AUTOMORPHIC FORMS ON PGSp(2)

YUVAL Z. FLICKER

ABSTRACT. Development of the theory of lifting of automorphic and admissible representations in a new case of great classical interest: Siegel automorphic forms, is reported. The self-contragredient representations of PGL(4) are determined as lifts of representations of rank two either symplectic PGSp(2) or orthogonal SO(4) groups. The method uses the global trace formula and the local fundamental lemma. The lifting is stated in terms of character relations. This permits us to introduce a definition of packets and quasi-packets of representations of PGSp(2). We deal with all representations, not only generic ones. Globally we obtain multiplicity one theorem for the discrete spectrum of PGSp(2), rigidity theorem for packets and quasi-packets, determine all counterexamples to the naive Ramanujan conjecture, and show that each packet which is not such a counterexample has precisely one generic member. The lifting from SO(4) to PGL(4) amounts to establishing a product of two representations of GL(2) with central characters whose product is 1. The rigidity theorem for SO(4) amounts to a strong rigidity statement for a pair of representations of $\mathrm{GL}(2,\mathbb{A})$.

This report summarizes results of [F1]. These consist of a definition of local and global packets and quasi-packets of admissible and automorphic representations of the projective group PGSp(2) of symplectic similitudes, and a description of the lifting from this group to PGL(4), as well as the complementary lifting from SO(4) to PGL(4), and the lifting to PGSp(2) from its elliptic endoscopic group PGL(2)×PGL(2). The packets are defined using the liftings. The liftings are predicted by homomorphisms of dual groups via Langlands' principle of functoriality, which we describe first. The classification of the automorphic representations of PGSp(2) has applications to the decomposition of the étale cohomology with compact supports and twisted coefficients of the Shimura varieties associated with GSp(2), but this will be discussed elsewhere.

1. HOMOMORPHISMS OF DUAL GROUPS

Let $\mathbf{G}$ be the projective general linear group PGL(4)=PSL(4) over a number field F. Our initial purpose is to determine the automorphic representations π of $\mathbf{G}(\mathbb{A})$, $\mathbb{A}$ is the ring of adeles of F, which are self-contragredient: $\pi \simeq \check{\pi}$, equivalently ([BZ1]), θ-invariant: $\pi \simeq {}^{\theta}\pi$. Here θ, $\theta(g) = J^{-1}{}^t g^{-1} J$, is the involution defined by $J = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}$, $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where ${}^t g$ denotes the transpose of $g \in \mathbf{G}$, and ${}^{\theta}\pi(g) = \pi(\theta(g))$. According to the principle of functoriality ([Bo], [A]) these automorphic representations are essentially described by representations of the Weil group W_F of F into the dual group $\widehat{G} = \mathrm{SL}(4,\mathbb{C})$ of $\mathbf{G}$ which are

Department of Mathematics, The Ohio State University, 231 W. 18th Ave., Columbus, OH 43210-1174; flicker@math.ohio-state.edu. 2000 MSC: 10D40, 10D30, 12A67, 12A85, 14G10, 22E55, 11F27, 11R42, 11S40.

$\hat\theta$-invariant, namely representations of W_F into centralizers $Z_{\widehat{G}}(\hat s\hat\theta)$ of $\mathrm{Int}(\hat s)\hat\theta$ in $\widehat{G}$. Here $\hat\theta$ is the dual involution $\hat\theta(\hat g) = J^{-1}{}^t\hat g^{-1}J$, and $\hat s$ is a semisimple element in $\widehat{G}$. These centralizers are called twisted (by $\hat s\hat\theta$) endoscopic groups ([KS]).

A twisted endoscopic group is called elliptic if it is not contained in a proper Levi subgroup of $\widehat{G}$. Representations of non elliptic endoscopic groups can be reduced by parabolic induction to known ones of smaller rank groups. For our $\widehat{G}$, up to conjugacy the elliptic twisted endoscopic groups are the symplectic group $\widehat{H} = Z_{\widehat{G}}(\hat\theta) = \mathrm{Sp}(2,\mathbb{C})$ and the special orthogonal group $\widehat{C} = Z_{\widehat{G}}(\hat s\hat\theta) =$ "$\mathrm{SO}(4,\mathbb{C})$" $= \left\{g \in \mathrm{SL}(4,\mathbb{C}); g\hat sJ^tg = \hat sJ = \left(\begin{smallmatrix} 0 & \omega \\ \omega^{-1} & 0\end{smallmatrix}\right)\right\}$, of all $A\otimes B = \left(\begin{smallmatrix} aB & bB \\ cB & dB\end{smallmatrix}\right)$; $\left(A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), B\right) \in (\mathrm{GL}(2,\mathbb{C})\times\mathrm{GL}(2,\mathbb{C}))/\mathbb{C}^\times$ which satisfy $\det A\cdot\det B = 1$. Here $z\in\mathbb{C}^\times$ embeds as (z,z^{-1}), $\hat s = \mathrm{diag}(-1,1,-1,1)$ and $\omega = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$.

The group $\widehat{H}$ is the dual group of the simple F-group $\mathbf{H} = \mathrm{PSp}(2) = \mathrm{PGSp}(2)$, the projective group of symplectic similitudes. It is the quotient of the group $\mathrm{GSp}(2)$ of $(g,\boldsymbol{\lambda}) \in \mathrm{GL}(4)\times\mathbb{G}_m$ with ${}^tgJg = \boldsymbol{\lambda}J$, by its center $\{(\boldsymbol{\lambda},\boldsymbol{\lambda}^2)\}\simeq\mathbb{G}_m$. Since $\boldsymbol{\lambda}$ is uniquely determined by g (we write $\boldsymbol{\lambda} = \boldsymbol{\lambda}(g)$), we view $\mathrm{GSp}(2)$ as a subgroup of $\mathrm{GL}(4)$ and $\mathrm{PGSp}(2)$ of $\mathrm{PGL}(4)$.

The group $\widehat{C}$ is the dual group of the special orthogonal group $\mathbf{C} =$"$\mathrm{SO}(4)$" of pairs $(g_1,g_2)\in(\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathbb{G}_m$ with $\det g_1 = \det g_2$. Here $z\in\mathbb{G}_m$ embeds as the central element (z,z). Also we write $((\mathrm{GL}(2)\times\mathrm{GL}(2))/\mathrm{GL}(1))'$ for $\mathbf{C}$, where the prime indicates that the two factors in $\mathrm{GL}(2)$ have equal determinants, and $(\dots)''$ if their product is 1.

The principle of functoriality suggests that automorphic discrete series representations of $\mathbf{H}(\mathbb{A})$ and $\mathbf{C}(\mathbb{A})$ parametrize (or lift to) the θ-invariant automorphic discrete series representations of the group $\mathbf{G}(\mathbb{A})$ of $\mathbb{A}$-valued points of $\mathbf{G}$. Our main purpose is to describe this parametrization, in particular define tensor products of two automorphic forms of $\mathrm{GL}(2,\mathbb{A})$ the product of whose central characters is 1, and especially describe the automorphic representations of the projective symplectic group of similitudes of rank two, $\mathrm{PGSp}(2,\mathbb{A})$, in terms of θ-invariant representations of $\mathrm{PGL}(4,\mathbb{A})$.

2. Unramified Lifting

We proceed to explain how the liftings are defined, first for unramified representations.

An irreducible admissible representation π of an adele group $\mathbf{G}(\mathbb{A})$ is the restricted tensor product $\otimes\pi_v$ of irreducible admissible ([BZ1]) representations π_v of the groups $\mathbf{G}(F_v)$ of F_v-points of $\mathbf{G}$, where F_v is the completion of F at the place v of F. Almost all the local components π_v are unramified, that is contain a (unique up to a scalar multiple) nonzero K_v-fixed vector. Here K_v is the standard maximal compact subgroup of $\mathbf{G}(F_v)$, namely the group $\mathbf{G}(R_v)$ of R_v-points, R_v being the ring of integers of the non archimedean local field F_v; $\mathbf{G}$ is defined over R_v at almost all v. For such v, an irreducible unramified $\mathbf{G}(F_v)$-module π_v is the unique unramified irreducible constituent in an unramified principle series representation $I(\eta_v)$, normalizedly induced ([BZ2]) from an unramified character η_v of the maximal torus $\mathbf{T}(F_v)$ of a Borel subgroup $\mathbf{B}(F_v)$ of $\mathbf{G}(F_v)$ (extended trivially to the unipotent radical $\mathbf{N}(F_v)$ of $\mathbf{B}(F_v)$). The space of $I(\eta_v)$ consists of the smooth functions $\phi : \mathbf{G}(F_v)\to\mathbb{C}$ with $\phi(ank) = (\delta_v^{1/2}\eta_v)(a)\phi(k)$, $k\in K_v$, $n\in\mathbf{N}(F_v)$, $a\in\mathbf{T}(F_v)$, $\delta_v(a) = \det[\mathrm{Ad}(a)|\,\mathrm{Lie}\,\mathbf{N}(F_v)]$, and the $\mathbf{G}(F_v)$-action is $(g\cdot\phi)(h) = \phi(hg)$, $g,h\in\mathbf{G}(F_v)$.

The character η_v is unramified, thus it factorizes as $\eta_v : \mathbf{T}(F_v)/\mathbf{T}(R_v) \to \mathbb{C}^\times$. As $\mathbf{T}(F_v)/\mathbf{T}(R_v) \simeq X_*(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$, η_v lies in $\mathrm{Hom}(X_*(\mathbf{T}), \mathbb{C}^\times) = \mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times)$, where $\widehat{T}$ is the maximal torus in the Borel subgroup $\widehat{B}$ of $\widehat{G}$, both fixed in the definition of the (complex) dual group $\widehat{G}$ ([Bo], [Ko]). Now $\mathrm{Hom}(X^*(\widehat{T}), \mathbb{C}^\times) = X_*(\widehat{T})$, and $X_*(\widehat{T}) \otimes \mathbb{C}^\times = \widehat{T} \subset \widehat{G}$, thus the unramified irreducible $\mathbf{G}(F_v)$-module π_v determines a conjugacy class $t(\pi_v) = t(I(\eta_v))$ (the "Langlands parameter") in $\widehat{G}$, represented by the image of η_v in $\widehat{T}$.

3. The Lifting λ_1 from SO(4) to PGL(4)

We next describe our results on our secondary lifting λ_1, to $\mathbf{G} = \mathrm{PGL}(4)$ from $\mathbf{C} = \mathrm{SO}(4) = ((\mathrm{GL}(2) \times \mathrm{GL}(2))/\mathrm{GL}(1))'$. An unramified (irreducible) $\mathrm{GL}(2, F_v)$-module π_{1v} is parametrized by a conjugacy class $t(\pi_{1v})$ in $\mathrm{GL}(2, \mathbb{C})$ (the Langlands parameter; its eigenvalues are called the Hecke eigenvalues of the representation). An unramified irreducible $\mathbf{C}(F_v)$-module $\pi_{1v} \times \pi_{2v}$ is parametrized by a class $t(\pi_{1v}) \times t(\pi_{2v})$ in $((\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}))/\mathbb{C}^\times)'' = \widehat{C} \subset \widehat{G}$. If π_{iv} is the unramified constituent of $I(\eta_{iv})$, $t(\pi_{iv}) = \mathrm{diag}(\eta_{i1}, \eta_{i2})$, $\eta_{11}\eta_{12}\eta_{21}\eta_{22} = 1$, $\eta_{ij} = \eta_{ijv}(\boldsymbol{\pi}_v)$, we define the "lift" $\pi_{1v} \boxtimes \pi_{2v} = \lambda_1(\pi_{1v} \times \pi_{2v})$ of $\pi_{1v} \times \pi_{2v}$ with respect to the dual group homomorphism $\lambda_1 : \widehat{C} = \mathrm{SO}(4, \mathbb{C}) \hookrightarrow \widehat{G} = \mathrm{SL}(4, \mathbb{C})$ (the natural embedding) to be the unramified irreducible constituent π_v of the $\mathrm{PGL}(4, F_v)$-module $I(\eta_v)$ parametrized by the class $t(\pi_v) = \mathrm{diag}(\eta_{11}\eta_{21}, \eta_{11}\eta_{22}, \eta_{12}\eta_{21}, \eta_{12}\eta_{22})$ in $\widehat{G} = \mathrm{SL}(4, \mathbb{C})$. In different notations, $\lambda_1(I(a_1, a_2) \times I(b_1, b_2)) = I(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$ where $a_i, b_i \in \mathbb{C}^\times$, provided that $a_1a_2b_1b_2 = 1$. Note that the inverse image under λ_1 of $I(a_1b_1, a_1b_2, b_1a_2, a_2b_2)$ consists only of $\chi I(a_1, a_2) \times \chi^{-1} I(b_1, b_2)$ and $\chi I(b_1, b_2) \times \chi^{-1} I(a_1, a_2)$ where χ is any character of $F_v^\times$. Thus, λ_1 is two-to-one unless $\pi_{1v} = \check{\pi}_{2v}$ (the contragredient of π_{2v}), where λ_1 is injective on the set of orbits of multiplication by χ in $\mathrm{Hom}(F_v^\times, \mathbb{C}^\times)$.

The rigidity theorem for discrete series automorphic representations of $\mathrm{GL}(n, \mathbb{A})$ asserts that discrete series automorphic representations $\pi_1 = \otimes \pi_{1v}$ and $\pi_2 = \otimes \pi_{2v}$ which have $\pi_{1v} \simeq \pi_{2v}$ for almost all places v of F are equivalent ([JS], [MW]). Moreover they are even equal, by the multiplicity one theorem for $\mathrm{GL}(n)$ ([Shal]). Representations of $\mathrm{PGL}(n, \mathbb{A})$ (or $\mathrm{PGL}(n, F_v)$) are simply representations of $\mathrm{GL}(n, \mathbb{A})$ (or $\mathrm{GL}(n, F_v)$) with trivial central character (since $H^1(F, \mathbb{G}_m) = \{0\}$), and the rigidity theorem applies then to $\mathrm{PGL}(n)$. Both multiplicity one theorem, and the rigidity theorem for packets (the latter asserts that $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ must lie in the same packet if $\pi_v \simeq \pi'_v$ for almost all v) hold for $\mathrm{SL}(2)$ ([F3]) and fail for $\mathrm{SL}(n)$, $n \geq 3$ ([Bla]).

The rigidity theorem holds for $\mathbf{C} = \mathrm{SO}(4)$; this is the content of the assertion that the lifting λ_1 is injective, made in the second paragraph of the following theorem. The first paragraph asserts that the lifting exists. By an elliptic representation we mean one whose character ([H]) is non zero on the set of elliptic elements.

Theorem (SO(4) to PGL(4)). *For every two discrete series automorphic representations $\pi_1 = \otimes \pi_{1v} \neq \pi_2 = \otimes \pi_{2v}$ of $\mathrm{GL}(2, \mathbb{A})$, whose central characters ω_1, ω_2 are equal, and whose components at three places v_1, v_2, v_3 are elliptic, there* **exists** *a (unique) discrete series automorphic representation $\pi = \lambda_1(\pi_1 \times \check{\pi}_2)$ of $\mathrm{PGL}(4, \mathbb{A})$ with $\pi_v = \lambda_1(\pi_{1v} \times \check{\pi}_{2v})$ for almost all v. Moreover, $\lambda_1(\pi_1 \times \check{\pi}_1)$ is the $\mathrm{PGL}(4, \mathbb{A})$-module normalizedly induced from the $\mathrm{PGL}(3, \mathbb{A})$-module $\mathrm{Sym}^2(\pi_1)$ on the Levi factor of the parabolic subgroup of type $(3, 1)$.*

Further, the global map λ_1 is **injective** *on the set of pairs $\pi_1 \times \check{\pi}_2$ with $\omega_1 = \omega_2$ up to the equivalence $\pi_1 \times \check{\pi}_2 \simeq \chi\pi_1 \times \chi^{-1}\check{\pi}_2$, χ a character of $\mathbb{A}^\times/F^\times$, and $\pi_1 \times \check{\pi}_2 \simeq \check{\pi}_2 \times \pi_1$.*

The injectivity means that if $\pi_1, \pi_2, \pi_1^0, \pi_2^0$ are discrete series automorphic representations of $\mathrm{GL}(2, \mathbb{A})$ with central characters $\omega_1, \omega_2, \omega_1^0, \omega_2^0$ satisfying $\omega_1\omega_2 = 1 = \omega_1^0\omega_2^0$, each of which has elliptic components at least at the three places v_1, v_2, v_3, and if for each v outside a fixed finite set of places of F there is a character χ_v of $F_v^\times$ such that the set $\{\pi_{1v}\chi_v, \pi_{2v}\chi_v^{-1}\}$ is equal to the set $\{\pi_{1v}^0, \pi_{2v}^0\}$ (up to equivalence of representations), then there is a character χ of $\mathbb{A}^\times/F^\times$ such that the set $\{\pi_1\chi, \pi_2\chi^{-1}\}$ is equal to the set $\{\pi_1^0, \pi_2^0\}$. In particular, starting with a pair π_1, π_2 of automorphic discrete series representations of $\mathrm{GL}(2, \mathbb{A})$ with $\omega_1\omega_2 = 1$, we cannot get another such pair by interchanging a set of their components π_{1v}, π_{2v} and multiplying π_{1v} by a local character and π_{2v} by its inverse, unless we interchange π_1, π_2 and multiply π_1 by a global character and π_2 by its inverse.

Our global results are complemented and strengthened by very precise local results. We define λ_1-lifting locally by means of character relations of the form: $\lambda_1(\pi_1 \times \check{\pi}_2) = \pi$ if $\operatorname{tr}\pi(f \times \theta) = \operatorname{tr}(\pi_1 \times \check{\pi}_2)(f_C)$ for all matching functions f, f_C. This definition is compatible with the one given above for purely induced π_1 and π_2 and unramified representations, and we have $\lambda_1(I_2(\mu, \mu') \times \check{\pi}_2) = I_4(\mu\check{\pi}_2, \mu'\check{\pi}_2)$ (the central character of the $\mathrm{GL}(2, F)$-module π_2 is $\mu\mu'$). The local and global results are closely analogous.

4. Special Cases of The Lifting λ_1

Let us describe some special cases of the lifting λ_1. When $\pi_2 = \check{\pi}_1$ is the contragredient of π_1, $\lambda_1(\pi_1 \times \check{\pi}_1)$ is the $\mathrm{PGL}(4, \mathbb{A})$-module normalizedly induced from the maximal parabolic of type (3,1) and the $\mathrm{PGL}(3, \mathbb{A})$-module $\mathrm{Sym}^2(\pi_1)$ on the GL(3)-factor of the Levi subgroup (extended trivially to the GL(1)-factor of the Levi, and to the unipotent radical). Here $\mathrm{Sym}^2(\pi_1)$ is the symmetric square lifting from GL(2) to PGL(3) ([F3]). Indeed, if the local component π_{1v} of π_1 at v is unramified then $t(\pi_{1v}) = \mathrm{diag}(a, b)$ (thus π_{1v} is a constituent of $I_2(a, b)$), $\pi_v = \lambda_1(\pi_{1v} \times \check{\pi}_{1v})$ has $t(\pi_v) = \mathrm{diag}(a/b, 1, 1, b/a)$ (thus π_v is a constituent of $I_4(I_3(a/b, 1, b/a), 1)$, and $I_3(a/b, 1, b/a)$ is the symmetric square lifting of $I_2(a, b)$). We write I_n to emphasize that the representation is of the group $\mathrm{GL}(n)$, and e.g. $I_{(3,1)}(\pi_3, \pi_1)$ to indicate the representation of GL(4) induced from its maximal parabolic subgroup of type (3,1). However, the results of [F3] are stronger, in lifting representations of $\mathrm{SL}(2, \mathbb{A})$ to $\mathrm{PGL}(3, \mathbb{A})$ and consequently providing new results such as multiplicity one for SL(2).

Although we do not obtain a new proof of the existence of the symmetric square lift of discrete series representations of $\mathrm{PGL}(2, \mathbb{A})$, we do obtain new character relations, of the θ-twisted character of $I_{(3,1)}(\mathrm{Sym}^2 \pi_2, 1)$ with that of $\pi_2 \times \check{\pi}_2$. Clearly in this case the lift λ_1 is injective: if $\lambda_1(\pi_1 \times \check{\pi}_2) = \lambda_1(\pi_0 \times \check{\pi}_0)(= I_{(3,1)}(\mathrm{Sym}^2(\pi_0), 1))$ then $\pi_1 = \pi_2 = \pi_0\chi$ for some character χ of $\mathbb{A}^\times/F^\times$.

In particular, if π_1 is the one dimensional representation $g \mapsto \chi(\det g)$ of $\mathrm{GL}(2, \mathbb{A})$, then $\lambda_1(\pi_1 \times \check{\pi}_1) = I_{(3,1)}(\mathbf{1}_3, 1)$ is the representation of $\mathrm{PGL}(4, \mathbb{A})$ normalizedly induced from the trivial representation of the maximal parabolic subgroup of type (3,1). An alternative purely local computation of this twisted character is developed in [FZ].

Let $\pi_1 = \pi(\mu)$ be a cuspidal monomial representation of $\mathrm{GL}(2, \mathbb{A})$ associated with a character μ of $\mathbb{A}_E^\times/E^\times$ where E is a quadratic extension of F (denote by σ the non trivial element of $\mathrm{Gal}(E/F)$). Then $\mathrm{Sym}^2\pi_1 = I_{(2,1)}(\pi(\mu/{}^\sigma\mu), \chi_{E/F})$, where $\chi_{E/F}$ is the quadratic character of $\mathbb{A}^\times/F^\times N_{E/F}\mathbb{A}_E^\times$ ($N_{E/F}$ is the norm map from E to F). Moreover, $\lambda_1(\pi(\mu)\times\check{\pi}(\mu))$ is the representation $I_{(2,1,1)}(\pi(\mu/{}^\sigma\mu), \chi_{E/F}, 1)$ induced from the parabolic subgroup of type (2,1,1) of PGL(4). The central character of the $\mathrm{GL}(2,\mathbb{A})$-module $\pi(\mu)$ is $\chi_{E/F}\cdot\mu|\mathbb{A}^\times$, for any character μ of $\mathbb{A}_E^\times/E^\times$. If $\pi(\mu)$ is a $\mathrm{PGL}(2,\mathbb{A})$-module we have that the restriction of μ to $\mathbb{A}_F^\times/F^\times$ is $\chi_{E/F}$, non trivial but trivial on $F^\times N_{E/F}\mathbb{A}_E^\times$.

If $\pi_1 = \pi_E(\mu_1)$, $\pi_2 = \pi_E(\mu_2)$, cuspidal monomial representations of $\mathrm{GL}(2,\mathbb{A})$ associated with characters μ_1, μ_2 of $\mathbb{A}_E^\times/E^\times$ where E is a quadratic extension of F such that the restriction of $\mu_1\mu_2$ to $\mathbb{A}^\times$ is 1, then $\lambda_1(\pi_E(\mu_1)\times\pi_E(\mu_2)) = I_{(2,2)}(\pi_E(\mu_1\overline{\mu}_2), \pi_E(\mu_1\mu_2))$. When $\mu_1 = \mu_2^{-1}$ we have $\pi(\mu_1\overline{\mu}_2) = \pi(\mu_1/\overline{\mu}_1)$ and $\pi(\mu_1\mu_2) = I(\chi_{E/F}, 1)$, thus

$$\lambda_1(\pi(\mu_1)\times\check{\pi}(\mu_1)) = I_{(2,1,1)}(\pi(\mu_1/\overline{\mu}_1), \chi_{E/F}, 1) = I_{(3,1)}(\mathrm{Sym}^2(\pi(\mu_1)), 1).$$

Another interesting special case is when π_1 is taken to be the trivial representation $\mathbf{1}_2$ of $\mathrm{PGL}(2,\mathbb{A})$ while π_2 is a cuspidal representation of $\mathrm{PGL}(2,\mathbb{A})$. Then $\lambda_1(\mathbf{1}_2\times\pi_2)$ is the discrete series non cuspidal $\mathrm{PGL}(4,\mathbb{A})$-module $J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$, the quotient of the normalizedly induced $I(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ from the parabolic of type (2,2) of PGL(4). Here $\nu(x) = |x|$. Indeed, $\mathbf{1}_2$ is the quotient of the induced $I(\nu^{1/2}, \nu^{-1/2})$. Thus $t(\lambda_1(\mathbf{1}_{2v}\times\pi_{2v}))$ is $(t(\nu_v^{1/2}\pi_{2v}), t(\nu_v^{-1/2}\pi_{2v}))$ and so $\lambda_1(\mathbf{1}_{2v}\times\pi_{2v})$ is the quotient $J(\nu_v^{1/2}\pi_{2v}, \nu_v^{-1/2}\pi_{2v})$ of the induced $I(\nu_v^{1/2}\pi_{2v}, \nu_v^{-1/2}\pi_{2v})$ for all v where π_{2v} is unramified. Hence it is $J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ globally by the rigidity theorem for this non cuspidal discrete series spectrum ([MW]). On the set of pairs $\pi_1\times\pi_2$ such that at least one of π_1 or π_2 is one dimensional, the lifting λ_1 is injective. Indeed, a discrete series representation of $\mathrm{GL}(2,\mathbb{A})$ with a one-dimensional component is necessarily one-dimensional. If π_2 is not cuspidal but rather trivial, then the quotient $J(\nu^{1/2}\mathbf{1}_2, \nu^{-1/2}\mathbf{1}_2)$ of $I_4(\nu^{1/2}\mathbf{1}_2, \nu^{-1/2}\mathbf{1}_2)$ is not discrete series, but the induced $I_4(\mathbf{1}_3)$ from the trivial representation of the (3,1)-parabolic; this is $\lambda_1(\mathbf{1}_2\times\mathbf{1}_2)$.

5. The Lifting λ from PGSp(2) to PGL(4)

We now turn to the study of our main lifting λ, and of the automorphic representations of the F-group $\mathbf{H} = \mathrm{PGSp}(2) = \mathrm{GSp}(2)/\mathbb{G}_m$. The center $\mathbb{G}_m$ of $\mathrm{GSp}(2) = \{g\in\mathrm{GL}(4);\ {}^tgJg = \boldsymbol{\lambda}J,\ \exists\boldsymbol{\lambda} = \boldsymbol{\lambda}(g)\in\mathbb{G}_m\}$ consists of the scalar matrices. Its dual group is $\widehat{H} = \mathrm{Sp}(2,\mathbb{C}) = Z_{\widehat{G}}(\widehat{\theta})\subset\widehat{G} = \mathrm{SL}(4,\mathbb{C})$, where $\widehat{\theta}(g) = J^{-1}\,{}^tg^{-1}J$. It has a single elliptic endoscopic group $\mathbf{C}_0$ different than $\mathbf{H}$ itself. Thus

$$\widehat{C}_0 = Z_{\widehat{H}}(\widehat{s}_0) = \left\{\begin{pmatrix} a & 0 & 0 & b\\ 0 & \alpha & \beta & 0\\ 0 & \gamma & \delta & 0\\ c & 0 & 0 & d\end{pmatrix}\in\widehat{H}\right\}\simeq\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}),$$

where $\widehat{s}_0 = \mathrm{diag}(-1,1,1,-1)$, and $\mathbf{C}_0 = \mathrm{PGL}(2)\times\mathrm{PGL}(2)$. Write λ_0 for the embedding $\widehat{C}_0\hookrightarrow\widehat{H}$, and λ for the embedding $\widehat{H}\hookrightarrow\widehat{G}$.

The embedding $\lambda_0 : \widehat{C}_0 = \mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\hookrightarrow\widehat{H} = \mathrm{Sp}(2,\mathbb{C})$ defines the "endoscopic" lifting $\lambda_0 : \pi_2(\mu_1,\mu_1^{-1})\times\pi_2(\mu_2,\mu_2^{-1})\mapsto\pi_{\mathrm{PGSp}(2)}(\mu_1,\mu_2)$. Here $\pi_2(\mu_i,\mu_i^{-1})$ is the unramified irreducible constituent of the normalizedly induced $\mathrm{PGL}(2,F_v)$-module $I(\mu_i,\mu_i^{-1})$ (μ_i

are unramified characters of $F_v^\times$, $i = 1, 2$); $\pi_{\mathrm{PGSp}(2)}(\mu_1, \mu_2)$ is the unramified irreducible constituent of the $\mathrm{PGSp}(2, F_v)$-module $I_{\mathrm{PGSp}(2)}(\mu_1, \mu_2)$ normalizedly induced from the character $n \cdot \mathrm{diag}(\alpha, \beta, \gamma, \delta) \mapsto \mu_1(\alpha/\gamma)\mu_2(\alpha/\beta)$ of the upper triangular subgroup of $\mathrm{PGSp}(2, F_v)$ (n is in the unipotent radical, $\alpha\delta = \beta\gamma$).

The embedding $\lambda : \widehat{H} = \mathrm{Sp}(2, \mathbb{C}) \hookrightarrow \mathrm{SL}(4, \mathbb{C}) = \widehat{G}$ defines the lifting λ which maps the unramified irreducible $\mathrm{PGSp}(2, F_v)$-module $\pi_{\mathrm{PGSp}(2)}(\mu_1, \mu_2)$ to the unramified irreducible $\mathrm{PGL}(4, F_v)$-module $\pi_4(\mu_1, \mu_2, \mu_2^{-1}, \mu_1^{-1})$.

The composition $\lambda \circ \lambda_0 : \widehat{C}_0 = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \to \widehat{G} = \mathrm{SL}(4, \mathbb{C})$ takes $\pi_2(\mu_1, \mu_1^{-1}) \times \pi_2(\mu_2, \mu_2^{-1})$ to $\pi_4(\mu_1, \mu_2, \mu_2^{-1}, \mu_1^{-1}) = \pi_4(\mu_1, \mu_1^{-1}, \mu_2, \mu_2^{-1})$, namely the unramified irreducible $\mathrm{PGL}(2, F_v) \times \mathrm{PGL}(2, F_v)$-module $\pi_2 \times \pi_2'$ to the unramified irreducible constituent $\pi_4(\pi_2, \pi_2')$ of the $\mathrm{PGL}(4, F_v)$-module $I_4(\pi_2, \pi_2')$ normalizedly induced from the representation $\pi_2 \otimes \pi_2'$ of the parabolic of type (2,2) of $\mathrm{PGL}(4, F_v)$ (extended trivially on the unipotent radical). For example $\lambda \circ \lambda_0$ takes the trivial $\mathrm{PGL}(2, F_v) \times \mathrm{PGL}(2, F_v)$-module $\mathbf{1}_2 \times \mathbf{1}_2$ to the unramified irreducible constituent $\pi_4(\mathbf{1}_2, \mathbf{1}_2)$ of $I_4(\mathbf{1}_2, \mathbf{1}_2)$, and $\mathbf{1}_2 \times \pi_2$ to $\pi_4(\mathbf{1}_2, \pi_2) = \pi_4(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$. This last π_4 is traditionally denoted by J.

The definition of lifting is extended from the case of unramified representations to that of any admissible representations. For this purpose we defined in [F5] norm maps from the θ-stable θ-regular conjugacy classes in $G = \mathbf{G}(F)$ to stable conjugacy classes in $H = \mathbf{H}(F)$, and from these to conjugacy classes in $\mathbf{C}_0(F)$, extending the norm maps on the split tori in these groups which are dual to the dual groups homomorphisms λ and λ_0. This is used to define a relation of matching functions f, f_H and f_{C_0} (they have suitably defined matching orbital integrals) and a dual relation of liftings of representations.

To express the lifting results we use the following notations for induced representations of $H = \mathrm{PGSp}(2, F)$. For characters μ_1, μ_2, σ of $F^\times$ with $\mu_1\mu_2\sigma^2 = 1$ we write $\mu_1 \times \mu_2 \rtimes \sigma$ for the H-module normalizedly induced from the character $mu \mapsto \mu_1(a)\mu_2(b)\sigma(\boldsymbol{\lambda})$, $u \in U$, $m = \mathrm{diag}(a, b, \boldsymbol{\lambda}/b, \boldsymbol{\lambda}/a)$, $a, b, \boldsymbol{\lambda} \in F^\times$, of the upper triangular minimal parabolic of H.

For a $\mathrm{GL}(2, F)$-module π_2 and character μ we write $\pi_2 \rtimes \mu$ for the $\mathrm{PGSp}(2, F)$-module normalizedly induced from the representation $p = mu \mapsto \pi_2(g)\mu(\boldsymbol{\lambda})$, $m = \mathrm{diag}(g, \boldsymbol{\lambda} w^t g^{-1} w)$, $u \in U_{(2)}$, $\boldsymbol{\lambda} \in F^\times$ (here the product of the central character ω of π_2 with μ^2 is 1) of the Siegel parabolic subgroup (whose unipotent radical $U_{(2)}$ is abelian).

We write $\mu \rtimes \pi_2$, if $\omega\mu = 1$, for the representation of $\mathrm{PGSp}(2, F)$ normalizedly induced from the representation $p = mu \mapsto \mu(a)\pi_2(g)$, $m = \mathrm{diag}(a, g, \boldsymbol{\lambda}(g)/a)$, $u \in U_{(1)}$, $\boldsymbol{\lambda}(g) = \det g$, of the Heisenberg parabolic subgroup (whose unipotent radical $U_{(1)}$ is a Heisenberg group).

These inductions are normalized by multiplying the inducing representation by the character $p \mapsto |\det(\mathrm{Ad}(p))|\,\mathrm{Lie}\,U|^{1/2}$, as usual. For example, $I_H(\mu_1, \mu_2) = \mu_1\mu_2 \times \mu_1/\mu_2 \rtimes \mu_1^{-1}$. Note that $\pi \rtimes \sigma \simeq \check{\pi} \rtimes \omega\sigma$ and $\mu(\pi \rtimes \sigma) = \pi \rtimes \mu\sigma$. Complete results describing reducibility of these induced representations, are recorded in [ST] – whose results and notations we use – following earlier work of [Ro2], [Sh2,3] and [W1]. Our lifting results explain the results recorded in [ST].

For properly induced representations, defining λ- and λ_0-liftings by character relations ($\lambda(\pi_H) = \pi_4$ if $\mathrm{tr}\,\pi_4(f \times \theta) = \mathrm{tr}\,\pi_H(f_H)$ for all matching f, f_H, and $\lambda_0(\pi_1 \times \pi_2) = \pi_H$ if $\mathrm{tr}\,\pi_H(f_H) = \mathrm{tr}(\pi_1 \times \pi_2)(f_{C_0})$ for all matching f_H, f_{C_0}), our preliminary results (obtained by local character evaluations), are that $\omega^{-1} \rtimes \pi_2$ λ-lifts to $\pi_4 = I_G(\pi_2, \check{\pi}_2)$, that $\mu\pi_2 \rtimes \mu^{-1}$

(here $\omega = 1$) λ-lifts to $\pi_4 = I_G(\mu, \pi_2, \mu^{-1})$, and that $I_2(\mu, \mu^{-1}) \times \pi_2$ λ_0-lifts from C_0 to $\mu\pi_2 \rtimes \mu^{-1}$ on $H = \mathrm{PGSp}(2, F)$.

Let χ be a character of $F^\times/F^{\times 2}$. It defines a one-dimensional representation $\chi_H(h) = \chi(\boldsymbol{\lambda}(h))$ of H, which λ-lifts to the one-dimensional representation $\chi(g) = \chi(\det g)$ of G (if $h = Ng$ then $\boldsymbol{\lambda}(h) = \det g$; on diagonal matrices $N(\mathrm{diag}(a, b, c, d)) = \mathrm{diag}(ab, ac, db, dc)$). The Steinberg representation of H λ-lifts to the Steinberg representation of G, and for any character χ of $F^\times$ with $\chi^2 = 1$ we have $\lambda(\chi_H \mathrm{St}_H) = \chi \mathrm{St}_G$.

6. Elliptic Representations

Our local liftings, λ, λ_0 and λ_1, are relations of representations, defined by means of character relations. Denote by $K(H)$ the Grothendieck group of H (integral linear combinations of equivalence classes of irreducible representations of H). Similarly we have $K(C)$, $K(C_0)$, $K(G)$. Denote by $K^\theta(G)$ the free abelian group of (equivalence classes of) representations of $G \rtimes \langle\theta\rangle$. It is the $\mathbb{Z}$-span of the irreducible θ-invariant representations of G, extended to $G \rtimes \langle\theta\rangle$. Denote by $I : K(C_0) \to K^\theta(G)$ the functor of normalized induction. Thus $I(\pi_0)$ denotes the representation of G normalizedly induced from π_0 ($\pi_0 \in K(C_0)$ is viewed as a representation of the parabolic subgroup of type $(2, 2)$ of G), extended to the semidirect product $G \rtimes \langle\theta\rangle$ (there are two extensions, differing by a sign, but there is a canonical choice, defined first for generic representations).

We define two homomorphisms λ_0^+, $\lambda_0^- : K(C_0) \to K(H)$ as follows. Note that $K(C_0) = K_I(C_0) \oplus K_\square(C_0)$, where $K_\square(C_0)$ is the span of the square integrable representations, while $K_I(C_0)$ is spanned by the $\pi_1 \times \pi_2$ where at least one of π_1, π_2 is induced and the other is induced or square integrable. We require that $\lambda_0^- = 0$ on $K_I(C_0)$ and $\lambda_0^+(K_I(C_0))$ lies in $K_I(H)$, the span of the induced representations of H, and that for each π_0 in $K_I(C_0)$ or $K_\square(C_0)$ we have

$$\operatorname{tr} \pi_0(f_{C_0}) = \operatorname{tr} \lambda_0^+(\pi_0)(f_H) - \operatorname{tr} \lambda_0^-(\pi_0)(f_H),$$
$$\operatorname{tr} I(\pi_0; f \times \theta) = \operatorname{tr} \lambda_0^+(\pi_0)(f_H) + \operatorname{tr} \lambda_0^-(\pi_0)(f_H),$$

for every triple (f, f_H, f_{C_0}) of matching functions on G, H, C_0.

We define a homomorphism $\lambda : K(H) \to K^\theta(G)$ and a subgroup $K_1(H)$ of $K(H)$, as follows. On $\lambda_0^+(K_\square(C_0)) + \lambda_0^-(K_\square(C_0))$ define λ by $\lambda \circ \lambda_0^+ = I$, $\lambda \circ \lambda_0^- = I$ on $K_\square(C_0)$. We require that $K_1(H)$ satisfies $K_1(H) \oplus [\lambda_0^+(K_\square(C_0)) + \lambda_0^-(K_\square(C_0))] = K(H)$ and that for any $\pi_H \in K_1(H)$ we have $\operatorname{tr} \lambda(\pi_H)(f \times \theta) = \operatorname{tr} \pi_H(f_H)$ for all matching (f, f_H). Further we define a homomorphism $\lambda_1 : K(C) \to K^\theta(G)$ by $\lambda_1(\pi_C) = \pi$ if for every matching pair (f, f_C) we have $\operatorname{tr} \pi(f \times \theta) = \operatorname{tr} \pi_C(f_C)$. Our local results assert the existence of the maps λ_0^+, λ_0^-, λ, λ_1, and the subgroup $K_1(H)$, and describe them and their properties in detail. For example, $\operatorname{Im} \lambda_0^+ \cap \operatorname{Im} \lambda_0^- = \{0\}$.

Thus our finer local lifting results concern elliptic representations (whose characters are non zero on the set of elliptic elements). They follow on using global techniques. Elliptic representations include the cuspidal ones (terminology of [BZ]. These are called "supercuspidal" by Harish-Chandra, who used the word "cuspidal" for what is currently named "discrete series" or "square integrable" representations).

Local Theorem (PGSp(2) to PGL(4)). *For any (unordered) pair π_1, π_2 of square integrable (irreducible) representations of* $\mathrm{PGL}(2,F)$ *there exists a (unique) pair π_H^+, π_H^- of tempered (square integrable if $\pi_1 \neq \pi_2$, cuspidal if $\pi_1 \neq \pi_2$ are cuspidal) representations of H with*

$$\operatorname{tr}(\pi_1 \times \pi_2)(f_{C_0}) = \operatorname{tr}\pi_H^+(f_H) - \operatorname{tr}\pi_H^-(f_H),$$
$$\operatorname{tr} I_G(\pi_1, \pi_2; f \times \theta) = \operatorname{tr}\pi_H^+(f_H) + \operatorname{tr}\pi_H^-(f_H)$$

for all triples (f, f_H, f_{C_0}) of matching functions.

If $\pi_1 = \pi_2$ is cuspidal, π_H^+ and π_H^- are the two inequivalent constituents of $1 \rtimes \pi_1$.

If $\pi_1 = \pi_2 = \sigma\,\mathrm{sp}_2$ where σ is a character of $F^\times$ with $\sigma^2 = 1$, then π_H^+ and π_H^- are the two tempered inequivalent constituents $\tau(\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2})$ and $\tau(\nu^{1/2}\mathbf{1}_2, \sigma\nu^{-1/2})$ of $1 \rtimes \sigma\,\mathrm{sp}_2$.

If $\pi_1 = \sigma\,\mathrm{sp}_2$, $\sigma^2 = 1$, and π_2 is cuspidal, then π_H^+ is the square integrable constituent $\delta(\sigma\nu^{1/2}\pi_2, \sigma\nu^{-1/2})$ of the induced $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2}$; π_H^- is cuspidal, denoted here by $\delta^-(\sigma\nu^{1/2}\pi_2, \sigma\nu^{-1/2})$.

If $\pi_1 = \sigma\,\mathrm{sp}_2$ and $\pi_2 = \xi\sigma\,\mathrm{sp}_2$, ξ $(\neq 1 = \xi^2)$ and σ $(\sigma^2 = 1)$ are characters of $F^\times$, then π_H^+ is the square integrable constituent $\delta(\xi\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2})$ of the induced $\xi\nu^{1/2}\,\mathrm{sp}_2 \rtimes \sigma\nu^{-1/2}$; π_H^- is cuspidal, denoted here by $\delta^-(\xi\nu^{1/2}\,\mathrm{sp}_2, \sigma\nu^{-1/2})$.

For every character σ of $F^\times/F^{\times 2}$ and square integrable π_2 there exists a non tempered representation $\pi_H^\times$ of H such that for all matching f, f_H, f_{C_0}

$$\operatorname{tr}(\sigma\mathbf{1}_2 \times \pi_2)(f_{C_0}) = \operatorname{tr}\pi_H^\times(f_H) + \operatorname{tr}\pi_H^-(f_H),$$
$$\operatorname{tr} I_G(\sigma\mathbf{1}_2, \pi_2; f \times \theta) = \operatorname{tr}\pi_H^\times(f_H) - \operatorname{tr}\pi_H^-(f_H).$$

Here $\pi_H^- = \pi_H^-(\sigma\,\mathrm{sp}_2 \times \pi_2)$ and $\pi_H^\times = L(\sigma\nu^{1/2}\pi_2, \sigma\nu^{-1/2})$.

For any characters ξ, σ of $F^\times/F^{\times 2}$ and matching f, f_H, f_{C_0} we have

$$\operatorname{tr}(\sigma\xi\mathbf{1}_2 \times \sigma\mathbf{1}_2)(f_{C_0}) = \operatorname{tr} L(\nu\xi, \xi \rtimes \sigma\nu^{-1/2})(f_H) - \operatorname{tr} X(\xi\nu^{1/2}\,\mathrm{sp}_2, \xi\sigma\nu^{-1/2})(f_H),$$
$$\operatorname{tr} I_G(\sigma\xi\mathbf{1}_2, \sigma\mathbf{1}_2; f \times \theta) = \operatorname{tr} L(\nu\xi, \xi \rtimes \sigma\nu^{-1/2})(f_H) + \operatorname{tr} X(\xi\nu^{1/2}\,\mathrm{sp}_2, \xi\sigma\nu^{-1/2})(f_H).$$

Here $X = \delta^-$ if $\xi \neq 1$ and $X = L$ if $\xi = 1$.

Any θ-invariant irreducible square integrable representation π of G which is not a λ_1-lift is a λ-lift of an irreducible square integrable representation π_H of H, thus $\operatorname{tr}\pi(f \times \theta) = \operatorname{tr}\pi_H(f_H)$ for all matching f, f_H. In particular, the square integrable (resp. non tempered) constituent $\delta(\xi\nu, \nu^{-1/2}\pi_2)$ (resp. $L(\xi\nu, \nu^{-1/2}\pi_2)$) of the induced representation $\xi\nu \rtimes \nu^{-1/2}\pi_2$ of H, where π_2 is a cuspidal (irreducible) representation of $\mathrm{GL}(2,F)$ *with central character $\xi \neq 1 = \xi^2$ and $\xi\pi_2 = \pi_2$, λ-lifts to the square integrable (resp. non tempered) constituent $S(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ (resp. $J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$) of the induced representation $I_G(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ of $G = \mathrm{PGL}(4,F)$.* □

These character relations permit us to introduce the notion of a packet of a(n irreducible) representation, and of a quasi-packet, over a local field. Thus we say that the *packet* of a representation π_H of H consists of π_H alone unless it is tempered of the form π_H^+ or π_H^- for some pair π_1, π_2 of (irreducible) square integrable representations of $\mathrm{PGL}(2,F)$, in which case the packet $\{\pi_H\}$ is defined to be $\{\pi_H^+, \pi_H^-\}$, and we write $\lambda_0(\pi_1 \times \pi_2) = \{\pi_H^+, \pi_H^-\}$

and $\lambda(\{\pi_H^+,\pi_H^-\}) = I_G(\pi_1,\pi_2)$. Further, we define a *quasi-packet* only for the non tempered (irreducible) representations $\pi_H^\times$ and $L = L(\nu\xi, \xi \rtimes \sigma\nu^{-1/2})$, to consist of $\{\pi_H^\times, \pi_H^-\}$ and $\{L, X\}$, $X = X(\xi\nu^{1/2}\,\mathrm{sp}_2, \xi\sigma\nu^{-1/2})$. We say that $\sigma\mathbf{1}_2 \times \pi_2$ λ_0-lifts to the quasi-packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2) = \{\pi_H^\times,\, \pi_H^-\}$, which in turns λ-lifts to $I_G(\sigma\mathbf{1}_2, \pi_2)$, and similarly, $\sigma\xi\mathbf{1}_2 \times \sigma\mathbf{1}_2$ λ_0-lifts to $\lambda_0(\sigma\xi\mathbf{1}_2 \times \sigma\mathbf{1}_2) = \{L, X\}$ which λ-lifts to $I_G(\sigma\xi\mathbf{1}_2, \sigma\mathbf{1}_2)$.

7. Automorphic Representations

With these local definitions we can state our global results. These global results are partial, since we work with test functions whose components are elliptic at least at three places, and consequently we cannot detect automorphic representations which do not have at least three components whose (θ-) characters are non zero on the (θ-) elliptic set. Thus we fix three places $\{v_1,\, v_2,\, v_3\}$ and discuss only $\pi_1 \times \pi_2$, π_H and $\pi = \pi_G$ whose components there are (θ-) elliptic. This constraint will be removed once the trace formulae identity is established for general test functions.

With this reservation, emphasized by a $*$-superscript in the following Global Theorem, the discrete series representations of $\mathbf{H}(\mathbb{A}) = \mathrm{PGSp}(2,\mathbb{A})$ can now be described by means of the liftings. They consist of two types, stable and unstable, and global packets and quasi-packets define a partition of the spectrum. To define a (global) [quasi-] packet $\{\pi_H\}$, fix a local [quasi-] packet $\{\pi_{Hv}\}$ at each place v of F, such that $\{\pi_{Hv}\}$ contains an unramified member π_{Hv}^0 (and then $\{\pi_{Hv}\}$ consists only of π_{Hv}^0 in case it is a packet) for almost all v. The [quasi-] *packet* $\{\pi_H\}$ is then defined to consist of all products $\otimes_v \pi'_{Hv}$ with π'_{Hv} in $\{\pi_{Hv}\}$ for all v, and $\pi'_{Hv} = \pi_{Hv}^0$ for almost all v. The [quasi-] packet $\{\pi_H\}$ of an automorphic representation π_H is defined by the local [quasi-] packets $\{\pi_{Hv}\}$ of the components π_{Hv} of π_H at almost all places.

The discrete spectrum of $\mathrm{PGSp}(2,\mathbb{A})$ will be described by means of the λ_0- and λ-liftings. We say that discrete series $\pi_1 \times \pi_2$ λ_0-*lifts* to a packet $\{\pi_H\}$ (or to a member thereof) if $\{\pi_{Hv}\} = \lambda_0(\pi_{1v} \times \pi_{2v})$ for almost all v, and that a packet $\{\pi_H\}$ (or a member of it) λ-*lifts* to an irreducible self-contragredient automorphic representation π if $\lambda(\{\pi_{Hv}\}) = \pi_v$ for almost all v. The *unstable* spectrum of $\mathrm{PGSp}(2,\mathbb{A})$ is defined to be the set of discrete series representations which are λ_0-lifts; its complement is the *stable* spectrum.

Global Theorem* (PGSp(2) to PGL(4)). *The packets and quasi-packets partition the discrete spectrum of the group* $\mathrm{PGSp}(2,\mathbb{A})$, *namely they satisfy the rigidity theorem: if* π_H *and* π'_H *are discrete series representations locally equivalent at almost all places then their packets or quasi packets are equal.*

The λ-lifting is a bijection between the set of packets (*resp. quasi-packets*) *of discrete series representations in the stable spectrum* (*of* $\mathrm{PGSp}(2,\mathbb{A})$) *and the set of self contragredient discrete series representations of* $\mathrm{PGL}(4,\mathbb{A})$ *which are one dimensional, or cuspidal and not a λ_1-lift from* $\mathbf{C}(\mathbb{A})$ (*resp. residual* $J(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$ *where* π_2 *is a cuspidal representation of* $\mathrm{GL}(2,\mathbb{A})$ *with central character* $\xi \neq 1 = \xi^2$ *and* $\xi\pi_2 = \pi_2$).

The λ_0-lifting is a bijection between the set of pairs of discrete series representations $\{\pi_1 \times \pi_2, \pi_2 \times \pi_1; \pi_1 \neq \pi_2\}$ *of* $\mathrm{PGL}(2,\mathbb{A}) \times \mathrm{PGL}(2,\mathbb{A})$, *and the set of packets and quasi-packets in the unstable spectrum of* $\mathrm{PGSp}(2,\mathbb{A})$. *The λ-lifting is a bijection from this last set to the set of automorphic representations* $I_G(\pi_1,\pi_2)$ *of* $\mathrm{PGL}(4,\mathbb{A})$, *normalizedly induced* *from discrete series $\pi_1 \times \pi_2$ $(\pi_1 \neq \pi_2)$ on the parabolic subgroup with Levi factor of type $(2,2)$. If $\pi_1 \times \pi_2$ is cuspidal, its λ_0-lift is a packet, otherwise: quasi-packet.*

Each member of a stable packet occurs in the discrete spectrum of $\mathrm{PGSp}(2,\mathbb{A})$ *with multiplicity one. The multiplicity $m(\pi_H)$ of a member $\pi_H = \otimes \pi_{Hv}$ of an unstable [quasi-]packet $\lambda_0(\pi_1 \times \pi_2)$ $(\pi_1 \neq \pi_2)$ is not ("stable", or) constant over the [quasi-]packet. If $\pi_1 \times \pi_2$ is cuspidal, it is $m(\pi_H) = \frac{1}{2}(1 + (-1)^{n(\pi_H)})$ $(\in \{0,1\})$, where $n(\pi_H)$ is the number of components π_{Hv}^- of π_H and each π_H with $m(\pi_H) = 1$ is cuspidal ($n(\pi_H)$ is bounded by the number of places v where both π_{1v} and π_{2v} are square integrable).*

The multiplicity $m(\pi_H)$ (in the discrete spectrum of $\mathrm{PGSp}(2,\mathbb{A})$*) of $\pi_H = \otimes \pi_{Hv}$ of a quasi-packet $\lambda_0(\sigma \mathbf{1}_2 \times \pi_2)$, where π_2 is a cuspidal representation of* $\mathrm{PGL}(2,\mathbb{A})$ *and σ is a character of $\mathbb{A}^\times / F^\times \mathbb{A}^{\times 2}$, is $\frac{1}{2}(1 + \varepsilon(\sigma\pi_2, \frac{1}{2})(-1)^{n(\pi_H)})$ $(= 0$ or $1)$, where $n(\pi_H)$ is the number of components π_{Hv}^- of π_H, and $\varepsilon(\sigma\pi_2, s)$ is the usual ε-factor which appears in the functional equation of the L-function $L(\pi_2, s)$. In particular $\pi_H^\times = \otimes \pi_{Hv}^\times$ $(n(\pi_H) = 0)$ is discrete series if and only if $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$.*

Finally we have $m(\pi_H) = \frac{1}{2}(1 + (-1)^{n(\pi_H)})$ for $\pi_H = \otimes \pi_{Hv}$ in $\lambda_0(\sigma\xi\mathbf{1}_2 \times \sigma\mathbf{1}_2)$ with $n(\pi_H)$ components $\pi_{Hv} = X_v$. Here $\pi_H = \otimes L_v$ $(n(\pi_H) = 0)$ is residual. □

8. Unstable Spectrum

Note that the quasi-packet $\lambda_0(\sigma\xi\mathbf{1} \times \sigma\mathbf{1}_2)$ is defined by the local quasi-packets $\{L_v = L(\nu_v\xi_v, \xi_v \rtimes \sigma_v\nu_v^{-1/2}),\ X_v = X(\xi_v\nu_v^{1/2}\,\mathrm{sp}_{2v}, \xi_v\sigma_v\nu_v^{-1/2})\}$ for every v, where ξ $(\neq 1)$, σ are characters of $\mathbb{A}^\times/F^\times$ with $\xi^2 = 1 = \sigma^2$ and ξ_v, σ_v are their components. When ξ_v, σ_v are unramified, this quasi-packet contains the unramified representation $\pi_{Hv}^0 = L_v$. Members of this quasi-packet have been studied by means of the theta correspondence by Howe and Piatetski-Shapiro, see, e.g., [PS1], Theorem 2.5. They attracted interest since they violate the naive generalization of the Ramanujan conjecture, which expects the components of a cuspidal representation to be tempered. (The form of the Ramanujan conjecture which is expected to be true asserts that the components of a cuspidal representation of PGSp(2,$\mathbb{A}$) which λ-lifts to a cuspidal representation of PGL(4,$\mathbb{A}$) are tempered.) Members of this quasi-packet are equivalent at almost all places to the quotient of the properly induced representation $\nu\xi \times \xi \rtimes \sigma\nu^{-1/2}$.

Let π_2 be a cuspidal representation of $\mathrm{PGL}(2,\mathbb{A})$ and σ a character of $\mathbb{A}^\times/F^\times\mathbb{A}^{\times 2}$. The packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ contains the constituent $\pi_H^\times = \otimes_v \pi_{Hv}^\times$ of the representation $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2} \simeq \sigma\nu^{-1/2}\pi_2 \rtimes \sigma\nu^{1/2}$ properly induced from an automorphic representation, hence it is automorphic by [L]. It is known that $\pi_H^\times$ is residual precisely when $L(\sigma\pi_2, \frac{1}{2}) \neq 0$; hence $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$ in this case.

Let $n(\pi_2)$ denote the number of square integrable components of π_2. The quasi-packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ thus consists of $2^{n(\pi_2)}$ (irreducible) representations. If $n(\pi_2) \geq 1$, half of them are discrete series, all cuspidal if $L(\sigma\pi_2, \frac{1}{2}) = 0$, all but one: $\pi_H^\times = \otimes_v \pi_{Hv}^\times$, are cuspidal if $L(\sigma\pi_2, \frac{1}{2}) \neq 0$. If $n(\pi_2) \geq 1$ and $L(\sigma\pi_2, \frac{1}{2}) = 0$, the automorphic non residual $\pi_H^\times$ is cuspidal when $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$.

If π_2 has no square integrable components ($n(\pi_2) = 0$), the packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ consists only of $\pi_H^\times$. This $\pi_H^\times$ is residual if $L(\sigma\pi_2, \frac{1}{2}) \neq 0$; cuspidal (by [PS1], Theorem 2.6 and [PS2], Theorem A.2) if $L(\sigma\pi_2, \frac{1}{2}) = 0$ and $\varepsilon(\sigma\pi_2, \frac{1}{2}) = 1$; or (automorphic but) not discrete series otherwise: $L(\sigma\pi_2, \frac{1}{2}) = 0$ and $\varepsilon(\sigma\pi_2, \frac{1}{2}) = -1$. In this last case the λ_0-lift of $\sigma\mathbf{1}_2 \times \pi_2$ is not a discrete series, and there is no discrete series λ-lifting to $I_G(\sigma\mathbf{1}_2, \pi_2)$.

At a place v where π_{2v} is induced $I(\mu_v, \mu_v^{-1})$, the packet $\pi_{Hv} = \lambda_0(\sigma_v\mathbf{1}_2 \times \pi_{2v})$ is the irreducible induced $\mu_v\sigma_v\mathbf{1}_2 \rtimes \mu_v^{-1}$, which λ-lifts to $I_G(\mu_v, \sigma_v\mathbf{1}_2, \mu_v^{-1})$, and *not* the irreducible induced

$$\sigma_v\mu_v\nu_v^{1/2} \times \sigma_v\mu_v^{-1}\nu_v^{1/2} \rtimes \sigma_v\nu_v^{-1/2} = \sigma_v\mu_v\nu_v^{1/2} \rtimes I(\mu_v^{-1}, \sigma_v\nu_v^{-1/2}),$$

which λ-lifts to the reducible induced $I_G(\mu_v, \sigma_v I(\nu_v^{1/2}, \nu_v^{-1/2}), \mu_v^{-1})$, which has the constituent $I_G(\mu_v, \sigma_v\mathbf{1}_2, \mu_v^{-1})$.

Members of the quasi-packet $\lambda_0(\sigma\mathbf{1}_2 \times \pi_2)$ were studied numerically by H. Saito and N. Kurokawa, and using the theta correspondence by Piatetski-Shapiro and others, see [PS1], Theorem 2.6. They attracted interest since they violate the naive generalization of the Ramanujan conjecture. They are equivalent at almost all places to the quotient of the properly induced representation $\sigma\nu^{1/2}\pi_2 \rtimes \sigma\nu^{-1/2}$.

A discrete series representation π_H with component $L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$ (whose packet consists of itself), where π_{2v} is a cuspidal representation with central character $\xi_v \neq 1 = \xi_v^2$ and $\xi_v\pi_{2v} = \pi_{2v}$, is in the packet of $L(\nu\xi, \nu^{-1/2}\pi_2)$. Here π_2 is cuspidal with central character $\xi \neq 1 = \xi^2$, hence $\xi\pi_2 = \pi_2$, whose components at v are π_{2v} and ξ_v. It λ-lifts to $J_G(\nu^{1/2}\pi_2, \nu^{-1/2}\pi_2)$. At v with $\xi_v = 1$ the component π_{2v} is induced. If $\pi_{2v} = I(\mu_v, \mu_v\xi_v)$, $\xi_v^2 = 1$ and $\mu_v^2 = 1$ (in particular whenever $\xi_v \neq 1$ and π_{2v} is not cuspidal), then $L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$ is $L_v = L(\nu_v\xi_v, \xi_v \rtimes \mu_v\nu_v^{-1/2})$, which λ-lifts to $I_G(\mu_v\mathbf{1}_2, \mu_v\xi_v\mathbf{1}_2)$, and its packet contains also $X_v = X(\nu_v^{1/2}\xi_v \,\mathrm{sp}_{2v}, \xi_v\mu_v\nu_v^{-1/2})$. Thus the packet of π_H is determined by $\{L_v, X_v\}$ at all v where $\pi_{2v} = I(\mu_v, \mu_v\xi_v)$, $\mu_v^2 = 1 = \xi_v^2$, and by the singleton $\{L_v = L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})\}$ at all other v, where π_{2v} is cuspidal, or $\xi_v = 1$ and $\pi_{2v} = I(\mu_v, \mu_v^{-1})$, $\mu_v^2 \neq 1$. Each member of this infinite packet occurs in the discrete series with multiplicity one, and is cuspidal, with the exception of $L(\nu\xi, \nu^{-1/2}\pi_2) = \otimes_v L(\nu_v\xi_v, \nu_v^{-1/2}\pi_{2v})$, which is residual ([Kim], Theorem 7.2). Members of the packet $L(\nu\xi, \nu^{-1/2}\pi_2)$ are considered in the Appendix of [PS1] and its corrigendum.

If π_1 and π_2 are cuspidal but there is no place v where both are square integrable, $\lambda_0(\pi_1 \times \pi_2)$ consists of a single irreducible cuspidal representation. This instance of the lifting λ_0 – where π_i are cuspidal – can also be studied using the theta correspondence for suitable dual reductive pairs (SO(4), PGSp(2)) for the isotropic and anisotropic forms of the orthogonal group, to describe further properties of the packets, such as their periods.

9. Generic Representations

Our proof of the existence of the lifting λ uses only the trace formula, orbital integrals and character relations. To compute the multiplicities in the discrete spectrum we use two independent techniques. The first is purely local. It uses only character relations. It is based on a twisted analogue of Rodier's result [Ro1] that the number of Whittaker models is captured as a coefficient of a regular nilpotent orbit in the germ expansion of the character. Such a technique was first used in [F4;II], Proof of Proposition 3.5, p. 47, to prove multiplicity one for the discrete spectrum of U(3). For further details see [F4;IV]. Our second proof is global. It relies on results of [GRS], [KRS], and [Sh1] from the theory of generic representations.

This second proof in our case of PGSp(2) is similar to the second proof of [F4;II], Proposition 3.5, p. 48, in the case of U(3). The 2nd proof of [F4;II] is also based on the theory of generic representations. But it is not complete. Indeed, the claim in Proposition 2.4(i) in reference [GP] to [F4;II], that "$L^2_{0,1}$ has multiplicity 1", is interpreted in [F4;II] as asserting that generic representations of U(3) occur in the discrete spectrum with multiplicity one. But it should be interpreted as asserting that irreducible π in $L^2_{0,1}$ have multiplicity one *only in the subspace* $L^2_{0,1}$ of the discrete spectrum. This assertion does not exclude the possibility of having a cuspidal π' perpendicular and equivalent to $\pi \subset L^2_{0,1}$. Multiplicity one for the generic spectrum of U(3) could be deduced from Proposition 2.4(i) of [GP] of [F4;II] using the statement that a locally generic representation is globally generic. (In fact this statement is equivalent to multiplicity one). In our case of PGSp(2) this statement follows from [KRS], [GRS], [Sh1]. A proof for U(3) (independent of the local proof of [F4;II], p. 47) still needs to be written down.

There is some overlap between our results on the existence of the λ-lifting and the work of [GRS]. However, the methods of [GRS] apply only to generic representations, while our methods apply to all representations of PGSp(2). Thus we can define packets, describe their structure, establish multiplicity one theorem and rigidity theorem for packets of PGSp(2), conclude that in a packet which λ-lifts to a generic self-contragredient representation of $\mathrm{PGL}(4,\mathbb{A})$ there is precisely one generic member, and we can also λ-lift the non generic non tempered (at almost all places) packets to residual self-contragredient representations of $\mathrm{PGL}(4,\mathbb{A})$. Our liftings are proven in terms of all places, not only almost all places. In addition we establish the lifting λ_1 from SO(4) to PGL(4), determine its fibers (that is, prove multiplicity one theorem for SO(4) and rigidity in the sense explained above), and show that any self-contragredient discrete series representation of $\mathrm{PGL}(4,\mathbb{A})$ which is not a λ-lift from $\mathrm{PGSp}(2,\mathbb{A})$ is a λ_1-lift from $\mathrm{SO}(4,\mathbb{A})$.

Our work is an analogue for (SO(4), PGSp(2), PGL(4)) of [F3], which dealt with the symmetric square lifting, thus with (PGL(2), SL(2), PGL(3)), and of [F4], which dealt with quadratic base change for the unitary group U(3,E/F), thus with (U(2,E/F), U(3,E/F), GL(3,E)). These works use the twisted – by transpose-inverse (and the Galois action in the unitary groups case) – trace formulae on PGL(4), PGL(3), GL(3,E). They are based on the fundamental lemma: [F5] in our case, [F3;V] and [F4;I] in the other cases. The technique employed in these last works benefited from work of Weissauer [W] and Kazhdan [K]. The present work, which deals with the applications of the fundamental lemma and the trace formula to character relations, liftings and the definition of packets, is analogous to [F3;II] and [F4;IV]. The trace formula identity is proven in [F3;III] and [F4;III] for all test functions. Here we deal only with test functions which have at least three elliptic components. The trace formulae identity for a general test function has not yet been proven in our case. There is no doubt that the method of [AC] could be used for that, as it has been applied in a general rank case of GL(n). It would be interesting to pursue the elementary techniques of [F3;III] and [F4;III], and [F2], which establishes the trace formulae identity for base change for GL(2) by elementary means, based on the usage of regular, Iwahori test functions. In particular our work does not develop the trace formula. It only uses a form of it.

Our approach to the lifting uses the trace formula, developed by Arthur, as envisaged by Langlands e.g. in his work on base change for GL(2). It is compatible with the conjectures of

Arthur [A]. Of course Siegel modular forms have been extensively studied by many authors (e.g. Siegel, Maass, Shimura, Andrianov,...) over a long period of time.

An important representation theoretic approach alternative to the trace formula, based on the theta correspondence, Weil representation, Howe's dual reductive pairs, L-functions and converse theorems, has been fruitfully developed in our context of the symplectic group by many authors, see, e.g., [PS], [KRS], [GRS].

Another – purely local – approach is developed in [FZ].

References.

[A] J. Arthur, On some problems suggested by the trace formula, *Lie group representations* II, Springer Lecture Notes 1041 (1984), 1-49; Unipotent automorphic representations: conjectures, in: Orbites unipotentes et representations, II, *Asterisque* 171-172 (1989), 13-71.

[AC] J. Arthur, L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals Math. Studies 120, Princeton, NJ, 1989. xiv+230 pp.

[BZ1] J. Bernstein, A. Zelevinskii, Representations of the group $\mathrm{GL}(n, F)$ where F is a nonarchimedean local field, *Russian Math. Surveys* 31 (1976), 1-68.

[BZ2] J. Bernstein, A. Zelevinsky, Induced representations of reductive p-adic groups, *Ann. Sci. École Norm. Sup.* 10 (1977), 441-472.

[Bla] D. Blasius, On multiplicities for $\mathrm{SL}(n)$, *Israel J. Math.* 88 (1994), 237-251.

[Bo] A. Borel, Automorphic L-functions, *Proc. Sympos. Pure Math.* 33 II, Amer. Math. Soc., Providence, R.I., (1979), 27-63.

[F1] Y. Flicker, Lifting automorphic forms of PGSp(2) and SO(4) to PGL(4).

[F2] Y. Flicker, Regular trace formula and base-change lifting, *Amer. J. Math.* 110 (1988), 739-764.

[F3] Y. Flicker, On the symmetric square. VI. Applications of a trace formula, *Trans. Amer. Math. Soc.* 330 (1992), 125-152; V. Unit elements, *Pacific J. Math.* 175 (1996), 507-526; VI. Total global comparison; *J. Funct. Analyse* 122 (1994), 255-278.

[F4] Y. Flicker, I. Elementary proof of a fundamental lemma for a unitary group, *Canad. J. Math.* 50 (1998), 74-98; II. Packets and liftings for U(3), *J. Analyse Math.* 50 (1988), 19-63; III. Base change trace identity for U(3), *J. Analyse Math.* 52 (1989), 39-52; IV. Germs of twisted characters and multiplicity one for U(3), preprint.

[F5] Y. Flicker, Matching of orbital integrals on GL(4) and GSp(2), *Mem. Amer. Math. Soc.* no. 655, vol. 137 (1999), 1-114.

[FZ] Y. Flicker, D. Zinoviev, Twisted character of a small representation of PGL(4), preprint.

[GRS] D. Ginzburg, S. Rallis, D. Soudry, On Generic Forms on $\mathrm{SO}(2n+1)$; Functorial Lift to $\mathrm{GL}(2n)$, Endoscopy and Base Change, *Internat. Math. Res. Notices* 14 (2001), 729-763.

[H] Harish-Chandra, Admissible invariant distributions on reductive p-adic groups, *Queen's Papers in Pure and Appl. Math.* 48 (1978), 281-346.

[JS] H. Jacquet, J. Shalika, On Euler products and the classification of automorphic forms II, *Amer. J. Math.* 103 (1981), 777-815.

[K] D. Kazhdan, On lifting, in *Lie groups representations II*, Springer Lecture Notes 1041 (1984), 209-249.

[Kim] H. Kim, Residual spectrum of odd orthogonal groups, *Internat. Math. Res. Notices* 17 (2001), 873-906.

[Ko] R. Kottwitz, Stable trace formula: cuspidal tempered terms, *Duke Math. J.* 51 (1984), 611-650.

[KS] R. Kottwitz, D. Shelstad, Foundations of twisted endoscopy, *Asterisque* 255 (1999), vi+190 pp.

[KRS] S. Kudla, S. Rallis, D. Soudry, On the degree 5 L-function for Sp(2), *Invent. Math.* 107 (1992), 483-541.

[L] R. Langlands, On the notion of an automorphic representation, *Proc. Sympos. Pure Math.* 33 I (1979), 203-207.

[MW] C. Moeglin, J.-L. Waldspurger, Le spectre résiduel de $\mathrm{GL}(n)$, *Ann. Sci. Ecole Norm. Sup.* 22 (1989), 605-674.

[PS1] I. Piatetski-Shapiro, On the Saito-Kurokawa lifting, *Invent. Math.* 71 (1983), 309-338; A remark on my paper: "On the Saito-Kurokawa lifting", *Invent. Math.* 76 (1984), 75-76.

[PS2] I. Piatetski-Shapiro, Work of Waldspurger, *Lie group representations* II, Springer Lecture Notes 1041 (1984), 280-302.

[Ro1] F. Rodier, Modèle de Whittaker et caractères de représentations, *Non-commutative harmonic analysis*, Springer Lecture Notes 466 (1975), 151-171.

[Ro2] F. Rodier, Sur les representations non ramifiees des groupes reductifs p-adiques; l'example de GSp(4), *Bull. Soc. Math. France* 116 (1988), 15-42.

[ST] P. Sally, M. Tadic, Induced representations and classifications for $\mathrm{GSp}(2,F)$ and $\mathrm{Sp}(2,F)$, *Mem. Soc. Math. France* 52 (1993), 75-133.

[Sh1] F. Shahidi, Local coefficients and normalization of intertwining operators for $\mathrm{GL}(n)$, *Compos. Math.* 48 (1983), 271-295.

[Sh2] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. of Math.* 132 (1990), 273-330.

[Sh3] F. Shahidi, Langlands' conjecture on Plancherel measures for p-adic groups, in *Harmonic analysis on reductive groups*, 277–295, Progr. Math., 101, Birkhäuser, Boston, MA, 1991.

[Shal] F. Shalika, A theorem on semisimple p-adic groups, *Ann. of Math.* 95 (1972), 226-242.

[W1] J.-L. Waldspurger, Un exercise sur $\mathrm{GSp}(4,F)$ et les representations de Weil, *Bull. Soc. Math. France* 115 (1987), 35-69.

[W2] J.-L. Waldspurger, Correspondances de Shimura et quaternions, *Forum Math.* 3 (1991), 219-307.

[W] R. Weissauer, A special case of the fundamental lemma, preprint.