ and -1 should be in S_O. Moreover, the multiplicity of -1 must be 4 or higher as well as the multiplicity of $+1$ for even N. We assume this in what follows.

In terms of Dynkin diagram, a Levi subgroup is obtained by scraping out a subset of nodes, while for non-Levi isotropy subgroups one should resort to the affine Dynkin diagram:



In other words, a non-Levi subgroup necessarily contains semisimple orthogonal block of even dimension rotating the eigenspace of -1 and, for even N, a semisimple orthogonal block rotating the eigenspace of $+1$.

With a class O, we associate an integer valued vector $\boldsymbol{n} = (n_i)_{i=1}^{\ell+2}$ subject to $\sum_{i=1}^{\ell+2} n_i = n$, and a complex valued vector $\boldsymbol{\mu} = (\mu_i)_{i=1}^{\ell+2}$. We assume that the coordinates of $\boldsymbol{\mu}$ are all invertible, with $\mu_i \neq \mu_j^{\pm1}$ for $i < j \leqslant \ell$ and $\mu_i^2 \neq 1$ for $1 \leqslant i \leqslant \ell$. Finally, we put $\mu_{\ell+1} = -1$ and $\mu_{\ell+2} = 1$. We reserve the special notation $m = n_{\ell+1}$ and $p = n_{\ell+2}$.

The initial point $o \subset O$ is fixed to the diagonal matrix with entries

$$\underbrace{\mu_1,\ldots,\mu_1}_{n_1},\ldots,\underbrace{\mu_\ell,\ldots,\mu_\ell}_{n_\ell},\underbrace{-1,\ldots,-1}_{m},\underbrace{1,\ldots,1}_{P},\underbrace{-1,\ldots,-1}_{m},\underbrace{\mu_\ell^{-1},\ldots,\mu_\ell^{-1}}_{n_\ell},\ldots,\underbrace{\mu_1^{-1},\ldots,\mu_1^{-1}}_{n_1},$$

where $P = 2p$ if $N = 2n$ and $P = 2p + 1$ if $N = 2n + 1$.

The stabilizer subgroup of the initial point $o \in O$ is the direct product

$$K = GL(n_1) \times \ldots \times GL(n_\ell) \times SO(2m) \times SO(P) \tag{2.5}$$

and it is determined solely by the vector $\boldsymbol{n}$. The integer ℓ counts the number of GL-blocks in K of dimension n_i, $i = 1, \ldots, \ell$, while m and p are the ranks of the orthogonal blocks in K corresponding to the eigenvalues -1 and $+1$, respectively. The specialization $n_1 = \ldots = n_\ell = 0$ is formally encoded by $\ell = 0$ and referred to as the *symmetric case*. Then (2.5) reduces to $SO(2m) \times SO(P)$, and the class $O \simeq K\backslash G$ to a symmetric space.

Let $\mathcal{M}_K$ denote the moduli space of conjugacy classes with the fixed isotropy subgroup (2.5), regarded as Poisson spaces. The set of all $\ell + 2$-tuples $\boldsymbol{\mu}$ such as specified above parameterize $\mathcal{M}_K$ (not uniquely). For even N one can also chose the alternative parametrization $\mu_{\ell+1} = 1$, $\mu_{\ell+2} = -1$, however it is compensated by the Poisson automorphism $A \mapsto -A$. Therefore, the subset $\hat{\mathcal{M}}_K$ of $\boldsymbol{\mu}$ with fixed $\mu_{\ell+1} = -1$ and $\mu_{\ell+2} = 1$ can be used for parametrization of $\mathcal{M}_K$.

The conjugacy class O associated with $\boldsymbol{\mu}$ and $\boldsymbol{n}$ is specified by the set of equations

$$(A - \mu_1) \ldots (A - \mu_\ell)(A + 1)(A - 1)(A - \mu_\ell^{-1}) \ldots (A - \mu_1^{-1}) = 0, \tag{2.6}$$

$$\mathrm{Tr}(A^k) = \sum_{i=1}^{\ell} n_i(\mu_i^k + \mu_i^{-k}) + 2m(-1)^k + P, \quad k = 1, \ldots, N, \tag{2.7}$$

where the matrix multiplication in the first line is understood. This system is polynomial in the matrix entries A_{ij} and defines an ideal of $\mathbb{C}[\mathrm{End}(\mathbb{C}^N)]$ vanishing on O.

Theorem 2.1. *The system of polynomial relations (2.6) and (2.7) along with the defining relations of the group (2.1) generates the defining ideal of the class $O \subset SO(N)$.*

Proof. The proof is similar to the symplectic case worked out in [2]. It boils down to checking the rank of Jacobian of the system of equations (2.6), (2.7), and (2.1). □

3 Quantum orthogonal groups

The quantum group $U_\hbar(\mathfrak{g})$ is a deformation of the universal enveloping algebra $U(\mathfrak{g})$ along the formal parameter $\hbar$ in the class of Hopf algebras, [14]. By definition, it is a topologically free $\mathbb{C}[[\hbar]]$-algebra. It also contains important subalgebras over $\mathbb{C}[q, q^{-1}] \subset \mathbb{C}[[\hbar]]$, $q = e^\hbar$, and its various localizations, which also appear in our presentation.

Let R and R^+ denote respectively the root system and the set of positive roots of the orthogonal Lie algebra $\mathfrak{g}$. Let $\Pi^+ = (\alpha_1, \alpha_1, \ldots, \alpha_n)$ be the set of simple positive roots.

They can be conveniently expressed through an orthogonal basis $(\varepsilon_i)_{i=1}^n$ with respect to the canonical symmetric inner form $(\,.\,,\,.\,)$ on the linear span of Π^+:

$$
\begin{array}{rclll}
\alpha_i &=& \varepsilon_i-\varepsilon_{i+1}, \quad i=1,\ldots,n-1, & \alpha_n=\varepsilon_{n-1}+\varepsilon_n, & \mathfrak{g}=\mathfrak{so}(2n),\\
\alpha_i &=& \varepsilon_i-\varepsilon_{i+1}, \quad i=1,\ldots,n-1, & \alpha_n=\varepsilon_n, & \mathfrak{g}=\mathfrak{so}(2n+1).
\end{array}
$$

Given a reductive subalgebra $\mathfrak{f}\subset\mathfrak{g}$ we label its root subsystem with subscript $\mathfrak{f}$, as well as the set of positive and simple positive roots: $R_\mathfrak{f}$, $R_\mathfrak{f}^+$, $\Pi_\mathfrak{f}^+$. We reserve the notation $\mathfrak{g}_k$ for the orthogonal subalgebra of rank $k\leqslant n$ corresponding to the positive roots $\Pi^+_{\mathfrak{g}_k}=\{\alpha_{n-k+1},\ldots,\alpha_n\}$.

Denote by $\mathfrak{h}$ the dual vector space to the linear span $\mathbb{C}\Pi^+$. The inner product establishes a linear isomorphism between the $\mathbb{C}\Pi^+$ and $\mathfrak{h}$. We define $h_\lambda\in\mathfrak{h}$ for every $\lambda\in\mathfrak{h}^*=\mathbb{C}\Pi^+$ to be its image under this isomorphism: $\mu(h_\lambda)=(\lambda,\mu)$ for all $h\in\mathfrak{h}$.

The vector space $\mathfrak{h}$ generates a commutative subalgebra $U_\hbar(\mathfrak{h})\subset U_\hbar(\mathfrak{g})$ called the Cartan subalgebra. The quantum group $U_\hbar(\mathfrak{g})$ is a $\mathbb{C}[[\hbar]]$-algebra generated by simple root vectors (Chevalley generators) e_μ, f_μ, and $h_\mu\in\mathfrak{h}$ (Cartan generators) $\mu\in\Pi^+$. The Cartan and Chevalley generators obey the commutation rule

$$
[h_\mu,e_\nu]=(\mu,\nu)e_\nu, \quad [h_\mu,f_\nu]=-(\mu,\nu)f_\nu.
$$

Positive Chevalley generators commute with negative to the subalgebra $U_\hbar(\mathfrak{h})$:

$$
[e_\mu,f_\nu]=\delta_{\mu,\nu}\frac{q^{h_\mu}-q^{-h_\mu}}{q-q^{-1}}, \quad \mu\in\Pi^+.
$$

Note with care that the denominator is independent of root, contrary to the usual definition with $q_\mu=e^{\hbar\frac{(\mu,\mu)}{2}}$, $\mu\in\Pi^+$, see e.g. [17]. The difference comes from a renormalization of the Chevalley generators, which also respects the Serre relations below. With our normalization, the vector representation of $U_\hbar\big(\mathfrak{so}(N)\big)$ on $\mathbb{C}^N$ is determined by the classical matrix assignment on the generators, independent of q.

Let $a_{ij}=\frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, $i,j=1,\ldots,n$, be the Cartan matrix and put $q_i:=q_{\alpha_i}$. Define $[z]_q=\frac{q^z-q^{-z}}{q-q^{-1}}$ for any complex z, and the q-binomial coefficients

$$
\left[\begin{array}{c} n\\ k\end{array}\right]_q=\frac{[n]_q!}{[k]_q![n-k]_q!}, \quad [0]_q!=1, \quad [n]_q!=[1]_q\cdot[2]_q\ldots[n]_q, \quad k\leqslant n\in\mathbb{N}.
$$

The positive Chevalley generators satisfy the quantum Serre relations

$$
\sum_{k=0}^{1-a_{ij}}(-1)^k\left[\begin{array}{c} 1-a_{ij}\\ k\end{array}\right]_{q_i} e_{\alpha_i}^{1-a_{ij}-k}e_{\alpha_j}e_{\alpha_i}^k=0.
$$

Similar relations holds for the negative Chevalley generators f_μ.

The comultiplication Δ and antipode γ are defined on the generators by

$$\Delta(h_\mu) = h_\mu \otimes 1 + 1 \otimes h_\mu, \quad \gamma(h_\mu) = -h_\mu,$$

$$\Delta(e_\mu) = e_\mu \otimes 1 + q^{h_\mu} \otimes e_\mu, \quad \gamma(e_\mu) = -q^{-h_\mu} e_\mu,$$

$$\Delta(f_\mu) = f_\mu \otimes q^{-h_\mu} + 1 \otimes f_\mu, \quad \gamma(f_\mu) = -f_\mu q^{h_\mu},$$

for all $\mu \in \Pi^+$. The counit homomorphism $\varepsilon \colon U_\hbar(\mathfrak{g}) \to \mathbb{C}[[\hbar]]$ annihilates e_μ, f_μ, h_μ.

Besides the Cartan subalgebra $U_\hbar(\mathfrak{h})$, the quantum group $U_\hbar(\mathfrak{g})$ contains the following Hopf subalgebras. The positive and negative Borel subalgebras $U_\hbar(\mathfrak{b}^\pm)$ are generated over $U_\hbar(\mathfrak{h})$ with respect to either left or right multiplication by $\{e_\mu\}_{\mu\in\Pi^+}$ and $\{f_\mu\}_{\mu\in\Pi^+}$, correspondingly. For any root subsystem in R the associated Levi subalgebra $U(\mathfrak{l})$ is quantized to a Hopf subalgebra $U_\hbar(\mathfrak{l})$, along with the parabolic subalgebras $U_\hbar(\mathfrak{p}^\pm)$ generated by $U_\hbar(\mathfrak{b}^\pm)$ over $U_\hbar(\mathfrak{l})$.

We shall also deal with the Hopf subalgebra $U_q(\mathfrak{g}) \subset U_\hbar(\mathfrak{g})$ generated by the Chevalley generators and the exponentials $t^\pm_{\alpha_i} = q^{\pm h_{\alpha_i}}$, $\alpha_i \in \Pi^+$, over the ring of scalars $\mathbb{C}[q, q^{-1}]$. We shall work with its extensions by fractions over the multiplicative system $\{q^l - 1\}_{l\in\mathbb{Z}}$ thus assuming that q is not a root of unity, when specialized to a complex number. The above mentioned subalgebras of $U_\hbar(\mathfrak{g})$ have their counterparts in $U_q(\mathfrak{g})$, and we use the subscript q to distinguish them from the corresponding $\mathbb{C}[[\hbar]]$-algebras.

Quantum counterparts of higher root vectors in $\mathfrak{g}$ are defined by

$$e_\mu = e_\nu e_\sigma - q^{(\nu,\sigma)} e_\nu e_\sigma, \quad f_\mu = e_\sigma e_\nu - q^{-(\nu,\sigma)} e_\sigma e_\nu, \quad \nu, \sigma, \mu = \nu + \sigma \in R^+.$$

This allows to define deformations of $\mathfrak{h}$-invariant subspaces of $\mathfrak{g}$, like $\mathfrak{g}$ itself, $\mathfrak{b}^\pm$ *etc.* There is an ordering on R^+ such that $(e_\mu)_{\mu\in R^+} \subset U_\hbar(\mathfrak{b}^+)$ and $(f_\mu)_{\mu\in R^+} \subset U_\hbar(\mathfrak{b}^-)$ generate a Poincare-Birkgoff-Witt (PBW) basis over $U_\hbar(\mathfrak{h})$, [17]. This basis establishes an $\mathfrak{h}$-invariant bijection between the quantized universal enveloping algebra and classical (upon $\hbar$-adic completion) and facilitates the use of same notation for $\mathfrak{h}$-submodules expressed through the PBW monomials. So, by $\mathfrak{g} \subset U_\hbar(\mathfrak{g})$ we understand the sum of $\mathfrak{h}$ and the linear span of $\{f_\mu, e_\mu\}_{\mu\in\Pi^+}$. Similar convention is adopted with regard of other $\mathfrak{h}$-submodules in $\mathfrak{g}$.

The triangular decomposition $\mathfrak{g} = \mathfrak{n}^-_\mathfrak{l} \oplus \mathfrak{l} \oplus \mathfrak{n}^+_\mathfrak{l}$ gives rise to the triangular factorization

$$U_\hbar(\mathfrak{g}) = U_\hbar(\mathfrak{n}^-_\mathfrak{l}) U_\hbar(\mathfrak{l}) U_\hbar(\mathfrak{n}^+_\mathfrak{l}), \tag{3.8}$$

where $U_\hbar(\mathfrak{n}^\pm_\mathfrak{l})$ are subalgebras in $U_\hbar(\mathfrak{b}^\pm)$ generated by the positive or negative root vectors from $\mathfrak{n}^\pm_\mathfrak{l}$, respectively. This factorization makes $U_\hbar(\mathfrak{g})$ a free $U_\hbar(\mathfrak{n}^-_\mathfrak{l}) - U_\hbar(\mathfrak{n}^+_\mathfrak{l})$-bimodule generated by $U_\hbar(\mathfrak{l})$. For the special case $\mathfrak{l} = \mathfrak{h}$, we drop the subscript from notation: $\mathfrak{n}^\pm = \mathfrak{n}^\pm_\mathfrak{h}$. Contrary to the classical universal enveloping algebras, $U_\hbar(\mathfrak{n}^\pm_\mathfrak{l})$ are not Hopf subalgebras in $U_\hbar(\mathfrak{g})$.

4 Auxiliary parabolic Verma module $\hat{M}_\lambda$

We adopt certain conventions concerning representations of quantum groups, which are similar to [2]. We assume that they are free modules over the ring of scalars and their rank will be referred to as dimension. Finite dimensional $U_\hbar(\mathfrak{g})$-modules are deformations of their classical counterparts, and we will drop the reference to the deformation parameter in order to simplify notation. For instance, the natural vector representation of $U_\hbar(\mathfrak{g})$ will be denoted simply by $\mathbb{C}^N$.

We shall deal with weight modules, and by $[V]_\alpha$ we mean the subspace of weight α in an $\mathfrak{h}$-invariant subspace V of such modules. We prefer additive parametrization of weights of $U_q(\mathfrak{g})$ facilitated by the embedding of $U_q(\mathfrak{g})$ in $U_\hbar(\mathfrak{g})$. Under this convention, the weights of $U_q(\mathfrak{g})$ belong to $\frac{1}{\hbar}\mathfrak{h}^*[[\hbar]]$. It is sufficient for us to confine them to $\hbar^{-1}\mathfrak{h}^*\oplus\mathfrak{h}^*$.

Let $L\subset G$ denote the Levi subgroup

$$L = GL(n_1)\times\ldots\times GL(n_\ell)\times GL(m)\times SO(P).$$

It is a maximal Levi subgroup of G among those contained in K, cf. (2.5) (the other one is obtained by reducing $SO(P)$ to $GL(p)$). By $\mathfrak{l}$ we denote the Lie algebra of L. It is a reductive subalgebra in $\mathfrak{g}$ of maximal rank n.

We denote by $\mathfrak{c}_\mathfrak{l}\subset\mathfrak{h}$ the center of $\mathfrak{l}$ and realize its dual $\mathfrak{c}^*_\mathfrak{l}$ as a subspace in $\mathfrak{h}^*$, via the canonical inner product. A element $\lambda\in\mathfrak{C}^*_\mathfrak{l}=\hbar^{-1}\mathfrak{c}^*_\mathfrak{l}\oplus\mathfrak{c}^*_\mathfrak{l}$ defines a one-dimensional representation of $U_q(\mathfrak{l})$ denoted by $\mathbb{C}_\lambda$. On the Cartan subalgebra, it acts by the assignment $q^{h_\alpha}\mapsto q^{(\alpha,\lambda)}$. Since $q=e^\hbar$, the pole is compensated, and the representation is correctly defined. It extends to $U_q(\mathfrak{p}^+)$ by nil on $\mathfrak{n}^+_\mathfrak{l}\subset\mathfrak{p}^+_\mathfrak{l}$. Denote by $\hat{M}_\lambda=U_q(\mathfrak{g})\otimes_{U_q(\mathfrak{p}^+)}\mathbb{C}_\lambda$ the parabolic Verma $U_q(\mathfrak{g})$-module induced from $\mathbb{C}_\lambda$, [18]. It plays an intermediate role in our construction: we are interested in its quotient module M_λ, which can be defined for certain values of λ.

Let us fix some notation for weight subspaces involved in our study. By $\mathfrak{c}^*_{\mathfrak{l},reg}$ we denote the set of all weights in $\mathfrak{c}^*_\mathfrak{l}$ that cannot be extended to characters of any reductive subalgebra in $\mathfrak{g}$ containing $\mathfrak{l}$. We denote by $\mathfrak{c}^*_\mathfrak{k}$ the subset in $\mathfrak{c}^*_\mathfrak{l}$ such that $q^{2(\alpha_{n-p},\lambda)}=-1$ for $\lambda\in\hbar^{-1}\mathfrak{c}^*_\mathfrak{k}$; its intersection with $\mathfrak{c}^*_{\mathfrak{l},reg}$ is designated by $\mathfrak{c}^*_{\mathfrak{k},reg}$. The generic point of $\mathfrak{c}^*_\mathfrak{k}$ yields the spectrum of the conjugacy class $K\backslash G$ in the classical limit. Finally, we denote by $\mathfrak{C}^*_\mathfrak{k}$ and $\mathfrak{C}^*_{\mathfrak{k},reg}$ the vector subspace in, respectively, $\hbar^{-1}\mathfrak{c}^*_\mathfrak{k}\oplus\mathfrak{c}^*_\mathfrak{l}$ and $\hbar^{-1}\mathfrak{c}^*_{\mathfrak{k},reg}\oplus\mathfrak{c}^*_\mathfrak{l}$ of weights λ satisfying $q^{2(\alpha,\lambda)}=-q^{\mp P}$, where the minus corresponds to $\mathfrak{g}=\mathfrak{so}(2n)$, $P=2p$ and plus to $\mathfrak{g}=\mathfrak{so}(2n+1)$, $P=2p+1$. The generic point of $\mathfrak{C}^*_{\mathfrak{k},reg}$ is a deformation of the spectrum S_O, $O=K\backslash G$.

Regarded as a $U_q(\mathfrak{h})$-module, $\hat{M}_\lambda$ is isomorphic to $U_q(\mathfrak{n}^+_\mathfrak{l})\otimes\mathbb{C}_\lambda$. It follows that, for all λ, $\hat{M}_\lambda$ are isomorphic as vector spaces. Let v_λ denote the image of $1\otimes 1$ in $\hat{M}_\lambda$. It generates $\hat{M}_\lambda$ over $U_\hbar(\mathfrak{g})$ and carries the highest weight λ. For any sequence of Chevalley generators

$f_{\alpha_{k_1}}, \dots, f_{\alpha_{k_m}}$ we call the product $f_{\alpha_{k_1}} \dots f_{\alpha_{k_m}} v_\lambda \in \hat{M}_\lambda$ a Chevalley monomial or simply monomial.

Along with $\hat{M}_\lambda$, we consider the *right* $U_q(\mathfrak{g})$-module $\hat{M}^\star_\lambda = \mathbb{C}_\lambda \otimes_{U_q(\mathfrak{p}^-)} U_q(\mathfrak{g})$. Here $\mathbb{C}_\lambda$ supports the 1-dimensional representation of $U_q(\mathfrak{p}^-)$ which extends the representation of $U_q(\mathfrak{l})$ by nil on $\mathfrak{n}^-_\mathfrak{l} \subset \mathfrak{p}^-_\mathfrak{l}$. As a $U_q(\mathfrak{h})$-module, it is isomorphic to $\mathbb{C}_\lambda \otimes U_q(\mathfrak{n}^+_\mathfrak{l})$, and its generator $v^\star_\lambda = 1 \otimes 1$ carries the lowest weight λ. Given a monomial $v = f_{\alpha_{k_1}} \dots f_{\alpha_{k_m}} v_\lambda$ we define $v^\star$ to be the monomial $v^\star_\lambda e_{\alpha_{k_m}} \dots e_{\alpha_{k_1}} \in \hat{M}^\star_\lambda$. There is a bilinear pairing (Shapovalov form) between $\hat{M}^\star_\lambda$ and $\hat{M}_\lambda$. It is determined by the following requirements: *i*) it is cyclic, i.e. $\langle xu, y\rangle = \langle x, uy\rangle$ for all $x \in \hat{M}^\star_\lambda$, $y \in \hat{M}_\lambda$, $u \in U_q(\mathfrak{g})$, *ii*) $v^\star_\lambda$ annihilates all vectors of weight lower than λ, *iii*) it is normalized to $\langle v^\star_\lambda, v_\lambda\rangle = 1$. We are not concerned with transformation properties of this form using it just as a calculation tool.

4.1 Some technical constructions

In this technical section we introduce some constructions which we use further on. They involve the quantum subgroup $U_q\big(\mathfrak{gl}(n)\big)$ in $U_\hbar(\mathfrak{g})$ corresponding to the roots $\{\alpha_i\}_{i=1}^{n-1}$. Its negative Chevalley generators obey the Serre relations

$$f^2_{\alpha_i} f_{\alpha_{i-1}} - (q + q^{-1}) f_{\alpha_i} f_{\alpha_{i-1}} f_{\alpha_i} + f_{\alpha_{i-1}} f^2_{\alpha_i} = 0, \quad [f_{\alpha_i}, f_{\alpha_j}] = 0, \tag{4.9}$$

for all feasible i, j, and $|i - j| > 1$ (the positive generators satisfy similar relations). They will be heavily used in what follows.

Below we fix the Levi subalgebra $\mathfrak{l} = \mathfrak{gl}(2) \oplus \mathfrak{so}(N-4)$, which corresponds to $\ell = 0$, $m = 2$, $p = n - 2 > 0$, and let $\hat{M}_\lambda$ be a parabolic Verma module. Note that f_α kills its generator v_λ unless $\alpha = \alpha_2$. Introduce the element $\omega = f_{\alpha_p} \dots f_{\alpha_2} v_\lambda \in \hat{M}_\lambda$ and $\omega = v_\lambda$ for $p = n - 2 = 0$. This vector participates in a basis, which we use for calculation of a singular vector in $\hat{M}_\lambda$ in subsequent sections. It is constructed solely out of the $\mathfrak{gl}(n)$-generators and features the following

Lemma 4.1. *Suppose that* $3 \leqslant p$. *Then* ω *is annihilated by* f_{α_i}, $3 \leqslant i \leqslant p$.

Proof. Assuming $3 \leqslant i < p$, we get

$$f_{\alpha_i}\omega = f_{\alpha_p} \dots f_{\alpha_i} f_{\alpha_{i+1}} f_{\alpha_i} \dots f_{\alpha_2} v_\lambda \sim f_{\alpha_p} \dots (f^2_{\alpha_i} f_{\alpha_{i+1}} + f_{\alpha_{i+1}} f^2_{\alpha_i}) \dots f_{\alpha_2} v_\lambda,$$

by the Serre relations (4.9). The rightmost dots contain generators with numbers strictly less than i. Since they commute with $f_{\alpha_{i+1}}$, it can be pushed to the right in the first summand, where it kills v_λ. The second summand is equal to $f_{\alpha_p} \dots f^2_{\alpha_i} f_{\alpha_{i-1}} \dots v_\lambda$. Again, using the Serre relation for $f^2_{\alpha_i} f_{\alpha_{i-1}}$ we can place at least one factor f_{α_i} on the right of $f_{\alpha_{i-1}}$ and push it freely further to the right. This kills the second summand.

If $i = p$, then we have $f_{\alpha_p}\omega = f^2_{\alpha_p} f_{\alpha_{p-1}} \dots f_{\alpha_2} v_\lambda$. Using (4.9), at least one copy of f_{α_p} can be pushed through $f_{\alpha_{p-1}}$ to the right and further on till it kills v_λ. □

Lemma 4.2. *The vector ω is annihilated by e_{α_i}, $i \neq p$.*

Proof. Obviously, ω is annihilated by e_{α_i}, if $i = 1$, $p+1$, $p+2 = n$. Applying e_{α_i} with $2 \leqslant i \leqslant p-1$ to ω yields $f_{\alpha_p} \dots f_{\alpha_{i+1}} \dots v_\lambda$ up to a scalar multiplier. Here the dots on the right stand for the generators with numbers strictly less than i. Since they commute with $f_{\alpha_{i+1}}$, the latter can be pushed to the right, where it kills v_λ. □

Remark that ω is a non-zero vector of weight $\lambda - \varepsilon_2 + \varepsilon_{n-1}$. Indeed, one can check that $\dim[\hat{M}_\lambda]_{\lambda-\varepsilon_2+\varepsilon_{n-1}} = 1$ and all other monomials of this weight turn zero.

Further we present another auxiliary construction, which also involves only the $\mathfrak{gl}(n)$-generators. Suppose that $3 \leqslant n$ and introduce vectors $y_k \in \hat{M}_\lambda$, $k = 2, \dots, n-1$, by

$$y_2 = [f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_2} v_\lambda, \quad y_k = f_{\alpha_k} \dots f_{\alpha_3} [f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_k} \dots f_{\alpha_2} v_\lambda, \quad 3 \leqslant k \leqslant n-1,$$

where $[X, Y]_a = XY - aYX$. Here and further on we set the parameter a equal to $q + q^{-1}$. Vectors y_k is an auxiliary technical construction due to the following.

Lemma 4.3. *For all $k = 2, \dots, n-1$, one has $y_k = 0$.*

Proof. For $k = 2$ we find $[f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_2} v_\lambda = f_{\alpha_1} f_{\alpha_2} f_{\alpha_2} v_\lambda - a f_{\alpha_2} f_{\alpha_1} f_{\alpha_2} v_\lambda = 0$ by the Serre relation (4.9). For higher k we use induction. Suppose the lemma is proved for some $k \geqslant 2$. Then

$$\begin{aligned} y_{k+1} &= f_{\alpha_{k+1}} f_{\alpha_k} \dots f_{\alpha_3} [f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_{k+1}} f_{\alpha_k} \dots f_{\alpha_2} v_\lambda \\ &= (f_{\alpha_{k+1}} f_{\alpha_k} f_{\alpha_{k+1}}) \dots f_{\alpha_3} [f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_k} \dots f_{\alpha_2} v_\lambda. \end{aligned}$$

The term in the brackets produces $a^{-1}(f^2_{\alpha_{k+1}} f_{\alpha_k} + f_{\alpha_k} f^2_{\alpha_{k+1}})$ through (4.9). The first term is zero by the induction assumption. The second term is zero too, because $f^2_{\alpha_{k+1}}$ can be pushed to the right till it meets the second copy of f_{α_k}. By the Serre relation, one factor $f_{\alpha_{k+1}}$ can be pushed through f_{α_k} to the right. Then it proceeds freely till it kills v_λ. This proves the statement. □

Remark that the case $N = 5$, $m = 4$, $p = 0$ is excluded from this construction, because $y_2 \neq 0$ then.

4.2 The module $\hat{M}_\lambda$ for $\mathfrak{l} = \mathfrak{gl}(2) \oplus \mathfrak{so}(P)$

A substantial part of our theory comes from the special case of symmetric conjugacy classes. That is accounted for the fact that the difference between K and L is confined in the orthogonal blocks of K. Because of that, we start from the symmetric case, when the stabilizer $\mathfrak{k}$ consists of two simple orthogonal blocks of ranks m and p. Furthermore, the general symmetric case can be readily derived from the specialization $m = 2$ (note that $\mathfrak{so}(4)$ is the smallest semisimple orthogonal algebra of even dimension). By this reason, we start with $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(N-4)$, $\mathfrak{l} = \mathfrak{gl}(2) \oplus \mathfrak{so}(N-4)$.

Observe that the stabilizer of the class is generated over the maximal Levi subalgebra by a pair of root vectors e_δ, f_δ, where

$$\delta = \alpha_1 + 2\sum_{i=2}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n, \quad \mathfrak{g} = \mathfrak{so}(2n), \quad \delta = \alpha_1 + 2\sum_{i=2}^{n} \alpha_i, \quad \mathfrak{g} = \mathfrak{so}(2n+1).$$

By this we mean that $\mathfrak{k} = \mathfrak{m}^- \oplus \mathfrak{l} \oplus \mathfrak{m}^+$, where $\mathfrak{m}^- = \mathrm{ad}(\mathfrak{l})(f_\delta)$ and $\mathfrak{m}^+ = \mathrm{ad}(\mathfrak{l})(e_\delta)$ are abelian Lie subalgebras. In terms of the universal enveloping algebras, $U(\mathfrak{k})$ features the triangular decomposition $U(\mathfrak{k}) = U(\mathfrak{m}^-) \times U(\mathfrak{l}) \times U(\mathfrak{m}^+)$.

In the symmetric case under consideration, the weight λ satisfies the conditions $(\alpha_i, \lambda) = 0$ for all i but $i = 2$. Therefore, $\hat{M}_\lambda$ is parameterized by (α_2, λ). Its highest weight vector v_λ is annihilated by all e_{α_i} and all f_{α_i} except for f_{α_2}. By the PBW property, the algebra $U_q(\mathfrak{n}_\mathfrak{l}^-)$ is generated by the root vectors

$$f_{\varepsilon_1 \pm \varepsilon_i}, \; f_{\varepsilon_2 \pm \varepsilon_i}, \; f_{\varepsilon_1 + \varepsilon_2}, \; f_{\varepsilon_1}, \; f_{\varepsilon_2} \in \mathfrak{n}_\mathfrak{l}^-,$$

where $i = 3, \ldots, n$, and f_{ε_1}, f_{ε_2} are present only when N is odd. Therefore, $[\hat{M}_\lambda]_{\lambda-\delta}$ has the basis of $N-3$ elements

$$f_{\varepsilon_1 \pm \varepsilon_i} f_{\varepsilon_2 \mp \varepsilon_i} v_\lambda, \quad f_{\varepsilon_1 + \varepsilon_2} v_\lambda, \quad f_{\varepsilon_1} f_{\varepsilon_2} v_\lambda.$$

The last term is present for odd N.

We intend to calculate a singular vector $v_{\lambda-\delta} \in [\hat{M}_\lambda]_{\lambda-\delta}$. Singular means that $v_{\lambda-\delta}$ lies in the kernel of all $e_\alpha \in \mathfrak{n}^+$. In order to facilitate the calculations, we need to choose a suitable basis. Notice that the subspace of weight $\lambda - \delta + \alpha_1 = \lambda - 2\varepsilon_2$ (the image $e_{\alpha_1}[\hat{M}_\lambda]_{\lambda-\delta}$ for generic λ) has a basis of elements

$$f_{\varepsilon_2 \pm \varepsilon_i} f_{\varepsilon_2 \mp \varepsilon_i} v_\lambda, \quad f_{\varepsilon_2} f_{\varepsilon_2} v_\lambda,$$

where the last term counts for odd N. Therefore, $\ker e_{\alpha_1}|_{[M_\lambda]_{\lambda-\delta}} = [\ker e_{\alpha_1}]_{\lambda-\delta}$ has dimension $n-1$.

We consider the PBW basis being not particularly convenient for our purposes and introduce another basis in $[\ker e_{\alpha_1}]_{\lambda-\delta}$. Define $n-1$ vectors $x_i \in [\hat{M}_\lambda]_{\lambda-\delta}$ by

$$
\begin{aligned}
x_2 &= [f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_3} \overset{<}{\ldots} f_{\alpha_n}(f_{\alpha_n}\omega),\\
x_i &= f_{\alpha_i} \overset{>}{\ldots} f_{\alpha_3}[f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_{i+1}} \overset{<}{\ldots} f_{\alpha_n}(f_{\alpha_n}\omega), \quad i = 3, \ldots, n,
\end{aligned}
$$

for $N = 2n+1$, and by

$$
\begin{aligned}
x_2 &= [f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_3} \overset{<}{\ldots} f_{\alpha_{n-2}}(f_{\alpha_{n-1}} f_{\alpha_n}\omega),\\
x_i &= f_{\alpha_i} \overset{>}{\ldots} f_{\alpha_3}[f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_{i+1}} \overset{<}{\ldots} f_{\alpha_{n-2}}(f_{\alpha_{n-1}} f_{\alpha_n}\omega), \quad i = 3, \ldots, n-2,\\
x_{n-1} &= f_{\alpha_{n-1}} f_{\alpha_{n-2}} \overset{>}{\ldots} f_{\alpha_3}[f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_n}\omega,\\
x_n &= f_{\alpha_n} f_{\alpha_{n-2}} \overset{>}{\ldots} f_{\alpha_3}[f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_{n-1}}\omega,
\end{aligned}
$$

for $N = 2n$. We emphasize that generators within the parenthesis stay there as i varies, while other generators are permuted as specified. Remark that the mere presence of f_{α_3} in these formulas implies $N > 8$. The definition extends to the case $N = 5, 7, 8$ by removing the extra generators from the formulas. So, for $N = 5$ we have only one vector

$$x_2 = [f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_2} v_\lambda,$$

for $N = 7$ we have two vectors

$$x_2 = [f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_3}(f_{\alpha_3}\omega), \quad x_3 = f_{\alpha_3}[f_{\alpha_1}, f_{\alpha_2}]_a(f_{\alpha_3}\omega), \quad \omega = f_{\alpha_2} v_\lambda.$$

There are three vectors for $N = 8$:

$$x_2 = [f_{\alpha_1}, f_{\alpha_2}]_a(f_{\alpha_3} f_{\alpha_4}\omega), \quad x_3 = f_{\alpha_3}[f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_4}\omega, \quad x_4 = f_{\alpha_4}[f_{\alpha_1}, f_{\alpha_2}]_a f_{\alpha_3}\omega, \quad \omega = f_{\alpha_2} v_\lambda.$$

The following lemma accounts for the choice of the commutator parameter a.

Lemma 4.4. *The vectors x_i, $i = 2, \ldots, n$, belong to* $\ker e_{\alpha_1} \subset \hat{M}_\lambda$.

Proof. Applying e_{α_1} to x_i we get

$$e_{\alpha_1} x_i \sim \ldots [q^{h_{\alpha_1}} - q^{-h_{\alpha_1}}, f_{\alpha_2}]_a \ldots \omega = \big((q^2 - q^{-2}) - a(q - q^{-1})\big) \ldots f_{\alpha_2} \ldots \omega = 0.$$

Indeed, observe that h_{α_1} commutes with everything between the commutator and ω, which has weight $\lambda - \varepsilon_2 + \varepsilon_{n-1}$ for $N \geqslant 7$ and $\lambda - \varepsilon_2$ for $N = 5$. This produces the vanishing scalar factor. $\square$

As we already mentioned, the total dimension of $[\hat{M}_\lambda]_{\lambda-\delta}$ is equal to $N-3$. Every vector x_i contains the commutator $[f_{\alpha_1}, f_{\alpha_2}]_a$ thus involving two Chevalley monomials. Overall

$\{x_i\}_{i=2}^n$ involve $2n-2$ monomials of weight $\lambda-\delta$. This is equal to $\dim[\hat{M}_\lambda]_{\lambda-\delta}$ for odd N, but greater by 1 for even N. However,

$$f_{\alpha_1}f_{\alpha_{n-1}}f_{\alpha_p}\dots f_{\alpha_2}f_{\alpha_n}\omega \sim f_{\alpha_1}f_{\alpha_n}f_{\alpha_{n-1}}f^2_{\alpha_p}\dots f^2_{\alpha_2}\omega \sim f_{\alpha_1}f_{\alpha_n}f_{\alpha_p}\dots f_{\alpha_2}f_{\alpha_{n-1}}\omega.$$

Therefore, there are effectively $2n-3$ Chevalley monomials participating in $\{x_i\}_{i=2}^n$ for $N=2n$, as required.

In fact, we will restrict to the subspace $[\ker e_{\alpha_1}]_{\lambda-\delta}$, so $n-1$ vectors x_i annihilated by e_{α_1} are just right to form a basis. The following lemma is crucial in proving that x_i are independent.

Introduce vectors $x'_i \in [\hat{M}_\lambda]_{\lambda-\delta+\alpha_i}$ for $i=2,\dots,n$ as follows: x'_2 is obtained from x_2 by replacing the commutator $[f_{\alpha_1},f_{\alpha_2}]_a$ with f_{α_1}; to get x'_i for $i>2$, we remove the leftmost copy of f_{α_i} from x_{α_i}. One can see that $e_{\alpha_i}x_i \sim x'_i$ for $i=2,\dots,n$.

Lemma 4.5. *For all $i=2,\dots,n$, $x'_i \neq 0$.*

Proof. Observe that $\dim[\hat{M}_\lambda]_{\lambda-\delta+\alpha_2}=1$ and $\dim[\hat{M}_\lambda]_{\lambda-\delta+\alpha_i}=2$, where $i=3,\dots,n$ (use the PBW basis for that). Also, notice that $\dim[\hat{M}_\lambda]_{\lambda-\delta+\alpha_i+\alpha_1}=1$ for such i. Consider the Chevalley monomials x''_i of weights $\lambda-\delta+\alpha_i+\alpha_1$, $i=3,\dots,n$, obtained from x'_i by replacing the commutator $[f_{\alpha_1},f_{\alpha_2}]_a$ with f_{α_2}. Using Lemma 4.2, one can easily calculate the matrix elements

$$\langle {x'}^\star_2, x'_2\rangle = \langle \omega^\star,\omega\rangle, \quad \langle {x''}^\star_i, x''_i\rangle = \frac{q^{(\alpha_2,\lambda)-1}-q^{-(\alpha_2,\lambda)+1}}{q-q^{-1}}\langle \omega^\star,\omega\rangle, \quad i>2,$$

and $\langle \omega^\star,\omega\rangle = \frac{q^{(\alpha_2,\lambda)}-q^{-(\alpha_2,\lambda)}}{q-q^{-1}}$, of the Shapovalov pairing. This calculation proves that x'_2 and x''_i do not vanish for generic λ and hence for all λ.

Further, there are exactly two ways to construct a monomial of weight $\lambda-\delta+\alpha_i$, $i=3,\dots,n$, out of x''_i: either placing f_{α_1} on the left or on the right of the leftmost f_{α_2}. This gives two independent monomials participating in x'_i, $i=3,\dots,n$. Consequently, they do not vanish. $\square$

Note that the vectors x_i can be labeled with the simple roots of the subalgebra $\mathfrak{g}_{n-1}=\mathfrak{g}_{p+1}\subset\mathfrak{g}$ via the assignment $\alpha_i\mapsto x_i$, $i=2,\dots,n$. The next proposition provides qualitative information about the action of positive Chevalley generators on the system $\{x_i\}\subset \ker e_{\alpha_1}$.

Proposition 4.6. *For all $\alpha,\alpha_i\in\Pi^+_{\mathfrak{g}_{p+1}}$ such that $(\alpha,\alpha_i)=0$ the generator e_α annihilates x_i. If $(\alpha_j,\alpha_i)\neq 0$, then $e_{\alpha_j}x_i\sim x'_j$.*

Proof. Suppose first that N *is even* and put $x_\mu=x_{n-1}$ and $x_\nu=x_n$. Up to a scalar multiplier, $e_\nu x_\mu$ is equal to y_{n-1}, which is zero due to Lemma 4.3. Further, observe that

$e_\mu x_i$ for $i<p$ contains the factor $f_{\alpha_p}f_\nu f_{\alpha_p}$ producing $f^2_{\alpha_p}f_\nu$ and $f_\nu f^2_{\alpha_p}$ via the Serre relation (4.9). In the first term, the generator f_ν goes freely to the right and kills v_λ. The second term gives rise to the factor $f_{\alpha_p}\omega$, which is nil by Lemma 4.1.

Due to the symmetry between μ and ν, this also proves $e_\mu x_\nu=0$ and $e_\mu x_i=0$ for $i<p$.

For $2\leqslant i<p$, the action of e_{α_i} on $x_\mu=f_\mu f_{\alpha_p}\dots f_{\alpha_{i+1}}f_{\alpha_i}\dots[f_{\alpha_1},f_{\alpha_2}]\dots\omega$ knocks out the factor f_{α_i} releasing $f_{\alpha_{i+1}}$ next to the left. It can be pushed to the right till it meets ω and annihilates it by Lemma 4.1. Hence $e_{\alpha_i}x_\mu=e_{\alpha_i}x_\nu=0$ for $2\leqslant i<p$. Similar effect is produced by the action of e_{α_i} on x_k for $3\leqslant i+1<k\leqslant p$. If $3\leqslant k+1<i\leqslant p$, the vector $e_{\alpha_i}x_k$ contains the factor $f_{\alpha_{i-1}}f_{\alpha_{i+1}}\dots\omega=\dots f_{\alpha_{i-1}}\omega$, which is zero due to Lemma 4.1. This completes the proof of the first assertion for even N.

Now suppose that N is *odd*. There is nothing to prove if $p=0$, as there is only one vector, x_2. So we assume $p>0$. If $3\leqslant k+1<i\leqslant n$, then the vector x_k has the structure $\dots[f_{\alpha_1},f_{\alpha_2}]f_{\alpha_{k+1}}\dots f_{\alpha_{i-1}}f_{\alpha_i}\dots(f_{\alpha_n}\omega)$. Observe that the generator e_{α_i} effectively acts only on the displayed copy of f_{α_i} (the other is hidden in ω). In view of Lemma 4.1, it true for $i<n$. If $i=n$, then $e_{\alpha_n}x_k$ still comprises the factor $f_{\alpha_{n-1}}f_{\alpha_n}\omega$. In any case, e_{α_i} knocks out the leftmost f_{α_i} and releases the factor $f_{\alpha_{i-1}}$ on the left, which goes freely to the right. It kills ω by Lemma 4.1 if $i<n$. Still it kills $f_{\alpha_n}\omega$ if $i=n$, and the proof is similar to Lemma 4.1.

If $3\leqslant i+1<k\leqslant n$, then $x_k=f_{\alpha_k}\dots f_{\alpha_{i+1}}f_{\alpha_i}\dots[f_{\alpha_1},f_{\alpha_2}]_a\dots f_{\alpha_n}\omega$, and $f_{\alpha_{i+1}}$ commutes with everything between f_{α_i} and f_{α_n}. Further reasoning is similar to the case $k+1<i$, with $f_{\alpha_{i-1}}$ replaced by $f_{\alpha_{i+1}}$. This completes the first part of the proposition for odd N.

The proof of the second statement becomes quite straightforward on examining the structure of x_i. This is left for the reader as an exercise. □

Corollary 4.7. *The vectors* $\{x_i\}_{i=2}^n$ *form a basis in* $[\ker e_{\alpha_1}]_{\lambda-\delta}$.

Proof. The proof is based on Lemma 4.5. By Proposition 4.6, the operator $E=\sum_{i=2}^n e_{\alpha_i}$ sends the linear span $X=\mathrm{Span}\{x_i\}\subset[\ker e_{\alpha_1}]_{\lambda-\delta}$ to the linear span $X'=\mathrm{Span}\{x'_i\}$. Since all $x'_i\neq 0$ have different weights and hence independent, a singular vector from X is exactly a non-zero element of $\ker E|_X$. We shall see in the next section (cf. Proposition 5.2) that $\ker E|_X=\{0\}$ for generic λ. As all $\hat{M}_\lambda$ are isomorphic as vector spaces, $\{x_i\}_{i=2}^n$ are independent. This proves the statement, since $\dim[\ker e_{\alpha_1}]_{\lambda-\delta}=n-1$. □

Remark that independence of $\{x_i\}_{i=2}^n$ affects uniqueness of singular vector $v_{\lambda-\delta}\in\mathrm{Span}\{x_i\}_{i=2}^n$, for special λ, but not its existence.

5 The module M_λ

In this section we construct the highest weight $U_q(\mathfrak{g})$-module M_λ that supports quantization of the class $K\backslash G$. We define it as a quotient by a proper submodule generated by a singular vector of certain weight. First we do it for symmetric $K\backslash G$ and afterwards extend the solution for general K.

5.1 The symmetric case

Consider the simplest symmetric case $m = 2$, $p = n - 2$. In other words, assume $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(P)$, and $\mathfrak{l} = \mathfrak{gl}(2) \oplus \mathfrak{so}(P)$.

Define a vector $v_{\lambda-\delta} \in [\hat{M}_\lambda]_{\lambda-\delta}$ by

$$v_{\lambda-\delta} = c_2 x_2 + \dots c_{n-2} x_{n-2} + c_{n-1} x_{n-1} + c_n x_n,$$

with the scalar coefficients c_i set to be

$$c_i = (-1)^i q^{n-i-\frac{1}{2}} + (-1)^i q^{-(n-i-\frac{1}{2})}, \quad 2 \leqslant i \leqslant n,$$

for $N = 2n + 1$ and

$$c_i = (-q)^{n-1-i} + (-q)^{-(n-1-i)}, \quad 2 \leqslant i \leqslant n-2, \quad c_{n-1} = c_n = 1,$$

for $N = 2n$. The $n - 1$ coefficients c_i satisfy the recurrent system of $n - 2$ equations

$$\begin{gathered} c_{i-1} + a c_i + c_{i+1} = c_{n-1} + c_n = 0,\ N = 2n + 1, \\ c_{i-1} + a c_i + c_{i+1} = c_{n-3} + a c_{n-2} + c_{n-1} + c_n = c_{n-2} + a c_{n-1} = c_{n-2} + a c_n = 0,\ N = 2n, \end{gathered} \tag{5.10}$$

where i varies from 3 to $n-3$ in the first line and from 3 to $n-1$ in the second. This system determines (c_i) up to a common multiplier.

Lemma 5.1. *The vector $v_{\lambda-\delta}$ is annihilated by e_α, $\alpha \in \Pi^+ - \{\alpha_2\}$.*

Proof. First of all, $v_{\lambda-\delta}$ is annihilated by e_{α_1}, due to Lemma 4.4. Further proof is based on Proposition 4.6, which translates $n - 2$ vector equations $e_{\alpha_i} v_{\lambda-\delta} = 0$ into $n - 2$ scalar equations $E_i = 0$, where E_i are the proportionality factors in $e_{\alpha_i} v_{\lambda-\delta} \sim x'_i$. This system of equations is written down in (5.10) for each parity of N. It determines (c_i) uniquely, up to a common factor. □

Recall that a vector in a $U_q(\mathfrak{g})$-module is called singular if it is annihilated by $\mathfrak{n}^+$. In a module with highest weight, singular vectors generate proper submodules.

Proposition 5.2. *Suppose that λ satisfies the condition*

$$q^{2(\alpha_2,\lambda)} = -q^{\mp P},$$

where the minus corresponds to $P = 2p$ and the plus to $P = 2p+1$. Then the vector $v_{\lambda-\delta} \in \hat{M}_\lambda$ is singular. Up to a scalar factor, it is a unique singular vector of weight $\lambda - \delta$ and it exists only if λ satisfies the above condition.

Proof. In view of Lemma 5.1, we only need to satisfy the condition $e_{\alpha_2} v_{\lambda-\delta} = 0$. From Corollary 4.7, we get $e_{\alpha_2} v_{\lambda-\delta} = E_2 x_2'$ for some scalar E_2. Evaluating E_2 and equating it to zero we get the equations on the weight λ:

$$q^{(\alpha_2,\lambda)} q^2 c_2 + q^{(\alpha_2,\lambda)} q c_3 + q^{(\alpha_2,\lambda)} q c_4 = q^{-(\alpha_2,\lambda)} q^{-2} c_2 + q^{-(\alpha_2,\lambda)} q^{-1} c_3 + q^{-(\alpha_2,\lambda)} q^{-1} c_4$$

for $N = 8$,

$$q^{(\alpha_2,\lambda)} q^2 c_2 + q^{(\alpha_2,\lambda)} q c_3 = q^{-(\alpha_2,\lambda)} q^{-2} c_2 + q^{-(\alpha_2,\lambda)} q^{-1} c_3$$

for $N = 2n > 8$ and $N = 2n+1 \geqslant 7$. If $N = 5$, then we find

$$q^{(\alpha_2,\lambda)}(1 - q^{-1}) = q^{-(\alpha_2,\lambda)}(1 - q).$$

Plugging in the expressions for c_i gives the result.

By Corollary 4.7, the vector $v_{\lambda-\delta} \neq 0$, hence it is singular. Clearly the solution is unique and exists only for λ satisfying the hypothesis of the proposition. □

5.2 The highest weight module M_λ for general $\mathfrak{k}$

In this section we abandon the simplifying ansatz $\ell = 0$, $m = 2$ and allow for general isotropy subgroup K, as in (2.5). Its Lie algebra and the maximal Levi subalgebra read

$$\mathfrak{k} = \mathfrak{gl}(n_1) \oplus \ldots \oplus \mathfrak{gl}(n_\ell) \oplus \mathfrak{so}(2m) \oplus \mathfrak{so}(P), \quad \mathfrak{l} = \mathfrak{gl}(n_1) \oplus \ldots \oplus \mathfrak{gl}(n_\ell) \oplus \mathfrak{gl}(m) \oplus \mathfrak{so}(P).$$

Consider the subalgebra $U_q(\mathfrak{g}_{p+2}) \subset U_q(\mathfrak{g})$ with the simple positive roots $(\alpha_{n-p-1}, \ldots, \alpha_n)$. Under this embedding the root α_2 goes over to α_{n-p}, and the rood δ reads

$$\delta = \alpha_{n-p-1} + 2\sum_{i=n-p}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n, \; \mathfrak{g} = \mathfrak{so}(2n), \quad \delta = \alpha_{n-p-1} + 2\sum_{i=n-p}^{n} \alpha_i, \; \mathfrak{g} = \mathfrak{so}(2n+1).$$

Assuming $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg}$, let $\hat{M}_\lambda$ be the parabolic Verma module over $U_q(\mathfrak{g})$. When restricted to a $\mathfrak{g}_{p+2}$-weight, λ satisfies the assumptions of Proposition 5.2. Therefore, there is a singular vector $v_{\lambda-\delta}$ in the $U_q(\mathfrak{g}_{p+2})$-submodule of $\hat{M}_\lambda$ generated by v_λ.

Theorem 5.3. *Suppose that $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg}$. Then $v_{\lambda-\delta} \in \hat{M}_\lambda$ is a unique singular vector of weight $\lambda - \delta$.*

Proof. The vector $v_{\lambda-\delta}$ is annihilated by e_α for $\alpha \in \Pi^+_{\mathfrak{g}_{p+2}}$, by to Proposition 5.2. Furthermore, it is constructed out of $f_\beta \in \Pi^+_{\mathfrak{g}_{p+2}}$, which commute with e_α for $\alpha \in \Pi^+ - \Pi^+_{\mathfrak{g}_{p+2}}$. Therefore, such e_α annihilate $v_{\lambda-\delta}$ too. This proves that $v_{\lambda-\delta}$ is singular in $\hat{M}_\lambda$. It is unique as it is so for $U_q(\mathfrak{g}_{p+2})$. □

The singular vector $v_{\lambda-\delta}$ generates a proper submodule in $\hat{M}_\lambda$, call it $\hat{M}_{\lambda-\delta}$. We denote by M_λ the quotient $\hat{M}_\lambda/\hat{M}_{\lambda-\delta}$. This is the key object of our construction and of our further study.

We are going to prove that the quantization of the conjugacy class $K\backslash G$ can be realized by linear operators on M_λ. To this end, we study the module structure of the tensor product $\mathbb{C}^N \otimes M_\lambda$ in the following section. Again we start with the symmetric case $\ell = 0$. By technical reasons we process separately the cases of even and odd N.

6 The $U_q(\mathfrak{g})$-module $\mathbb{C}^N \otimes M_\lambda$ in the symmetric case

In this section, we study the $U_q(\mathfrak{g})$-module $\mathbb{C}^N \otimes M_\lambda$. We do it "by hands" and have to develop a special diagram technique, which takes the bulk of this section. To a large extent, the difference between Levi and non-Levi classes is concentrated in the "symmetric part" of the stabilizer, so we consider this case first, as we did in the preceding sections.

We assume that the isotropy subalgebra consists of two orthogonal blocks of rank m and p. When restricted to the Levi subalgebra, the vector representation $\mathbb{C}^N$ splits into three irreducible sub-representations, $\mathbb{C}^N = \mathbb{C}^m \oplus \mathbb{C}^P \oplus \mathbb{C}^m$, where $P = 2p$ for the D-series and $P = 2p + 1$ for the B-series.

We fix the standard basis $\{w_i\}_{i=1}^N \subset \mathbb{C}^N$ of columns with the only non-zero entry in the i-th position. The highest weights of the irreducible $U_q(\mathfrak{l})$-submodules in $\mathbb{C}^N$ are ε_1, ε_{m+1}, $-\varepsilon_m$, and the corresponding weight vectors are w_1, w_{m+1}, w_{N+1-m}. For generic λ, the tensor product $\mathbb{C}^N \otimes \hat{M}_\lambda$ splits into the direct sum of three $U_q(\mathfrak{g})$-modules $\mathbb{C}^N \otimes \hat{M}_\lambda = \hat{M}_1 \oplus \hat{M}_2 \oplus \hat{M}_2$, of highest weights $\nu_1 = \lambda + \varepsilon_1$, $\nu_2 = \lambda + \varepsilon_{m+1}$, and $\nu_3 = \lambda - \varepsilon_m$. In the classical limit, the two $\mathfrak{l}$-modules generated by w_1 and w_{N+1-m} glue up to a single irreducible $\mathfrak{k}$-module. In the quantum situation, for $\lambda \in \mathfrak{C}^*_{\mathfrak{k}}$, the projection $\mathbb{C}^N \otimes \hat{M}_\lambda \to \mathbb{C}^N \otimes M_\lambda$ kills $\hat{M}_3$, so that $\mathbb{C}^N \otimes M_\lambda = M_1 \oplus M_2$ with M_i being the images of $\hat{M}_i$. This results in a degree reduction of the minimal polynomial for the quantum coordinate matrix (a similar effect is produced on the classical coordinate matrix by the transition from $L\backslash G$ to $K\backslash G$). Our nearest goal is to prove the decomposition $\mathbb{C}^N \otimes M_\lambda = M_1 \oplus M_2$, the key step in our construction.

The vector $u_{\nu_1} = w_1 \otimes v_\lambda$ carries the highest weight $\lambda + \varepsilon_1$ and generates the submodule $\hat{M}_1$. The singular vector of weight $\lambda + \varepsilon_{m+1}$ generating $\hat{M}_2$ reads

$$u_{\nu_2} \quad = \quad \frac{q^{(\alpha,\lambda)} - q^{-(\alpha,\lambda)}}{q - q^{-1}} w_{m+1} \otimes v_\lambda + (-q)^{-1} w_m \otimes f_{\alpha_m} v_\lambda + \ldots + (-q)^{-m} w_1 \otimes f_{\alpha_1} \ldots f_{\alpha_m} v_\lambda .$$

It is calculated for the symplectic case in [2] and still valid for orthogonal $\mathfrak{g}$, as it involves only generators of $\mathfrak{gl}(n) \subset \mathfrak{g}$. Note that u_{ν_2} is also singular in $\mathbb{C}^N \otimes M_\lambda$, as it does not vanish there.

The following fact is established in [2].

Lemma 6.1. *The singular vector u_{ν_2} is equal to $q^{-m} \frac{q^{(\alpha,\lambda)+m} - q^{-(\alpha,\lambda)-m}}{q - q^{-1}} w_{m+1} \otimes v_\lambda$ modulo $\hat{M}_1$.*

The action of a positive (negative) Chevalley generator on the standard basis $\{w_i\}_{i=1}^N$ features the following property: the line $\mathbb{C}w_i$ is either annihilated or mapped onto the line $\mathbb{C}w_k$ for some k. It is convenient to depict such an action graphically. Below we consider negative generators, since positive can be obtained by reversing the arrows.

Up to an invertible scalar multiplier, the action of the family $\{f_\alpha\}_{\alpha \in \Pi^+} \subset U_q\big(\mathfrak{so}(2n+1)\big)$ on the standard basis $\{w_i\}_{i=1}^{2n+1}$ in $\mathbb{C}^{2n+1}$ is encoded in the following scheme:

$$w_{2n+1} \xleftarrow{f_{\alpha_1}} w_{2n} \xleftarrow{f_{\alpha_2}} \cdots \xleftarrow{f_{\alpha_{n-1}}} w_{n+2} \xleftarrow{f_{\alpha_n}} w_{n+1} \xleftarrow{f_{\alpha_n}} w_n \xleftarrow{f_{\alpha_{n-1}}} \cdots \xleftarrow{f_{\alpha_2}} w_2 \xleftarrow{f_{\alpha_1}} w_1$$

This diagram has simple linear structure without branching and cycles. The diagram for $\mathfrak{g} = \mathfrak{so}(2n)$ is more complicated:

$$\begin{tikzcd} w_{2n} & w_{2n-1} \arrow[l, "f_{\alpha_1}"'] & \cdots \arrow[l, "f_{\alpha_2}"'] & w_{n+2} \arrow[l, "f_{\alpha_{n-2}}"'] & w_{n+1} \arrow[l, "f_{\alpha_{n-1}}"'] & w_n \arrow[ll, bend right, "f_{\alpha_n}"'] & w_{n-1} \arrow[l, "f_{\alpha_{n-1}}"'] \arrow[ll, bend right, "f_{\alpha_n}"'] & \cdots \arrow[l, "f_{\alpha_{n-2}}"'] & w_2 \arrow[l, "f_{\alpha_2}"'] & w_1 \arrow[l, "f_{\alpha_1}"'] \end{tikzcd}$$

Reversing the arrows one gets the diagrams for positive Chevalley generators.

As we said, the arrows designate the action up to a non-zero scalar, which is equal to -1 on the left part of the diagram and $+1$ on the right part. More exactly, moving along the diagram from left to right, the first occurrence of f_α produces -1, while the second gives $+1$. In the matrix language, the non-zero entries above and below the skew diagonal are $+1$ and -1, respectively. Along with this sign rule, the above graphs for $\{f_\alpha\}$ and $\{e_\alpha\}$ determine the representation of $U_q(\mathfrak{g})$ on $\mathbb{C}^N$.

Further we adopt the following convention. If a vector v is proportional to a vector u with a scalar coefficient $c \neq 0$, i.e. $v = cu$, we write $v \simeq u$ and say that v is equivalent to u. If the difference $v - cu$ belongs to a vector space W, we write $v \simeq u \bmod W$.

In order to study the action of $\{f_\alpha\}_{\alpha\in\Pi^+}$ on the tensor product $\mathbb{C}^N \otimes \hat{M}_\lambda$, we develop our diagram technique further. First of all, we transpose the above diagrams of the natural representation to columns, so the arrows become vertical and oriented downward.

Suppose $(v^i)_{i=1}^l \in \hat{M}_\lambda$ is a finite sequence of vectors. We associate a horizontal graph with nodes $(v^i)_{i=1}^l$ and arrows designating the action of $\{f_\alpha\}_{\alpha\in\Pi^+}$, in a similar fashion as vertical but with the following difference: it involves *not all* possible arrows $v^k \leftarrow v^i$ but only those of our interest. We still assume that the chosen arrows are isomorphisms of lines spanned by v^i. This implies that, up to a non-zero scalar factor, the nodes are *determined* by the subset of maximal nodes (with no inward arrows) and by the set of arrows. In our case, there will be only one maximal vector $v^1 = v_\lambda$, so other nodes are determined by arrows. This implies that a horizontal graph is connected.

Let $\mathrm{Arr}(w_k)$ denote the set of negative Chevalley generators whose arrows are directed from w_k. Similarly, $\mathrm{Arr}(v^i)$ denote the set of generators whose arrows are directed from v^i. For instance, this set consists of only one element for $k > 2n+1$ and is empty for $k = 2n+1$, in the series B. We say that arrow f has length k if it sends node i to the node $i+k$. All vertical arrows but f_{α_n} for even N have length 1.

We construct a "tensor product" of vertical and horizontal graphs to be a diagram with the nodes $w_k \otimes v^i$, $k = 1, \dots, N$, $i = 1, \dots, l$. The factor w_k marks the rows from top to bottom, while v^i marks the columns from right to left. The vertical and horizontal arrows display the action of the designated Chevalley generators on the tensor factors, up to a scalar multiplier. Under the assumptions made, such diagrams provide information about the action of $\{f_\alpha\}_{\alpha\in\Pi^+}$ not just on the tensor factors but on the entire tensor product $\mathbb{C}^N \otimes \hat{M}_\lambda$, in the following sense:

Proposition 6.2. *Suppose a horizontal arrow designates the action of a Chevalley generator f_α on v^i. If $f_\alpha \notin \mathrm{Arr}(w_k)$, then $f_\alpha(w_k \otimes v^i) \simeq w_k \otimes (f_\alpha v^i)$, otherwise $f_\alpha(w_k \otimes v^i) \simeq w_k \otimes f_\alpha v^i$ modulo $\mathbb{C} f_\alpha w_k \otimes v^i$.*

Proof. This statement immediately follows from the assumptions and quasi-primitivity of the Chevalley generators (cf. the comultiplication in Section 3). In particular, $f_\alpha \notin \mathrm{Arr}(w_k)$ if and only if $f_\alpha w_k = 0$, hence the first alternative. The second alternative is also based on quasi-primitivity of f_α. □

Corollary 6.3. *Suppose that $f_\alpha \in \mathrm{Arr}(v^i) - \mathrm{Arr}(w_k)$ for some i and k. Suppose that for $k_1 \leqslant j \leqslant k$ the nodes $w_j \otimes v^i$ lie in a submodule $\hat{M} \subset \mathbb{C}^N \otimes \hat{M}_\lambda$ and all arrows from $\mathrm{Arr}(v^j)$ have length 1. Then $\mathrm{Span}\{w_j \otimes f_\alpha v^i\}_{j=k_1}^k \subset \hat{M}$.*

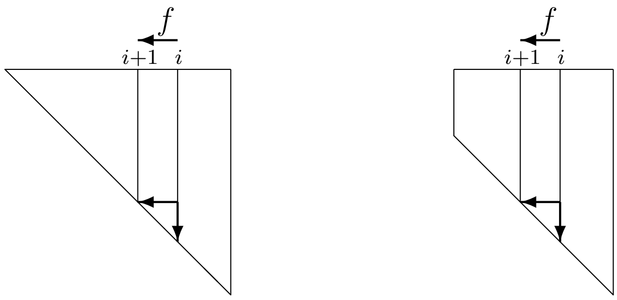
Proof. We have $f_\alpha(w_k \otimes v^i) \simeq w_k \otimes f_\alpha v^i$, as $f_\alpha w_k = 0$. Therefore, $\mathrm{Span}\{f_\alpha(w_j \otimes v^i)\}_{j=k_1}^k = \mathrm{Span}\{w_j \otimes f_\alpha v^i\}_{j=k_1}^k$ modulo $\mathrm{Span}\{w_j \otimes v^i\}_{j=k_1}^k$. Now the proof is immediate. □

Suppose there are segments of vertical nodes (w_k), $k \in I_v = [k_1, k_2]$, and horizontal nodes (v^i), $i \in I_h = [i_1, i_2]$, such that: *all* vertical arrows directed from w_k, $k \in I'_v = [k_1, k_2 - 1]$ are of length 1; for each $i \in I'_h = [i_1, i_2 - 1]$ *there is* a horizontal arrow of length 1. In particular, $\mathrm{Arr}(w_k)$ consists of one element for all $k \in I'_v$. Let us denote the subset of these selected horizontal arrows by A_h. Consider the subgraph with nodes $(w_k \otimes v^i)$, $(k, i) \in I_v \times I_h$, the vertical arrows from $\mathrm{Arr}(w_k)$, $k \in I'_v$, and the horizontal arrows from the selected subset A_h. We call such a subgraph *simple rectangle*. Topologically, it is simply connected. In particular, the entire diagram may be simple.

Specifically the diagrams of interest will be defined in the next section. Here we establish a general fact, which will be used in what follows.

Lemma 6.4. *Suppose that a diagram D contains an equilateral rectangular triangle T (leveled by top and right edges) belonging to a simple rectangle in D. Suppose that the right edge of T belongs to a submodule $\hat{M} \subset \mathbb{C}^N \otimes \hat{M}_\lambda$. Then the entire triangle T belongs to $\hat{M}$.*

Proof. Without loss of generality we may assume that D is simple and T sits in the north-east corner of D, i.e. $w_1 \otimes v^1 \in \hat{M}$ is its maximal node. Suppose that its edge contains t nodes. We do induction on column's number i, which is illustrated by the following figure (on the left).



By the hypothesis, the column $\{w_k \otimes v^1\}_{k=1}^t$ lies in $\hat{M}$. Suppose that the statement is proved for $1 \leqslant i < t$. Let $f \in U_q(\mathfrak{g})$ be the operator assigned to the horizontal arrow $v^{i+1} \leftarrow v^i$. Each node $w_k \otimes v^i$, $k = 1, \ldots, t - i$, is sent by f to $w_k \otimes v^{i+1}$ possibly modulo $\mathbb{C}w_{k+1} \otimes v^i$, which belongs to $\hat{M}$ by the induction assumption. Therefore, the $i + 1$-st column of T does belong to $\hat{M}$. ∎

Remark that the triangle can be replaced with a trapezoid obtained by cutting off the left end of T with vertical line, as shown on the figure (on the right).

The sequence (v^i) is assumed to be finite and contain the unique minimal node (no outward arrows). This node is in the focus of our interest. In what follows, it carries the weight $\lambda - \delta$, and the whole diagram yields a path (paths) to it from the maximal node v_λ. This way, a diagram is associated with every non-vanishing Chevalley monomial in $[\hat{M}_\lambda]_{\lambda-\delta}$ participating in the singular vector $v_{\lambda-\delta}$.

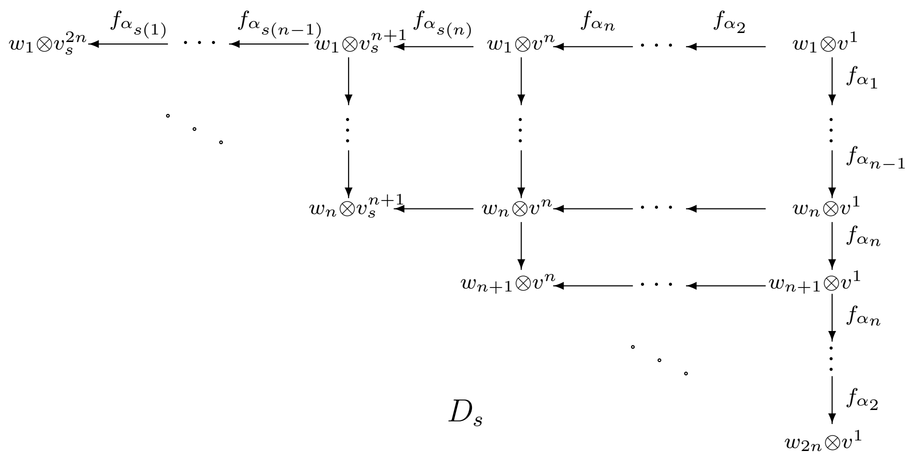
6.1 Series B, symmetric case $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(2n-3)$

Suppose first that $m = 2$. Later on we drop this restriction. Given a permutation s of $1, \ldots, n$, we define Chevalley monomials $v_s^i \in \hat{M}_\lambda$, $i = 1, \ldots, 2n$, through the graph

$$\begin{array}{ccccccccccc} \circ & \xleftarrow{f_{\alpha_{s(1)}}} & \cdots & \xleftarrow{f_{\alpha_{s(n-1)}}} & \circ & \xleftarrow{f_{\alpha_{s(n)}}} & \circ & \xleftarrow{f_{\alpha_n}} \cdots \xleftarrow{f_{\alpha_3}} & \circ & \xleftarrow{f_{\alpha_2}} & \circ \\ v_s^{2n} & & & & v_s^{n+1} & & v_s^n & & & & v_s^1 = v_\lambda \end{array}$$

In particular, the minimal node of the graph is $v_s^{2n} = f_{\alpha_{s(1)}} \ldots f_{\alpha_{s(n)}} f_{\alpha_n} \ldots f_{\alpha_2} v_\lambda \in [\hat{M}_\lambda]_{\lambda-\delta}$. For $s = \mathrm{id}$ we omit the subscript s and denote v_s^i simply by v^i.

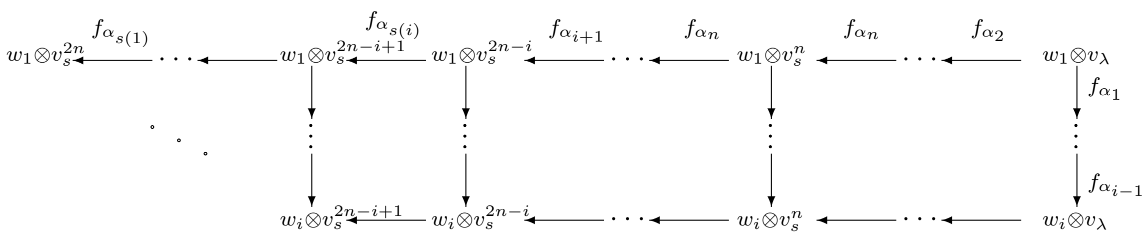
With a permutation s of $1, \ldots, n$ such that $v_s^{2n} \neq 0$ we associate a diagram D_s, as explained in the preceding section. Essential are the nodes $\{w_k \otimes v^i\}$ with $i + k \leqslant 2n + 1$, so we display only this triangular part of D_s:



D_s

Note that first n columns (counting from the right) in all D_s are the same.

Denote by $D'_{\mathrm{id}} \subset D_{\mathrm{id}}$ the sub-graph above the principal diagonal, i.e. consisting of nodes $\{w_k \otimes v^j\}$ such that $k + j \leqslant 2n$. Given $s \neq \mathrm{id}$ let i the highest of $1, \ldots, n$ displaced by s, i.e. $s(i) \neq i$. Denote by $D'_s \subset D_s$ the trapezoid of nodes $\{w_k \otimes v_s^j\}$ obeying $k + j \leqslant 2n + 1$, $k \leqslant i$.



D'_s, $\quad s \neq \mathrm{id}$

Let $\hat{M}$ denote the $U_q(\mathfrak{g})$-submodule $\hat{M}_1 + \hat{M}_2 \subset \mathbb{C}^N \otimes \hat{M}_\lambda$.

Lemma 6.5. *Suppose that $q^{2p+5} \neq -1$. Then D'_s lies in $\hat{M}$.*

Proof. First assume that $s = \mathrm{id}$. The statement is trivial for the node $u_{\nu_1} = w_1 \otimes v_\lambda$ generating $\hat{M}_1$. By Lemma 6.1, $u_{\nu_2} \simeq w_2 \otimes v_\lambda \mod \hat{M}_1$, under the assumption $q^{2p+5} \neq -1$. Hence the $w_2 \otimes v_\lambda$ belongs to $\hat{M}$ too. Further observe that the rightmost column of D'_{id} lies in $\hat{M}$. For this part of D'_{id}, the vertical arrows actually depict the action on the whole tensor square, as they kill v_λ. Notice that D_s is simple. One is left to apply Lemma 6.4 to the triangle $T = D'_{\mathrm{id}}$.

Now we consider the case $s \neq \mathrm{id}$ and let i be the highest integer displaced by s. Observe that the right rectangular part of D'_{id} up to column $2n - i$ is the same as in D_{id} and belongs to D'_{id}. Hence it lies in $\hat{M}$, as already proved. Since $s(i) \neq i$, the horizontal arrow $f_{\alpha_{s(i)}} \in \mathrm{Arr}(v_s^{2n-i})$ is distinct from vertical f_{α_i} constituting $\mathrm{Arr}(w_i)$ (suppressed in the graph). By Corollary 6.3, column $2n - i + 1$ of D'_{id} belongs to $\hat{M}$. The remaining part of D'_s is a triangle bounded on the right with column $2n - i + 1$. By Lemma 6.4, it belongs to $\hat{M}$. $\square$

Now we are ready to prove the following

Proposition 6.6. *Suppose that $q^{2m+2p+1} \neq -1$. Then the tensor product $\mathbb{C}^N \otimes M_\lambda$ splits into the direct sum $M_1 \oplus M_2$.*

Proof. Observe that the sum $M_1 + M_2$ is direct, as M_i carry different eigenvalues $\mu_1 = -q^{2p+1}$, $\mu_2 = q^{-2m}$ of the intertwiner $\mathcal{Q}$, see (7.14) below. We prove the statement if we show that $\mathbb{C}^N \otimes v_\lambda \subset M = M_1 \oplus M_2$. First consider the case $m = 2$. Then w_{2n} is the highest weight vector of the $\mathfrak{l}$-submodule of weight $-\varepsilon_2$ in $\mathbb{C}^N$. Our strategy is to reach $w_{2n} \otimes v_\lambda$ from $w_1 \otimes v_\lambda$ passing through all $w_i \otimes v_\lambda$ in between and staying within M. Then we get $w_{2n+1} \otimes v_\lambda \in M$ by applying f_{α_1} to $w_{2n} \otimes v_\lambda$.

By Lemma 6.5, the diagonal of D_{id} right over the principal diagonal lies in $\hat{M}$. Notice that the vertical and horizontal arrows applied to all nodes in this diagonal coincide. Therefore, up to a non-zero scalar factor, the elements on the principal diagonal are all equivalent modulo $\hat{M}$. For instance, apply f_{α_1} to $w_1 \otimes v^{2n-1} \in \hat{M}$ and get $q^2 w_2 \otimes v^{2n-1} + w_1 \otimes v^{2n}$, hence $w_1 \otimes v^{2n} = -q^2 w_2 \otimes v^{2n-1} \mod \hat{M}$. Moving further down the principal diagonal, we find $w_1 \otimes v^{2n} \simeq w_{2n} \otimes v_\lambda \mod \hat{M}$. Now notice that $x_2 = v^{2n} - a v_s^{2n}$, where s is the transposition $(1, 2)$. Observe that all other x_i are linear combinations of v_s^{2n} for certain $s \neq \mathrm{id}$. Since $v_s^{2n} \in \hat{M}$ by Lemma 6.5, $w_1 \otimes v_{\lambda-\delta} \simeq w_{2n} \otimes v_\lambda$ modulo $\hat{M}$. Hence $w_{2n} \otimes v_\lambda \in M$. Applying f_{α_1} to $w_{2n} \otimes v_\lambda$ we get $w_{2n+1} \otimes v_\lambda \in M$.

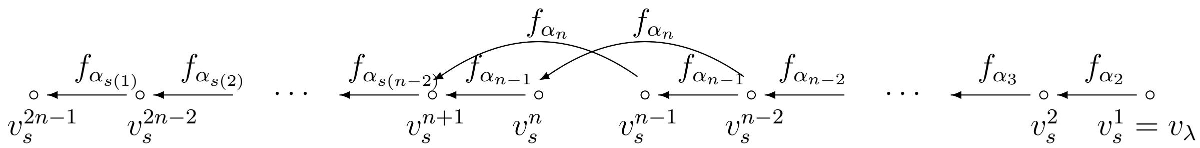
We have proved the inclusion $\mathbb{C}^N \otimes v_\lambda \subset M$ under the assumption $m = 2$. Now we drop this restriction. First of all, $w_i \otimes v_\lambda = f_{\alpha_{i-1}}(w_{i-1} \otimes v_\lambda)$ for $i = 2, \ldots, m$, hence $\mathbb{C}^m \otimes v_\lambda \subset M_1 \subset M$. The transition from $w_m \otimes v_\lambda$ to $w_{m+1} \otimes v_\lambda$ is facilitated by Lemma 6.1 and is similar to the case $m = 2$. Namely, $w_{m+1} \otimes v_\lambda \simeq u_{\nu_2} \in M_2 \subset M$ modulo M under the assumption $q^{2m+2p+1} \neq -1$. This reduces the proof to $m = 2$-case. Acting by $\{f_\alpha\}_{\alpha \in \mathfrak{g}_{p+2}}$ on $w_{m-1} \otimes v_\lambda$ we check that $\mathbb{C}^{2p+1} \otimes v_\lambda \subset M$ and $w_{N+1-m} \otimes v_\lambda \in M$. Then we descent from $w_{N+1-m} \otimes v_\lambda$ to $w_{2n+1} \otimes v_\lambda$ using the negative Borel subalgebra of $U_q(\mathfrak{l})$. This way we complete the inclusion $\mathbb{C}^N \otimes M_\lambda \subset M$. $\square$

The assumption of Proposition 6.6 should be viewed as a condition on q. It is fulfilled for an open set including $q = 1$ and therefore over the formal series in $\hbar$ with $q = e^\hbar$. Observe that $q^{2m+2p+1} = -1$ if and only if the eigenvalues of $\mathcal{Q}$ coincide, cf. (7.14). This is accountable by Lemma 6.1, because for such q we get the inclusion $\hat{M}_2 \subset \hat{M}_1$. Similar effect takes place for $\mathfrak{g} = \mathfrak{so}(2n)$, see the next section.

6.2 Series D, symmetric case $\mathfrak{k} = \mathfrak{so}(4) \oplus \mathfrak{so}(2n-4)$

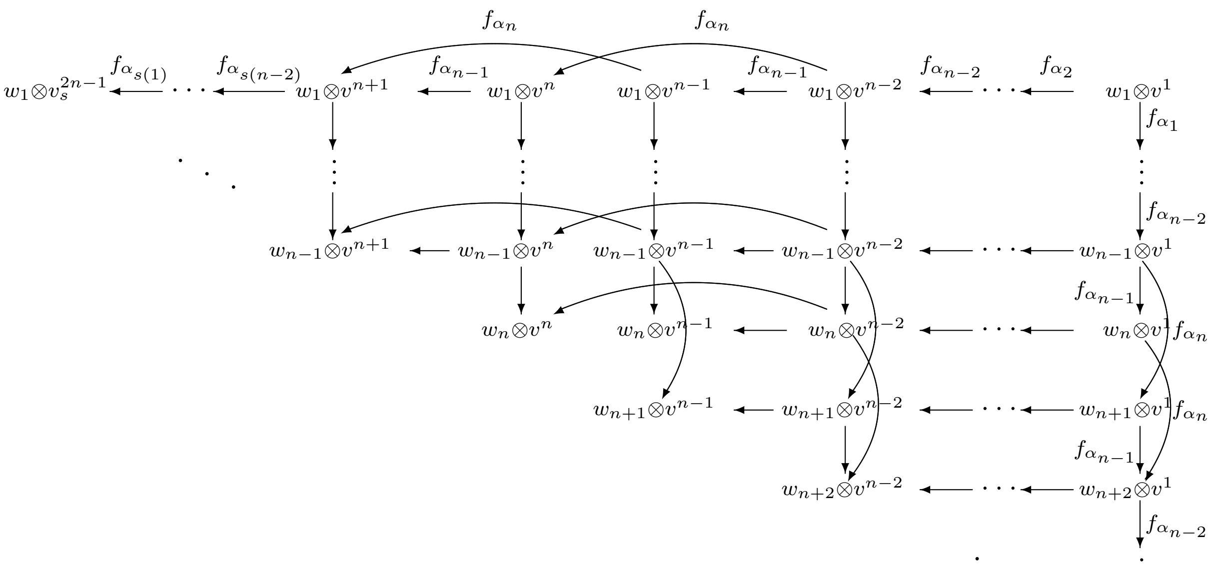
We have to consider two types of diagrams for $\mathfrak{g} = \mathfrak{so}(2n)$. One of them accounts for Chevalley monomials constituting the vectors x_i, $i = 2, \ldots, n-2$. The other corresponds to monomials entering the "tail" vectors x_{n-1} and x_n.

Let us start with the first type. Given a permutation s of $1, \ldots, n-2$ we define Chevalley monomials $v_s^i \in \hat{M}_\lambda$ for $i = 1, \ldots, 2n-1$ through the graph



The minimal element of this sequence is $v_s^{2n-1} = f_{\alpha_{s(1)}} \ldots f_{\alpha_{s(n-2)}} f_{\alpha_n} \ldots f_{\alpha_2} v_\lambda$. The first $n+1$ terms are independent of s. As for odd N, we drop the subscript s from v_s^i for $s = \mathrm{id}$.

With every permutation s such that $v_s^{2n-1} \neq 0$ we associate the diagram D_s. As before, we restrict consideration to the triangular part of it, retaining only $w_k \otimes v_s^i$ with $k + i \leqslant 2n$.



D_s

Only the part of D_s to the left of column $n+1$ depends on s. We have emphasized this by omitting the subscript s in the right part.

The diagrams D_s account for Chevalley monomials participating in $\{x_i\}_{i=2}^{n-2}$. The vectors x_{n-1} and x_n involve different diagrams. Due to the symmetry between x_{n-1} and x_n we consider only x_n. Define the set $\{v^i\}_{i=1}^{2n-1} \subset M_\lambda$ as follows. The first $n-1$ vectors are as before: $v^1 = v_\lambda$, $v^i = f_{\alpha_i} v^{i-1}$, $i = 2, \ldots, n-1$. The other n vectors are set to be

$$v^{n-1+k} = f_{\alpha_k} v^{n+k-2}, \quad k = 1, \ldots, n.$$

The arrows $v^{i-1} \leftarrow v^i$ are uniquely determined by the set of nodes $\{v^i\}$. Given a permutation s of $1, \ldots, n-2$ we define the set $\{v_s^i\}_{i=1}^{2n-1}$ by $v_s^i = v^i$, $i = 1, \ldots, n-1$ and

$$v_s^{n-1+k} = f_{\alpha_{s(k)}} v_s^{n+k-2}, \quad k = 1, \ldots, n.$$

In fact, for most s, these vectors are zero. Of all s we only need the transposition $(1,2)$. This is sufficient for x_n, which comprises two Chevalley monomials, due to the factor $[f_{\alpha_1}, f_{\alpha_2}]_a$ in it.

$$D^n_{\mathrm{id}}$$

$$\begin{tikzcd}
w_1 \otimes v^{2n-1} & \cdots \arrow[l, "f_{\alpha_n}"'] & w_1 \otimes v^{n+1} \arrow[l, "f_{\alpha_2}"'] \arrow[d] & w_1 \otimes v^{n-1} \arrow[l, "f_{\alpha_1}"'] \arrow[d] & w_1 \otimes v^{n-2} \arrow[l, "f_{\alpha_{n-1}}"'] \arrow[d] & \cdots \arrow[l, "f_{\alpha_{n-2}}"'] & w_1 \otimes v^1 \arrow[l, "f_{\alpha_2}"'] \arrow[d, "f_{\alpha_1}"] \\
 & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d, "f_{\alpha_{n-2}}"] \\
 & & w_{n-1} \otimes v^{n+1} & w_{n-1} \otimes v^{n-1} \arrow[l] & w_{n-1} \otimes v^{n-2} \arrow[l] & \cdots \arrow[l] & w_{n-1} \otimes v^1 \arrow[l]
\end{tikzcd}$$

Here we display only the part which is relevant to our study. In what follows, we have to mind the arrows $f_{\alpha_{n-1}}, f_{\alpha_n} \in \mathrm{Arr}(w_{n-1})$, which are directed from the bottom line. Note that the right square of $(n-1) \times (n-1)$ nodes is the same in D_s^n for all s. It is also a sub-graph in D_{id}.

Denote by $D'_{\mathrm{id}} \subset D_{\mathrm{id}}$ the sub-graph above the principal diagonal, i.e. $\{w_k \otimes v^j\}$ such that $k + j \leqslant 2n-1$. For $s \neq \mathrm{id}$, let i be maximal of $1, \ldots, n-2$ displaced by s. We denote by $D'_s \subset D_s$ the trapezoid rested on line i, i.e. $\{w_k \otimes v^j\}$ subject to $k + j \leqslant 2n-1$ and $k \leqslant i$.

Lemma 6.7. *Suppose that $q^{4-2p} \neq -1$. Then D'_s and $D_s^{n\prime}$ lie in $\hat{M}$.*

Proof. The situation is slightly different from the settings of Lemma 6.4 (odd N), as the diagrams D_s are not simple. Applying similar arguments as in the proof of Lemma 6.5 we check that the trapezoid of D'_{id} up to column $w_i \otimes v^{n-2}$ lies in $\hat{M}$. The generator $f_{\alpha_{n-1}}$ sends the nodes of this column one step to the left modulo maybe one step down. Since the node $w_n \otimes v^{n-2}$ is sent strictly leftward, column $n-2$ of D'_{id} is mapped onto column $n-1$, modulo its column $n-2$, which does belong to $\hat{M}$. Therefore, column $n-1$ of D'_{id} lies in $\hat{M}$. The bottom node of column n of $D'_{\rm id}$ is $w_{n-1}\otimes v^n$. Modulo $w_{n+1}\otimes v^{n-2}\in \hat M$, it is the image of $w_{n-1}\otimes v^{n-2}\in \hat M$ under the arrow f_{α_n}. Hence $w_{n-1}\otimes v^n\in \hat M$. The nodes higher in this column are also obtained from column $n-2$ via f_{α_n}, which now acts strictly leftward. Therefore, the right part of $D'_{\rm id}$ lies in $\hat M$ up to column n column. The remaining part of $D'_{\rm id}$ to the left of column n inclusive is a triangle in a simple rectangle (just ignore the leftmost f_{α_n}) and falls under Lemma 6.4.

Now suppose that $s\neq {\rm id}$ and and let i be the highest integer displaced by s. Contrary to $s={\rm id}$, this case is pretty similar to Proposition 6.6 for odd N. Notice that right rectangular part of D'_s up to column $2n-i-1$ is the same as in $D'_{\rm id}\subset D_{\rm id}$ and lies in $\hat M$ as argued. Since $f_{\alpha_{s(i)}}\neq f_{\alpha_i}$, column $2n-i$ of $D'_{\rm id}$ lies in $\hat M$, by Corollary 6.3. The remaining part of $D'_{\rm id}$ is the triangle bounded by column $2n-i$ on the right. It belongs to $\hat M$ by Lemma 6.4.

The proof for $D^{n}_s{}'$ for $s={\rm id},(1,2)$ is similar to D'_s with $s\neq {\rm id}$. The key observation is that right rectangular part up to column $n-1$ is a sub-graph in $D'_{\rm id}$ and hence lie in $\hat M$. Further arguments are based on Lemma 6.3 applied to column $n-1$ of D'_s. $\square$

Now we are ready to prove the main result of this section.

Proposition 6.8. *Suppose that $q^{2m-2p}\neq -1$. Then the tensor product $\mathbb{C}^N\otimes M_\lambda$ splits to the direct sum $M_1\oplus M_2$.*

Proof. The invariant operator $\mathcal{Q}$ turns scalar multiplier on $\hat M_1$ and on $\hat M_2$, with the eigenvalues $\mu_1=-q^{2p}$, $\mu_2=q^{-2m}$, cf. (7.14). Hence the sum is direct. As for $\mathfrak{g}=\mathfrak{so}(2n+1)$, we need to show that $M=M_1\oplus M_2$ exhausts all of $\mathbb{C}^N\otimes M_\lambda$, and it is sufficient to prove the inclusion $\mathbb{C}^N\otimes v_\lambda\subset M$.

First we consider the case $m=2$. Then w_{2n-1} is the highest weight vector of the $\mathfrak{l}$-submodule $\mathbb{C}^m\subset\mathbb{C}^N$ of highest weight $-\varepsilon_2$. As before, we intend to reach $w_{2n-1}\otimes v_\lambda$ from $w_1\otimes v_\lambda$ through all $w_i\otimes v_\lambda$ in between staying within M. Then we get $w_{2n}\otimes v_\lambda\in M$ by applying f_{α_1} to $w_{2n-1}\otimes v_\lambda$.

By Lemma 6.7, the diagonal of $D_{\rm id}$ over the principal diagonal lies in $\hat M$, as it belongs to $D'_{\rm id}$. The vertical and horizontal arrows applied to this diagonal coincide. Therefore, up to a non-zero scalar factor, the elements on the principal diagonal are all equivalent modulo $\hat M$. In particular, $w_1\otimes v^{2n-1}\simeq w_{2n-1}\otimes v_\lambda \mod \hat M$. Now notice that $x_2=v^{2n-1}-av^{2n-1}_s$, where s is the transposition $1\leftrightarrow 2$. Observe that all other x_i, $i<n-1$ are linear combinations of v^{2n-1}_s for certain $s\neq{\rm id}$. Since $v^{2n-1}_s\in\hat M$ for such s by Lemma 6.7, all x_i with $i<n-1$ belong to $\hat M$. The vector x_n is a combination of v^{2n-1} and v^{2n-1}_s, where s is the transposition $(1,2)$. In view of Lemma 6.7 we conclude that $w_1\otimes x_n\in\hat M$. Due to the symmetry between x_{n-1} and x_n, we conclude that $w_1\otimes x_{n-1}\in\hat M$ too.

Since the singular vector $v_{\lambda-\delta}$ is a linear combination of x_i, $i=2,\ldots,n$, the vector $w_{2n-1}\otimes v_\lambda$ is equivalent to $w_1\otimes v_{\lambda-\delta}$ modulo $\hat M$. Hence $w_{2n-1}\otimes v_\lambda\in M$ as required.

Now we lift the restriction $m = 2$. This is done similarly to the $\mathfrak{g} = \mathfrak{so}(2n+1)$-case. Using $f_{\alpha_1}, \ldots, f_{\alpha_{m-1}}$ from the Levi subalgebra we get $w_i \otimes v_\lambda \in M$ for $i = 1, \ldots, m$. Lemma 6.1 facilitates transition to the vector $w_{m+1} \otimes v_\lambda \in M$, under the assumption $q^{2m-2p} \neq -1$. Further we pass to the subalgebra $U_q(\mathfrak{g}_{p+2})$ and reduce consideration to the case $m = 2$. This results in $\mathbb{C}^{2p} \otimes v_\lambda \subset M$ and $w_{n+p+1} \otimes v_\lambda \in M$. Finally, using the Levi generators $f_{\alpha_1}, \ldots, f_{\alpha_{m-1}}$, we descend from $w_{n+p+1} \otimes v_\lambda \in M$ to $w_{2n} \otimes v_\lambda \in M$. This completes the proof. $\square$

6.3 The module $\mathbb{C}^N \otimes M_\lambda$, general $\mathfrak{k}$

The symmetric case worked out in detail in the preceding sections will serve as an illustration to the case of general $\mathfrak{k}$ considered below. However, our strategy will be slightly different, in order to save the effort of calculating singular vectors in $\mathbb{C}^N \otimes M_\lambda$. We pay a price for that by getting a weaker result about the structure of $\mathbb{C}^N \otimes M_\lambda$. Namely, instead of direct sum decomposition of $\mathbb{C}^N \otimes M_\lambda$ we construct a filtration by highest weight modules. Still it is sufficient for our purposes, as all we need to know is the spectrum of the quantum coordinate matrix $\mathcal{Q}$, see (7.11). Under certain conditions, it can be extracted from the graded module associated with filtration as well as from direct sum decomposition.

We have irreducible decomposition

$$\mathbb{C}^N = \mathbb{C}^{n_1} \oplus \ldots \oplus \mathbb{C}^{n_\ell} \oplus \mathbb{C}^m \oplus \mathbb{C}^P \oplus \mathbb{C}^m \oplus \mathbb{C}^{n_\ell} \oplus \ldots \oplus \mathbb{C}^{n_1}$$

of the natural $\mathfrak{g}$-representation $\mathbb{C}^N$ into $\mathfrak{l}$-blocks. We enumerate them from left to right as W_i, $i = 1, \ldots, 2\ell + 3$. This decomposition is compatible with the standard basis $\{w_i\}$, and the basis element with the lowest number falling into the block is its highest weight vector. Let ν_i, $i = 1, \ldots, 2\ell + 3$, be the highest weights of the irreducible blocks and let $w_{\nu_i} \in W_i$ denote their highest weight vectors. As we said, they form a subset of the standard basis. Explicitly, the highest weights of the blocks are $\nu_i = \varepsilon_{n_1+\ldots+n_{i-1}+1}$ for $i = 1, \ldots, \ell + 2$ and $\nu_{2\ell+4-i} = -\varepsilon_{n_1+\ldots+n_i}$ for $i = 1, \ldots, \ell + 1$.

For generic λ this decomposition gives rise to the decomposition $\mathbb{C}^N \otimes \hat{M}_\lambda = \oplus_{i=1}^{2\ell+3} \hat{M}_i$ where each $\hat{M}_i$ is the parabolic Verma module induced from $W_i \subset \mathbb{C}^N$. Let M_i denote its image under the projection to $\mathbb{C}^N \otimes M_\lambda$.

Transition to the isotropy subalgebra $\mathfrak{k} \supset \mathfrak{l}$ merges two copies of $\mathbb{C}^m$ up into a single irreducible $\mathfrak{k}$-submodule. As a result, $M_{\ell+3}$ should disappear from $\mathbb{C}^N \otimes M_\lambda$. We saw this effect for $\ell = 0$ and we expect it for general $\mathfrak{k}$. However, constructing the direct sum decomposition of $\mathbb{C}^N \otimes M_\lambda$ along the same lines requires the knowledge of singular vectors for all $\hat{M}_i$. Instead of finding them, we work with a filtration, which construction is much easier. We do not even check that each graded component, apart from the $\ell + 3$-d, survives in the projection $\mathbb{C}^N \otimes \hat{M}_\lambda \to \mathbb{C}^N \otimes M_\lambda$. We just need to make sure that the simple divisor corresponding to the quotient $V_{\ell+3}/V_{\ell+2}$ drops from the minimal polynomial of $\mathcal{Q}$.

For all $j = 1, \ldots, 2\ell + 3$ we denote by $\hat{V}_j$ the submodule in $\mathbb{C}^N \otimes \hat{M}_\lambda$ generated by $\{w_{\nu_i} \otimes v_\lambda\}_{i=1,\ldots,j}$. Let V_j denote their images in $\mathbb{C}^N \otimes M_\lambda$. We have the obvious inclusions $\hat{V}_{j-1} \subset \hat{V}_j$, $V_{j-1} \subset V_j$. It is convenient to set $\hat{V}_0$ and V_0 to $\{0\}$.

Proposition 6.9. *The submodules $\{0\} = V_0 \subset V_1 \subset \ldots \subset V_{2\ell+3}$ form an ascending filtration of $\mathbb{C}^N \otimes M_\lambda$. For each $k = 1, \ldots, 2\ell + 3$, the graded component V_k/V_{k-1} is either $\{0\}$ or generated by (the image of) $w_{\nu_k} \otimes v_\lambda$, which is the highest weight vector in V_k/V_{k-1}. In particular, $V_{\ell+2} = V_{\ell+3}$.*

Proof. Our strategy is similar to the proof of Propositions 6.6 and 6.8. We mean to show that $\oplus_{i=1}^k W_i \otimes v_\lambda \subset V_k$ implying $\mathbb{C}^N \otimes v_\lambda \subset V_{2\ell+3}$ for $k = 2\ell+3$. Then $e_\alpha(w_{\nu_k} \otimes v_\lambda) = 0 \mod V_{k-1}$, i.e. $w_{\nu_k} \otimes v_\lambda$ is a singular vector in V_{k-1}/V_k if not zero. Since V_{k-1}/V_k is generated by $w_{\nu_k} \otimes v_\lambda$, it is the highest weight vector. This will imply $\mathbb{C}^N \otimes v_\lambda \subset V_{2\ell+3}$ and $V_{2\ell+3} = \mathbb{C}^N \otimes v_\lambda \subset M_\lambda$.

Thus, we wish to prove that $W_k \otimes v_\lambda \subset V_k$. This is true for $k = 0$ if we set $W_0 = \{0\}$. Suppose we have done this for some $k \geqslant 0$. By construction, $w_{\nu_{k+1}} \otimes v_\lambda \in V_{k+1}$. Consecutively applying the negative Chevalley generators from the appropriate block of $U_q(\mathfrak{l})$ we conclude that $W_{\nu_{k+1}} \otimes v_\lambda \subset V_{k+1}$. Induction on k proves $W_k \otimes v_\lambda \subset V_k$ for all k.

Finally, the equality $V_{l+2} = V_{l+3}$ follows from the inclusion $W_{l+3} \otimes v_\lambda \subset V_{l+2}$, and this boils down to the symmetric case studied in Propositions 6.6 and 6.8: it is sufficient to apply the negative Chevalley generators of $\mathfrak{g}_{p+2} \subset \mathfrak{g}$ to $w_{n-p-1} \otimes v_\lambda \in V_{l+2}$, in order to get $w_{N-n+p+1} \otimes v_\lambda$. The latter vector generates V_{l+3} modulo V_{l+2}. Hence $V_{l+3} = V_{l+2}$, and the proof is complete. $\square$

7 The matrix of quantum coordinate functions

Similarly to classical conjugacy classes, their quantum counterparts are described through a matrix A of non-commutative "coordinate functions" or its image $\mathcal{Q} \in \mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{g})$, which should be regarded as "restriction" of A to the quantum group G_q. In this section we study algebraic properties of $\mathcal{Q}$.

The operator $\mathcal{Q}$ is defined through the universal R-matrix $\mathcal{R}$, which is an invertible element of (completed) tensor square of $U_\hbar(\mathfrak{g})$:

$$\mathcal{Q} = (\pi \otimes \mathrm{id})(\mathcal{R}_{12}\mathcal{R}) \in \mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{g}). \tag{7.11}$$

Here π is the representation homomorphism $U_\hbar(\mathfrak{g}) \to \mathrm{End}(\mathbb{C}^N)$. The matrix $\mathcal{Q}$ commutes with $(\pi \otimes \mathrm{id}) \circ \Delta(u)$ for all $u \in U_q(\mathfrak{g})$ producing an invariant operator on $\mathbb{C}^N \otimes V$ for every $U_q(\mathfrak{g})$-module V.

Let ρ denote the half-sum of all positive roots $\rho = \frac{1}{2}\sum_{\alpha\in \mathrm{R}_+}\alpha$. In the orthogonal basis of weights $\{\varepsilon_i\}$, it reads

$$\rho = \sum_{i=1}^{n}\rho_i\varepsilon_i, \quad \rho_i = \rho_1 - (i-1), \quad \rho_1 = \left\{\begin{array}{lllll} n-\frac{1}{2} & \text{for} & \mathfrak{g} & = & \mathfrak{so}(2n+1), \\ n-1 & \text{for} & \mathfrak{g} & = & \mathfrak{so}(2n). \end{array}\right.$$

Regarded as an operator on $\mathbb{C}^N\otimes \hat{M}_\lambda$, the element $\mathcal{Q}$ satisfies a polynomial equation with the roots

$$q^{2(\lambda+\rho,\nu_i)-2(\rho,\varepsilon_1)+(\nu_i,\nu_i)-1} = \left\{\begin{array}{rl} q^{2(\lambda,\nu_i)+2(\rho,\nu_i-\varepsilon_1)} & \text{for} \quad p>0, \\ q^{-2n} & \text{for} \quad p=1, \quad i=\ell+2, \quad \mathfrak{g}=\mathfrak{so}(2n+1). \end{array}\right. .$$

where ν_i, $i=1,\ldots,2\ell+3$ are the highest weights of the irreducible $\mathfrak{l}$-submodules in $\mathbb{C}^N$, [3]. The bottom line corresponds to zero ν_i, which is present only for odd N if $p=0$.

Assuming $\lambda\in\mathfrak{C}^*_{\mathfrak{l},reg}$, put $\Lambda_i = (\lambda,\varepsilon_{n_1+\ldots+n_{i-1}+1}) = (\lambda,\varepsilon_{n_1+\ldots+n_i})$ for $i=1,\ldots,\ell+2$ (recall that $n_{\ell+1}=m$ and $n_{\ell+2}=p$, by our convention). The weight λ depends on the parameters (Λ_i), with $\Lambda_{\ell+2}=0$. Define the vector $\boldsymbol{\mu}$ by

$$\mu_i = q^{2\Lambda_i-2(n_1+\ldots+n_{i-1})}, \quad i=1,\ldots,\ell+2. \tag{7.12}$$

The eigenvalues of $\mathcal{Q}$ on $\mathrm{End}(\mathbb{C}^N\otimes\hat{M}_\lambda)$ are expressed through $\boldsymbol{\mu}$ by

$$\mu_i, \quad \mu_i^{-1}q^{-4\rho_1+2(n_i-1)} = \mu_i^{-1}q^{-2N+2(n_i+1)}, \quad i=1,\ldots,\ell+1, \quad \mu_{\ell+2}. \tag{7.13}$$

It is known that the operator $\mathcal{Q}$ on $\mathbb{C}^N\otimes\hat{M}_\lambda$ satisfies a polynomial equation of degree $2\ell+3$, [3]. Formula (7.13) implies that, at generic point $\lambda\in\mathfrak{C}^*_{\mathfrak{l},reg}$, the roots of the polynomial are pairwise distinct for almost all q. Hence $\mathcal{Q}$ is semisimple for almost all q at generic λ. In particular, the eigenvalues $\mu_{\ell+1}$, $\mu_{\ell+2}$, and $\mu_{\ell+3}$ read

$$\begin{array}{lllllllll} \mu_{\ell+1} & = & -q^{2(m-n)}, & \mu_{\ell+2} & = & q^{-2(n-p)}, & \mu_{\ell+3} & = & -q^{4(m-n)+2}, \quad N=2n, \\ \mu_{\ell+1} & = & -q^{4p+2m-2n+1}, & \mu_{\ell+2} & = & q^{-2(n-p)}, & \mu_{\ell+3} & = & -q^{2p-4n+4m+1}, \quad N=2n+1. \end{array} \tag{7.14}$$

Note that $\mu_{\ell+1}$ may be equal to $\mu_{\ell+3}$ only for even N and $m+1=n$, which case is excluded from our consideration. In other words, the minimal polynomial of $\mathcal{Q}$ remains semisimple on $\mathbb{C}^N\otimes\hat{M}_\lambda$ for almost all q upon specialization of λ to generic point of $\mathfrak{C}^*_{\mathfrak{k},reg}$. Therefore $\mathcal{Q}$ is semisimple on $\mathbb{C}^N\otimes\hat{M}_\lambda$ and hence on $\mathbb{C}^N\otimes M_\lambda$ for an open set in $\mathfrak{C}^*_{\mathfrak{k},reg}$, for almost all q.

We call a weight $\lambda\in\mathfrak{C}^*_{\mathfrak{k},reg}$ *admissible* if the vector $\boldsymbol{\mu}$ belongs to $\hat{\mathcal{M}}_K$ modulo $\hbar$. Recall that $\hat{\mathcal{M}}_K$ parameterizes the moduli space $\mathcal{M}_K$ of conjugacy classes with given K. By definition, admissibility is determined by the singular part $\hbar^{-1}\mathfrak{c}^*_{\mathfrak{k},reg}\subset\mathfrak{C}^*_{\mathfrak{k},reg}$. Clearly admissible weights are dense in $\mathfrak{C}^*_{\mathfrak{k},reg}$.

Proposition 7.1. *For admissible $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg}$ the operator $\mathcal{Q}$ satisfies a polynomial equation of degree $2\ell+2$ on $\mathbb{C}^N \otimes M_\lambda$ with the roots*

$$\mu_i, \quad \mu_i^{-1} q^{-4n+2(n_i-1)}, \quad i = 1, \ldots, \ell, \quad \mu_{\ell+1}, \quad \mu_{\ell+2}. \tag{7.15}$$

Proof. The operator $\mathcal{Q} \in \mathrm{End}(\mathbb{C} \otimes \hat{M}_\lambda)$ is semisimple for almost all q at admissible λ, and the roots (7.13) are pairwise distinct. Therefore, the projection of $\mathcal{Q}$ to $\mathrm{End}(\mathbb{C} \otimes M_\lambda)$ is semisimple for almost all q and satisfies the same polynomial equation. The eigenvalue $\mu_{\ell+3}$ drops from the spectrum of $\mathcal{Q}$ on $\mathbb{C} \otimes M_\lambda$ by Proposition 6.9, hence the simple divisor $\mathcal{Q} - \mu_{\ell+3}$ is invertible and can be canceled from the polynomial. $\square$

It can be shown that the polynomial from Proposition 7.1 is minimal for generic $\lambda \in \mathfrak{C}^*_{\mathfrak{k},reg}$ and $q \in \mathbb{C}$ and cannot be reduced further. Equivalently, only $V_{\ell+3}/V_{\ell+2}$ vanishes from $\oplus_{i=1}^{2\ell+3} V_i/V_{i-1}$. We do not focus on this issue here.

The matrix $\mathcal{Q}$ produces the center of $U_q(\mathfrak{g})$ via the q-trace construction. For any invariant matrix $X \in \mathrm{End}(\mathbb{C}^N) \otimes \mathcal{A}$ with the entries in a $U_q(\mathfrak{g})$-module $\mathcal{A}$ one can define an invariant element

$$\mathrm{Tr}_q(X) := \mathrm{Tr}\big(q^{2h_\rho} X\big) \in \mathcal{A}. \tag{7.16}$$

Recall that h_ρ is an element from $\mathfrak{h}$ such that $\alpha(h_\rho) = (\alpha, \rho)$ for all $\alpha \in \mathfrak{h}^*$. The q-trace, when applied $X = \mathcal{Q}^k$, $k \in \mathbb{Z}_+$, gives a series of central elements of $U_q(\mathfrak{g})$. We will use the shortcut notation τ_k for $\mathrm{Tr}_q(\mathcal{Q}^k)$, $k \in \mathbb{Z}_+$.

A module M of highest weight λ defines a one dimensional representation χ_λ of the center of $U_q(\mathfrak{g})$, which assigns a scalar to each τ_ℓ:

$$\chi^\lambda(\tau_k) = \sum_\nu q^{2k(\lambda+\rho,\nu)-2k(\rho,\varepsilon_1)} \prod_{\alpha \in \mathrm{R}_+} \frac{q^{(\lambda+\nu+\rho,\alpha)} - q^{-(\lambda+\nu+\rho,\alpha)}}{q^{(\lambda+\rho,\alpha)} - q^{-(\lambda+\rho,\alpha)}}. \tag{7.17}$$

The summation is taken over weights ν of the module $\mathbb{C}^N$. Restriction of λ to $\mathfrak{C}^*_{\mathfrak{k},reg}$ makes the right hand side a function of $\boldsymbol{\mu}$ defined in (7.12). We denote this function by $\vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu})$, where $\boldsymbol{n} = (n_1, \ldots, n_\ell, m, p)$ is the integer valued vector of multiplicities. In the limit $\hbar \to 0$ the function $\vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu})$ goes over into the right hand side of (2.7), where $\mu_i = \lim_{\hbar \to 0} q^{2(\lambda,\nu_i)}$, $i = 1, \ldots, \ell$.

8 Quantum conjugacy classes of non-Levi type

By quantization of a commutative $\mathbb{C}$-algebra $\mathcal{A}$ we understand a $\mathbb{C}[[\hbar]]$-algebra $\mathcal{A}_\hbar$, which is free as a $\mathbb{C}[[\hbar]]$-module and $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar \simeq \mathcal{A}$ as a $\mathbb{C}$-algebra. Below we describe the quantization of $\mathbb{C}[G]$ along the Poisson bracket (2.4).

Recall from [19] that the image of the universal R-matrix of the quantum group $U_\hbar(\mathfrak{g})$ in the defining representation is equal, up to a scalar factor, to

$$R = \sum_{i,j=1}^{N} q^{\delta_{ij}-\delta_{ij'}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i>j}}^{N} (e_{ij} \otimes e_{ji} - q^{\rho_i-\rho_j} e_{ij} \otimes e_{i'j'}).$$

The coefficients ρ_i are defined as $\rho_{n+1} = 0$, $\rho_i = -\rho_{i'} = (\rho, \varepsilon_i) = n + \frac{1}{2} - i$ for $N = 2n+1$ and $\rho_i = -\rho_{i'} = (\rho, \varepsilon_i) = n - i$ for $N = 2n$, where i runs over $1, \ldots, n$.

Denote by S the $U_\hbar(\mathfrak{g})$-invariant operator $PR \in \mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathbb{C}^N)$, where P is the ordinary flip of $\mathbb{C}^N \otimes \mathbb{C}^N$. This matrix has three invariant projectors to its eigenspaces, among which there is a one-dimensional projector κ to the trivial $U_\hbar(\mathfrak{g})$-submodule, proportional to $\sum_{i,j=1}^{N} q^{\rho_i-\rho_j} e_{i'j} \otimes e_{ij'}$.

Denote by $\mathbb{C}_\hbar[G]$ the associative algebra generated by the entries of a matrix $A = ||A_{ij}||_{i,j=1}^{N} \in \mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}_\hbar[G]$ modulo the relations

$$S_{12}A_2S_{12}A_2 = A_2S_{12}A_2S_{12}, \quad A_2S_{12}A_2\kappa = q^{-N+1}\kappa = \kappa A_2S_{12}A_2. \tag{8.18}$$

These relations are understood in $\mathrm{End}(\mathbb{C}^N) \otimes \mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}_\hbar[G]$, and the indices distinguish the two copies of $\mathrm{End}(\mathbb{C}^N)$, in the usual way. Note that the factor q^{-N+1} before κ is missing in [3].

The algebra $\mathbb{C}_\hbar[G]$ is a quantization of $\mathbb{C}[G]$ along the Poisson bracket (2.4). It carries a $U_\hbar(\mathfrak{g})$-action, which is a deformation of the conjugation action of $U(\mathfrak{g})$ on $\mathbb{C}[G]$. This action is determined by the requirements that A commutes with $(\pi \otimes \mathrm{id}) \circ \Delta U_\hbar(\mathfrak{g})$ in the tensor product $\mathrm{End}(\mathbb{C}^N) \otimes \mathbb{C}_\hbar[G] \rtimes U_\hbar(\mathfrak{g})$, where $\pi \colon U_\hbar(\mathfrak{g}) \to \mathrm{End}(\mathbb{C}^N)$ is the representation homomorphism. It is important that $\mathbb{C}_\hbar[G]$ can be realized as a $U_\hbar(\mathfrak{g})$-invariant subalgebra in $U_q(\mathfrak{g})$, with respect to the adjoint action. The embedding is implemented by the assignment

$$A \mapsto \mathcal{Q} \in \mathrm{End}(\mathbb{C}^N) \otimes U_q(\mathfrak{g}).$$

The following properties of $\mathbb{C}_\hbar[G]$ will be of importance. Denote by $I_\hbar(G) \subset \mathbb{C}_\hbar[G]$ the subalgebra of $U_\hbar(\mathfrak{g})$-invariants, which also coincides with the center of $\mathbb{C}_\hbar[G]$. For $N = 2n+1$ it is generated by the q-traces $\mathrm{Tr}_q(\mathcal{A}^l)$, $l = 1, \ldots, N$. Not all traces are independent, but that is immaterial for our presentation. Traces of A^l are not enough for $N = 2n$, and one should add one more invariant τ^- in order to get entire $I_\hbar(G)$. On a module of highest weight λ, this invariant returns $\chi^\lambda(\tau^-) = \prod_{i=1}^{n}(q^{2(\lambda+\rho,\varepsilon_i)} - q^{-2(\lambda+\rho,\varepsilon_i)})$, see Proposition 7.4, [3]. However, it vanishes on modules with highest weight $\lambda \in \mathfrak{C}^*_{\mathfrak{l}}$, so we take no care of it.

Theorem 8.1. *Suppose that $\lambda = \mathfrak{C}^*_{\mathfrak{k},reg}$ is admissible, and let $\boldsymbol{\mu}$ be as in (7.12). The quotient of $\mathbb{C}_\hbar[G]$ by the ideal of relations*

$$\prod_{i=1}^{\ell}(\mathcal{Q} - \mu_i) \times (\mathcal{Q} - \mu_{\ell+1})(\mathcal{Q} - \mu_{\ell+2}) \times \prod_{i=\ell}^{1}(\mathcal{Q} - \mu_i^{-1}q^{-2N+2(n_i+1)}) = 0, \tag{8.19}$$

$$\mathrm{Tr}_q(\mathcal{Q}^k) = \vartheta^k_{\boldsymbol{n},q}(\boldsymbol{\mu}) \tag{8.20}$$

is an equivariant quantization of the class $\lim_{\hbar\to 0}\boldsymbol{\mu} = \boldsymbol{\mu}^0 = \hat{\mathcal{M}}_K$*. It is the image of* $\mathbb{C}_\hbar[G]$ *in the algebra of endomorphisms of the* $U_q(\mathfrak{g})$*-module* M_λ*.*

Proof. The proof is similar to [2] and [3], where it is discussed at length. It is based on equivariant homomorphism of $\mathbb{C}_\hbar[G]$ to the regular part of $\mathrm{End}(M_\lambda)$, where the module M_λ is extended over the Laurent series in $\hbar$. This homomorphism factors through a homomorphism of $\mathbb{C}_\hbar[G]/I_\lambda$, where I_λ the ideal generated by the kernel of the central character χ^λ. This ideal is defined by the relations (8.20). The $U_\hbar(\mathfrak{g})$-algebra $\mathbb{C}_\hbar[G]/I_\lambda$ is a direct sum of isotypical components of finite rank over $\mathbb{C}[[\hbar]]$, by the quantum Richardson theorem, [20]. Therefore, the image of $\mathbb{C}_\hbar[G]$ under this homomorphism is free over $\mathbb{C}[[\hbar]]$. The kernel of $\mathbb{C}_\hbar[G]/I_\lambda$ contains the ideal generated by the entries of the matrix polynomial in the left-hand-side of (8.19). In the classical limit, this ideal goes over to the defining ideal of the conjugacy class, by Theorem 2.1. Hence it coincides with the kernel. □

Theorem 8.1 describes quantization in terms of the matrix $\mathcal{Q}$, which is the image of the matrix A. To obtain the description in terms of A, one should replace $\mathcal{Q}$ with A in (8.19) and (8.20) and add the relations (8.18).

The quantization we have constructed is equivariant with respect to the standard or Drinfeld-Jimbo quantum group $U_\hbar(\mathfrak{g})$. Other quantum groups are obtained from standard $U_\hbar(\mathfrak{g})$ by twist, [21]. Formulas (8.19) and (8.20) are valid for any quantum group $U_\hbar(\mathfrak{g})$ upon the following modifications. The matrix $\mathcal{Q}$ is expressed through the universal R-matrix as usual. The q-trace should be redefined as $\mathrm{Tr}_q(X) = q^{1+2\rho_1}\mathrm{Tr}\Big(\pi\big(\gamma^{-1}(\mathcal{R}_1)\mathcal{R}_2\big)X\Big) = q^N\mathrm{Tr}\Big(\pi\big(\gamma^{-1}(\mathcal{R}_1)\mathcal{R}_2\big)X\Big)$. This can be verified along the lines of [22].

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