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ON PERIODIC SELF-HOMEOMORPHISMS OF CLOSED ORIENTABLE SURFACES DETERMINED BY THEIR ORDERS

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Abstract. The fundamentals for the topological classification of periodic orientation preserving self-homeomorphisms of a closed orientable topological surface X_g of genus $g \geq 2$ have been established, by Nielsen, in the thirties of the last century. Recently, Hirose has shown that the order $N \geq 3g$ of a cyclic action on closed surface of genus $g \neq 4, 6, 9, 10, 12$ determines this action up to topological conjugation. Actually in such case, we have very few possibilities for such orders; $N = 4g + 2, 4g, 3g$ or $3g + 3$ and for the first three cases such actions actually exist for arbitrary g while the last exists if and only if $g \not\equiv 2 \pmod 3$. Motivated by this phenomenon, we call the order N of a cyclic action of $G = \mathrm{Z}_N$ on $X = X_g$ to be rigid if for any other cyclic action G' of order N on X, G and G' are conjugate by certain orientation preserving self-homeomorphism of X. It seems that rigidity property, observed by Hirose for mentioned orders of cyclic actions, is a rather rare phenomenon. Here, apart of it, we consider and study another related property of periodic cyclic actions called *weak rigidity*. We say that the cyclic action G on X is *weakly rigid* if any other cyclic action G' of the same order with singular orbits of the same size is conjugate to it by a homeomorphism of X. Using combinatorial techniques, we characterise a large class of weakly rigid cyclic actions with three singular orbits.

1. Introduction

The fundamentals for the topological classification of periodic orientation preserving self-homeomorphisms of a closed topological surface X_g of genus $g \geq 2$ have been established, by Nielsen, in the thirties of the last century. Certain classification has been given also by Yokayama [15] and, recently, Hirose [5] has shown that an order $N \geq 3g$ of a cyclic action on a closed surface of genus $g \neq 4, 6, 9, 10, 12$ is uniquely determined up to a topological conjugation. Actually in such case, we have very few possibilities for such orders; $N = 4g + 2, 4g, 3g$ or $3g + 3$ and for the first three cases such configuration actually exists for arbitrary g while the last exists if and only if $g \not\equiv 2 \pmod 3$. The order N of a cyclic action of $G = \mathrm{Z}_N$ on $X = X_g$ is said to be topologically rigid if for any other cyclic action G' of order N on X, G and G' are conjugate by certain orientation-preserving self-homeomorphism not necessarily periodic.

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It seems that rigidity property, observed by Hirose for mentioned orders of cyclic actions, is a rather rare phenomenon and here, apart of it, we consider and study another related property of periodic cyclic actions called *weak rigidity.* We say that the cyclic action G on X is *weakly rigid* if any other cyclic action G' of the same order with singular orbits of the same size is conjugate to it by an orientation preserving homeomorphism of X. Using combinatorial techniques, we characterise a large class of weakly rigid cyclic actions having three singular orbits and the orbit genus zero; such actions are examples of quasi-platonic or Belyi actions well known in the literature.

The class of actions considered in the paper, apart of being proved to be weakly rigid, has some extra pleasant features. Namely, due to the geometrization theorem of Nielsen and the Riemann uniformization theorem they can be realized as holomorphic actions on complex algebraic curves. Due to a celebrated theorem of Belyi [2], these curves can be defined over the algebraic numbers. By results of Greenberg [3] and Singerman [13] these curves are unique. Another result due to Singerman [14] states that they have one or two real form. Finally due to Kock-Singerman [10] these forms can be defined over the algebraic reals. To find explicitly equation of these forms does not seems to be hopeless.

Another interesting problem, not considered here, concerns relations between rigidity and weak rigidity of periodic cyclic actions. Namely rigidity implies, obviously, weak rigidity but, and though some explicit examples show the converse is not true, it seems that such exceptions are rather rare and there is a series of natural problems concerning, roughly speaking, the number of them and the estimation of the number of weakly rigid actions of given order on a closed surface of given genus. A sequel concerning these and other problems is planned. We use multiplicative notation for cyclic groups throughout the whole paper.

2. Description of the approach

We shall use the following ingredients. Throughout of this section X will denote a closed orientable surface of genus $g \geq 2$.

2.1. **Nielsen's geometrization.** Let φ be an orientation-preserving self-homeomorphism of X of finite order N. There exists a structure of a Riemann surface on X (we still denote the resulting Riemann surface also by X by abuse of language) so that φ is a conformal automorphism. In the rest of the section, we assume X to have such a Riemann surface structure when necessary.

2.2. **Riemann uniformization theorem and elementary covering theory.** A closed orientable Riemann surface X is isomorphic to the orbit space $\mathcal{H}/\Gamma$ of the hyperbolic upper half plane $\mathcal{H}$ with the holomorphic structure inherited from $\mathcal{H}$, where Γ

has signature $(h; -)$. Such a group Γ has the presentation

$$\langle a_1, b_1, \ldots, a_h, b_h \mid [a_1, b_1] \ldots [a_h, b_h] \rangle. \tag{1}$$

Furthermore having an automorphism φ of X, we have an isomorphism $\langle\varphi\rangle \cong \Lambda/\Gamma = \langle x \mid x^N = 1\rangle$ for some Fuchsian group Λ, say with signature $(h; m_1, \ldots, m_r)$, which means that it has the presentation

$$\langle x_1, \ldots, x_r, a_1, b_1, \ldots, a_h, b_h \,|\, x_1^{m_1}, \ldots, x_r^{m_r}, x_1 \ldots x_r[a_1, b_1] \ldots [a_h, b_h] \rangle. \tag{2}$$

Particular role in our considerations will play triangle groups which are Fuchsian groups with signatures $(0; m_1, m_2, m_3)$ which will be also abbreviated to (m_1, m_2, m_3) that is the ones algebraically isomorphic to

$$\langle x_1, x_2, x_3 \mid x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = x_1x_2x_3 = 1\rangle \tag{3}$$

Furthermore, the Hurwitz-Riemann formula says in this special situation

$$2(g-1) = N\left(1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3}\right) \tag{4}$$

2.3. **Harvey criterion.** Let Λ be a Fuchsian group with signature $(h; m_1, \ldots, m_r)$ and let $M = \mathrm{lcm}(m_1, \ldots, m_r)$. Then there exists a smooth epimorphism from Λ onto $\langle x \mid x^N = 1\rangle$ if and only if

$$\begin{cases} \text{(a)} & M = \mathrm{lcm}(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_r) \text{ for all } i; \\ \text{(b)} & M \text{ divides } N \text{ and if } h = 0 \text{ then } M = N; \\ \text{(c)} & r \neq 1 \text{ and if } h = 0 \text{ then } r \geq 3; \\ \text{(d)} & \text{if } N \text{ is even then the number of periods } m_i \text{ such that} \\ & N/m_i \text{ is odd is also even.} \end{cases} \tag{5}$$

2.4. **Maclachlan decomposition.** (Hidalgo [4]) In the case that $r = 3$, condition (a) in (5) above for the triple (m_1, m_2, m_3) is equivalent to have the following canonical decomposition

$$\begin{cases} m_1 &= AA_2A_3 \\ m_2 &= AA_1A_3 \\ m_3 &= AA_1A_2 \end{cases} \tag{6}$$

where $A = \gcd(m_1, m_2, m_3)$, $A_k = \gcd(m_i/A, m_j/A)$, for $k \neq i, j$. Note that the integers A_i are pairwise relatively prime and, by (b) in (5), that $N = AA_1A_2A_3$. Condition (d) in (5), states that $N = AA_1A_2A_3$ even is equivalent to have just one of A_i even. This decomposition has been discovered in [4] and the collection A, A_1, A_2, A_3 will be called *Maclachlan decomposition* of (m_1, m_2, m_3) and we shall call the triple (m_1, m_2, m_3) or the quadruple (A, A_1, A_2, A_3) *admissible.*

Furthermore, this signature is said to be *rigid* if the corresponding cyclic action, say of order N, is rigid in the sense described in the Introduction which in turn means that any two smooth epimorphism $\theta_1, \theta_2 : \Lambda \to \langle x \mid x^N = 1\rangle$ are equivalent in the sense described in Subsection 2.5.

2.5. **Few words on topological conjugacy.** Two orientation-preserving self-homeomorphisms φ_1, φ_2 of a surface X are topologically equivalent if they are conjugate by an orientation-preserving self-homeomorphism f of X. For periodic case in described above hyperbolization this is equivalent to classify up to conjugation by an orientation-preserving self-homeomorphism of conformal automorphisms. The general definition is that two conformal actions G_1, G_2 on Riemann surfaces X_1, X_2 given by surface-kernel epimorphisms $\theta_1 : \Lambda_1 \to G_1$ and $\theta_2 : \Lambda_2 \to G_2$ (means that their kernels are torsion free) are topologically equivalent if and only if the diagram

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\Phi} & \Lambda_2 \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\theta_2} \\ G_1 & \xrightarrow{\Psi} & G_2 \end{array}$$

commutes for some isomorphisms $\Phi : \Lambda_1 \to \Lambda_2, \Psi : G_1 \to G_2$. The Nielsen isomorphism theorem asserts that Φ can be chosen to be the conjugation by a self-homeomorphism f of $\mathcal{H}$ and, throughout the paper, we understand that f preserves orientation.

3. Rigid and weakly rigid actions

Let Λ be a Fuchsian group with the presentation (3). For a smooth epimorphism $\theta : \Lambda \to \langle a |\ a^N = 1\rangle$ we have

$$\theta(x_1) = a^{mN/m_1},\ \theta(x_2) = a^{kN/m_2},\ \theta(x_1) = a^{lN/m_3},$$

where $\gcd(m, m_1) = \gcd(k, m_2) = \gcd(l, m_3) = 1$ and

$$m\left(\frac{N}{m_1}\right) + k\left(\frac{N}{m_2}\right) + l\left(\frac{N}{m_3}\right) \equiv 0 \pmod N.$$

Let m' be the inversion of m modulo m_1. Let also A_1' be the maximal divisor of $A_1 = N/m_1$ coprime to m_1. Then by the Chinese Remainder Theorem there exists α such that $\alpha \equiv m' \pmod{m_1}$ and $\alpha \equiv 1 \pmod{A_1'}$. Then $\gcd(\alpha, N) = 1$ and

$$(a^{mN/m_1})^\alpha = (a^{N/m_1})^{m\alpha} = a^{N/m_1}$$

and so we see that, up to a powering of fixed generator of $\langle a |\ a^N = 1\rangle$, our θ is defined by

$$\theta(x_1) = a^{N/m_1},\ \theta(x_2) = a^{kN/m_2},\ \theta(x_1) = a^{lN/m_3}, \tag{7}$$

where $\gcd(k, m_2) = \gcd(l, m_3) = 1$ and

$$\frac{N}{m_1} + k\left(\frac{N}{m_2}\right) + l\left(\frac{N}{m_3}\right) \equiv 0 \pmod N.$$

Let $\mathcal{K} = \{k < m_2 \mid \gcd(k, m_2) = 1\}$. Then $K = \varphi(m_2)$ is its cardinality, where φ is the Euler function. Let also $\mathcal{L}$ be the subset of $\mathcal{K}$ consisting of those k for which

$$a^{N/m_1 + kN/m_2} \tag{8}$$

has order m_3 and let L be its cardinality. We see that with these notation we have

Lemma 3.1. *There are just L surface-kernel epimorphisms $\theta : \Lambda \to \langle a \mid a^N = 1\rangle$, where Λ is a Fuchsian group with the presentation* (3) for which $\theta(x_1) = a^{N/m_1}$. □

Now let $\mathcal{S}$ be the stabilizer of a^{N/m_1} in $\mathrm{Aut}\langle a\rangle = \mathbb{Z}_N^*$. Then for its cardinality S we have

Lemma 3.2. *$S = \varphi(N)/\varphi(m_1)$.*

Proof. Indeed

$$\begin{aligned} \varphi(m_1) &= \left|\mathrm{Orb}_{\mathbb{Z}_N^*}(a^{N/m_1})\right| \\ &= [\mathbb{Z}_N^* : \mathrm{Stab}_{\mathbb{Z}_N^*}(a^{N/m_1})] \\ &= \frac{\varphi(N)}{\left|\mathrm{Stab}_{\mathbb{Z}_N^*}(a^{N/m_1})\right|} \end{aligned}$$

and so the assertion. □

Lemma 3.3. *Each element of $\mathcal{S}$ acts on $\mathcal{K}$ without fixed points and so in particular the group $\mathcal{S}$ acts faithfully on $\mathcal{K}$.*

Proof. Let us take $k \in \mathcal{K}$ and $s, s' \in \mathcal{S}$. Recall that these mean $\gcd(k, m_2) = 1$ and $\gcd(s', N) = \gcd(s, N) = 1$. Furthermore, from the definition of $\mathcal{S}$ we have $(a^{N/m_1})^s = a^{N/m_1}$ which in turn gives $N(s-1)/m_1 \equiv 0\ (N)$ which in particulary means that m_1 divides $s-1$ and similarly for s' and in particular m_1 divides $s' - s$. Assume that

$$\left(a^{Nk/m_2}\right)^s = \left(a^{Nk/m_2}\right)^{s'}.$$

Then

$$N\frac{k(s-s')}{m_2} \equiv 0\ (N).$$

We see that m_2 divides $k(s-s')$ and therefore m_2 divides $s - s'$ which give $s \equiv s'$ (mod N). □

For counting L it will be crucial the decomposition (6). For, we have

$$N = \mathrm{lcm}(m_1, m_2, m_3) = \mathrm{lcm}(m_1, m_2) = \mathrm{lcm}(m_1, m_3) = \mathrm{lcm}(m_2, m_3),$$

where $N = AA_1A_2A_3$, $N/m_i = A_i$. Furthermore we know by (5) that if N is even, then the number of those A_i which are odd is even and therefore, since all A_i can not be even, only one of them is even. With these notations, elements (8) become

$$a^{A_1 + kA_2}$$

and so the set $\mathcal{L}$ becomes

$$\mathcal{L} = \{k < AA_1A_3 \mid \gcd(A_1 + kA_2, N) = A_3,\ \gcd(k, AA_1A_3) = 1\}. \tag{9}$$

We define $\psi(1) = 1$ and given a decomposition $n = p_1^{\alpha_1} \dots p_r^{\alpha_r} > 1$

$$\psi(n) = \prod_{i=1}^{r} (p_i - 2)p_i^{\alpha_i - 1}. \tag{10}$$

With this definition we have

Theorem 3.4. *Let C be the biggest divisor of A coprime with $A_1A_2A_3$ and let $B = A/C$. Then*

$$L = \varphi(A_1B)\psi(C). \tag{11}$$

Proof. Let $C = p_1^{\gamma_1} \cdots p_r^{\gamma_r}$ and let $B = B_1B_2B_3$, where for $i = 1, 2, 3$ each prime dividing B_i divides A_i. Then the numbers $A_1B_1,\ A_2B_2,\ A_3B_3,\ C$ are pairwise coprime and so

$$\mathbb{Z}_N \cong \mathbb{Z}_{A_1B_1} \oplus \mathbb{Z}_{A_2B_2} \oplus \mathbb{Z}_{A_3B_3} \oplus \mathbb{Z}_{p_1^{\gamma_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{\gamma_r}}.$$

and

$$\mathbb{Z}_{AA_1A_3} \cong \mathbb{Z}_{A_1B_1} \oplus \mathbb{Z}_{B_2} \oplus \mathbb{Z}_{A_3B_3} \oplus \mathbb{Z}_{p_1^{\gamma_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{\gamma_r}}.$$

Hence every element $x \in \mathbb{Z}_{AA_1A_3}$ can be represented as a sequence

$$(x_1, x_2, x_3, x_1', \dots, x_r')$$

with $x_1 \in \mathbb{Z}_{A_1B_1},\ x_2 \in \mathbb{Z}_{B_2},\ x_3 \in \mathbb{Z}_{A_3B_3}$ and $x_j' \in \mathbb{Z}_{p_j^{\gamma_j}}$. Similarly the elements of $\mathbb{Z}_N$ are represented. Moreover if x is invertible in $\mathbb{Z}^*_{AA_1B_1}$ then all components of the sequence corresponding to x are invertible in suitable rings.

Now, the correspondence $x \to A_1 + xA_2$ is an injection from $\mathbb{Z}^*_{AA_1A_3}$ to $\mathbb{Z}_N$. Let $(A_1 + x_1A_2, A_1 + x_2A_2, A_1 + x_3A_2, A_1 + x_1'A_2, \dots, A_1 + x_r'A_2)$ be the sequence corresponding to $A_1 + xA_2$ with entries taken modulo a suitable number. We need to know how many images of the function satisfy the condition $\gcd(A_1 + xA_2, N) = A_3$ that is how many corresponding sequences have all components invertible beside the third one which has to be of the form $A_1 + x_3A_2 = tA_3$, where $\gcd(t, B_3) = 1$.

Note first that all possible values of t satisfyjng the above condition is achievable that is we have $\varphi(B_3)$ values of it. Next, for $x \in Z^*_{AA_1A_3}$ we have $\gcd(A_1 + xA_2, A_1B_1A_2B_2) = 1$, so for a fixed tA_3 we have $\phi(A_1B_1) \times \phi(B_2)$ possibilities for the first two entries of the sequence.

Finally let us consider $A_1 + x_j'A_2$. Write $x_j' = y_j + p_jz_j$, where $1 \le x_1 \le p_j - 1$ and $0 \le z_j \le p_j^{\gamma_j - 1} - 1$. Then $A_1 + x_j'A_2 = (A_1 + y_jA_2) + p_jz_1A_2$ and for exactly one value of y_j we have $A_1 + y_jA_2 \equiv 0 \pmod{p_j}$. Therefore we have $(p_j - 2)p_j^{\gamma_j - 1}$ values of x_j' modulo $p_j^{\gamma_j}$ such that $A_1 + x_j'A_2$ is coprime to p_j.

Summarising we have

$$\begin{aligned} L &= \varphi(B_3)\cdot\varphi(A_1B_1)\cdot\varphi(B_2)\cdot\prod_{i=1}^{r}(p_i-2)p_i^{\gamma_i-1} \\ &= \varphi(A_1B_1B_2B_3)\psi(C) \\ &= \varphi(A_1B)\psi(C). \end{aligned}$$

□

Remark 3.5. Observe surprising analogy of the function (10) with the classical Euler-function, both in this what concern its algebraic properties and the explicit formula

$$\varphi(n)=\prod_{i=1}^{r}(p_i-1)p_i^{\alpha_i-1}$$

for it as well. We see that both of them can be seen as particular cases of function

$$\varphi_k(n)=\prod_{i=1}^{r}(p_i-k)p_i^{\alpha_i-1}$$

which can be defined for arbitrary k.

As a corollary we obtain our first main result.

Theorem 3.6. *Let C be the biggest divisor of A coprime with $A_1A_2A_3$ and let $B=A/C$. Then $L=S$ if and only if $B\in\{1,2\}$ and $C\in\{1,3\}$.*

Proof. Let $B=B_1B_2B_3$, where B_i are defined in the proof of Theorem 3.4. Then

$$\begin{aligned} S &= \frac{\varphi(N)}{\varphi(m_1)} \\ &= \frac{\varphi(A_1B_1)\varphi(A_2B_2)\varphi(A_3B_3)\varphi(C)}{\varphi(AA_2A_3)} \\ &= \frac{\varphi(A_1B_1)\varphi(A_2B_2)\varphi(A_3B_3)\varphi(C)}{\varphi(B_1)\varphi(A_2B_2)\varphi(A_3B_3)\varphi(C)} \\ &= \frac{\varphi(A_1B_1)}{\varphi(B_1)}. \end{aligned}$$

Since $\varphi(A_1B)\geq\varphi(A_1B_1)$, we have $C\in\{1,3\}$ and $\varphi(B_1)=1$ i.e. $B_1\in\{1,2\}$. If $B_1=2$ then for $B>B_1$ we get a contradiction as in this case $\varphi(A_1B)>\varphi(A_1B_1)$. So $B=B_1$. If $B_1=1$ then we get $\varphi(A_1B)=\varphi(A_1)$ which happens when $B\in\{1,2\}$. □

Lemma 3.7. *Let A, A_1, A_2, A_3 be the Maclachlan decomposition* 2.4 *of an admissible triple (m_1,m_2,m_3) for N with $m_1=m_2$ and let $S=1$. Then $1<L\leq 6$ if and only if it is given in* Table 1 *or in* Table 2.

A_1	A_2	A	A_3	C	B	L	N
1	1	5	1	5	1	3	5
1	1	7	1	7	1	5	7
1	1	9	1	9	1	3	9
1	1	15	1	15	1	3	15

Table 1. $L \leq 6$ for $m_1 = m_2 = m_3 = N$

Proof. Observe that $S = 1$ if and only if if $m_1 = N$ by the Lemma 3.2. So $m_1 = m_2 = N$. The list of all cases for which $L \leq 6$ can be easily derived from the formula (11). First of all, it follows from it that $C \in \{1, 3, 5, 7, 9, 15\}$ (note that C cannot be even). After fixing C we take the condition $C\varphi(A_1 B) \leq 6$ to be satisfied.

Let first $m_3 = N$ and let us list all cases giving a triple (N, N, N) with $1 < L \leq 6$. Here we have $A = N, A_1 = A_2 = A_3 = 1, C = N, B = 1$ and so $1 < L \leq 6$ if and only if it is given in Table 1.

Now let $m = m_3 < N$. In this case we have $A_1 = A_2 = 1$ and so $B_1 = B_2 = 1$. The only condition on A_3 is that it must be divisible by primes dividing $B = B_3$ if $B > 1$, and coprime to C. So we have all cases listed in Table 2. □

Theorem 3.8. *If $S = L$ for an admissible triple (m_1, m_2, m_3) then it defines a weakly rigid action. The converse holds for such triples except $(5, 5, 5), (9, 9, 9)$, $(15, 15, 15)$, for which we have weakly rigid action with $S < L = 3$ and the following cases for which we have rigid action with $S = 1, L = 2$:*

$$\begin{array}{lll}
(N, N, 3), & \text{for } N = 9t, & t \in \mathbb{N}, \\
(N, N, 4), & \text{for } N = 16t, & t \in \mathbb{N}, \\
(N, N, 6), & \text{for } N = 36t\ , & t \in \mathbb{N}, \\
(N, N, 12), & \text{for } N = 48t, & t \in \mathbb{N},\ 3 \nmid t.
\end{array}$$

Proof. For $L = S$, the rigidity follows from the faithfulness of the action of $\mathcal{S}$ on $\mathcal{L}$, which we proved in the Lemma 3.3.

The converse is a bit more involved since distinct elements of $\mathcal{L}$ may produce topologically equivalent actions. This is however not the case if m_i are pairwise distinct and so $S < L$ implies non-rigidity in this case. It is also true if $L > 6$. So assume that $L \leq 6$ and not all m_i are distinct, say $m_1 = m_2$. Then they are equal to N and so in particular $S = 1$ and the list of all configuration of N, m_1, m_2, m_3 are given in the Lemma 3.7. The case $m_3 = N$ is easy; here one can show that $(5, 5, 5), (9, 9, 9)$ and

A_1	A_2	A	A_3 $\gcd(A_3, C) = 1$	C	B	L	N divisible by
1	1	3,4,6	$\frac{N}{3}, \frac{N}{4}, \frac{N}{6}$	1	3,4,6	2	9,8,36
		5,8,10,12	$\frac{N}{5}, \frac{N}{8}, \frac{N}{10}, \frac{N}{12}$		5,8,10,12	4	25,16,100 ,72
		7,9,18	$\frac{N}{7}, \frac{N}{9}, \frac{N}{18}$		7,9,18	6	49,27,108
		12	$\frac{N}{12}$	3	4	2	24
		15,24,30	$\frac{N}{15}, \frac{N}{24}, \frac{N}{30}$		5,8,10	4	45,48,300
		21	$\frac{N}{21}$		7	6	147
		5	$\frac{N}{5}$	5	1	3	5
		15, 20, 30	$\frac{N}{15}, \frac{N}{20}, \frac{N}{30}$		3,4,6	6	45,40,180
		7	$\frac{N}{7}$	7	1	5	7
		9,18	$\frac{N}{9}, \frac{N}{18}$	9	1,2	3	9,36
		36	$\frac{N}{36}$		4	6	72
		15,30	$\frac{N}{15}, \frac{N}{30}$	15	1,2	3	15,60
		60	$\frac{N}{60}$		4	6	120

TABLE 2. $L \leq 6$ for $m_1 = m_2 = N, m_3 < N$

$(15, 15, 15)$ are rigid signatures, while $(7, 7, 7)$ allows two nonequivalent actions corresponding to $(a, a, a^5), (a, a^2, a^4)$. The case $m_3 < N$ described in Table 2 is a little bit more involved. First note, that there is no rigid action for $L > 2$. Hence we have to consider only the cases listed in the first part of Table 3. One can easily prove that the triples $(N, N, 3)$ and $(N, N, 6)$ determine rigid configuration. Let us consider the triple $(N, N, 8)$, $N = 8t$. We have two actions

$$(a, a^{2t-1}, a^{6t}), \quad (a, a^{6t-1}, a^{2t}).$$

m_1	m_2	m_3	N divsible by	L
N	N	3	9	2
		4	8	2
		6	36	2
		12	24	2
N	N	5	25	4
		8	16	4
		10	100	4
		12	72	4
N	N	7	49	6
		9	27	6
		18	108	6

TABLE 3. $L \leq 6$ for $m_1 = m_2 = N, m_3 < N$

Now the function induced by the correspondence $a \to a^{6t-1}$ moves a^{2t-1} to $a^{(2t-1)(6t-1)} = a^{4t^2+1}$. If t is even then this element is equal to a and we have equivalence of both actions. If t is odd then $t^2 \equiv 1 \pmod 8$ hence $4t^2+1 \equiv 5 \pmod 8$, which means that $a^{4t^2+1} \neq a$ and so both actions are not equivalent.

Finally take the triple $(N, N, 12)$, $N = 24t$. Note that in this case $3 \nmid t$ as by Table 2 $A_3 = N - m_3$ is coprime to $C = 3$. We have now four possibilities for actions

$$(a, a^{2t-1}, a^{22t}), \quad (a, a^{10t-1}, a^{14t}), \quad (a, a^{14t-1}, a^{10t}), \quad (a, a^{22t-1}, a^{2t}),$$

but only two of them are of the type $(N, N, 12)$. In fact, if $t \equiv 1 \pmod 3$, then $10t-1, 22t-1 \equiv 0 \pmod 3$ and if $t \equiv 2 \pmod 3$, then $2t-1, 14t-1 \equiv 0 \pmod 3$. Let us consider the first case. The function induced by the correspondence $a \to a^{14t-1}$ moves a^{2t-1} to $a^{(2t-1)(14t-1)} = a^{4t^2+8t+1}$. Since $4t^2+8t+1 = 4t(t+2)+1$, we see that only for even t we have $4t^2+8t+1 \equiv 1 \pmod{24t}$. Hence, only in this case both actions are equivalent. In the end let $t \equiv 2 \pmod 3$ and take the function induced by the correspondence $a \to a^{22t-1}$. It moves a^{10t-1} to $a^{(22t-1)(10t-1)} = a^{4t^2+16t+1}$. Again, since $4t^2+16t+1 = 4t(t+4)+1 \equiv 1 \pmod{24t}$ only for even t. $\square$

Corollary 3.9. *Let* Z_N *be a cyclic action with signature* (m_1, m_2, m_3) *and let* A, A_1, A_2, A_3 *be Maclachlan decomposition* 2.4 *of* (m_1, m_2, m_3). *Then the action is rigid if and only if one the following happen*

(1) $A \in \{1, 2\}$,
(2) $A \in \{3, 6\}$, *and* $\gcd(A_1A_2A_3, 3) = 1$

(3) $A \in \{5, 9, 15\}$, *and* $A_1 = A_2 = A_3 = 1$
(4) $A = 3$, $A_1 = A_2 = 1$, *and* $A_3 \equiv 0 \pmod 3$
(5) $A = 4$, $A_1 = A_2 = 1$, *and* $A_3 \equiv 0 \pmod 4$
(6) $A = 6$, $A_1 = A_2 = 1$, *and* $A_3 \equiv 0 \pmod 6$
(7) $A = 12$, $A_1 = A_2 = 1$, $A_3 \equiv 0 \pmod 4$, *and* $\gcd(A_3/4, 3) = 1$.

In particular the signature (m_1, m_2, m_3) *is no rigid for* $A \notin \{1, 2, 3, 4, 5, 6, 12\}$. □

4. Examples, equations, problems and remarks

4.1. **Rigid signature vs rigid order.** Let X be a closed orientable surface of genus $g \geq 2$ which we left fixed. The notion of rigid cyclic action on X leads us to define the concept of rigid order N for g and, similarly, the notion of weakly rigid action give rise to the concept of rigid signature for g which next allow to define the concept of weakly rigid order which means that all admissible signatures are rigid. In this subsection we shall consider cyclic actions with the orbit genus zero and having three singular orbits of the lengths m_1, m_2, m_3, calling such actions *triangular*. Let $N_1, \ldots, N_k$ be all possible orders of all such actions on X. Now $N = N_i$ can fail to be rigid order for two reasons. The first is that there may exist few distinct admissible signatures and, in principle, some of them may be rigid and the other not. The second, more subtle reason for nonrigidity of N for genus g, is that although there may exists just one admissible signature, this signature is not rigid. Mentioned results of Hirose mean that the cyclic actions of orders $4g + 1, 4g, 3g + 3, 3g$ on closed orientable surfaces of genus g, where $g \geq 12$ and additionally in the last case $g \not\equiv 2 \pmod 3$, are rigid. All of these phenomena, which show that rigidity of cyclic actions is essentially coarser than week rigidity indeed, are well illustrated in Tables 4 and 5; the rider will easily deduce rigidity of signatures in the last column using definitions.

4.2. **Cyclic actions with fixed-points free self-homeomorphisms.** The signature (m_1, m_2, m_3) is said to be (g, N)-admissible if there exists a self-homeomorphism φ of order N acting on a closed orientable surface X of genus g so that X/φ is the sphere ramified over three points with ramification indices m_1, m_2, m_3. An interesting case we have for $(g, N) = (11, 30)$ in Table 5 since these g, N are the smallest values for which exists an action with fixed-point free acting self-homeomorphism.

There are much more cyclic action containing fixed point acting self-homeomorphisms (the next to $(g, N) = (11, 30)$ are rigid actions for $(g, N) = (16, 42), (25, 60)$ with signatures $(21, 14, 6)$ and $(20, 15, 12)$). But particularly interesting is $N = 210$ which allow such actions of Z_N on surfaces of three consecutive genera $g = 95, 96, 97$. The corresponding ramification data are $(70, 42, 15), (70, 30, 21), (42, 35, 30)$ all of which are rigid since $A = 1$ for all of them. In addition there are nor other $(95, 210)$-admissible signatures and so 210 is the rigid order for $g = 95$. For $g = 96$, we have one more $(96, 210)$-admissible signatures $(210, 42, 15)$ which gives rigid action. In fact, let G be a cyclic group of order 210. Let us represent elements of G as 4-tuples that is elements of the direct product $G = \langle a_2\rangle \times \langle a_3\rangle \times \langle a_5\rangle \times \langle a_7\rangle$, where a_i has order i. Now for a fixed element x of order 210, say $x = (a_2, a_3, a_5, a_7)$ we have exactly one element y of order 42 such that xy has order 15, namely $y = (a_2, a_3, 1, a_7^{-1})$. So we see that $N = 210$ is the weakly rigid order for $g = 96$. Finally we have three more $(97, 210)$-admissible signatures $(210, 30, 21), (210, 70, 15), (210, 105, 14)$ whose Maclachlan decomposition are respectively $(3, 1, 7, 10), (5, 1, 3, 14), (7, 1, 2, 15)$. The first case defines rigid action as for $x = (a_2, a_3, a_5, a_7)$ again we have exactly one element $y = (a_2, a_3, a_5^{-1}, 1)$ of order 30 such that xy has order 21. The last two cases are not rigid. In the case $(210, 70, 15)$ for $x = (a_2, a_3, a_5, a_7)$ we have exactly three elements y of order 70 such that xy has order 15, namely $y = (a_2, 1, a_5^k, a_7^{-1})$, $k = 1, 2, 3$. In the case $(210, 105, 14)$ for $x = (a_2, a_3, a_5, a_7)$ we have exactly five elements y of order 105 such that xy has order 14, namely $y = (1, a_3^{-1}, a_5^{-1}, a_7^k)$, $k = 1, 2, 3, 4, 5$. So for $g = 97$, $N = 210$ is not weakly rigid order. So, all three phenomena: rigidity, weak rigidity and not weak rigidity can occur for cyclic action allowing fixed point free self-homeomorphisms for the same order on surfaces of three consecutive genera.

4.3. **Infinite series of non-rigidity examples.**

Example 1. Let p be an odd prime and let $\mathbb{Z}_{p^n}$ be an action on a surface X defined by admissible triple (m_1, m_2, m_3), and assume that $(m_1 \geq m_2 \geq m_3)$. Then $m_1 = m_2 = p^n$ and $m_3 = p^m$ for some $m \leq n$.

Let us first assume $m = n$. Then

$$(a, a, a^{-2}) \text{ and } (a, a^2, a^{-3})$$

are non-equivalent under the action of $\mathrm{Z}_{p^n}^* \rtimes \mathrm{S}_3$. These two triples correspond to the algebraic curves

$$C_1 := y^{p^n} = x(x-1) \text{ and } C_2 := y^{p^n} = x(x-1)^2.$$

Now let $m < n$. Then for arbitrary $q < p, (a, a^{qp^n-1}, a^{qp^n})$ is defining triple. These triples for $q = 1$ and q satisfying congruence

$$(p^{n-m} - 1)(qp^{n-m} = 1) \not\equiv 1 \pmod{p^n}$$

are not equivalent under the action of $\mathrm{Z}_{p^n}^* \rtimes \mathrm{Z}_2$. These triples correspond to the algebraic curves

$$C_1 := y^{p^n} = x(x-1)^{qp^n-1}$$

Example 2. Let p, q be distinct odd primes with $p < q$ and let Z_{pq} be an action on a surface X defined by the admissible triple (m_1, m_2, m_3). Then for $(m_1, m_2, m_3) = (p, q, pq)$ the action is rigid while for $(m_1, m_2, m_3) = (pq, pq, pq)$ we have two generating triples of elements from Z_{pq}

$$(a, a, a^{-2}) \text{ and } (a, a^2, a^{-3})$$

Genus	Order	Signature	Rigidity of order	Rigidity of signature
2	10	$(10,5,2)$	rigid	
	8	$(8,8,2)$	rigid	
	6	$(6,6,3)$	rigid	
	5	$(5,5,5)$	rigid	
3	14	$(14,7,2)$	rigid	
	12	$(12,4,3)$	weakly-rigid	
		$(12,12,2)$		
	9	$(9,9,3)$	rigid	
	8	$(8,8,4)$	rigid	
	7	$(7,7,7)$	non-rigid	
4	18	$(18,9,2)$	rigid	
	16	$(16,16,2)$	rigid	
	15	$(15,5,3)$	rigid	
	12	$(12,6,4)$	weakly-rigid	
		$(12,12,3)$		
	10	$(10,10,5)$	non-rigid	
	9	$(9,9,9)$	rigid	
5	22	$(22,11,2)$	rigid	
	20	$(20,20,2)$	rigid	
	15	$(15,15,3)$	rigid	
	12	$(12,12,6)$	rigid	
	11	$(11,11,11)$	non-rigid	
6	26	$(26,13,2)$	rigid	
	24	$(24,24,2)$	rigid	
	21	$(21,7,3)$	rigid	
	20	$(20,5,4)$	rigid	
	18	$(18,18,3)$	rigid	
	16	$(16,16,4)$	rigid	
	15	$(15,15,5)$	non-rigid	
	14	$(14,14,7)$	non-rigid	
	13	$(13,13,13)$	non-rigid	
7	30	$(30,15,2)$	rigid	
	28	$(28,28,2)$	rigid	
	24	$(24,8,3)$	rigid	
	21	$(21,21,3)$	rigid	
	20	$(20,10,4)$	rigid	
	18	$(18,9,6)$	non-rigid	
	16	$(16,16,8)$	non-rigid	
	15	$(15,15,15)$	rigid	

TABLE 4. Rigidity of triangular cyclic actions on surfaces of low genera

Genus	Order	Signature	Rigidity of order	Rigidity of signature
8	34	$(34, 17, 2)$	rigid	
	32	$(32, 32, 2)$	rigid	
	24	$(24, 24, 3)$	rigid	
	20	$(20, 20, 5)$	non-rigid	
	18	$(18, 18, 9)$	non-rigid	
	17	$(17, 17, 17)$	non-rigid	
9	38	$(38, 19, 2)$	rigid	
	36	$(36, 36, 2)$	rigid	
	30	$(30, 10, 3)$	rigid	
	28	$(28, 7, 4)$	rigid	
	27	$(27, 27, 3)$	rigid	
	24	$(24, 8, 6)$	non-rigid	rigid
		$(24, 24, 4)$		
	21	$(21, 21, 7)$	non-rigid	
	20	$(20, 20, 10)$	non-rigid	
	19	$(19, 19, 19)$	non-rigid	
10	42	$(42, 21, 2)$	rigid	
	40	$(40, 40, 2)$	rigid	
	33	$(33, 11, 3)$	rigid	
	30	$(30, 6, 5)$	weakly-rigid	
		$(30, 30, 3)$		
	28	$(28, 14, 4)$	rigid	
	25	$(25, 25, 5)$	non-rigid	
	24	$(24, 24, 6)$	non-rigid	rigid
		$(24, 12, 8)$		
	22	$(22, 22, 11)$	non-rigid	
	21	$(21, 21, 21)$	non-rigid	
11	46	$(46, 23, 2)$	rigid	
	44	$(44, 44, 2)$	rigid	
	33	$(33, 33, 3)$	rigid	
	30	$(15, 10, 6)$	rigid	
	24	$(24, 24, 12)$	non-rigid	
	23	$(23, 23, 23)$	non-rigid	

TABLE 5. Rigidity of triangular cyclic actions on surfaces of low genera

which define non-equivalent actions for $p \geq 5$. By direct calculus we obtain non-rigidity for $p = 3$ except for $q = 5$ $(q = 7)$. These two triples correspond to the algebraic curves

$$C_1 := y^{pq} = x(x-1) \text{ and } C_2 := y^{pq} = x(x-1)^2.$$

4.4. **Conformal rigidity of weakly rigid actions.** Observe that for $A = 1, 2$ no non-finitely maximal signature from Greenberg [3] and Singerman [13] lists is admissible. This means that the corresponding cyclic group Z_N of self-homeomorphisms of the corresponding topological surface X_g can not be finitely extended. In particular this means that there is just one conformal structure on X for which Z_N is the full group of conformal automorphisms. So the topological rigidity implies conformal rigidity. In the case $A = 1$, explicit projective equations were obtained in [4].

4.5. **Real forms.** Observe that all our surfaces are also symmetric, due to the result of Singerman [14], since the map $a \mapsto a^{-1}, b \mapsto b^{-1}$ induces an automorphism of any abelian group generated by a, b and also the map $a \mapsto b^{-1}, b \mapsto a^{-1}$ induce an automorphism if a and b have the same order. In our case our surfaces have two or one conjugacy classes of symmetries according to N being even or odd respectively.

4.6. **Open problems.**

(1) The preceding subsections allow us to deduce that X, with the unique conformal structure making Z_N the full group of its conformal automorphisms, is a symmetric Riemann surface, with one or two conjugacy classes of symmetries to which correspond one or two real forms for its defining equations according to N being odd or even. We do not consider the problem of finding these forms for equations given in [4] for $A = 1$, since a sequel concerning the general case of arbitrary A is planned.

(2) Consider the numbers $R(g)$ and $WR(g)$ of all rigid and weakly rigid actions on a closed orientable surface of genus $g \geq 2$. This problem consist in finding a most precise upper bounds for them and investigate asymptotic behaviour of the ratios $A(g)/R(g), A(g)/WR(g)$ and $R(g)/A(g), WR(g)/A(g)$, where $A(g)$ denotes the number of all quasi-platonic actions.

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