1. Mathematische Arbeitstagung
(Neue Serie)

9. - 15. Juni 1993

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
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Germany

MPI / 93-57

Inhalt

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Program of the Mathematische Arbeitstagung 1993 (I)

Wednesday, June 9, 1993

3:30 – 4:15 p.m.	Opening and first program discussion
5:00 – 6:00 p.m.	YU. I. MANIN (Max-Planck-Institut für Mathematik) Report on Mirror Symmetries
8:00 – 10:00 p.m.	Chamber Music Concert: Knopp-Melançon Duo Festsaal der Universität, Hauptgebäude (entrance from "Am Hof" street across from Bouvier bookstore)

Thursday, June 10, 1993

10:15 – 11:15 a.m.	M. KONTSEVICH: A_∞–algebras in mirror symmetry
12:00 – 1:00 p.m.	W. NAHM: Conformal field theories in 2 dimensions and $\mathcal{W}$–algebras
5:00 – 6:00 p.m.	D. HARBATER: Proof of Abhyankar's conjecture on π_1 of curves in characteristic p

Friday, June 11, 1993

9:50 – 10:10 a.m.	Program discussion for the Saturday and Sunday lectures
10:15 – 11:15 a.m.	N.-P. SKORUPPA: New methods in modular forms and Jacobi forms
1:00 – 8:00 p.m.	Boat trip to Bad Breisig Departure with "Carmen Sylva" near the Kennedy Bridge

All lectures will take place in the "Großer Hörsaal", Wegelerstraße 10.

There will be *tea breaks* on Wednesday from 4:15 to 5 p.m. and on other days (except Friday) from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 20 marks.

Lists of participants and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!

All Arbeitstagung participants and those accompanying them are warmly invited to the chamber music program on Wednesday evening.

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Program of the Mathematische Arbeitstagung 1993 (II)

Saturday, June 12, 1993

10:15 – 11:15 a.m.	M. WODZICKI: Algebraic K-theory of trace class operators
12:00 – 13:00 p.m.	M. RAPOPORT: p-adic period domains
5:00 – 6:00 p.m.	T. ODA: Moduli spaces of curves with pro-nilpotent level structure

Sunday, June 13, 1993

10:00 – 10:15 a.m.	Program discussion for the Monday and Tuesday lectures
10:15 – 11:15 a.m.	G. WÜSTHOLZ: Diophantine approximation in $\mathbf{P}^n$
12:00 – 13:00 p.m.	B. HUNT: Algebraic-geometric characterization of locally symmetric spaces
5:00 – 6:00 p.m.	M. YOSHIDA: Intersection theory for twisted homology and hypergeometric functions

All lectures will take place in the "Großer Hörsaal", Wegelerstraße 10.

There are daily *tea breaks* from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 20 marks.

Lists of participants and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!

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Program of the Mathematische Arbeitstagung 1993 (III)

Monday, June 14, 1993

10:15 – 11:15 a.m.	J.-M. KANTOR: Counting lattice points in convex polyhedra
12:00 – 13:00 p.m.	B. LEEB: Metrics of non-positive curvature on 3-manifolds
3:00 – 4:00 p.m.	A. SCHWARZ: New geometric and algebraic structures in string theory
5:00 – 6:00 p.m.	H. ESNAULT: Higher Kodaira-Spencer maps
8:00 – 11 (?) p.m.	Rector's Party Festsaal der Universität, Hauptgebäude (entrance from "Am Hof" street across from Bouvier bookstore)

Tuesday, June 15, 1993

10:15 – 11:15 a.m.	K. KÖHLER: Analytic torsion and higher direct images in arithmetic K-theory
12:00 – 13:00 p.m.	Y. RUAN: Symplectic Donaldson type invariants
5:00 – 6:00 p.m.	F. OORT: Subvarieties of moduli spaces

All lectures will take place in the "Großer Hörsaal", Wegelerstraße 10.

There are daily *tea breaks* from 11:15 to 12 a.m. and 4:15 to 5 p.m. At this time also *mail* will be distributed and you will have the opportunity to pay your *Tagungsbeitrag* of 20 marks.

Lists of participants and other information will be available at the entrance of the lecture room. All participants are requested to put their name on the list!

All Arbeitstagung participants and those accompanying them are warmly invited to attend the Rector's party on Monday evening.

Titel: REPORT ON THE MIRROR CONJECTURE

Autor: YU. I. MANIN

Adresse: MAX-PLANCK INSTITUTE FÜR MATH., BONN

1. INTRODUCTION. Mirror conjecture was discovered by physicists studying σ-models targeted on Calabi-Yau spaces. Roughly speaking, it says that a characteristic series whose coefficients count the numbers of rational curves of various degrees on a Calabi-Yau threefold can be identified with another function which expresses the infinitesimal variation of the Hodge structures of another ("mirror") family of Calabi-Yau threefolds. The aim of the talk was to describe this conjecture and to report what kind of evidence supports it.

2. CALABI-YAU MANIFOLDS. Algebro-geometrically, Calabi-Yau manifolds can be described as a subclass of algebraic projective manifolds V^d with $\Omega^d_V \cong \mathcal{O}_V$. More precisely, every such V admits a finite non-ramified covering $\tilde{V}$ such that

a) $\tilde{V} \cong \prod A_i \times \prod S_j \times \prod V_k$, where A_i, S_j, V_k are indecomposable (i.e. no finite non-ramified covering is a direct product;

b) A_i are (simple) abelian varieties;

c) S_j are complex symplectic, i.e. admit an everywhere non-degenerate holomorphic 2-form, but are not abelian;

d) V_k are neither abelian, nor symplectic.

If $\tilde{V}$ does not contain A_i - or S_j - type factors, V is called Calabi-Yau. The simplest examples of A (resp. S, resp. V) are: plane cubic curves, (resp. quartic surfaces, resp. quintics in $\mathbf{P}^5$). Most classification questions about CY-manifolds are open, e.g. one does not even know whether the number of classes of CY-threefolds up to homeomorphism is finite or not.

3. CURVE COUNT. Let V be an algebraic (or compact complex) manifold. We are interested in understanding the space of holomorphic maps $Map(P^1, V)$. Let $[\varphi] \in Map$ corresponds to $\varphi: P^1 \to V$. In the non-obstructed ("general position") case we have $\dim_{[\varphi]} Map = \deg \varphi^*(-K_V) + \dim V$. In particular, for $K_V = 0$ and $\dim V = 3$, rational curves in general position on V must be rigid. Projective curve count on a quintic threefold V_5 made H. Clemens to conjecture that $n_d := \# \{ C \mid C \subset V_5 \text{ rational}, \deg C = d \}$ is finite and non-zero for all d.

More generally, let V be a CY-threefold, K the Kähler cone in $\mathrm{Pic}\,V\otimes R$, $\mathcal{V}$ an open subset in $\mathrm{Pic}\,V\otimes R+iK$. For any $H\in\mathcal{V}$, identify the tangent space to $\mathcal{V}$ at H with $\mathrm{Pic}\,V\otimes C$. For any $E_1, E_2, E_3 \in \mathcal{T}_{\mathcal{V},H}$ define a map $F_V: S^3\mathcal{T}_{\mathcal{V}} \to \mathcal{O}_{\mathcal{V}}$:

$$F_V:\ (E_1\cdot E_2\cdot E_3)_H \longmapsto \langle E_1\cdot E_2\cdot E_3\rangle + {}^{\prime\prime}\sum_C{}^{\prime\prime}\langle C,E_1\rangle\langle C,E_2\rangle\langle C,E_3\rangle\,\frac{e^{2\pi i\langle C,H\rangle}}{1-e^{2\pi i\langle C,H\rangle}}.$$

In particular, for a quintic put $E=$ a generator of $\mathrm{Pic}\,V$ (positive one), $H=tE$, $\mathrm{Im}\,t>0$; $q=e^{2\pi i t}$. Then F_V can be written as a characteristic series for (n_d):

$$F_V=F_V(q)=\langle E^3\rangle+{}^{\prime\prime}\sum_{d\geq 1}{}^{\prime\prime}\, n_d\, d^3\frac{q^d}{1-q^d}.$$

Quotes around Σ in the formulas signal that some kind of general position argument is need in order to define without restriction how to sum over rational curves. Essentially, one should use Gromov-Witten construction: deform (V,H) into a pair consisting of the symplectic manifold $(V,\omega=\mathrm{Im}\,H)$, and an almost complex structure J' which is ω-tamed. Then one should count (J',ω)-holomorphic (or perturbed holomorphic) maps $(S^2, J_0)\to(V,J')$. See additional information in the talks by Ruan and Kontsevich.

4. PERIODS. Let now $\pi: W \to Z$ be a locally complete family of CY-threefolds. Put $\mathcal{L} := \pi_*(\Omega^3_{W/Z})$. There exists another cubic form

$$G_W : S^3 T_Z \to \mathcal{L}_Z^{-2}$$

which is the symbol of three times iterated Gauss-Manin connection (mapping $\pi_*(\Omega^3_{W/Z})$ to $R^3\pi_*(\mathcal{O}_W)$).

5. Pre-mirror data. Define pre-mirror data as a quintuple $(V, W; \mathcal{V}, Z, \omega_0, q)$ where $V, W, \mathcal{V}, Z$ are defined earlier; ω_0 is a generator of $\mathcal{L}$ over Z, and q is an isomorphism $\mathcal{V} \leftrightarrow Z$. (In particular, $\dim Z = h^{12}_W = h^{11}_V = \dim \mathcal{V}$).

6. MIRROR DATA. These data are called mirror data if $q^* G_W = F_V$.

7. MIRROR CONJECTURE. One expects that every CY-threefold with $h^{12}_V > 0$ (deformable) can be included into a mirror data. The evidence for this conjecture includes the following:

- computer check that most of computed pairs of Hodge numbers (h^{11}_V, h^{12}_V) follow the expected symmetric pattern.

• a systematic way to construct many pre-mirror data for CY realized as hypersurfaces in toric Fano manifolds.

• comparison of the first coefficients of F_V and G_W in some cases where the number of curves can be directly calculated.

However, "mathematical" reasons for mirror identities remain highly mysterious, even on a heuristic level. Our community must try to absorb the physical arguments leading to this remarkable picture, only a small part of which was described in the talk. The following references contain further bibliography:

REFERENCES.

1. Essays on Mirror Symmetry, ed. by Yau, World Scientific, 1992.

2. D. Morrison. Mirror Symmetry and rational curves on quintic threefolds: A guide for mathematicians. Preprint Duke.

3. V. Batyrev. Dual Polyhedra and Mirror Symmetry for Calabi-Yau hypersurfaces in Toric Varieties, Preprint Essen.

Titel: A_∞-algebras in Mirror Symmetry
Autor: Maxim Kontsevich
Seite: 1
Adresse: MPI, Bonn

Mirror Symmetry gives a correspondence between symplectic and complex manifolds. We propose a way to "explain" MS by identifying both moduli spaces of complex and symplectic manifolds with $c_1 = 0$ with the moduli space of A_∞-algebras. Our guess leads to the following prediction: there should be a kind of twistor correspondence from Lagrangian subvarieties on one side to complexes of holomorphic vector bundles on the other side.

In Sections 1-3 we are trying to give more or less precise formulation of the Mirror Conjecture. This part can be considered as a complement to the talk of Yu.Manin. In Sections 4-6 we show that A_∞-algebras arise naturally from complicated structures of MS. We hope that our point of view will lead to the proof of the Mirror Conjecture in some future.

We want to mention that only few results here are proved rigorously.

1. Basic example. (After P.Candelas et al., see [Y])

Let V be a generic quintic in $\mathbf{C}P^4$. Denote by n_d the "number" of rational curves of degree d on V (see Sect.2 and talks of Yu.Manin and Y.Ruan for the meaning of quote-marks). Consider the following generating function in variable t):

$$F(t) = \frac{5}{6}t^3 + \sum_{d\geq 1} n_d \mathrm{Li}_3(e^{td}), \qquad \text{where} \quad \mathrm{Li}_3(x) = \sum_{k\geq 1}\frac{x^k}{k^3}, \quad \mathrm{Re}(t) < t_0 = -7.590...$$

Here t can be considered as a local coordinate on the moduli space of symplectic manifolds. We endow V by 2-form $t\times$(the pullback of the Fubini-Studi symplectic form on $\mathbf{C}P^4$).

On the other hand, let $W = W(\lambda)$ be 1-parameter family of Calabi-Yau 3-folds obtained by the resolution of singularities of

$$\{(x_1 : x_2 : x_3 : x_4 : x_5) \in P^4 | \sum x_i^5 - \lambda x_1x_2x_3x_4x_5 = 0\}/(\mathbf{Z}/5\mathbf{Z})^3$$

where $(\mathbf{Z}/5\mathbf{Z})^3$ is the group of transformations $x_i \mapsto \xi_i x_i$, $\xi_i^5 = 1$, $\prod \xi_i = 1$ preserving the volume element on the quintic. We have the variation of 4-dimensional Hodge structures $\mathrm{H}^3(W(\lambda);\mathbf{C})$ over one parameter λ. For example, one of the periods is $\displaystyle\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\lambda^{-5n}$.

The conjectured mirror relation between V and W is the following. Let us consider the trivial 4-dimensional complex vector bundle with 4 linearly independent sections $e_1, \dots, e_4$ over 1-dimensional base $\{t \in \mathbf{C} | \mathrm{Re}(t) < t_0,\ 0 < \mathrm{Im}(t) < 2\pi\}$. Introduce connection ∇ by

$$\dot{e}_1 = e_2,\ \dot{e}_2 = F'''(t)e_3,\ \dot{e}_3 = e_4,\ \dot{e}_4 = 0, \qquad \dot{e}_i := \frac{\nabla(e_i)}{dt}$$

and the Hodge filtration $0 \subset \langle e_1\rangle \subset \langle e_1e_2\rangle \subset \langle e_1e_2e_3\rangle \subset \langle e_1e_2e_3e_4\rangle$. This gives a complex variation of Hodge structures which is conjecturally eqivalent (in some coordinates $\lambda = \lambda(t)$) to the variation $\mathrm{H}^3(W(\lambda);\mathbf{C})$. One can show that t as a function of λ is equal to the ratio of two periods.

2. Gromov-Witten invariants.

Let (V,ω) be a compact *semi-positive* symplectic manifold. Semipositivity means that there exists an almost complex structure on V compatible with ω such that the canonical 2-form representing $c_1(V)$ is non-negative. In this case we expect that invariants which we call Gromov-Witten invariants are defined. These invariants depend on homology class $\beta \in \mathrm{H}_2(V;\mathbf{Z})$ and a pair (g,n) of non-negative integers satisfying inequality $2 - 2g - n < 0$,

$$I_{g,n;\beta} \in \mathrm{H}_{\mathrm{even}}\left(\overline{\mathcal{M}}_{g,n} \times V^n;\mathbf{Z}\right) .$$

Here $\overline{\mathcal{M}}_{g,n}$ denotes the coarse Deligne-Mumford compactification of the moduli space of n-punctured complex curves of genus g.

Denote by $\mathcal{X}_{g,n;\beta}$ the space of equivalence classes of $(C, x_1, \ldots, x_n, \phi)$ where C is a smooth complex curve of genus g, x_i are distinct points on C and $\phi : C \to V$ is a pseudo-holomorphic map (=the solution of a generic perturbation of the Cauchy-Riemann equation). There is a natural map from $\mathcal{X}_{g,n;\beta}$ to $\mathcal{M}_{g,n} \times V^n$ by associating with (C, x_*, ϕ) the equivalence class of (C, x_*) and the sequence of points $(\phi(x_1), \ldots, \phi(x_n))$.

It seems that using results of M.Gromov, D.McDuff and Y.Ruan one can construct a natural compactification $\overline{\mathcal{X}}_{g,n;\beta}$ which a) maps to $\overline{\mathcal{M}}_{g,n} \times V^n$ and b) has a finite Whitney stratification by even-dimensional oriented strata. We define $I_{g,n;\beta}$ to be the image of the fundamental class of $\overline{\mathcal{X}}_{g,n;\beta}$.

3. Potential and related differential-geometric structures.

Here we consider semi-positive (V, ω) as above. Denote by $\mathrm{H} := \oplus \mathrm{H}^k(V; \mathbf{C})$ the total cohomology space of V considered as a super-vector space and also as a complex flat super-manifold. Following E.Witten [W] we introduce a function Φ on H by the formula

$$\Phi(\gamma) = \sum_{\beta \in \mathrm{H}_2(V;\mathbf{Z})} e^{-\int_\beta \omega} \sum_{n=3}^{\infty} \frac{1}{n!} \int_{I_{g,n;\beta}} 1_{\overline{\mathcal{M}}_{0,n}} \otimes \gamma \otimes \cdots \otimes \gamma \,.$$

Here γ denotes a non-homogeneous cohomology class on V, $1_{\overline{\mathcal{M}}_{0,n}}$ is a zero-dimensional cohomology class 1 on $\overline{\mathcal{M}}_{0,n}$.

We expect that this series is absolutely convergent in some open domain in H if the cohomology class $[\omega] \in h^2(V, \mathbf{R})$ is sufficiently large.

Function Φ must satisfy a remarkable system of non-linear differential equations of the third order. Let us choose a base x_α of the space H. Denote by $g = (g_{\alpha\beta})$, $g_{\alpha\beta} = \int_V \alpha \wedge \beta$ the matrix of the Poincaré pairing, $(g^{\alpha\beta}) = g^{-1}$ will be the inverse matrix. For all $\alpha, \beta, \gamma, \delta$

$$\boxed{\sum_{\epsilon,\epsilon'} \frac{\partial^3 \Phi}{\partial x_\alpha \partial x_\beta \partial x_\epsilon} g^{\epsilon\epsilon'} \frac{\partial^3 \Phi}{\partial x_\gamma \partial x_\delta \partial x_{\epsilon'}} = \sum_{\epsilon,\epsilon'} \frac{\partial^3 \Phi}{\partial x_\alpha \partial x_\gamma \partial x_\epsilon} g^{\epsilon\epsilon'} \frac{\partial^3 \Phi}{\partial x_\beta \partial x_\delta \partial x_{\epsilon'}}}$$

We show now why this equation should be satisfied at point $0 \in \mathrm{H}$ (for simplicity). Third derivatives $\partial_{\alpha\beta\gamma}\Phi$ at zero point count number of rational pseudo-holomorphic curves in V passing trough 3 cycles in V with the weights $exp(-\text{area})$. Let us fix now 4 points $x_1, \ldots, x_4$ on $\mathbf{C}P^1$ and 4 cycles $C_1, \ldots C_4$ on V. We can count also the number of maps ϕ such that $\phi(x_i) \in C_i$. This number will not change under the permutations of indices and will not depend on the cross-ratio of (x_i). As the cross-ratio tends to infinity the curves must degenerate to 2 copies of $\mathbf{C}P^1$ glued at some point. At the limit we obtain the problem of counting pairs of maps $\phi_1, \phi_2 : \mathbf{C}P^1 \to V$ with restrictions

$$\phi_1(0) \in C_1,\ \phi_1(1) \in C_2,\ \phi_2(0) \in C_3, \phi_2(1) \in C_4,\ \phi_1(\infty) = \phi_2(\infty)\,.$$

It is the same as maps $\phi_1 \times \phi_2$ from $\mathbf{C}P^1$ to $V \times V$ with certain restrictions on the values at $0, 1, \infty$. Using Künneth decomposition of the homology class of diagonal we reduce the question back to V. Hence we obtain the equation at zero.

The equation above was studied by B.Dubrovin [D]. He discovered that it is a completely integrable system, and it is equivalent to the following classical problem: find curvilinear coordinates in the standard flat Euclidean space $\mathbf{R}^n$ in which the metric is be diagonal.

This equation can be reformulated as the condition of associativity of the algebra given by structure constants $A^\gamma_{\alpha\beta} := \sum_{\gamma'} g^{\gamma\gamma'} \partial_{\alpha\beta\gamma'}\Phi$. In invariant terms it means that there exists a new commutative associative multiplication on H (Quantum Ring) depending on the point of H.

Let us introduce a connection ∇ on the tangent bundle to H by the formula $\nabla = \nabla_0 + A$ where ∇_0 is the standard trivial connection. One can deduce from the main equation that A is flat.

Now we are ready to formulate the general form of the Mirror Conjecture. Suppose that $c_1(V) = 0$ and V carries at least one integrable complex structure compatible with ω. For each such complex structure we have a Hodge decomposition $\mathrm{H} = \oplus \mathrm{H}^{p,q}$. We expect that all cycles $I_{g,n;\beta}$ are Hodge cycles (algebraic cycles for the algebraic case $[\omega] \in \mathrm{H}^2(V;\mathbf{Z})$) of dimensions independent on β. It follows that the restriction of ∇ to the submanifold $\mathrm{H}^{1,1}$ of H maps $\mathrm{H}^{p,q}$ to $\mathrm{H}^{p+1,q+1} \otimes \Omega^1(\mathrm{H}^{1,1})$.

We introduce filtrations $\oplus_{p\leq p_0}\mathrm{H}^{p,q}$ on the trivial bundles $\oplus_{p-q \text{ is fixed}}\mathrm{H}^{p,q}$ on $\mathrm{H}^{1,1}$. Hence we have flat connections and filtrations. One can prove using formal arguments with Hodge-Tate groups that the equivalence classes of this variations of Hodge structures are not changing under deformations of the complex structure on V used for the Hodge decomposition.

Mirror Conjecture. *These variations of Hodge structures are algebro-geometric. Sometimes they are Hodge structures on all cohomology groups of mirror manifolds.*

We wrote *sometimes* because there are examples of rigid Calabi-Yau manifolds which can't be dual to projective manifolds.

In the case of quintic V in CP^4 the function Φ is the sum of 2 terms: the contribution of the maps to points of V and the contribution of rational curves (and their multiple covers). We introduce coordinates t_i, $i = 0,\ldots 3$ in one-dimensional spaces $\mathrm{H}^{i,i}(V)$ and odd coordinates ξ_j, η_j, $j = 1,102$ in $\mathrm{H}^3(V)$. In these coordinates we have (up to adding a polynomial of degree 2)

$$\Phi(t_i,\xi_j,\eta_j) = \frac{5}{6}\sum_{i+j+k=3} t_it_jt_k + t_0\sum_j \xi_j\eta_j + \sum_{d\geq 1} n_d \mathrm{Li}_3(e^{t_1 d}) .$$

One can deduce from this formula the Candelas example.

4. Extended moduli spaces.

When we restrict the flat bundle to the subspace $\mathrm{H}^{1,1}$ a lot of information will be lost. It seems very reasonable to extend the moduli space of symplectic structures to the whole domain in H in which the potential Φ is defined. Hence the the tangent space to the extended moduli space is equal to $\mathrm{H} = \oplus \mathrm{H}^k$.

Now we want to construct some extended moduli space $\mathcal{M}$ for complex Calabi-Yau W containing the ordinary moduli space $Mod(W)$. The natural candidate to the tangent bundle to $\mathcal{M}$ at classical points $Mod(W)$ should be equal to the direct sum $\oplus \mathrm{H}^p(W, \wedge^q T)$. This problem was already dicussed by E.Witten (see his paper in [Y]).

Our guess is that $\oplus \mathrm{H}^p(W,\wedge^q T)$ can be interpreted as total Hochschild cohomology of the sheaf $\mathcal{O}_W$ of holomorphic functions on W.

The usual definition of Hochschild cohomology needs an associative algebra A and bimodule over it. The second Hochschild cohomology of A with coefficients in A classifies infinitesimal deformations of A. One can define HH also for graded differential algebras as well. We can replace now the sheaf of algebras $\mathcal{O}_W$ by the sheaf of differential algebras $(\Omega^{0,*}, \overline{\partial})$. This sheaf has cohomology only in degree 0. Let us pass to the differential algebra of sections, i.e. global $\overline{\partial}$-forms.

Theorem. *There exists a natural isomorphism* $HH(\Omega^{0,*}(W), \overline{\partial}) \cong \oplus H^p(W, \wedge^q T)$

Another approach to the definition of Hochschild cohomology of the sheaf $\mathcal{O}$ of holomorphic functions on algebraic manifolds was developed by M.Gerstenhaber and D.Shack.

The main question now is to give some interpretation in terms of deformations of *all* Hochschild cohomology groups of differentail graded associative algebras.

5. A_∞-algebras and their deformations.

A_∞-algebras were introduced by J.Stasheff in 1964 in [S]. We will give two equivalent definitions.

First definition. Let (x_i, ξ_j) be a set of indeterminates divided into two parts (even and odd variables). Denote by $\mathcal{A} := \overline{\mathbf{C}\langle x_*, \xi_*\rangle}$ the completed free associative algebra generated by x_*, ξ_*. $\mathcal{A}$ consists of all (infinite) formal linear combinations of monomials. We consider $\mathcal{A}$ as $\mathbf{Z}/2\mathbf{Z}$-graded algebra.

By definition A_∞-algebra is continuous map $D : \mathcal{A} \longrightarrow \mathcal{A}$ satisfying conditions:

(1) $\deg(Df) = \deg f + 1 \bmod 2$,
(2) $D(fg) = D(f)g + (-1)^{\deg f} f D(g)$,
(3) $D^2 = 0$.

It is enough to define D on generators because we have super-Leibniz rule 2). For example let A be an associative algebra and c_{ij}^k be the structure constants of A in some base. We define $\mathcal{A}$ to be generated by pure odd indeterminates ξ_i and define D by $D\xi_k = \sum c_{ij}^k \xi_j \xi_i$. The associativity of A is equivalent to the condition 3).

We will say that two A_∞-algebras $(\mathcal{A}, D)$ and $(\mathcal{A}', D')$ are equivalent if there exists a continuous isomorphism between $\mathbf{Z}/2\mathbf{Z}$-graded algebras $\mathcal{A}$ and $\mathcal{A}'$ identifying D and D'.

Second definition. Let A be $\mathbf{Z}/2\mathbf{Z}$-graded vector space. The structure of A_∞-algebra on A is the infinite sequence of linear maps $m_k : A^{\otimes k} \longrightarrow A$ satisfying (higher) associativity conditions:

(1) $m_1^2 = 0$, (we can consider m_1 as a differential and (A, m_1) as a complex),
(2) $m_1(m_2(a \otimes b) = m_2(m_1(a) \otimes b) \pm m_2(a \otimes m_1(b))$, ($m_2$ is a morphism of complexes),
(3) $m_3(m_1(a) \otimes b \otimes c) \pm m_3(a \otimes m_1(b) \otimes c) \pm m_3(a \otimes b \otimes m_1(c)) \pm m_1(m_3(a \otimes b \otimes c) =$

$$= m_2(m_2(a \otimes b) \otimes c) - m_2(a \otimes m_2(b \otimes c)), \ (m_2 \text{ is associative up to homotopy}),$$

(4) and so on...

One can pass from the second definition to the first one by defining the space of generators to be A^* with the reversed parity and D on generators to be equal $\sum m_k^*$.

It is very easy to see that infinitesimal deformations of a differntial graded algebra considered as an A_∞-algebra is equal to total Hochschild cohomology of A.

We expect that the formal moduli space of A_∞-algebras near Calabi-Yau manifolds is smooth.

There are several reasons to beleive that the moduli $\mathcal{M}$ of A_∞-algebras are relevant for the situation of Mirror Symmetry:

(1) on the tangent space to $\mathcal{M}$ at each point there exists a natural associative commutative multiplication (cup-product on Hochschild cohomology),
(2) there exists a natural bundle with flat connection on $\mathcal{M}$ with the fiber equal to the periodic cyclic homology of A_∞-algebras (see [G]),
(3) any finite-dimensional A_∞-algebra with some additional data (a scalar product compatible with all higher multiplications) gives cohomology classes of $\mathcal{M}_{g,n}$ (see [K]).

Now we describe natural A_∞-structure arising in the framework of symplectic geometry.

6. Fukaya's A_∞-category.

At the end of last year Kenji Fukaya introduced an A_∞-category for semi-positive symplectic manifolds for the purposes of Donaldson invariants of 4-manifolds. First of all, A_∞-category is not a category in the strict sense. It consists of objects, the space of morphisms between any two objects and a ladder of (higher) compositions of morphisms. The axioms are such that the set of morphisms of any object into itself will be A_∞-algebra (second definition).

Let (V, ω) will be semi-positive symplectic manifold with sufficiently large $[\omega]$. Objects of A_∞-category $F(V, \omega)$ are Lagrangian submanifolds of V. Morphisms are defined only when Lagrangian submanifolds are in general position, $F(\mathcal{L}, \mathcal{L}') := \mathbf{C}^{\mathcal{L} \cap \mathcal{L}'}$ with $\mathbf{Z}/2\mathbf{Z}$-grading arising from the Maslov index.

The differential on $F(\mathcal{L}, \mathcal{L}')$ will be a modified Floer differential. The matrix coefficient of differential associated with two intersection points $p_1, p_2 \in \mathcal{L} \cap \mathcal{L}'$ is the number of pseudo-holomorphic discs $\phi : D \longrightarrow V$ with

$$\phi(-1) = p_1, \ \phi(1) = p_2, \ \phi(\text{upper boundary}) \subset \mathcal{L}, \ \phi(\text{lower boundary}) \subset \mathcal{L}'$$

counted with the weight $\exp(-\text{area of } D)$.

Higher multiplications are defined analogously using maps from polygons to V.

One can show (not rigorously) that the cap-product on the Hochschild cohomology of this category coinsides with the quantum multiplication on the ordinary cohomology $\mathrm{H}(V, \mathbf{C})$.

We don't know at the moment any mathematical definition of an A_∞-algebra $A(V, \omega)$ such that Fukaya's category is equivalent to the A_∞-category of modules over it. Also we expect that one has to extend in some way $F(V, \omega)$ (fro example, consider local systems over Lagrangian submanifolds) and obtain an equivalence between (extended) $F(V, \omega)$ and the derived category of coherent sheaves on the dual complex manifold W.

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Conformal Field Theories in Two Dimensions and their W-Algebras

Conformal quantum field theories in two dimensions have been invented to describe aspects of nature, but they already have multiple uses in mathematics. Almost by definition, they contain the theory of projective highest weight representations of $\mathrm{Diff}(S^1)$ and of the group of continuous maps from S^1 to any compact Lie group G, but even in this case only they explain in a simple way why the characters are modular functions. Concerning finite simple groups, the monster has been constructed as automorphism group of such a theory. In topology, they provide invariants of 3-manifolds and knots. In algebraic geometry, they allow much insight into Calabi-Yau manifolds, as has been shown by the discovery of mirror symmetry.

Quantum field theories often are defined by functional integrals. Those integrals can be computed numerically, though this is very expensive and the existence of the relevant limits often remains an open problem. They also can be characterized axiomatically. Systems satisfying the axioms can be constructed in favorable cases, though to some extent their relation to the integral expressions remains guesswork.

Let us start with the second approach. First, consider one dimensional functional integrals, i.e. integrals over spaces $C(t_a,t_b)$ of continuous functions defined on closed intervals $[t_a,t_b]$ of the real line. The image space of the functions is arbitrary. The integrand should be local in the sense that

$$I(f_{13}) = I(f_{12})\, I(f_{23}) \, ,$$

where $f_{ij} \in C(t_i,t_j)$, $t_1 \leq t_2 \leq t_3$, and f_{12}, f_{23} are the restrictions of f_{13} to the respective intervals. Formally, this yields

$$\int_{f_{13}\in C(t_1,t_3)} I(f_{13}) = \int dx \int_{\substack{f_{12}\in C(t_1,t_2)\\ f_{12}(t_2)=x}} I(f_{12}) \int_{\substack{f_{23}\in C(t_2,t_3)\\ f_{23}(t_2)=x}} I(f_{23})$$

The two integrals on the right are functions of x. When the last one takes values in a space V, then the previous one naturally takes values in the dual space V^*. Let us assume that the functional integrand is invariant under translations of the real axis, such that V does not depend on t_2. If we also have invariance under an orientation change of the real line, then V and V^* can be identified. In physics this is often the case, but not always.
When we specify functions of $f_{13}(t_1)$ and $f_{13}(t_3)$ in V and V^*, resp., the equation above yields the following natural axioms for one dimensional functional integrals: We have a category with one object (corresponding to the points of the real line). The arrows are the non-negative real numbers t (corresponding to the intervals), with their additive semigroup structure. Another category has V as single object, the arrows are the linear maps of V. The functional integral is a functor from the first category to the second.

This formulation immediately generalizes to higher dimensions. For two dimensional functional integrals we consider the category which has one dimensional closed manifolds as objects (including the empty one) and two dimensional manifolds with corresponding boundary components as arrows.
A new feature is the possibility of any number of boundary components.
The corresponding function spaces obviously should be tensor products of the basic functions space V corresponding to a single circle. Thus the second category has as objects the tensor product powers of V and as arrows the corresponding linear maps. Again we get a functor which should satisfy some obvious properties.
In general, functional integrals in n dimensions require the glueing of bounded submanifolds along pieces of the boundary, such that one has to consider submanifolds of codimension 1,2, etc. However, for conformally invariant theories in two dimensions, this is not necessary and the framework sketched above is adequate (G. Segal, unpublished notes).
In this case, the arrows joining two circles form annuli with a complex structure. Their semigroup is the complexification of $Diff(S^1)$. The

categories have to be extended to allow for projective representations.

The complexification of the Lie algebra of $Diff_C(S^1)$ decomposes into two copies of the Virasoro algebra, corresponding to holomorphic and anti-holomorphic vector fields in neighbourhoods of S^1. Both contain natural rotation generators L_0, L_0', which give a grading on V. To get a good limiting behaviour for infinitesimally small annuli, the spectra of L_0 and L_0' must be bounded from below.

Arbitrary 2-manifolds can be constructed by glueing annuli to a standard sphere with three removed circles. We remove circles of the Riemann sphere around $0, \infty$, and z, and consider the map $V \times V \to V$ indexed by z, which is given by the functor of the conformal theory. This functor can be considered as a map

$$V \times \mathbb{C}^* \to GL(V).$$

More precisely, the resulting linear transformation of V is unbounded, but range and domain can be described easily. If the circle around 0 is filled, then z can be taken to zero, which yields an identification of V with certain operators on V. With this identification, the map $V \times V \to V$ can be regarded as an operator product expansion (OPE). The representation of $Diff_C(S^1)$ on V together with the OPE give a complete description of the conformal theory.

The subspace V_{hol} of V which is invariant under the anti-holomorphic part of $Diff_C(S^1)$ is called the space of holomorphic fields. The OPE of holomorphic fields is again holomorphic. The corresponding restriction of the OPE is called W-algebra. It takes the form

$$\varphi(z)\chi = \sum_{k\in\mathbb{Z}} z^k \psi_k$$

Invariance under L_0 yields $k = h(\psi_k) - h(\varphi) - h(\chi)$, where $L_0\varphi = h(\varphi)$ etc. Thus the sum over k is bounded from below.

By translational invariance one finds

$$\varphi(z)\chi(z') = \sum_k (z-z')^k \psi_k(z') .$$

Taking z and z' to the unit circle from opposite sides yields distributional limits. The two distributions resulting from the choice of the sides agree for $z \neq z'$ and are linear combinations of derivatives of Dirac's delta distribution. The difference defines a Lie bracket of $\varphi(z)$ and $\chi(z')$, which is in one-to-one correspondence with the singular part of the OPE.

A special case are the affine Kac-Moody algebras, which have the form

$$[j_a(z)\, j_b(z')] = f_{ab}{}^c\, j_c(z)\, \delta(z-z') + \delta_{ab}\, \delta'(z-z').$$

Together with such a Lie algebra, the OPE of holomophic fields defines a so-called normal ordered product

$$:\varphi\chi: \; = \psi_0 \,.$$

The remaining ψ_k can be obtained from normal ordered products of derivative fields. The Lie bracket of normal ordered products can be calculated in terms of the Lie brackets of their factors. In most cases of interest, all of V_{hol} is generated by a finite dimensional subspace V_{simple} in terms of derivatives and normal ordered products. The action of the diffeomorphisms on V_{simple} is uniquely determined in terms of the grading by L_0, such that the complete holomorphic part of the conformal theory is determined by the grading of a basis of V_{simple} and the finitely many structure constants which describe the Lie algebra of these basis elements. One obtains a conformal theory, when these structure constants are compatible with the Jacobi identity of the Lie bracket.
The generators of $\mathrm{Diff}_C(S^1)$ themselves yield a holomorphic field T with $h(T)=2$. Its Fourier components satisfy the Virasoro algebra. If one adds one additional simple field of grade >6, the Jacobi identities only allow a finite number of solutions for the structure constants. For grade 8 e.g. there are 9 solutions, one of which yields a central extension of the Virasoro algebra with $c=-3164/23$.
The space V can be considered as a representation space for the W-algebra V_{hol} and its anti-holomorphic counterpart. In many cases, the W-algebras only have finitely many irreducible representations, which yields a finitely reducible V. Such conformal theories are called rational. The characters of their representations are traces of q^{L_0} over irreducible subspaces of V. They can be related to the image of the complex torus of modulus τ under the functor of the conformal theory, where $q=\exp(2\pi i\tau)$. As a result, the characters of a rational theory form a representation of the modular group,

$$\chi_i\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\begin{pmatrix}a&b\\c&d\end{pmatrix}_{ij} \chi_j(\tau), \quad \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL(2,Z)$$

(Nahm 1990). This explains the behaviour of the characters of the affine Kac-Moody algebras and of the Virasoro algebra.
The resulting representation of the modular group together with its special

basis allows the construction of topological quantum field theories in three dimensions. Here one has a category of closed 2-manifolds with arrows given by topological 3-manifolds, and a functor to a category of finite dimensional vector spaces, which carry representations of the mapping class groups of the corresponding 2-manifolds. The functor can be explicitly evaluated by constructing the 3-manifolds by Dehn twists around knots in S^3 and using the corresponding elements of the modular group (Witten 1989). For SU(2) Kac-Moody algebras, the results agree with perturbative evaluations of the corresponding functional integrals (Freed and Gompf 1991).

For a large class of rational conformal theories, the characters have expressions of the form

$$\chi_i \quad \sum_{m_k \geq 0} \frac{q^{mBm+b_i m}}{(q)_{m_1}\ldots(q)_{m_r}}, \quad (q)_m=(1-q)\ldots(1-q^m)$$

(Nahm et al. 1993).

Here the rxr matrix B labels the theory and the vectors b_i the representations. For the representation of the W-algebra in V_{hol} itself, one has b=0. Apart from some power of q, these characters are modular functions. In particular, this yields

$$c = L(\zeta_i), \quad \zeta_i = \prod_j (1-\zeta_j)^{2B_{ij}}.$$

Here c is given by the central extension of $\mathrm{Diff}_C(S^1)$ and is rational for rational theories. Thus conformal theories yield identities for the Rogers dilogarithm L. The arguments lie in the number field generated by the matrix elements of the representation of the modular group. Since conformal theories yield invariants of 3-manifolds, one can hope for an interesting duality between their dilogarithm identities and the classification of 3-manifolds by their hyperbolic volume (Thurston 1982), which also is given by a sum of positive values of a dilogarithmic function.

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Consider the following general problem: If $\mathcal{U}$ is an affine curve over an algebraically closed field k of characteristic p, what is $\pi_1(\mathcal{U})$? And in particular, find the set $\pi_A(\mathcal{U})$ of Galois groups of finite étale Galois covers of $\mathcal{U}$. Thus $\pi_A(\mathcal{U})$ is the set of finite quotients of the profinite group $\pi_1(\mathcal{U})$.

If we allow $p=0$ and take $k=\mathbb{C}$, the answer is known by topology. Namely, if $\mathcal{U} = X - \{\xi_0, \dots, \xi_r\}$ where X is smooth and projective of genus g then $\pi_1(\mathcal{U})$ is free on $2g+r$ generators and $\pi_A(\mathcal{U})$ is the set of finite groups with $2g+r$ generators. From now on take $p>0$. Then the above description of π_1 is false. For example the affine line is no longer simply connected, since there are Artin-Schreier covers such as $y^p - y = x$. In 1957, Abhyankar posed the following conjecture [Ab1], for characteristic p:

Abhyankar's Conjecture ("AC") $\pi_A(\mathcal{U}) = \{G \mid$ every prime-to-p quotient of G has $2g+r$ generators$\}$.

Equivalently, $\pi_A(\mathcal{U}) = \{G \mid G/p(G)$ has $2g+r$ generators$\}$, where $p(G)$ is the subgroup of G generated by all the p-subgroups of G.

One inclusion follows from later work of Grothendieck [Gr, XIII, Cor. 2.12]. Namely, the prime-to-p part of π_1 is the free pro-prime-to-p group on $2g+r$ generators. This implies $\subset$ in AC. The same result in SGA1 also asserts that the tame fundamental group is a quotient of the corresponding π_1 over $\mathbb{C}$, and so is small. This shows that most groups referred to in AC cannot arise via a tame cover (i.e. a cover whose completion is tamely ramified over every $\xi_i \in X$). But one can pose the following

<u>Strong Abhyankar Conjecture</u> ("SAC"): If $G/p(G)$ has $2g+r$ generators, then G is the Galois group of a finite étale Galois cover of $\mathcal{U}$ which is tame except possibly over ξ_0.

The purpose of this talk is to discuss a proof of this SAC (and hence of AC).

<u>Remarks</u>: 1) SAC = AC over $\mathbb{A}^1$.

2) AC shows that π_A depends only on the values of g and r. But this is definitely <u>false</u> for π_1.

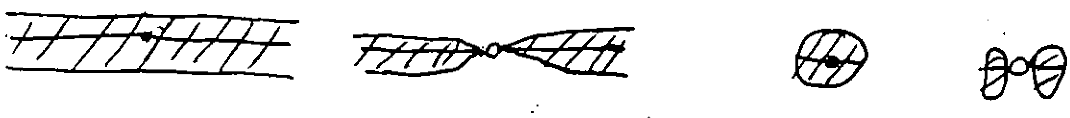
3) The above is only in the affine case, and there are many differences in the projective case. One result there, by my student Kate Stevenson is: Say G is generated by G_1 and G_2, and that G_i occurs over some projective curve of genus g_i. Then G occurs over some projective curve of genus $g = g_1 + g_2$.

Historically, much work was on the case $\mathcal{U} = \mathbb{A}^1$. There, AC says that $\pi_A = \{\text{quasi-p-groups}\}$, i.e. groups G with $G = p(G)$. In the past decade or so, Nori (cf. [Ka]) and Abhyankar (e.g. [Ab2]) realized many simple groups over $\mathbb{A}^1$. In 1990, Serre [Se] proved the case of G solvable, over $\mathbb{A}^1$, and also showed one can extend by a solvable group. Then, in 1992, Raynaud proved the full AC over $\mathbb{A}^1$, using rigid geometry [Ra].

For more general affine curves, I had shown partial results on AC in 1991 using formal geometry [Ha2], which is related to rigid geometry. Then this year ('93) I showed the full AC, and SAC, using Raynaud's result and formal geometry [Ha3]. I had also used similar methods earlier in showing that every finite group is a Galois group over $\mathbb{Q}_p(x)$ [Ha1].

The idea of formal or rigid geometry is to obtain more open sets than one has in the Zariski topology, so that cut-and-paste constructions become possible. Algebraically, consider the following simple situation: Let R be a Dedekind domain, and $f \in R$ an element lying in a single maximal ideal $\mathfrak{m}$. Let M be a finitely generated projective R-module. Then M induces M_1 over $R_1 = R[1/f]$, M_2 over $R_2 = \hat{R}_{\mathfrak{m}}$, and a common M_0 over $R_0 = \hat{K}_{\mathfrak{m}} = \mathrm{frac}(\hat{R}_{\mathfrak{m}})$, and these are also finitely generated and projective. Conversely, given M_1 and M_2 with an isomorphism over R_0, there is a unique M over R inducing them. Moreover, this remains true if we replace R, R_1, R_2, R_0 by $R[[t]], R_1[[t]], R_2[[t]], R_0[[t]]$, by [Ha2]. Geometrically, $X = \mathrm{Spec}\, R$ is a curve, $X_1 = \mathrm{Spec}\, R_1 = X - \{\mathfrak{m}\}$, $X_2 = \mathrm{Spec}\, \hat{\mathcal{O}}_{X,\mathfrak{m}} = \mathrm{Spec}\, R_2$ (a "small disc" about the point $\mathfrak{m}$ on X), and $X_0 = \mathrm{Spec}\, R_0$ is a "punctured small disc" about $\mathfrak{m}$. By adjoining power series in t, we are taking thickenings of these curves, i.e. tubular neighborhoods in some surface. We have these pictures:

and thickenings:



The assertion is that given modules over X_1 and X_2 that agree over X_0, they were induced by a unique module over X. And this also holds for the thickenings. In fact this is an equivalence of categories. So since it is true for modules, it is also true for algebras (which are modules with extra maps satisfying some diagrams) and for Galois covers. Also, the above still holds if X is projective and in the thickened case even for irreducible thickened curves with reducible closed fibres. Using this patching result, one can obtain covers of a thickened curve by constructing them locally.

The above is the formal geometric view. For the rigid view, delete the closed fibre, obtaining a curve $\mathfrak{X}$ over the metrized field $k((t))$. View this curve as like a Riemann surface. The corresponding patching picture is:

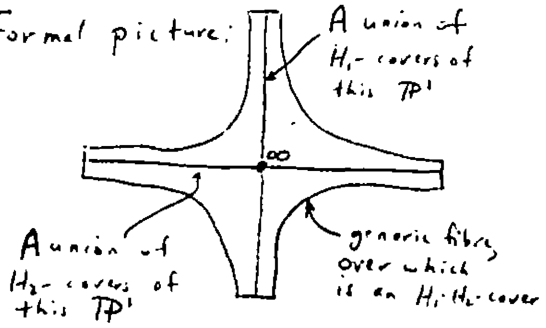
Thus given modules (or algebras, or covers) over $\mathfrak{X}_1$ and $\mathfrak{X}_2$ that agree over $\mathfrak{X}_0$, they can be patched to an object over $\mathfrak{X}$.

In Raynaud's proof of AC over $\mathbb{A}^1$, he proceeds by induction on the order of the quasi-p-group G. Let P be a Sylow p-subgroup of G, and let $H_1, \ldots, H_m$ be the proper quasi-p subgroups of G with a Sylow p-subgroup in P. There are three cases:

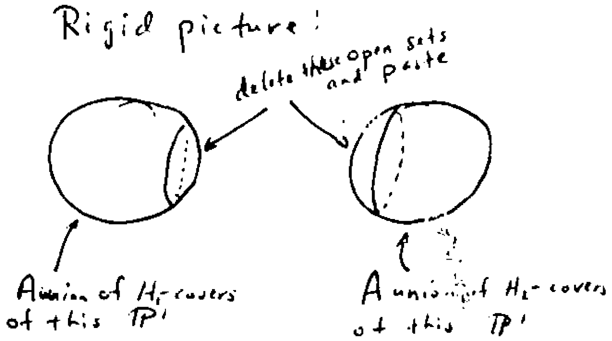
<u>Case I</u>: G has a non-trivial normal p-subgroup N. Then G/N is smaller, and so occurs by induction. We are done by Serre's result on extensions by solvable groups. [Se].

<u>Case II</u>: $H_1, \ldots, H_m$ generate G. By induction, each H_i occurs over $\mathbb{A}^1$. Taking a union of copies of an H_1-cover, indexed by the cosets of H_1 in $H_1 H_2$ we obtain an induced (disconnected) $H_1 \cdot H_2$-cover of the line. Do this also for H_2. Using the hypothesis on the subgroups H_i, these may be pasted, yielding an irreducible $H_1 \cdot H_2$-cover of $\mathbb{A}^1$. Now repeat successively with $H_3, \ldots, H_m$. Since $H_1, \ldots, H_m$ generate G, we eventually obtain a G-cover of $\mathbb{A}^1$. Here, this pasting may be viewed either via formal or rigid geometry:

Formal picture:



Rigid picture:



Case III: Otherwise. In this case, Raynaud's proof is somewhat similar in spirit to Grothendieck's proof in the prime-to-p case. The proof here requires the use of semi-stable reduction and the failure of the conditions of cases I and II.

For general affine curves, there are two steps: ① Reduction to the case of $\mathbb{P}^1 - \{0,\infty\}$; ② Proof of that case. For ①, take a general $U = X - \{\xi_0, \dots, \xi_r\}$ and G, with $G/p(G)$ having $2g+r$ generators. We want G over U. We can reduce to the case that G is a semi-direct product $G = p(G) \rtimes F$, with F of order prime to p. Now F occurs over U by [Gr]; and letting C be an inertia group over ξ_0, $E = p(G) \rtimes C$ occurs over $\mathbb{P}^1 - \{0, \infty\}$ by the conjecture in that case. Pasting these together, and using $G = E \cdot F$, we get G over U. For ②, by hypothesis, $1 \to Q \to G \to C \to 1$ with $Q = p(G)$, and by [Ra] there is a Q-cover of $\mathbb{A}^1$. We may assume that $H = P \cdot C$ is a semi-direct product, for some Sylow p-subgroup $P \subset G$. Pasting an appropriate H-cover of the thickened line to the Q-cover, one obtains the desired G-cover. In both ① and ②, the constructions yield not just AC but also SAC.

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Titel:

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NEW METHODS IN MODULAR FORMS AND JACOBI FORMS

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The following three theorems represent typical examples of recent applications of the theory of modular forms or Jacobi forms to other parts of mathematics or to physics.

Theorem [E-S]. *Let c be one of the values of the table below, and let H_c the associated set of rational numbers. Then there exists a set of holomorphic functions $\xi_{c,h}$ ($h \in H_c$), defined on the upper half plane, which satisfy the following conditions:*

(1) *the functions $\xi_{c,h}$ are non-zero modular functions for some congruence subgroup of $\mathrm{SL}(2,\mathbf{Z})$, and the space of functions spanned by the $\xi_{c,h}$ ($h \in H_c$) is an irreducible $\mathrm{SL}(2,\mathbf{Z})$-module via the action $(A,\xi(\tau)) \mapsto \xi(A\tau)$,*
(2) *for each $h \in H_c$ one has $\xi_{c,h}(\tau) = \mathcal{O}(q^{-\tilde{c}/24})$ as $\mathrm{Im}(\tau)$ tends to infinity, where $q = e^{2\pi i\tau}$ and $\tilde{c} = c - 24 \min H_c$,*
(3) *for each $h \in H_c$ the function $q^{-(h-\frac{c}{24})}\xi_{c,h}$ is periodic with period 1, and its Fourier coefficients are rational numbers.*

Moreover, up to multiplication by scalars, the functions $\xi_{c,h}$ ($h \in H_c$) are uniquely determined by these three conditions.

c	H_c
$-\frac{444}{11}$	$-\frac{1}{11}\{0,9,10,12,14,15,16,17,18,19\}$
$-\frac{1420}{17}$	$-\frac{1}{17}\{0,27,30,37,39,46,48,49,50,52,53,55,57,58,59,60\}$
$-\frac{3164}{23}$	$-\frac{1}{23}\{0,54,67,81,91,94,98,103,111,112,116,$
	$118,119,120,122,124,125,129,130,131,132,133\}$

Theorem [S]. *If a positive fundamental discriminant $D \equiv 1 \mod 8$ is a congruent number, i.e. the area of a right triangle with rational sides, then $\nu_+(D,r) = \nu_-(D,r)$, where r denotes any solution to $r^2 \equiv D \mod 128$ and*

$$\nu_\pm(D,r) = \#\{(a,b,c) \in \mathbf{Z}^3 \mid b^2 - 4ac = D,\ b^2 < D,\ \pm a > 0,$$
$$a \equiv 3\frac{b+r}{2} \mod 32,\ 3c \equiv \frac{b-r}{2} \mod 32\}.$$

Theorem [C-S-Z]. *Denote by $\epsilon(m)$ the number of modular elliptic curves with conductor $\leq m$, counted up to isogeny (i.e. the number of newforms of weight 2 on $\Gamma_0(m')$ with rational eigenvalues under all Hecke operators, where m' runs through all positive integers $\leq m$), . Then, for $m \leq 1000$, one has*

$$\epsilon(m) \approx \frac{7}{13}\frac{m^{3/2}}{\log m}.$$

The first theorem is the answer to a typical problem of physicists working in conformal field theory: Each c-value of the given table represents the central charge of a rational model of a certain $\mathcal{W}$-algebra, and the sets H_c represent the sets of conformal dimensions of these models (cf. W. Nahm's talk at this Arbeitstagung). It is in general difficult to compute the conformal characters of a given rational model of a $\mathcal{W}$-algebra directly from physical informations. However,

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one knows that they satisfy the "axioms" (1) – (3) given in the theorem. Thus the theorem actually states that, for the three specific models considered here, the conformal characters are uniquely determined by the central charge and the set of conformal dimensions. (Is that true in general, say, if one restricts to "generic" $\mathcal{W}$-algebras ?) Moreover, the proof of the theorem gives explicit, closed formulas for the conformal characters [E-S].

It should be noted that the theorem, or rather the functions which it characterizes, may deserve more interest as one might expect at the first glance. First of all, it is possibly only the prototype of a whole series of analogous theorems for other rational models of $\mathcal{W}$-algebras. Secondly, it is known by examples that conformal characters are very special modular functions with remarkable features. Sometimes they have analytic expressions which lead to identities of Roger-Ramanujan-type, which in turn are closely related to identities for the dilogarithm (cf. [N-R-T] for such examples). The modular functions $\xi_{c,h}$ characterized by the theorem appear to be promising in this respect too: among some other remarkable features they have for instance interesting product expansions [E-S]. In this sense, the first theorem is perhaps only a tiny indication of new methods which may come up in the future in the interaction of conformal field theory and modular forms.

The second theorem is, in a different statement, known as Tunnell's theorem [T]: Tunnell gave easily computable, necessary conditions for a number D to be a congruent number. The criterion given here is different from Tunnell's. It is a prototype of an infinity of analogous theorems which can be produced systematically in an algorithmic way as we shall indicate below (cf. [S4] for more details).

The third theorem, as stated, is, of course, not really a theorem: one would have to make precise the "$\approx$". What it means (still vaguely) is that if you plot the graph of $\epsilon(m)$ then it just looks very close to the graph of the function on the right hand side of the approximate identity. For more precise information cf. [C-S-Z].

There is one point which is common to all three theorems: there is no conceptual proof for them, or, to state it more precisely, in all three cases the assertions are reduced to problems which have to be solved computationally.

For the third theorem it is obvious what has to done for proving the statement: compute the newforms of weight 2 on $\Gamma_0(m') = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ m'\mathbf{Z} & \mathbf{Z} \end{pmatrix} \cap \mathrm{SL}(2,\mathbf{Z})$ for all $m' \leq m$. The approximate identity was indeed obtained exactly like this (as a side result of an algorithmic project of tabulating all newforms in a given range [C-S-Z]).

The first theorem is reduced, using several more or less deep theorems from the representation theory of $\mathrm{SL}(2,\mathbf{Z})$ and from the theory of modular forms, to essentually the problem of computing a basis for the space of all modular forms of weight $k = k(c)$ on a certain subgroup $\Gamma = \Gamma(c)$ (or rather a certain subspace of this space which is cut out by certain representation theoretic considerations [E-S]).

The second theorem is reduced, following Tunnell's original reasoning, to the problem of finding explicit formulas for the values $L(f, D, 1)$. Here f is the unique (normalized) cusp form of weight 2 on $\Gamma_0(32)$, and $L(f, D, s)$ is the L-series of f twisted by D, i.e.

$$L(f, D, s) = \sum_{n=1}^{\infty} a_f(n)\chi_D(n)n^{-s},$$

where the $a_f(n)$ are the Fourier coefficients of f, and where $\chi_D(n)$ is the quadratic character mod D which, for any odd prime p, satisfies $\chi_D(p)$ = usual Legendre symbol "D over p". Indeed, the standard compactification of the Riemann surface $\Gamma_0(32)\backslash\mathfrak{H}$ ($\mathfrak{H}$ = upper half plane in $\mathbf{C}$) equals $E(\mathbf{C})$ for a certain elliptic curve E defined over $\mathbf{Q}$ (a Weierstrass model of E is given by $y^2 = x^3 - x$). Translating the question of D beeing a congruent number or not into an equation it is not hard to show that this question is equivalent to the question whether $E_D(\mathbf{Q})$, the set of $\mathbf{Q}$-rational point on the twisted elliptic curve E_D, is infinite or not. A well-known theorem

(Coates-Wiles) shows that $E(\mathbb{Q})$ infinite implies $L(E_D, 1) = 0$, where $L(E_D, s)$ is the L-series of E_D. By Eichler-Shimura theory one has $L(E_D, s) = L(f, D, s)$. Thus, the problem is reduced to finding formulas for $L(f, D, 1)$.

At this stage Jacobi forms come into play (in contrast to Tunnell's original theorem where he used modular forms of half integral weight). We sketch the main features of this theory.

Denote the space of cusp forms of (integral) weight k on $\Gamma_0(m)$ by $S_k(\Gamma_0(m))$, and let $S^{\epsilon}_{k,m}$ be the space of Jacobi (cusp) forms of weight k, (integral) index $m > 0$ and signature ϵ. The members of $S^{\epsilon}_{k,m}$ are called "holomorphic" for $\epsilon = -1$ and "skew-holomorphic" for $\epsilon = +1$. (For a precise definition of Jacobi forms cf. any of [S1] up to [S4].) The main point needed here is that Jacobi forms are functions in two variables, having a Fourier development, where the Fourier coefficients $C_\phi(D, r)$ of a Jacobi form $\phi \in S^{\epsilon}_{k,m}$ are indexed by discriminants D and integers r satisfying $D\epsilon > 0$ and $r^2 \equiv D \bmod 4m$, and where these coefficients depend on r only modulo $2m$. As for modular forms on $\Gamma_0(m)$ one has Hecke operators $T(l)$ ($l = 1, 2, \ldots, \gcd(l, m) = 1$) acting on $S^{\epsilon}_{k,m}$, and one has the notion of a "newform": A Jacobi form (or modular form) is called newform if it is an eigenform of all $T(l)$ and if it is uniquely determined (in $S^{\epsilon}_{k,m}$ respectively $S_k(\Gamma_0(m))$) by all its eigenvalues. One has the following two basic theorems:

Theorem [S-Z]. *For each newform $\phi \in S_{k,m} := S^{+}_{k,m} \oplus S^{-}_{k,m}$ there exists a newform $f \in S_{2k-2}(\Gamma_0(m))$ which has the same eigenvalues under all Hecke operators $T(l)$ and vice versa.*

Theorem [G-K-Z]. *Let $\phi \in S_{k,m}$ and $f \in S_{2k-2}(\Gamma_0(m))$ be newforms with the same set of eigenvalues under all Hecke operators $T(l)$. Then, for any fundamental discriminant D and any integer r such that $r^2 \equiv D \mod 4m$, one has*

$$|c_\phi(D, r)|^2 = c(k, m)\frac{|\phi|^2}{|f|^2}|D|^{k-3/2}\, L(f, D, k-1)$$

with a constant $c(k, m)$ depending only on k and m and $|\phi|^2$, $|f|^2$ denoting Petersson scalar products.

(For a proof of the latter theorem for the case $\epsilon = -1$ cf. [G-K-Z, Corollary 1 in sec.II.3]; for the case $\epsilon = +1$ use [S3, Proposition 1] from which the asserted identity can be deduced analogously to the reasoning in [G-K-Z]).

These two theorems suggest that Jacobi forms are very similar to modular forms of half-integral weight. Indeed, there is a very close relationship and various theorems concerning modular forms of half-integral weight can easily be translated to Jacobi forms: for instance, the last theorem is nothing else but an adaption of Waldspurger's theorem to Jacobi forms. The point in using Jacobi forms instead of modular forms of half integral weight is that the latter have certain technical deficiencies which make them hard to use in general: For instance, a really satisfactory analogue of the one before the last theorem is only known for essentially squarefree m; it is not even clear in general how many modular forms of half intgeral weight in given space "lift" to a given newform. In contrast to this Jacobi forms are quite easy and natural tu use: one might view them too as a "new method" in modular forms.

Returning to the congruent number problem, we see from the preceeding considerations that the question for the values $L(f, D, 1)$ is reduced to the question for formulas for the Fourier coefficients $C_\phi(D, r)$ of the (unique) newform in $S_{2,32}$.

Summing up, we have seen that the proofs of the three theorems stated in the beginning are in each case reduced to the problem of computing certain modular forms or Jacobi forms or finding closed formulas for them.

For presenting a last "new method" we sketch how one can arrive at explicit formulas for the Fourier coefficients of modular forms and Jacobi forms. This method is based on an idea which is, as far as it concerns modular forms, not at all new: The idea is due to Manin [M1], [M2]. It is roughly the idea that modular forms can be characterized by their periods, and hence that there must be formulas expressing the Fourier coefficients of of modular forms in terms of their periods: a prototype of such a formula is the "Manin reciprocity law" ([M2]). This idea was taken up by many authors (key word: "modular symbols"). Modular symbols have been used to create the tables [Cr] and parts of the tables [C-S-Z], periods of special modular forms have been computed in [K-Z], [A], interesting formulas for the action of Hecke operators on modular symbols have been given recently in [Z], [Me].

For simplicity assume $\Gamma = \Gamma_0(m)$ and $k = 2$. Consider the following exact sequence of Γ-modules (with respect to the obvious actions)

$$0 \to V \xrightarrow{i} \mathbf{C}[\mathbf{P}_1(\mathbf{Q})] \xrightarrow{\deg} \mathbf{C} \to 0.$$

We set

$$Co(\Gamma) := \ker\left(\mathrm{H}_0(\Gamma, V) \xrightarrow{i_*} \mathrm{H}_0(\Gamma, \mathbf{C}[\mathbf{P}_1(\mathbf{Q})])\right),$$

where $\mathrm{H}_0(\Gamma, W)$, for any Γ-module W, is the space of Γ-co-invariants of W. Let g be the diagonal matrix with diagonal entries -1 and $+1$. Since Γ is normalized by g the natural action of g on elements of $\mathbf{C}[\mathbf{P}_1(\mathbf{Q})]$ induces an involution on $Co(\Gamma)$: denote by $Co(\Gamma)^{\pm}$ the corresponding ± 1 eigenspaces.

There is an obvious action of the Hecke operators $T(l)$ on $Co(\Gamma)$, which can be described explicitly in terms of representatives of co-invariant classes by simple formulas. Furthermore, there is an obvious notion of integration of holomorphic or anti-holomorphic modular forms along elements of $Co(\Gamma)$. This integration defines a perfect pairing between $S_k(\Gamma) \oplus \overline{S}_k(\Gamma)$ (the second space denoting the space of anti-holomorphic cusp forms) and $Co(\Gamma)$ [S4, Theorem 2]. On $S_k(\Gamma) \oplus \overline{S}_k(\Gamma)$ we have the perfect pairing defined by taking wedge products and integrating along $\Gamma\backslash\mathfrak{H}$, and on $Co(\Gamma)$ we have the (via integration along co-invariants) dual pairing $\rho \#_\Gamma \sigma$. Using this "intersection number" one has the following theorem:

Theorem[S5]. *Let $\sigma_0 \in Co(\Gamma)$, let $\epsilon = \pm 1$. Then the association*

$$\sigma \mapsto \sum_{l=1}^{\infty} ((T(l)\sigma_0) \#_\Gamma \sigma)\, q^l$$

defines a Hecke-equivariant map $L_{\sigma_0}: Co(\Gamma)^\epsilon \to S_2(\Gamma)$. There exist co-invariants σ_0 such that L_{σ_0} is an isomorphism.

This theorem is a simple formal consequence of the fact that integration along co-invariants defines a perfect pairing and the most basic facts from Hecke theory. It is meaningless with respect to the problem of computing modular forms as long as there are no explicit formulas for the intersection numbers $\rho \#_\Gamma \sigma$. Fortunately, such explicit formulas do exist: roughly, they can be obtained by constructing suitable (non-holomorphic) kernel functions k_σ for the functional $f \mapsto$ the integral of f along $\sigma \in Co(\Gamma)$ and computing the integrals of these kernel functions along co-invariants [S5].

The kernel functions which show up in these considerations show also up as the basic components of the so-called theta kernels which yield the correspondance between Jacobi and modular newforms mentioned above [S2]. Analyzing this more carefully one obtains a theorem for Jacobi forms which is analogous to the last one, i.e. one obtains maps from $Co(\Gamma_0(m))$ into $S_{2,m}$ which can be explicitly described using the intersection number of co-invariants, and among these there exist maps which are surjective [S4], [S5]. Thus, this theory yields explicit formulas for the Fourier coefficients of Jacobi forms too, which maybe used, e.g., to produce explicit formulas for the values $L(f, D, 1)$ for newforms f on any given $\Gamma_0(m)$.

To give a flavour of the kind of explicit formulas for the intersection numbers $\rho \#_\Gamma \sigma$ let $Q(X, Y)$ be a binary, integral, indefinite quadratic form whose discriminant is not a perfect square, let $\lambda_\pm$ denote the two (real) roots of the equation $Q(X,1)=0$, and denote by σ_Q be the coinvariant such that, for any $f \in S_2(\Gamma)$, the integral of f and $\overline{f}$ along σ_Q equals the integral of $\omega := f(\tau)Q(\tau,1)\,d\tau$ and $\overline{\omega}$ along $\Gamma_Q\backslash\gamma$, where γ is the hyperbolic line in the upper half plane connecting λ_+ to λ_-, and where Γ_Q is the stabilizer in Γ of this line. Then, for $\sigma = \sigma_Q$, one has

$$\rho\#_\Gamma\sigma = \text{const.}\cdot \sum_{Z\in\Gamma Z_2} Z_1\#Z,$$

where const. is an easy constant depending only on Γ, where $Z_2 = (\lambda_+) - (\lambda_-) \in \mathbf{C}[\mathbf{P}_1(\mathbf{R})]$, where ΓZ_2 is the orbit of Z_2 under the natural action of Γ on $\mathbf{C}[\mathbf{P}_1(\mathbf{R})]$, where $Z_1 \in \mathbf{C}[\mathbf{P}_1(\mathbf{Q})]$ is a representative of the co-invarinant class ρ, and where we use

$$Z_1\#Z_2 = \frac{1}{2}\sum_{r,s\in\mathbf{R}} m_r n_s \operatorname{sign}(s-r), \qquad \left(Z_1 = \sum_{r\in\mathbf{P}_1(\mathbf{R})} m_r(r),\ Z_2 = \sum_{s\in\mathbf{P}_1(\mathbf{R})} n_s(s)\ \in \mathbf{C}[\mathbf{P}_1(\mathbf{R})]\right).$$

The latter is the natural intersection number that one would associate to Z_1, Z_2 viewed as formal linear combinations of hyperbolic lines in the upper half plane. It is easy to check that the sum in the formula for $\rho\#_\Gamma\sigma$ is finite. A similar formula for the intersection number $\rho\#_\Gamma\sigma$ holds true if $Z_2 \in \mathbf{C}[\mathbf{P}_1(\mathbf{Q})]$ is a representative of σ; however, in this case the sum in the above formula for $\rho\#_\Gamma\sigma$ is no longer finite and has to be "renormalized" [S5].

Finally we mention that the above reasoning can be performed for any group Γ which admits a reasonable Hecke theory and for arbitrary integral weight [S5]. By specializing the resulting formulas for the Fourier coefficients of modular forms one reobtains the formulas given in [M2], [K-Z], [A], [Z], [Me]. It is also possible to use the above ideas to produce closed formulas for modular forms of half integral weight.

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Titel: Algebraic K-theory of Trace Class Operators
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Notation: $\mathcal{B} = \mathcal{B}(H)$ the ring of bounded linear operators on the standard Hilbert space H

$\mathcal{F}$ 2-sided ideal of finite rank operators

$\mathcal{K}$ 2-sided ideal of compact operators

By Calkin - von Neumann (1941, Ann. Math.) any proper nonzero 2-sided ideal $J \subset \mathcal{B}$ contains $\mathcal{F}$ and is contained in $\mathcal{K}$.

Let $A \in \mathcal{B}$ be an operator invertible modulo J. Then A defines the class

(1) $$[A] \in K_1(\mathcal{B}/J) \overset{\partial}{\rightarrow} K_0(\mathcal{B}, J).$$

<u>Theorem</u>. For any proper nonzero ideal $J \subset \mathcal{B}$ one has

$$\mathbb{Z} = K_0(\mathcal{B}, \mathcal{F}) \xrightarrow{\sim} K_0(\mathcal{B}, J)$$

and

$$K_{-1}(\mathcal{B}, J) = 0. \quad \square$$

Thus, (1) associates with A a certain integer. That integer is, of course, the index of A.

Unlike $K_0(\mathcal{B}, J)$, the higher index groups $K_i(\mathcal{B}, J)$ depend on J. Let us consider two extreme cases:

$(\mathcal{J} = \mathcal{F})$ $\quad K_*(\mathcal{B}, \mathcal{F}) \simeq K_*(\mathbb{C})$ the isomorphism being induced by the inclusion $\mathbb{C} \hookrightarrow \mathcal{F}$

$$1 \longmapsto \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix} \in \mathcal{F}$$

$(\mathcal{J} = \mathcal{K})$ $\quad K_*(\mathcal{B}, \mathcal{K}) \simeq K_*^{top}(\mathbb{C})$ $\quad$ (A. Suslin + M.W., 1990).

Therefore, the groups $K_*(\mathcal{B}, \mathcal{J})$ can be thought of as interpolating between $K_*(\mathbb{C})$ and $K_*^{top}(\mathbb{C})$.

Definition. We shall say that $\mathcal{J}$ is a Banach ideal if $\mathcal{J}$ admits a complete norm such that the composition tri-linear map $\mathcal{B} \times \mathcal{J} \times \mathcal{B} \longrightarrow \mathcal{B}$ is continuous.

Main Theorem. Let $\mathcal{J}$ be an arbitrary Banach ideal in $\mathcal{B}$. Then:

1) there are 5-term exact sequences

$$0 \to HC_{2j-1}(\mathcal{B}, \mathcal{J}) \longrightarrow K_{2j}(\mathcal{B}, \mathcal{J}) \longrightarrow \mathbb{Z} \longrightarrow HC_{2j-2}(\mathcal{B}, \mathcal{J}) \longrightarrow K_{2j-1}(\mathcal{B}, \mathcal{J}) \to 0$$

$(j \in \mathbb{Z})$ which are natural in $\mathcal{J}$, and

2) there exists a natural spectral sequence

$$(2) \qquad E^k_{pq} \Longrightarrow K_{p+q}(\mathcal{B}, \mathcal{J})$$

whose E^1-term is given by

$$E^1_{pq} = \begin{cases} H_{q-p-1}(\mathcal{B}; J^{p+1})_{\mathbb{Z}/(p+1)\mathbb{Z}} & q-1 \geq p \geq 0 \\ \mathbb{Z} & p = q \\ 0 & \text{otherwise} \end{cases} \quad \square$$

Here, as everywhere else, HC_* and H_* stand for the cyclic and Hochschild homology groups of $\mathbb{Q}$-algebras and corresponding bimodules. The action of $\mathbb{Z}/n\mathbb{Z}$ on $H_*(\mathcal{B}; J^n)$ is described as follows.

Let $B_* = B_*(\mathcal{B}; J)$ denote the Bar resolution $\mathcal{B} \otimes_{\mathbb{Q}} J \xleftarrow{b'} \mathcal{B} \otimes_{\mathbb{Q}} \mathcal{B} \otimes_{\mathbb{Q}} J \xleftarrow{b'} \cdots$. There are n natural quasiisomorphisms

$$\underbrace{B_* \otimes_{\mathcal{B}} B_* \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} B_* \otimes_{\mathcal{B}}}_{(n \text{ times})} \overset{\nu_1}{\underset{\nu_n}{\rightrightarrows}} C_*(\mathcal{B}; J^n)$$

where $C_*(\mathcal{B}; J^n)$ denotes the Hochschild complex of the bimodule J^n over the $\underline{\mathbb{Q}\text{-algebra}}$ $\mathcal{B}$. They give rise to the $\mathbb{Z}/n\mathbb{Z}$-action on $H_*(\mathcal{B}; J^n)$.

A similar action has been previously considered for ideals in free algebras by Quillen (1988).

Comments and remarks :

1) Regulators. The canonical comparison map $K_i(\mathbb{C}) \to K_i^{top}(\mathbb{C})$ is zero for $i > 0$. This, combined with Main Theorem (M.T.) gives natural "regulator" maps:

$$\rho_{2j}^{J} : K_{2j}(\mathbb{C}) \longrightarrow HC_{2j-1}(\mathcal{B}, J)$$

and

$$\rho_{2j-1}^{J} : K_{2j-1}(\mathbb{C}) \longrightarrow HC_{2j-2}(\mathcal{B}, J) / \mathit{Image}\ \mathbb{Z}$$

$(j \in \mathbb{Z})$ which are induced by the inclusion $\mathcal{F} \hookrightarrow J$.

2) The Main Theorem holds for the following class of ideals $J \subset \mathcal{B}$ which is much larger than the class of Banach ideals:

define the tensor multiplier of an ideal I as

$$TM(I) = \{ T \in \mathcal{B} \mid T \otimes_{\mathbb{C}} A \in I(H^{\tilde{\otimes} 2}) \ \ \forall A \in I(H) \}$$

and $J_\infty := \bigcup_{n \geq 1} J^{1/n}$ (thanks to Dixmier (1951), we know that every 2-sided ideal in a von Neumann algebra has a unique n-th root).

Let us call an ideal $J \subset \mathcal{B}$ <u>convenient</u> if the inclusion $\mathcal{F} \hookrightarrow TM(J_\infty)$ induces the zero map $\mathbb{C} = H_0(\mathcal{B};\mathcal{F}) \xrightarrow{0} H_0(\mathcal{B}; TM(J_\infty))$ and the inclusion $TM(J_\infty) \hookrightarrow \mathcal{K}$ induces a surjective map

$$K_{-2}(TM(J_\infty)) \twoheadrightarrow K_{-2}(\mathcal{K}) = \mathbb{Z}.$$

Then the assertion of M.T. is true for any convenient ideal. One can show that every Banach ideal is convenient.

3) The spectral sequence (2) exists for <u>any</u> ideal $J \subset \mathcal{B}$. In particular, $E^2_{**} = E^\infty_{**}$ = Absolute Deligne cohomology of Spec $\mathbb{C}$. However, the exact relation between (2) and $K_*(\mathcal{B}, J)$ for an <u>arbitrary</u> J is not clear.

4) <u>Schatten and related ideals</u>.

Let $C_p = \{ A \in \mathcal{K} \mid \sum_{i=1}^{\infty} \lambda_i (AA^*)^{p/2} < \infty \}$ $(p > 0)$

and $C_{p-} = \bigcup_{q<p} C_q$. One has, in particular,

C_1 = trace class operators

C_2 = Hilbert-Schmidt operators.

Corollary.

$$(n < p \leq n+1) \qquad K_i(\mathcal{B}, C_{p-}) \simeq \begin{cases} \mathbb{Z} & i = 2j \leq 2n \\ 0 & i = 2j+1 \leq 2n-1 \\ \mathbb{C}/\mathrm{Image}\ \mathbb{Z} & i = 2n+1 \end{cases}$$

$$(n < p < n+1) \qquad K_i(\mathcal{B}, C_{p-}) \xrightarrow{\sim} K_i(\mathcal{B}, C_p) \quad \text{for } i \leq 2n+1$$

$$(p = n) \qquad K_i(\mathcal{B}, C_n) \simeq \begin{cases} \mathbb{Z} & i = 2j \leq 2n-2 \\ 0 & i = 2j+1 \leq 2n-3 \\ (\mathbb{C}/\mathrm{Image}\ \mathbb{Z}) \oplus C_1^0/[C_2, C_2] & \text{for } i = 2n-1 \end{cases}$$

(here $C_1^0 = \{\text{trace zero operators}\}$; Gary Weiss showed (1979) that $\dim C_1^0/[C_2, C_2] > \aleph_0$, Nigel Kalton (1989) gave a complete description of $[C_2, C_2]$).

5) Fix an ideal $J \subset \mathcal{B}$. Let us define a functor

$$K_*^J : \{\text{unital } \mathbb{C}\text{-algebras}\} \longrightarrow \{\text{graded ab. groups}\}$$

by setting $K_*^J(A) = K_*(A \otimes_{\mathbb{C}} \mathcal{B}, A \otimes_{\mathbb{C}} J)$.

For the smallest ideal, namely $J = \mathcal{F}$, we get the algebraic K-theory. On the other hand, the following theorem shows that for suitable choices of J,

$K_*^J(\cdot)$ can behave as the usual topological K-theory.

Theorem. Let us assume that $J = J^2$ and $C_p \otimes_{\mathbb{C}} J \subset J(H^{\tilde{\otimes} 2})$ for some $p > 0$. Then $K_*^J(\cdot)$ is a strictly excisive and Bott periodic K-theory. $\square$

An extension of M.i. to the case of arbitrary coefficient C-algebra reads as follows.

Theorem. Let J be a Banach operator ideal. Then there exists a natural in J long exact sequence of functors

$$\cdots \to K_{-1}^{J_\infty} \to HC_{2j-1}^J \to K_{2j}^J \to K_0^{J_\infty} \to HC_{2j-2}^J \to$$
$$\to K_{2j-1}^J \to K_{-1}^{J_\infty} \to HC_{2j-3}^J \to \cdots$$
$$(j \in \mathbb{Z}).$$

Here $HC_*^J(A) := HC_*(A \otimes_{\mathbb{C}} \mathcal{B}, A \otimes_{\mathbb{C}} J)$, A a unital $\mathbb{C}$-algebra and HC_* means the cyclic homology over $\mathbb{Q}$.

$\square$

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In joint work with Th. Zink we investigated p-adic analogues of the classical period domains in cases corresponding to p-divisible groups.

Recall that the classical period domains arise in the following manner. Let G be a reductive algebraic group over $\mathbb{R}$ and let $h_0 : \mathbb{C}^\times \to G$ be a homomorphism of real algebraic groups factoring through a compact Cartan subgroup. Let D be the set of all conjugates of h_0. Then for every representation (V, ρ) of G any $h \in D$ equips V with a real polarizable Hodge structure. Associating to h the corresponding Hodge filtration $\mathcal{F}_h^\bullet$ identifies D with an open subset of a suitable flag variety $\mathcal{F}$. In this way D becomes a complex manifold, the <u>period domain associated to G and the conjugacy class of</u> h_0.

To formulate the p-adic analogue of this construction we fix the following notations. Let $\mathbb{C}$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Let

K_0 be the completion of the maximal unramified subextension of $\mathbb{Q}_p$, with its Frobenius automorphism σ. In the following $K \subset \mathbb{C}$ will always be a finite extension of K_0.

An <u>isocrystal</u> is a pair consisting of a finite-dimensional K_0-vector space V_0 and a σ-linear bijective map from V_0 into itself. A <u>filtered isocrystal over</u> K is an isocrystal (V_0, Φ) together with a finite decreasing separating and exhaustive filtration $\mathcal{F}^\bullet$ of the K-vector space $V_0 \otimes_{K_0} K$. A filtered isocrystal over K is called <u>weakly admissible</u> (Fontaine) if for all Φ-stable subspaces V_0'

$$\sum_i i \cdot \dim gr^i_{\mathcal{F}^\bullet \cap V'} (V_0' \otimes_{K_0} K) \leq \operatorname{ord} \det (\Phi \, ; V_0')$$

and if equality holds for $V_0' = V_0$. Faltings has shown that the tensor product of two weakly admissible filtered isocrystals over K is again one.

<u>Conjecture</u> (Fontaine): (i) To every weakly admissible filtered isocrystal over K one can associate a p-adic Galois representation of $\mathrm{Gal}(\bar{K}/K)$ of the same dimension as V_0, compatible with $\oplus$, $\otimes$ and duals.

(ii) If $(V_0, \Phi, \mathcal{F}^\cdot)$ is such that (V_0, Φ) is the isocrystal associated to a p-divisible group over $\overline{\mathbb{F}}_p$ and such that $\mathcal{F}^0 = V_0 \otimes_{K_0} K$ and $\mathcal{F}^2 = (0)$ then $(V_0, \Phi, \mathcal{F}^\cdot)$ is the filtered isocrystal associated to a p-divisible group with good reduction over K and the associated p-adic Galois representation is afforded by its rational Tate module.

To form the p-adic period domain, let

G = reductive algebraic group.

$\mu : \mathbb{G}_m \longrightarrow G_{\mathbb{C}}$, up to conjugacy.

$\mathbb{F}$ corresponding flag variety, defined over some finite extension $E \subset \mathbb{C}$ of $\mathbb{Q}_p$.

$b \in G(K_0)$.

Any point $x \in \mathbb{F}(K)$, where K contains E, defines for any representation (V, ϱ) of G a filtered isocrystal over K, <u>via</u>

$$V_0 = V \otimes_{\mathbb{Q}_p} K_0 , \quad \Phi = \varrho(b) \cdot (id_V \otimes \sigma), \quad \mathcal{F}^\cdot = \mathcal{F}^\cdot_{\varrho(x)} .$$

The point x is called <u>weakly admissible</u> if this filtered isocrystal is weakly admissible for every (V, ϱ).

Let $\mathbb{F}^{wa}$ be the subset of weakly admissible points of $\mathbb{F}\otimes_E(EK_0)$ (this set depends on b). It is easy to see that $\mathbb{F}^{wa}$ is an admissible open subset of $\mathbb{F}\otimes_E(EK_0)$ in the sense of rigid-analytic geometry. It is the <u>p-adic period domain associated to</u> (G, the conjugacy class of μ, b). Rigid spaces of this kind were also considered in a similar way by Faltings.

Fix (G, μ, b) as above. Let

$J(\mathbb{Q}_p) = \{ g \in G(K_0) ; \ \sigma(g) = b \cdot g \cdot b^{-1} \}$. It acts on $\mathbb{F}^{wa}$.

Then this is the set of $\mathbb{Q}_p$-rational points of an inner form of a Levi subgroup of G. The element $b \in G(K_0)$ is called <u>basic</u> if J is an inner form of G itself. <u>In this case</u> we conjecture that $\mathbb{F}^{wa}$ coincides with the following subset of $\mathbb{F}\otimes_E(EK_0)$ which was in essence considered by van der Put / Voskuil. Let $T \subset J$ be a torus and let $\mathbb{F}(T)^{ss}$ be the set of points of $\mathbb{F}\otimes_E(EK_0)$ which are semi-stable in the sense of geometric invariant theory w.r.t. the restriction to T of the action of J on $\mathbb{F}\otimes_E(EK_0)$ and its lifting to the anticanonical (ample) line bundle. The subset in question is given as

$$\bigcap_T \mathcal{F}(T)^{ss}.$$

Now assume that $G \subset GL(V)$ is a classical group and that (V_0, Φ) is the isocrystal of a p-divisible group over $\overline{\mathbb{F}}_p$ and that μ describes filtrations $\mathcal{F}^\bullet$ with $\mathcal{F}^0 = V_0 \otimes \mathbb{C}$, $\mathcal{F}^2 = (0)$. It is possible to associate to (μ, b) an inner form G' of G. The work of Zink and myself produces for every open compact subgroup $K' \subset G'(\mathbb{Q}_p)$ a rigid space $M_{K'}$ with an action of $J(\mathbb{Q}_p)$ and an equivariant étale morphism

$$\pi_{K'} : M_{K'} \longrightarrow \mathcal{F}^{wa}.$$

and such that $G'(\mathbb{Q}_p)$ acts on this tower as a group of Hecke correspondences. If conjecture (ii) of Fontaine above (p. 3) were true, then $\pi_{K'}$ would be surjective. Our construction first produces for certain K' a formal scheme $\mathcal{M}_{K'}$ over $\mathrm{Spf}(O_{E.K_0})$ as the solution to a moduli problem of p-divisible groups whose reduction modulo p belongs to the isogeny class defined by (V_0, Φ). Then $M_{K'}$ is the associated rigid space and $\pi_{K'}$ is the period morphism (essentially due to Grothendieck). The case of a general open compact subgroup K' is obtained by imposing level structures on the universal p-divisible group.

Our construction generalizes a similar construction of Drinfeld. Other special cases were treated previously (but by different methods) by Dwork and by Gross and Hopkins.

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Titel: The universal monodromy representations on the pro-nilpotent fundamental groups of algebraic curves

Seite: 1

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§ 1. Formulation of the problem

(1.1) The universal monodromy representation on the fundamental groups.

Let $\mathcal{M}_g$ be the moduli stack of smooth complete algebraic curves of genus g, and $\mathcal{C}_g \xrightarrow{\pi} \mathcal{M}_g$ be the universal family over $\mathcal{M}_g$. Let $\mu : \operatorname{Spec} \Omega \to \mathcal{M}_g$ be a geometric point and C_μ be the geometric fiber at μ. Then we have the universal monodromy representation

$$\rho_g : \pi_1(\mathcal{M}_g, \mu) \longrightarrow \operatorname{Out} \pi_1(C_\mu).$$

Here $\pi_1(\mathcal{M}_g, \mu)$ is the fundamental group of $\mathcal{M}_g$, and $\operatorname{Out} \pi_1(C_\mu)$ is the outer automorphism of $\pi_1(C_\mu)$:

$$\operatorname{Out} \pi_1(C_\mu) = \operatorname{Aut} \pi_1(C_\mu) / \operatorname{Inn} \pi_1(C_\mu).$$

The group $\operatorname{Out} \pi_1(C_\mu)$ itself is quite huge, almost intractable. Here is a branching point how to proceed. We take a way directed by the papers of Ihara [ ] and Deligne [ ], i.e. consider the pro-nilpotent completion of $\pi_1(C_\mu)$.

(1.2) Truncation by the weight filtration.

Fix a prime number l. Let $\pi_1(C_\mu)_l$ be the pro-l completion of $\pi_1(C_\mu)$. For a complete curve C_μ, the weight filtration $\{W_{-m}(\pi_1(C_\mu)_l)\}_{m \geq 1}$ on $\pi_1(C_\mu)_l$ is identical with the lower central series defined inductively by

$$W_{-1}\pi_1(C_\mu)_l = \pi_1(C_\mu)_l \;;\; W_{-(m+1)}\pi_1(C_\mu)_l = \overline{[\pi_1(C_\mu)_l, W_{-m}\pi_1(C_\mu)_l]} \quad (m \geq 1)$$

Here $[H, K]$ is the commutator subgroups of the subgroups H, K of $\pi_1(C_\mu)_l$, and $\overline{}$ the closure.

These $W_{-m}\pi_1(C_\mu)_\ell$ are characteristic subgroups of $\pi_1(C_\mu)_\ell$, and $\pi_1(C_\mu)_\ell$ is a characteristic quotient of $\pi_1(C_\mu)$. Therefore there is a canonical homomorphism $\mathrm{Out}\,\pi_1(C_\mu) \longrightarrow \mathrm{Out}\,\pi_1(C_\mu)_\ell / \{W_{-m-1}\pi_1\}$. Composing this with ρ_g, we have a truncated monodromy representation:

$$\rho_g(\text{class } m) : \pi_1(\mathcal{M}_g, \mu) \longrightarrow \mathrm{Out}\{\pi_1(C_\mu)_\ell / W_{-(m+1)}\pi_1(C_\mu)_\ell\}.$$

We denote by $\pi_1^{geom}(\mathcal{M}_g)$ the subgroup $\pi_1(\mathcal{M}_g \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu)$ in $\pi_1(\mathcal{M}_g, \mu)$, and by $\rho_g^{geom}(\text{class } m)$ the restriction of ρ_g to $\pi_1^{geom}(\mathcal{M}_g)$.

Now we want to squeeze out the geometric part from the truncated monodromy representation $\rho_g(\text{class } m)$.

Since $\pi_1^{geom}(\mathcal{M}_g) \triangleleft \pi_1(\mathcal{M}_g, \mu)$, $\mathrm{Im}\{\rho_g^{geom}(\text{class } m)\}$ is a normal subgroup of $\mathrm{Im}\{\rho_g(\text{class } m)\}$. On passing to the quotients, we have a natural homomorphism

$$\psi_m : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cong \frac{\pi_1(\mathcal{M}_g, \mu)}{\pi_1^{geom}(\mathcal{M}_g)} \longrightarrow \frac{\mathrm{Im}\{\rho_g(\text{class } m)\}}{\mathrm{Im}\{\rho_g^{geom}(\text{class } m)\}}$$

<u>Definition</u> Set

$$K(\mathcal{M}_g; \text{class } m) := \overline{\mathbb{Q}}^{\,\mathrm{Ker}(\psi_m)}.$$

The fields $\{K(\mathcal{M}_g; \text{class } m)\}_{m \geq 0}$ makes a tower of field extensions such that for each $m \geq 1$, $\mathrm{Gal}(K(\mathcal{M}_g; \text{class } m+1)/K(\mathcal{M}_g; \text{class } m))$ is a finitely generated $\mathbb{Z}_\ell$-module. $K(\mathcal{M}_g; \text{class } 0) = \mathbb{Q}$ and $K(\mathcal{M}_g; \text{class } 1) = \mathbb{Q}(\zeta_{\ell^\infty})$, the ℓ-cyclotomic extension of $\mathbb{Q}$.

Here is a theorem.

§2. <u>A theorem and conjectures</u>

Theorem If $k \geq 1$, then the extension
$$K(\mathcal{M}_g;\ \text{class } 2k)\,/\,K(\mathcal{M}_g;\ \text{class } 2k-1)$$
is a finite (l-) extension.

Conjecture A. If $k \geq 1$, then
$$K(\mathcal{M}_g;\ \text{class } 2k) = K(\mathcal{M}_g;\ \text{class } 2k-1).$$

Here is the second conjecture.

Let $\mathcal{M}_{0,n}$ be the moduli space of ordered set of n-distinct points in $\mathbb{P}^1$. $\mathcal{M}_{0,3} = \operatorname{Spec} \mathbb{Q}$, $\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$ over $\mathbb{Q}$. $\mathcal{M}_{0,5} = \mathbb{P}^2 - \{\ \}$. Let $\mathcal{M}_{0,4} \to \mathcal{M}_{0,3}$ be the forgetful morphism of the last point. Let
$$\rho_{0,3} : \underset{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}{\underset{\|}{\pi_1(\mathcal{M}_{0,3})}} \longrightarrow \operatorname{Out}\{\pi_1(C_\mu)_l \,/\, W_{-(m+1)}\pi_1(C_\mu)_l\}$$
be the truncated monodromy representation in this situation. Since $C_\mu = \mathbb{P}^1 - \{0, 1, \infty\}/\overline{\mathbb{Q}}$, the weight filtration $\{W_{-m}\pi_1(C_\mu)_l\}_{m \geq 1}$ is non-trivial only even m. Set
$$K(\mathcal{M}_{0,3} : \text{class } m) := \overline{\mathbb{Q}}^{\operatorname{Ker}\{\rho_{0,3}(\text{class } m)\}}.$$
Then by definition
$$K(\mathcal{M}_{0,3} : \text{class } 2k) = K(\mathcal{M}_{0,3};\ \text{class } 2k-1),$$
if $k \geq 1$.

§3. Some results on the Teichmüller groups

Let Π_1 be the surface group of genus g, and Γ_g be the Teichmüller group of genus g. Let $\operatorname{Out}^+(\Pi_1)$ be the orientation preserving outer automorphism group of the surface group Π_1.

Then the Nielsen isomorphism $\Gamma_g \xrightarrow{\simeq} \mathrm{Out}^+(\Pi_1)$ is identified with the universal monodromy homomorphisms associated to the universal family $C_g^{an} \to \mathcal{M}_g^{an}$. Here $\mathcal{M}_g^{an}$ is the analytic stack of compact Riemann surfaces of genus g.

Let $\{W_{-m}(\Pi_1)\}_{m\geq 1}$ be the lower central filtration on Π_1, and set

$$\Gamma_g[m] \underset{\mathrm{def}}{:=} \ker\left(\Gamma_g \longrightarrow \mathrm{Out}\{\Pi_1/W_{-(m+1)}(\Pi_1)\}\right).$$

Hence $\Gamma_g[0]=\Gamma_g$, $\Gamma_g[0]/\Gamma_g[1] \cong Sp_{2g}(\mathbb{Z})$, and $\Gamma_g[m]/\Gamma_g[m+1]$ is a free $\mathbb{Z}$-module of finite rank.

$\Gamma_g[1]/\Gamma_g[2]$ is determined by D. Johnson [ ], $\Gamma_g[2]/\Gamma_g[3]$ by S. Morita [ ], and $\Gamma_g[3]/\Gamma_g[4] \otimes_{\mathbb{Z}} \mathbb{Q}$ by M. Asada & H. Nakamura. By comparison theorem and a simply group theoretical description of $\mathrm{Gal}(K(\mathcal{M}_g;\text{class } m+1)/K(\mathcal{M}_g;\text{class } m))$ implies the following

<u>Proposition</u> (i) $K(\mathcal{M}_g;\text{class }1) = K(\mathcal{M}_g;\text{class }2) = K(\mathcal{M}_g;\text{class }3)$ if l is odd

(ii) $K(\mathcal{M}_g;\text{class }3) = K(\mathcal{M}_g;\text{class }4)$ if $l >> 2$.

To push forward the investigation of the Nielsen (monodromy) isomorphism, we are developping some combinatorial machine using graph of groups of Bass-Serre.

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In the talk we gave a report on joint work with G. Faltings on the generalization of results of W. Schmidt to forms of higher degree.
Let K be a number field of degree d and $\mathcal{O}_K$ its ring of integers. We fix a form $0 \neq F \in \mathcal{O}_K[X_0, \dots, X_n]$ of degree r such that $F(x) \neq 0$ for all $x \in \mathbb{Z}^{n+1}$. Then we have the Liouville estimate

$$|F(x)| \gg \|x\|^{-r(d-1)}$$

where $\|x\| = \max |x_i|$ for $x = (x_0, \dots, x_n)$. The right hand side depends in particular on d and the basic question is to get firstly a bound independent of the degree d and secondly to find the infimum for the exponents such that the inequality above with $-r(d-1)$ replaced by the exponent has infinitely many solutions in integral vectors. More generally we are considering the following question. Let $L \supseteq K$ be number fields, $\mathcal{P}$ a finite set of places of L and for each $w \in \mathcal{P}$ let I_w be a finite set. For $w \in \mathcal{P}, \alpha \in I_w$ let $f_{w,\alpha}$ be a form in $T_0, \dots, T_n$ with coefficients in L and $c_{w,\alpha} \geq 0$ a real number. Then we consider the set of diophantine inequalities

(*) $\quad \|f_{w,\alpha}(x)\|_w < H(x)^{-c_{w,\alpha}} \quad , w \in \mathcal{S}, \alpha \in I_w.$

Here $H(x)$ is the absolute Weil height of $x=(x_0,\dots,x_n) \in \mathbb{P}^n(K)$ and $\|f_{w,\alpha}(x)\|_w = |f_{w,\alpha}(x)|_w / \|x\|_w^{d_{w,\alpha}}$ with $\|x\|_w = \max|x_i|_w$ and $d_{w,\alpha}$ the degree of $f_{w,\alpha}$.

Particular cases of (*) were considered by Thue, Siegel, Dyson, Roth and by W. Schmidt.

The starting point for our studies was Faltings' product theorem which leads to a new proof of the subspace theorem of Schmidt and gives then a condition under which the inequalities (*) have only finitely many solutions.

Let $R = \bigoplus_{n\geq 0} R_n$ be a graded Noetherian algebra generated by R_1 over a field $K = R_0$ equipped with a separated decreasing filtration $F = (F^i)$ by graded ideals with $F^0 = R$, $F^1 \subset R_+ = \bigoplus_{n>0} R_n$ and with $F^i \cdot F^j \subseteq F^{i+j}$. More generally let $M = \bigoplus_{n\geq 0} M_n$ be a graded R-module with a separated filtration $F = (F^j M)$ by graded submodules such that $F^i \cdot F^j M \subseteq F^{i+j} M$. We further assume that $gr^F M$ is finitely generated over $gr^F R$. Then we obtain probability measures ρ_n given by

$$\rho_n([x,\infty)) = \max_{j/n \geq x} (\dim F^j M_n / \dim M_n).$$

One shows that $\rho_n \to \rho_\infty$ for $n \to \infty$ and a measure ρ_∞. We let $E(\rho_\infty)$ be its expectation value. In general it is difficult to relate $E(\rho_\infty)$ for M and for the $E(\rho_\infty)$ for a quotient M/M'. This is one reason to consider more simply filtered K-vector spaces (V, F) where

$$V = F^{p_0} \supset F^{p_1} \supset \cdots \supset F^{p_m} \supset F^{p_{m+1}} = 0$$

and where $0 \leq p_0 < p_1 < \cdots < p_{m+1}$ are weights. We put

$$\mu(V) = \sum p_j \dim(F^{p_j}/F^{p_{j+1}}) / \dim V$$

and call V semistable if $\mu(V') \leq \mu(V)$ whenever $0 \neq V' \subset V$. More generally let $L \supseteq K$ be a finite extension and suppose that the filtration F^{p_j} is defined only over L which means that $F^{p_j} \subseteq V_L = V \otimes_K L$. then we get again an invariant $\mu(V)$. Instead of one filtration we can also work with a finite set of filtrations $\{F_v\}$ and get invariants $\mu_v(V)$ and $\mu(V) = \sum \mu_v(V)$. If $\mu(V') \leq \mu(V)$ for $0 \neq V' \subset V$ in this situation we say that V is jointly semistable. The two main properties of semistability is the existence of the Harder-Narasimhan Filtration and that the result of Narasimhan and Seshadri also holds in this context. This means that the tensor product of semistable vector spaces is again semistable.

We apply this in the following way. Let $V = \Gamma(\mathbb{P}^n_K, \mathcal{O}(1))$. The $f_{w,\alpha}$ define for each $w \in \mathfrak{I}$ a filtration $F_w = (F_w^{p_j})$ such that $F_w^{p_j}$ is the vector space generated over L by the $f_{w,\alpha}$ with $c_{w,\alpha} \geq p_j$. For $w \notin \mathfrak{I}$ we let F_w be the trivial filtration with $\mu_w(V) = 0$. Then we put $\mu(V) = \sum_w \mu_w(V)$.

Theorem 1. Suppose that the $f_{w,\alpha}$ are all linear and define a jointly semistable filtration. If $\mu(V) > [L : \mathbb{Q}]$ then $(*)$ has only finitely many solutions $x \in \mathbb{P}(V)(K)$.

If the filtration is not jointly semistable then we let $W > 0$ be the first step in the Harder-Narasimhan filtration.

Theorem 2. Suppose that $\mu(W) > [L : \mathbb{Q}]$. Then all but finitely many solutions $x \in \mathbb{P}(V)(K)$ of $(*)$ are contained in $\mathbb{P}(V/W)(K)$.

We come now to the case when the $f_{w,\alpha}$ are no longer linear. Then we may assume that $\deg f_{w,\alpha} = r$ for all w, α. They define for each w a filtration F_w on $R = L[T_0, \dots, T_n]$. Namely F_w^p is the ideal generated by $\prod_\alpha f_{w,\alpha}^{p_\alpha}$ with $\sum_\alpha p_\alpha c_{w,\alpha} \geq p$.

Then we get probability measures $\rho_{w,\infty}$ and expectation values $E(\rho_{w,\infty})$. We put

$$E = \sum_w E(\rho_{w,\infty}).$$

Theorem 3. Suppose that $E > [L : \mathbb{Q}]$. Then the solutions of (*) in $\mathbb{P}^n(K)$ are contained in finitely many hypersurfaces. The same holds if $\mathbb{P}^n$ is replaced by a closed subscheme and if the measures $\rho_{w,\infty}$ are with respect to the coordinate ring of the subscheme.

To go from the linear to the non-linear situation one uses the r-fold Segre embedding

$$i : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$

given by monomials of degree r. Then $i^*\mathcal{O}(1) = \mathcal{O}(r)$ and one applies Theorem 2.

As a corollary we find that in the situation right at the beginning the solutions of

$$\| F(x) \| < H(x)^{-r(n+1)}$$

ly in finitely many hypersurfaces.

Titel: Algebraic-geometric characterization of loc. symm. spaces

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I. A well-known theorem of Yau gives a characterisation of quotients of the complex ball $\mathbb{B}_n = \{ \underline{x} \in \mathbb{C}^n \,/\, \sum |x_i|^2 < 1 \}$ by a discretely acting, fixed point free group $\Gamma \subset \mathrm{Aut}(\mathbb{B}_n) \cong PSU(n,1)$:

<u>Theorem (Yau)</u>: A smooth projective variety X with K_X ample is a quotient of the complex ball $\Leftrightarrow$

$$(*)_n \qquad c_1^n(X) = \frac{2(n+1)}{n} c_1^{n-2}(X) \cdot c_2(X).$$

This theorem was applied by Hirzebruch and others to determine when a branched cover $X \to \mathbb{P}^n$ of projective space is a ball quotient, especially the case when the branch locus in $\mathbb{P}^n$ is an arrangement $\mathcal{A}$ of hyperplanes. There was a priori one and in hindsight another advantage here:

1) The Chern numbers can be calculated as functions of the degree and the combinatorial data of the arrangement, allowing one to easily state conditions on an arrangement for it to allow $c_1^n(X) = \frac{2(n+1)}{n} c_1^{n-2} c_2(X)$.

2) The variety X given as branched cover $X \to \mathbb{P}^n$ comes equipped with a special set of divisors, namely the ramification divisor $D = \sum D_i \subset X$.

For $D \subset X$ ample, by applying the adjunction formula, it turned out that the equality $(*)$ is equivalent to $\underline{2}$ equalities of Chern numbers on the components D_i:

i) the equality $(*)_{n-1}$ for each D_i,

ii) a corresponding equality of the Chern numbers of the normal bundle of D_i in X (relative Hirzebruch proportionality).

It is natural to say D_i is a <u>subball quotient</u> if it fulfills i) and ii). Then from $(*)_n$ one gets:

$$\exists_{D \text{ ample}},\ D = \sum D_i,\ \text{each } D_i \text{ a subball quotient} \Rightarrow X \text{ ball quotient}.$$

The advantage of this formulation is that it no longer uses Chern numbers, so gives a candidate for a corresponding statement for more general bounded symmetric domains.

II. Let $G_{\mathbb{Q}}$ be an algebraic group of hermitian type, i.e. such that the corresponding symmetric space $\mathcal{D} = G_{\mathbb{Q}}(\mathbb{R})/K$ (K maximal compact) is hermitian symmetric of the non-compact type, in other words $\mathcal{D}$ is a bounded symmetric domain. Let $\Gamma \subset G_{\mathbb{Q}}$ be an arithmetic group (for a faithful rational representation $\rho: G_{\mathbb{Q}} \longrightarrow GL(V_{\mathbb{Q}})$, $\rho^{-1}(GL(V_{\mathbb{Z}}))$ and Γ are commensurable). Then the quotient $X_{\Gamma} = \Gamma \backslash \mathcal{D}$ is called an <u>arithmetic quotient</u> (of the bounded symmetric domain $\mathcal{D}$). It is a complex manifold if Γ is torsion free (which we hence forth assume), and compact $\Leftrightarrow \operatorname{rank}_{\mathbb{Q}} G_{\mathbb{Q}} = 0$. Let X_{Γ}^* denote the Baily-Borel compactification, a normal algebraic variety. It is a disjoint union $X_{\Gamma}^* = X_{\Gamma} \cup V_1 \cup \dots \cup V_N$, and each V_i is an arithmetic quotient of $\mathbb{Q}$-rank strictly less than that of X_{Γ}. Let $\overline{X_{\Gamma}}$ denote a toroidal compactification of X_{Γ} dominating X_{Γ}^* such that • $\overline{X_{\Gamma}}$ is projective, • $\overline{X_{\Gamma}} \setminus X_{\Gamma} = \Delta_{\Gamma}$ is a normal crossings divisor. Whereas X_{Γ}^* is unique, $\overline{X_{\Gamma}}$ is unique only up to birational transformations.

Let $(G_i)_{\mathbb{Q}}$ be hermitian subgroups, i.e. $(G_i)_{\mathbb{Q}} \subset G_{\mathbb{Q}}$ such that the corresponding subspace $\mathcal{D}_i = (G_i)_{\mathbb{Q}}(\mathbb{R})/K_i$ is a hermitian symmetric subspace. Let $\Gamma_i := \Gamma \cap (G_i)_{\mathbb{Q}}$.

<u>Definition:</u> The subvariety $D_i := \Gamma_i \backslash \mathcal{D}_i \subset X_{\Gamma}$ is called a <u>modular</u> <u>subvariety</u> of the arithmetic quotient X_{Γ}.

These modular subvarieties have the property that the closures D_i^* in X_{Γ}^* are Baily-Borel embeddings of the D_i. Similarly, the closures $\overline{D_i}$ in $\overline{X_{\Gamma}}$ are toroidal embeddings of the D_i. Suppose we are given some finite set $\{D_i\}_{i \in I}$ of such modular subvarieties. Then we think of $\overline{X_{\Gamma}}$ as the tripel $(\overline{X_{\Gamma}}, \Delta_{\Gamma}, \overline{D_{\Gamma}})$, $D_{\Gamma} = \Sigma D_i$. This tripel will fulfill certain conditions.

III. Conversely, given a projective variety $\overline{Y}$, a normal crossings divisor $\Pi = \sum \Pi_\lambda$ and a subvariety $\overline{E} = \sum \overline{E}_i$ we introduce a series of conditions on the tripel $(\overline{Y}, \Pi, \overline{E})$, which will be fulfilled for an arithmetic quotient $(\overline{X}_\Gamma, \Delta_\Gamma, \overline{D}_\Gamma)$ as above. Set $Y = \overline{Y} \setminus \Pi$,

Assumptions:

1) The components E_i of $E = \overline{E} \cap Y$ fulfill the following:
   1.1.: Each E_i is an arithmetic quotient $E_i = \Gamma_i \setminus \mathcal{D}_i$, corresponding to an algebraic group $(G_i)_{\mathbb{Q}} \supset \Gamma_i$.
   1.2.: There is an algebraic group $G_{\mathbb{Q}}$ with all $(G_i)_{\mathbb{Q}} < G_{\mathbb{Q}}$
   1.3.: In each intersection $E_i \cap E_j$ there are modular subvarieties E_{ij}, maximal in both E_i and E_j.

2). There exists a singular compactification $\overline{Y} \longrightarrow Y^*$ such that
   2.1: $Y^* \setminus Y = W_1 \cup \cdots \cup W_N$, W_λ an arithmetic quotient of $\mathcal{D}_\lambda$ which is isomorphic to a rational boundary component of $\mathcal{D} = G_{\mathbb{Q}}(\mathbb{R})/K$.
   2.2: The closure E_i^* of E_i in Y^* is a Baily-Borel compactification of E_i.

3) The tripel (Y, Π, E) fulfills:
   3.1: $\pi_1(E \cup (\Pi_\varepsilon \setminus \Pi)) = \pi_1(Y)$ (here ε denotes a tubular neighborhood of Π in Y) and $E \cup (\Pi_\varepsilon \setminus \Pi)$ is connected,
   OR:
   $\overline{E} + \Pi$ is ample on Y, dimension $(E_i) \geq 2$
   3.2: $\pi_1((\Pi_\lambda)_\varepsilon \setminus \Pi_\lambda)$ is $\simeq$ arithmetic subgroup Γ_λ of a $\mathbb{Q}$-parabolic $P_\lambda \subset G_{\mathbb{Q}}$.
   3.3. If $W_\lambda \subset \overline{E}_i \setminus E_i$, then P_λ and $(G_i)_{\mathbb{Q}}$ are associated (roughly: $(G_i)_{\mathbb{Q}}$ contains the $\mathbb{Q}$-split part of the semisimple component of the Levi subgroup of P_λ).

Theorem: Let $\overline{Y}$ be a projective, smooth, minimal variety. Then the following are equivalent:

a) $\bar{Y}$ is the (toroidal) compactification of an arithmetic quotient of a bounded symmetric domain,

b) There exists $\bar{E} = \sum \bar{E}_i$, $\Pi = \sum \Pi_j$ such that $(\bar{Y}, \Pi, \bar{E})$ fulfill the assumptions above.

IV.

Remarks: 1) Both directions of this theorem are roughly equally difficult. In fact, a) $\Rightarrow$ b) applied to ball quotients (for which the E_i are themselves divisors) yields the following:

Corollary: An arithmetic ball quotient has Picard number $\rho \geq 2$.

This results from the fact that $\rho = 1$ implies any effective divisor is ample, so by Lefschetz theory, induces a surjection on the fundamental groups, while by the theorem for the E_i we have injections $\pi_1(E_i) \subset \pi_1(Y)$.

2) There are different types of characterisations of such arithmetic quotients:

- differential geometric: (Ballmann-Burns-Spatzier for Riemannian case, Yau, Kobayashi-Ochiai, Simpson in the hermitian case.)
- topological: the strong rigidity results of Siu can be formulated: a Kähler manifold V is isomorphic to an arithmetic quotient (actually arithmetic is not necessary here) $\Leftrightarrow$ V is homotopically equivalent to some X_Γ.

Our result could be formulated in either of the two following ways

- an algebro-geometric characterisation,
- the existence and strong rigidity of configurations of modular subvarieties,
- A Lefschetz-theorem for strong rigidity of arithmetic quotients of bounded symmetric domains.

Titel: Intersection theory for twisted cycles and hypergeometric functions

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I am interested in hypergeometric functions (HGF) defined by (the so called hypergeometric) integrals:

$\int_0^1 t^{p-1}(1-t)^{q-1}\,dt$: Beta ftn

$\int t^*(1-t)^*(t-x)^*\,dt$: The HGF

$\int_{x_i}^{x_{i+1}} \prod_{j=1}^{n}(t-x_j)^{\alpha_j}\,dt$: Appell-Lauricella HGF

$\int \prod_{j=1}^{n} l_j(x,t)^{\alpha_j}\,dt_1\wedge\cdots\wedge dt_r$: HGF of type $(r+1, n+1)$

$(l_j := \sum_{i=0}^{r} x_{ij} t_i,\ t_0 = 1)$.

HGF's are studied from various aspects by many authors. Today I concentrate my interests on "domains of integration": $(0,1)$, (x_i, x_{i+1}), ... You may think why these simple things can be worth studying. Let me give you one typical example.

Example $r=2$, $n=5$, $\alpha_j = -1/2$. type $(3,6)$.

I have 6 lines (one is at infinity) $\{l_j = 0\}$ on $\mathbb{P}^2 \ni (t_0:t_1:t_2)$. The integral is the unique holomorphic 2-form on the double covering S of $\mathbb{P}^2$ branching along these 6 lines, a K3 surface. Since I can fix 4 lines arbitrary, only 2 lines can move, so I get 4-dim. family of K3 surfaces. You should integrate the 2-form on "transcendental cycles" on the surface, otherwise the integral is 0. Then the integral gives a period of S, which is the key to study this family. You surely want to know the structure of the transc. cycles, especially the <u>intersection form</u>.

I knew 3 method to do it,

(1) Construct a basis explicitly and counts intersection points. It is made in [MSY]. ... Stupidly honest way.

(2) Find a set of generators of the monodromy group of the HGF [MSTY1] and find a monod.-inv~~ar~~ariant symmetric matrix [MSTY2].

Since the inters. matrix is invariant under the monod. grp, we can find it in this way. But we can do it only by using computer.

(3) due to J. Stienstra. using Nikulin's theorem on lattices, not so easy, depend heavily on the K3 lattice.

Don't you think it funny? Our surface S is completely determined by 6 lines and the ramification index, which is 2. Don't you think all the happenings upstairs can be understood and computed down-stairs?

Main Theorem [KY] Yes, we can, for general r, n and α_j.

Recall the definition of HGF's. They are defined by integrals. What "integration" means? Put

$$T := \mathbb{C}^r - \bigcup_1^n \{l_j = 0\}.$$

Our form $\prod l_j(t)^{\alpha_j} dt_1 \wedge \dots \wedge dt_r$ is not single-valued, nevertheless we want to think our integral as a pairing between "cohomology" and "homology." One way is to consider a covering space of T where the form may be single-valued. But this is the very thing we did in the Example and we are trapped. We should stick to the ground floor T and should twist ∂ and d instead. This is the twisted (co)homology I am going to recall briefly now.

Multi-valued ftn $u = \prod_1^n l_j(t)^{\alpha_j}$ on T defines a representation $\pi_1(T) \to \mathbb{C}^\times$; it defines a local system $\mathcal{S}$ with stalk $\mathbb{Z}[c_j^\pm]$, where $c_j := \exp 2\pi i \alpha_j$.

(Homology) $C_p(\mathcal{S}) := \{$topological chain along which a branch of u is attached$\}$
$C_p^{lf}(\mathcal{S}) := \{$locally finite " $\}$

boundary operator $\partial_\mathcal{S}$ is defined accordingly.

$$H_p(T,\mathcal{S}) := \ker(C_p(\mathcal{S}) \xrightarrow{\partial_\mathcal{S}} C_{p-1}(\mathcal{S}))/\mathrm{Im}(C_{p+1}(\mathcal{S}) \to C_p(\mathcal{S})),\ H_p^{lf}(T,\mathcal{S}) = \cdots$$

(Cohomology). $\omega := du/u$, $\nabla := d + \omega\wedge$.

$$0 \to \mathcal{L} \hookrightarrow \mathcal{E}^0 \xrightarrow{\nabla} \mathcal{E}^1 \xrightarrow{\nabla} \cdots,\quad \mathcal{L} = \ker(\mathcal{E}^0 \xrightarrow{\nabla} \mathcal{E}^1):\text{local system}$$
$$= \underline{\text{collection of branches of } u^{-1}}$$

Notice that $\mathcal{L} \simeq$ dual of $\mathcal{S} \otimes \mathbb{C}$.

$H^p(T, \mathcal{L}) := \ker(\Gamma(\mathcal{E}^p) \xrightarrow{\nabla}\ ) / \mathrm{Im}(\Gamma(\mathcal{E}^{p-1}) \xrightarrow{\nabla}\ )$, $H^p_c(T, \mathcal{L})$: compact supp.

(Integration) perfect pairing

$$H^{(lf)}_p(T, \mathcal{S}) \otimes \mathbb{C} \times H^p_{(c)}(T, \mathcal{L}) \to \mathbb{C} \qquad (\gamma, \varphi) \mapsto \int_\gamma u \varphi$$

Stokes' theorem reads $\int_\gamma u \nabla \varphi = \int_{\partial_\mathcal{S} \gamma} u \varphi$.

(Intersection form).

$$H^{lf}_p(T, \mathcal{S}^\vee) \times H_{2r}(T, \mathcal{S}) \to \mathbb{Z}[c_j^\pm].$$

This is the theme we discuss today.

In order to make the story simple, I assume very strong conditions.

1). l_j : linear
2) $B := \{l_j = 0\} \cup \{\infty\text{-plane}\}$: normally crossings) ⇒ one can assume l_j is define over $\mathbb{R}$.
3) $\alpha_j, \sum_1^n \alpha_j \notin \mathbb{Z}$.

<u>Known facts</u>. Many about cohomology. e.g. [D], [Aom], [KN], [ESV]

- $H^p(T, \mathcal{S}) \cong H^p(\Gamma(\mathbb{P}^r, \Omega^\bullet(\log B)), \nabla)$,
- $H^p = H_p = 0, \quad p \neq r$.
- $H^{lf}_r(T, \mathcal{S}) \otimes \mathbb{Q}$ is spanned by cycles with supports on bounded chambers of $\mathbb{R}^r - B \cap \mathbb{R}^r$.
- $\exists$ reg : $H^{lf}_r(T, \mathcal{S}) \otimes \mathbb{Q}[\frac{1}{c_j - 1}] \xrightarrow{\sim} H_r(T, \mathcal{S}) \otimes \mathbb{Q}[\frac{1}{c_j - 1}]$,

called the regularization, which you shall know soon what it is.

<u>Intersection Theory</u>. In order to fix bases of H^{lf}_r and H_r, we consider them over $R := \mathbb{Q}[c_j^\pm, \frac{1}{c_j - 1}]$; write $\mathcal{S}$ for $\mathcal{S} \otimes R$.

So our intersection form is

$$H_r(T, \mathcal{S}) \times H^{lf}_r(T, \mathcal{S}^\vee) \longrightarrow R.$$

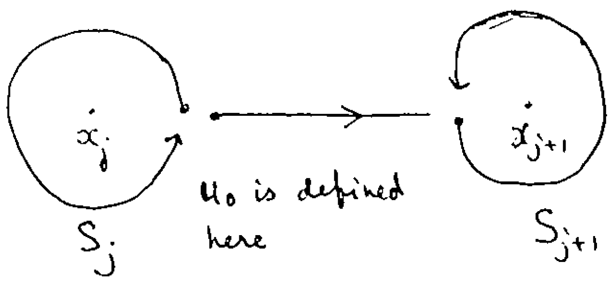
$\boxed{r = 1}$ $\qquad u = \prod_1^n (t - x_j)^{\alpha_j} \qquad x_1 < x_2 < \cdots < x_n$

Fix a branch, say u_0, of u on the lower half plane. On each oriented segment (x_j, x_{j+1}), attach u_0 and call it $D_j \in H^{lf}_1(T, \mathcal{S})$.

Define $D_j^\vee \in H_1^{lf}(T, \mathcal{S}^\vee)$ in the same way using u_0^{-1} instead. The regularization $\operatorname{reg} D_j \in H_1(T, \mathcal{S})$ of D_j is defined as follows:

$$\operatorname{reg} D_j := \frac{S_j}{c_j - 1} \otimes u_0 + \overrightarrow{(x_j + \varepsilon, x_{j+1} - \varepsilon)} \otimes u_0 - \frac{S_{j+1}}{c_{j+1} - 1} \otimes u_0 ,$$

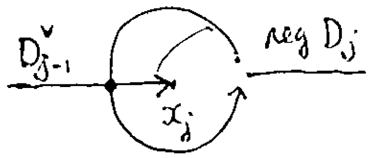
where



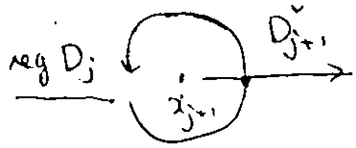
Rem. Since $\partial S_j = (c_j - 1)[x_j + \varepsilon] \otimes u_0(x_j + \varepsilon)$, we have $\partial \operatorname{reg} D_j = 0$; $\operatorname{reg} D_j$ is a cycle.

Now we are ready to compute the intersection number $\operatorname{reg} D_j \bullet D_k^\vee$.

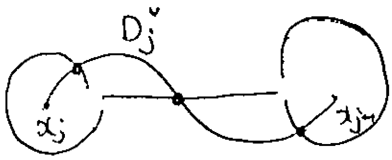
Put $d_j := c_j - 1$, $d_{jk\cdots} = c_j \cdot c_k \cdots - 1$



$$\therefore \operatorname{reg} D_j \cdot D_{j-1}^\vee = \frac{c_j}{d_j},$$



$$\therefore \operatorname{reg} D_j \cdot D_{j+1}^\vee = \frac{1}{d_j},$$

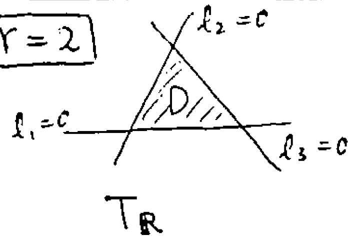


$$\therefore \operatorname{reg} D_j \cdot D_j^\vee = -\left(\frac{1}{d_j} + 1 + \frac{1}{d_{j+1}}\right) = -\frac{d_{12}}{d_1 d_2},$$

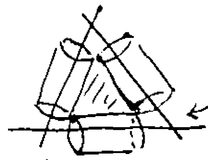
0 otherwise. Therefore we get

$$I(2, n+1; \alpha) = (\operatorname{reg} D_j, D_k^\vee) = \begin{pmatrix} -\frac{d_{12}}{d_1 d_2} & \frac{1}{d_2} & & 0 \\ \frac{c_2}{d_2} & -\frac{d_{23}}{d_2 d_3} & \ddots & \\ & \ddots & \ddots & \frac{1}{d_{n-1}} \\ 0 & & \frac{c_{n-1}}{d_{n-1}} & -\frac{d_{n-1\,n}}{d_{n-1} d_n} \end{pmatrix}.$$

$\boxed{r=2}$



reg D is made in $T = T_{\mathbb{R}} + \sqrt{-1}\,T_{\mathbb{R}}$. Please guess by the following picture

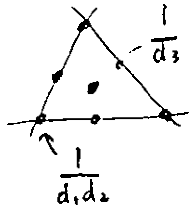


here $S^1 \times S^1$ is attached.

Deform the triangle in such a way that it meets reg D transversally exactly at barycenters of the faces of the triangle. We get

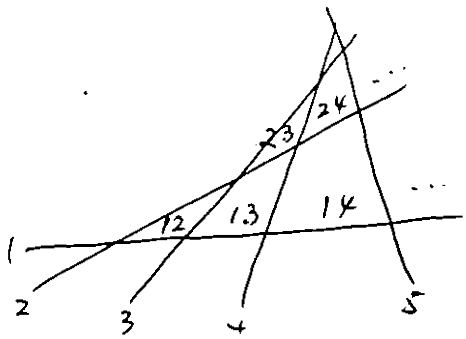
$$\mathrm{reg}\, D \cdot D^{\vee} = 1 + \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} + \frac{1}{d_1 d_2} + \frac{1}{d_2 d_3} + \frac{1}{d_3 d_1}$$

$$= d_{123} / d_1 d_2 d_3 ,$$



Striking fact is, it is equal to $\begin{vmatrix} -\frac{d_{12}}{d_1 d_2} & \frac{1}{d_2} \\ \frac{c_2}{d_2} & -\frac{d_{23}}{d_2 d_3} \end{vmatrix}$.

It is not a coincidence. If I index the bounded chambers in $\mathbb{R}^2 - \cup\{\ell_j = c\}$ as in the picture, we have



Th [KY]

$$\mathrm{reg}\, D_{ij} \cdot D_{k\ell}^{\vee} = \begin{pmatrix} i & k & \ell \\ j & & \end{pmatrix}\text{-minor}$$

of $I(2, n+1; \alpha)$

$\boxed{r \geq 2}$ Same theorem ; $ij\cdots$, $k\ell\cdots$.

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Counting lattice points in convex polyhedra

Titel: Counting lattice points in convex polyhedra
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There has been a recent interest for the (classical) question of counting lattice points in convex compact polyhedra (polytopes), due to

– the development of the study of toric varieties which constitute a bridge between convex polytopes with, say, integral lattices, and algebraic geometry

– some particular cases show very interesting properties, for example the case of reflexive polytopes with only one interior point corresponds to "mirror-symmetry"

– Extensions to more general situations lead to fascinating new problems: in group theory, computation of multiplicities (V. Guillemin), proof of Verlinde's formula, ...

Our work with Askold Khovanskii tries to better understand the link between discrete and continuous invariants (as, in algebraic geometry, the Riemann-Roch theorem) inside the framework of polytopes with no help from toric varieties.

Other recent work (Mc Mullen's theorem of Hard Lefschetz type, Morelli's [M]) suggest that a general K (and #)-theory of polytopes might be on the way.

Also, our results are closely related, for particular polytopes ("pyramids"), with the study of generalized Dedekind sums

I Chains and valuations, Ehrhart's polynomials

We consider polytopes in $\mathbb{R}^d$, with vertices in a given lattice, taken here to be $\mathbb{Z}^d$ to simplify. Let $\mathcal{P}_d$ be the set of such polytopes. It is useful to replace any polytope Δ by its characteristic function 1_Δ and to consider

$$\mathcal{L}_d = \{ f : \mathbb{R}^d \to \mathbb{R},\ f = \sum m_i 1_{\Delta_i},\ \text{finite sum},\ \begin{matrix} m_i \in \mathbb{Z} \\ \Delta_i \in \mathcal{P}_d \end{matrix} \}$$

Any such f can be written in many ways, but the obvious relations (for Δ convex union of two polytopes) are the only ones, so this allows to identify $\mathcal{L}_d$ with the group generated by $\mathcal{P}_d$ modulo these obvious relations of pasting and cutting. We do not add translation equivalence (which would lead to the analogous of the scissor group in case of all polytopes)

Definition : A *valuation* on $\mathcal{P}_d$ with values in a group A is an additive map

$$\varphi : \mathcal{L}_d \longrightarrow A$$

or, equivalently, a map on $\mathcal{P}_d$ satisfying

$$\varphi(\Delta_1 \cup \Delta_2) = \varphi(\Delta_1) + \varphi(\Delta_2) - \varphi(\Delta_1 \cap \Delta_2)$$

if the Δ's are in $\mathcal{P}_d$ as well as $\Delta_1 \cup \Delta_2$

Definition:

The valuation φ is *translation-invariant* if

$$\varphi(a + \Delta) = \varphi(\Delta) \quad \forall \Delta \in \mathcal{P}_d$$

c/ The valuation φ a <u>polynomial Valuation</u> of degree less or equal to m if for any Δ

$$a \longrightarrow \varphi(a+\Delta)$$

is a polynomial of degree less or equal to m.

We give a sketch of an elementary (geometric) proof of the following results:

<u>Theorem 1</u>

1) If φ is a polynomial valuation of degree less or equal to m for any elements $\Delta_1, \dots \Delta_k$ in $\mathcal{P}_d$ the function

$$\varphi(n_1\Delta_1 + n_2\Delta_2 + \cdots n_k\Delta_k)$$

is a polynomial in $(n_1, n_2, \dots n_k)$ of total degree $\leq m$

2) In particular, taking $k = 1$, $\varphi(\Delta) = \#(\Delta) = \text{card}(\Delta \cap \mathbb{Z}^d)$ one gets a polynomial

$$\varphi(n\Delta) = a_0 + a_1(\Delta)n + \cdots + a_d(\Delta)n^d, \quad n \geq 0$$

and moreover taking the value of this polynomial at $(-n)$:

$$(2) \qquad \varphi(-n\Delta) = (-1)^d \#(n\Delta^\circ) \qquad n \geq 0$$

This polynomial is called <u>Ehrhart polynomial</u>. The subject of our talk is mainly the study of the a_n's. The proof of theorem 1 is due to [KP], the original proof to Mc Mullen for $m=1$

Formula (2) is called "Duality formula".
It is elementary to show that

$$a_d(\Delta) = \text{vol}\,\Delta$$

and, using (2)

$$a_{d-1}(\Delta) = \mu_{d-1}(\Delta)/2 \qquad (d-1)\text{-relative measure}$$

of Δ.

The proof uses elementary constructions on $\mathcal{L}_d$: Euler measure, and convolution associated with it, corresponding to Minkowski sums of polytopes.

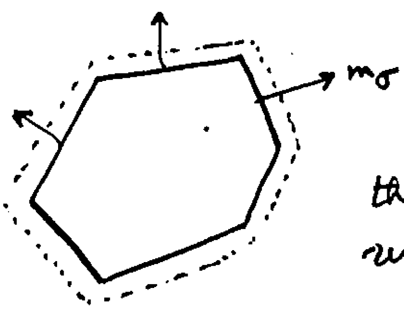
II Combinatorial Riemann-Roch theorem (smooth case)

In general, Riemann-Roch type theorems relate discrete and continuous invariants. So it is natural to consider, for a given polytope Δ, the discrete invariant $\#(\Delta)$ and the volume of Δ as well as volumes of deformed Δ's.

The theory of toric varieties interprets $\#(\Delta)$ as the integral of the cup product of two characteristic classes, but one of them (Todd class) is hard to compute.

Let Δ be a polytope with associated primitive fan Σ; i.e. that is all cones in Σ are primitive (they are generated by parts of $\mathbb{Z}$- basis of $\mathbb{Z}^d$). This amounts to say that the toric variety X_Σ associated to Σ is smooth.

Corresponding to each cone σ of Σ^1 set of one-dimensional cones of Σ, there is a primitive vector m_σ and a facet of Δ (face of codimension 1) which we translate parallel to itself (orthogonal to m_σ) by h_σ measured along σ.



Because of the hypothesis the combinatorial structure remains constant.

Let $V(h)$ be the volume of the deformed polytope $\Delta(h)$

Call
$$T_\sigma = \frac{\partial/\partial h_\sigma}{1-\exp(-\partial/\partial h_\sigma)}$$

Infinite order operator, and
$$T_\Sigma = \prod_{\sigma\in\Sigma^1} T_\sigma$$

the Todd operator associated with Σ.

Then

Theorem 3 (Combinatorial Riemann-Roch theorem)
$$T_\Sigma(V(h))\Big|_{h=0} = \#(\Delta)$$

if Σ is primitive.

Translating in the language of toric varieties, this theorem gives the Riemann-Roch theorem for non singular toric varieties.

The proof uses "decomposition" of Δ into cones, and elementary summation of exponential functions (or integration of them). In dimension one it gives an elementary proof of Euler-MacLaurin formula!

Reference: [KK]$_1$, [KP]

III General case

Definition: let Δ be a polytope, Σ the fan corresponding to it, Σ^i the set of cones of dimension i of Σ.

Δ (resp. Σ) is said to be k-primitive if

$\Sigma^0, \Sigma^1, \dots \Sigma^{k-1}$ are primitive (made of primitive cones)

Σ^k is simplicial (made of simplicial cones)

let $V(h)$ defined as above by moving the facets of Δ (a precise definition uses support functions), T_Σ the Todd operator corresponding to rays of Σ', as before. One has

$$\#(\Delta) = a_0 + a_1(\Delta) + \cdots + a_d(\Delta)$$ from Ehrhart's polynomial

$$\tilde{T}_{\Sigma}(V(h))\Big|_{h=0} = b_0 + b_1(\Delta) + \cdots + b_d(\Delta) :$$ graded decomposition under dilation

<u>Theorem</u> $b_d(\Delta) = a_d(\Delta) = \mathrm{vol}(\Delta)$

$$b_{d-k+1}(\Delta) = a_{d-k+1}(\Delta)$$

$$b_{d-k}(\Delta) = a_{d-k}(\Delta) + \sum_F \mu_{d-k}(F)\; t_k(\hat{F})$$

where F is a face of codimension k of Δ, t_k a discrete invariant associated to any cone of dimension k (here the cone corresponding to F in $\mathbb{R}^{d*}$) in $\mathbb{R}^{d*}$ with respect to $\mathbb{Z}^{d*}$

In particular, because all cones in $\mathbb{R}^2$ are simplicial, this theorem gives a complete formula for $a_{d-2}(\Delta)$ for any polytope Δ in $\mathbb{R}^d$, and a complete formula for $\#(\Delta)$ for any polytope Δ in $\mathbb{R}^3$ and $\mathbb{R}^4$. The formula involves <u>defects</u> (of type Dedekind sums) already computed by J. Pommersheim [P] explained here in a natural way using the canonical subdivision of a fan in primitive ones as in Oda's book.

[KK]$_1$ J.M. Kantor, A. Khovanskii Integral points in convex polyhedra, ... IHES/M/93/37, June 93

[KK]$_2$ J.M. Kantor, A. Khovanski Note aux C.R.A.S. juin 1993.

[KP] A. Khovanskii, S. Pukhlikov Theorem of Riemann-Roch ... Algebra i Analyz, 4, 92, 4, p. 188-216

[M] R. Morelli: Pick's theorem and the Todd class of a variety, to appear, Advances in Math.

[P] J. Pommersheim Toric varieties, ... Math. Ann. 93, 1-24.

Titel: Nonpositively curved metrics on 3-manifolds
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According to Thurstons Hyperbolisation Conjecture, a closed 3-manifold admits a metric of constant negative sectional curvature as soon as the basic topological obstructions [P] are not present, i.e. if it is irreducible, atoroidal and has infinite fundamental group. Thurston proved his conjecture for Haken manifolds [Th 1].

We ask the analogous question complementary to hyperbolisation for nonpositive sectional curvature (npc): Which manifolds in the class $\mathcal{M}$ of closed, orientable, irreducible, toroidal 3-manifolds admit npc metrics? Not all of them do, as the example of non-trivial circle bundles over orientable surfaces shows. We give examples of existence and nonexistence, but do not see a handy general criterion.

Recall the topological decomposition of $M \in \mathcal{M}$ due to Jaco, Shalen and Johannson [JS],[Jo]: There is an up to isotopy unique minimal family of disjoint embedded incompressible 2-tori $T_1, \ldots, T_n$ ($n \geq 0$) cutting M into pieces which are Seifert or atoroidal. We consider the case where M is non-Seifert, i.e. $n \geq 1$. (The Seifert case is well-understood [S].) Then the atoroidal pieces are hyperbolic by Thurstons Uniformisation Theorem [Th1], and also the Seifert pieces carry npc geometries modelled on $\mathbb{H}^2 \times \mathbb{R}$ (generic case) or $\mathbb{R}^3$. We ask whether a simultaneous weak geometrisation of all pieces by putting a npc metric on M is possible.

If M does carry a npc metric g, then the T_i can be made totally geodesic by an isotopy. The key observation is that g displays some rigidity on the Seifert pieces (i.e. splits locally as a product $\mathbb{R}\times\ldots$), whereas hyperbolic pieces are completely flexible.

Existence-Thm: If in the topological decomposition of $M\in\mathcal{M}$ occurs at least one hyperbolic piece, then M carries a npc metric.

The delicate case is the case of graph-manifolds, i.e. 3-manifolds glued from Seifert pieces only. Then any npc metric splits a.e. locally isometrically as a product, and this rigidity leads to obstructions.

Nonexistence-Examples: We give examples of graph-manifolds with arbitrarily many Seifert components and linear gluing graph which do not admit npc metrics.

The case of arbitrary dimension / actions of 3-manifold groups on npc spaces: Let $M^3\in\mathcal{M}$ be a manifold which does not admit a npc metric. Consider isometric actions of π_1M^3 on CAT(0)-spaces (these are metric spaces with npc in the sense of Aleksandrov-Toponogov distance comparison; think e.g. of complete, 1-conn., npc Riemannian manifolds). We show that π_1M^3 cannot act like a subgroup of the decktransformation group of a closed manifold:

Thm: There is no discrete, properly discontinuous action of the above kind without parabolic elements.

This leads to a new obstruction for npc metrics:

Cor (Obstruction-Thm.): Let N^n be a closed manifold, $M^3\in\mathcal{M}$ not admitting npc metrics. If there is an injective homom $\pi_1M^3\to\pi_1N^n$, then N^n does not admit a npc metric.

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Titel: New geometric and algebraic structures in string theory

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Seite: 1

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1. BV-algebras [3], [4], [5]
2. Geometry of Batalin-Vilkovisky quantization procedure. Gauge independence. Classification of odd symplectic manifolds and SP-manifolds. [1]
3. Semiclassical approximation in BV-quantization. Expression in terms of Reidemeister torsion [2]
4. BV-algebras in topological conformal field theory [3], [4], [5]
5. Application to string field theory [6]

1. Let us suppose that $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra. We say that an operator α acting on $\mathcal{A}$ is a second order (super) derivation if the expression

$$[\alpha, \hat{a}]b - \alpha(a)b$$

is a (super)derivation on $\mathcal{A}$ for fixed $a \in \mathcal{A}$. (Here $\hat{a}$ denotes an operator of multiplication by a, $[\ ,\ ]$ stands for the supercommutator.) One can define a BV-algebra as a supercommutative associative algebra $\mathcal{A}$ equipped with a parity reversing second order derivation Δ satisfying $\Delta^2 = 0$.

Theorem. One can introduce the structure of Lie superalgebra on $\mathcal{A}$ by the formula

$$\{f, g\} = (-1)^{\varepsilon(f)}\Delta(fg) - (-1)^{\varepsilon(f)}(\Delta f)g - f\Delta g,$$

where $\varepsilon(f)$ stands for the parity of f. The operator $\{f, \ \}$ is a (super) derivation. (In other words a BV-algebra can be considered as a Gerstenhaber algebra.)

2. BV-algebras appear naturally in many problems of quantum field theory and pure mathematics. Their name is related to the Batalin-Vilkovisky procedure of quantization of gauge theories. Mathematical formulation of this procedure requires the consideration of SP-manifolds, i.e. supermanifolds equipped with a parity reversing differential operator Δ of second order satisfying $\Delta^2=0$ and having a non-degenerate principal symbol. The space of superfunctions on an SP-manifold has a structure of BV-algebra. One can describe SP-manifolds also as (super) manifolds with a locally flat SP-structure where $SP = S \cap P$. (Here S denotes the group of volume preserving linear transformations of $\mathbb{R}^{n,n}$ and P is a group of linear transformations of $\mathbb{R}^{n,n}$ preserving an odd bilinear form $x_i \xi^i$ where $x_1, \ldots, x_n$ are even and $\xi^1, \ldots, \xi^n$ are odd coordinates in $\mathbb{R}^{n,n}$). In such a way every SP-manifold is an odd symplectic manifold (manifold with locally flat P-structure). Therefore

one can introduce a notion of Lagrangian submanifold of an SP-manifold.

Theorem. Let us suppose that M is a compact SP-manifold, L_1 and L_2 are homologous Lagrangian submanifolds of M, F_1 and F_2 are such functions on M that $\Delta F_1 = \Delta F_2 = 0$ and $F_2 - F_1$ can be represented in the form ΔK. Then

$$\int_{L_1} F_1 \, d\lambda_1 = \int_{L_2} F_2 \, d\lambda_2$$

(Here $d\lambda_i$ denotes the volume element induced on Lagrangian submanifold L_i.)

The proof is based on classification theorems for odd symplectic manifolds and SP-manifolds.

The choice of Lagrangian submanifold corresponds in BV-quantization to a choice of gauge condition. The theorem above can be considered as a theorem about independence of physical quantities of the gauge condition.

3. Let us consider semiclassical approximation in BV-formalism, i.e. the asymptotic behavior of

$$\int_L \exp(-\hbar^{-1} S) \, d\lambda$$

as $\hbar \to 0$. Here L is a Lagrangian submanifold and we suppose that $\Delta(\exp(-\hbar^{-1}S))=0$. Let us denote the set of critical points of S by Y. The odd symplectic structure in the SP-manifold $\mathcal{M}$ induces an odd presymplectic structure in Y. Under certain conditions one can introduce the structure of an SP-manifold in the space Y' obtained from Y by factorization with respect to null-vectors of symplectic form. (The volume element in Y' is determined as a Reidemeister torsion of the Hessian of S.)

Theorem.

$$\int_L \exp(-\hbar^{-1}S)\,d\lambda \approx \hbar^{(z_1-z_2)/2} \exp(-\hbar^{-1}S_0)\,\mathrm{vol}(C),$$

where vol(C) stands for a volume of a Lagrangian submanifold of Y', S_0 denotes the value of S on Y and $\dim Y=(z_1,z_2)$.

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Titel: Higher Kodaira-Spencer maps

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(joint work with Eckart Viehweg).

Let $f: X \to S$ be a proper flat family of smooth varieties defined over an algebraically closed field k of characteristic zero. We consider a sheaf A of $f^{-1}\mathcal{O}_S$ Lie algebras, together with an exact sequence of $f^{-1}\mathcal{O}_S$ sheaves

$$(E) \quad 0 \to A \to \tilde{A} \to f^{-1}T_S \to 0 \quad (T = \text{tangent sheaf}),$$

such that the $f^{-1}\mathcal{O}_S$ Lie bracket $A \otimes_{f^{-1}\mathcal{O}_S} A \to A$ is induced by a k Lie bracket coming from

$$(L) \quad A \otimes_k \tilde{A} \to A.$$

We denote by $\tau_S : T_S \to R^1 f_* A$ the edge morphism of (E) ($=$ Kodaira-Spencer map).

For example, $A = T_{X/S}$ (the relative tangent sheaf), $\tilde{A} = \tilde{T} := \{(t, s) \in T_X \times f^{-1}T_S$, Image t in $f^*T_S = f^{-1}T_S \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X =$ Image $s\}$. ($\tilde{T}$ was considered in [BS]). (Or, if E is a vector bundle on X, A is its Atiyah algebra, or an extension of $T_{X/S}$

by $\mathrm{End}^\circ \mathcal{E}$, then $\mathcal{A}$ is the extension of $\widetilde{T}$ by $\mathrm{End}\,\mathcal{E}$).

In this situation, we construct:

① a complex $A^\cdot(n)$ (of length n) of $f^{-1}\mathcal{O}_S$ sheaves on the n-fold product $X\times_S\cdots\times_S X$ ($\xrightarrow{f} S$), such that each diagonal embedding $\Delta: X\times_S\cdots\times_S X$ ($(n-1)$ times) $\to X\times_S\cdots\times_S X$ (n times) induces a map

(*) $R^{n-1}f_*A^\cdot(n-1) \to R^n f_* A^\cdot(n)$.

② a Σ_n (= symmetric group in n letters) action on $A^\cdot(n)$ such that (*) induces a map

$$\left(R^{n-1}f_*A^\cdot(n-1)\right)^{-\Sigma_{n-1}} \longrightarrow \left(R^n f_* A^\cdot(n)\right)^{-\Sigma_n}$$

with cokernel lying in $\left(R^n f_* A^0(n)\right)^{-\Sigma_n}$, where $(\ )^{-G}$ for any subgroup $G\subset\Sigma_n$ is the space on which $\sigma\in G$ acts via sgn (so for $H\subset G$, $(\ )^{-G}\to(\ )^{-H}$ splits via $\frac{1}{|G:H|}\sum_\sigma \mathrm{sgn}\,\sigma\cdot\sigma$, where σ runs through representants of G/H)

③ a sheaf $\left(R^n f_* A^\cdot(n)\right)^{-\Sigma_n,-\Sigma_{n-1},\dots,-\Sigma_2}$ (a quotient of $\left(R^n f_* A^\cdot(n)\right)^{-\Sigma_n}$).

The theorem says:

<u>Theorem</u> ([EV]).

a) There is a n-extension (E_n) on $X \times_S \cdots \times_S X$ (n times) of $f^{-1}(T_S \otimes_k \cdots \otimes_k T_S)$ (n-times) by $A^{\cdot}(n)$

b) Assume that either $R^p f_* A = 0$ for $p > 1$ (for example for bundles on a curve) or $f_* A = 0$ (for example no infinitesimal automorphism). Then the connecting morphism

$$T_S \otimes_k \cdots \otimes_k T_S \ (n \text{ times}) \longrightarrow R^n f_* A^{\cdot}(n)$$

of (E_n) gives rise to a $\underbrace{\text{diagram}}_{\text{(commutative)}}$ of left $\mathcal{O}_S$-modules:

$$\begin{array}{ccc}
0 & & \\
\uparrow & & \\
S^n T_S & \xrightarrow{S^n \varphi_S} & S^n R^1 f_* A \\
\uparrow & & \uparrow \\
\mathcal{D}_S^n / \mathcal{O}_S & \xrightarrow{\varphi_{n,S}} & \left(R^n f_* A^{\cdot}(n)\right)^{-\Sigma_n, \ldots, -\Sigma_2} \\
\uparrow & & \uparrow \\
\mathcal{D}_S^{n-1} / \mathcal{O}_S & \longrightarrow & \left(R^{n-1} f_* A^{\cdot}(n-1)\right)^{-\Sigma_{n-1}, \ldots, -\Sigma_2} \\
\uparrow & & \\
0 & &
\end{array}$$

c) If φ_S is surjective, $\varphi_{n,S}$ is surjective as well. If $f_* A = 0$ and φ_S is injective, $\varphi_{n,S}$ is injective as well.

Here $\mathcal{D}_S^{\le n}$ is the sheaf of differential operators of order $\le n$ on S.

This work grew out of an attempt to understand [BG] in which Beilinson and Ginzburg announced a description of $(\mathcal{D}_S^{n}/\mathcal{O}_S)^{\text{dual}}$ on the moduli stack of vector bundles on a curve in terms of some antiinvariant forms on the blow up of $X \times \dots \times X$ along some diagonals.

Schechtman and Hinich try "operads and differential graded Lie algebras" to obtain similar results.

Ran ("Derivations of moduli") made a short note on a similar subject from the viewpoint of deformations.

Finally, P. Deligne, in two letters to the author, explains a sketch of a programme, how to conceive the moduli of objects whose deformations are "controlled" by a differential graded algebra L.

[BG]: Beilinson – Ginzburg: International Math. Research Notices, Duke, 4 (1992), 63–73

[BS]: Beilinson – Schechtman: Comm. Math. Physics 118 (1988) 651–701

[EV]: Esnault – Viehweg: Higher Kodaira – Spencer classes, 43 pages.

Titel: Analytic torsion forms and direct images in arithmetic K-theory
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Let $\pi:M\to B$ be a fibration of complex manifolds equipped with hermitian metrics and compact fibre Z. Analytic torsion forms are forms modulo ∂- and $\bar\partial$-coboundaries on B, associated to hermitian holomorphic vector bundles E on M. The degree zero part of these forms (thus, a function on B) is equal to the complex Ray-Singer torsion of the fibres, which is a regularized determinant of the Kodaira-Hodge Laplacian [RS].
The main motivation for the investigation of the analytic torsion forms is the construction of a direct image in hermitian K-theory. This K-theory was developed by Gillet and Soulé in the context of Arakelov arithmetic geometry. We will briefly discribe this theory, but the main results presented here are discribed in terms of complex differential geometry.

I) Hermitian K-theory

Let X be a complex manifold with hermitian metric g. Let A(X) denote the space of real forms on X which are sums of forms of type (p,p) and let $\partial,\bar\partial$ be the holomorphic and antiholomorphic derivation operators. We define

$$\tilde{A}(X):=A(X)/(\mathrm{Im}\,\partial+\mathrm{Im}\,\bar\partial)$$

Let E be a holomorphic vector bundle with hermitian metric h. Then there exists a unique hermitian holomorphic connection ∇ on E. Let Ω denote its curvature. For a Chern-Weil polynom P, we denote by

$$P(E,h) := P(-\tfrac{1}{2\pi i}\Omega)\in A(X)$$

the Chern-Weil form which represents the corresponding characteristic class. For a short exact sequence $\mathcal{E}:0\to E'\to E\to E''\to 0$ equipped with arbitrary metrics h',h,h" let $\tilde{P}(\mathcal{E},h,h',h'')\in\tilde{A}(X)$ be the Bott-Chern secondary class, which was defined axiomatically in [BGS1]. In particular,

$$\frac{\partial\bar\partial}{2\pi i}\tilde{P}(\mathcal{E},h,h',h'')=P(E,h)-P(E',h')-P(E'',h'').$$

The group $\hat{K}_0(X)$ of virtuel hermitian vector bundles is then defined as the Quotient of the free abelian group generated by the triples (E,h,η) (where η is in $\tilde{A}(X)$) by the relation

$$(E',h',\eta')+(E'',h'',\eta'')\sim(E,h,\eta'+\eta''+\widetilde{ch}(\mathcal{E},h,h',h''))$$

for each exact sequence $\mathcal{E}$. We will denote by $\tilde{P}(E,h,h')$ the Bott-Chern classes associated to the very short exact sequence $0\to E\to E\to 0$.

II) Analytic torsion forms

Now let $\pi:M\to B$ be a Kählerian fibration of complex manifolds and let (E,h) be a hermitian vector bundle on M. Let ω be the Kähler form on M and g^{TZ} be the restriction of the Kähler metric to the compact fibre Z. Suppose that the direct image sheaves $R\pi_*E$ have constant dimension, i.e. that they are locally free. The sheaves

$R\pi_*E$ at $x \in B$ are given by the cohomology $H(Z_x,E)$ in the fibre. For this situation, Bismut and the author constructed analytic torsion forms $T_{\pi,\omega} \in \tilde{A}(B)$ with the following two properties:

Theorem 1 (Bismut,K. [BK]): Let $h^{R\pi_*E}$ be the via Hodge theory induced metric on $R\pi_*E$. Then the following double transgression formula holds

$$\frac{\partial\bar\partial}{2\pi i} T_{\pi,\omega}(E,h) = ch(R\pi_*E,h^{R\pi_*E}) - \int_Z Td(TZ,g^{TZ})ch(E,h).$$

This theorem is a refined, "hermitian" version of the Grothendieck-Riemann-Roch theorem. Let now ω,ω' be two Kähler forms on M and h,h' two metrics on E. Let g^{TZ}, g'^{TZ} and $h^{R\pi_*E}, h'^{R\pi_*E}$ be the corresponding metrics. The principal result of [BK] is the following

Theorem 2 (Bismut,K.): The following anomaly formula holds

$$T_{\pi,\omega'}(E,h') - T_{\pi,\omega}(E,h) = \widetilde{ch}(R\pi_*E,h^{R\pi_*E},h'^{R\pi_*E})$$

$$- \int_Z (\widetilde{Td}(TZ,g^{TZ},g'^{TZ})ch(E,h) + Td(TZ,g'^{TZ})\widetilde{ch}(E,h,h'))$$

modulo - and -coboundaries.

The difficult part is the description of the dependance on ω. In particular, one notices that T depends only on g^{TZ} and not on the full Kähler metric on M. Following Gillet and Soulé [GS1], one may now define a direct image

$$\pi_!: K(X) \to K(Y)$$

setting

$$\pi_!(E,h,\eta) = \sum_i (-1)^i (R^i\pi_*E,h^{R\pi_*E},0) + (0,0,T_{\pi,\omega}(E,h) + (\eta\, Td(TZ,g^{TZ}))).$$

Let $f:B \to pt$ be the projection of B to a point. The exact sequence

$$0 \leftrightarrows TZ \leftrightarrows TM \leftrightarrows \pi^*TB \leftrightarrows 0$$

associates a Bott-Chern class $Td(TM,TB,g^{TM},g^{TB})$ to the Todd polynomial. Bismut and Berthomieu proved the following result

Theorem 3 (Bismut,Berthomieu [BB]): The following formula holds

$$T_{f\circ\pi,\omega}(E,h) + \int_M Td(TM,TB,g^{TM},g^{TB})ch(E,h)$$
$$= T_{f,\omega}(R\pi_*E,h^{R\pi_*E}) + \int_B Td(TB,g^{TB})\, T_{\pi,\omega}(E,h)$$

This result was obtained by an "adiabatic limit" technique, i.e. by multiplying the metric on B by a constant K and investigating the behaviour of the analytic torsion when K tends to infinity. Theorem 3 gives the composition rule in degree 0 for the direct image described above.

III) Application to Arakelov arithmetic geometry

We shall give only a brief introduction to the constructions of Gillet and Soulé in arithmetic geometry. Details may be found in [S]. Let $\mathcal{X}$ be an arithmetic variety, i.e. a regular scheme, quasi-projective and flat over $\mathbb{Z}$, smooth over the generic point $\mathbb{Q}$. Let $X := \mathcal{X}\otimes\mathbb{C}$ be the fibre at infinity. We equip X with a Kähler metric. Gillet and Soulé constructed an intersection theory $\widehat{CH}(\mathcal{X})_{\mathbb{Q}}$ of $(\mathcal{X},g)$ using Green currents on (X,g). Let $CH(\mathcal{X})_{\mathbb{Q}}$ denote the classical Chow theory and let H(X) be the Dolbeault cohomology. Then there is an exact sequence

$$H(X)\xrightarrow{a}\widehat{CH}(\mathcal{X})\xrightarrow{(\zeta,\omega)}CH(\mathcal{X})\oplus(\ker\bar{\partial}\partial\cap A(X)).$$

A smooth map $\pi:\mathcal{X}\to\mathcal{Y}$ induces a direct image $\pi_*:\widehat{CH}(\mathcal{X})_{\mathbb{Q}}\to\widehat{CH}(\mathcal{Y})_{\mathbb{Q}}$. Gillet and Soule constructed characteristic classes $\hat{P}:\hat{K}_0(X)\to\widehat{CH}(\mathcal{X})_{\mathbb{Q}}$, which verify the properties

$$\omega(\hat{P}(E,h,\eta))=P(E,h)-\frac{\partial\bar{\partial}}{2\pi i}\,\eta$$

and

$$\hat{P}(E,h,\eta)-\hat{P}(E,h',\eta)=P(E,h,h').$$

The Chern character induces an isomorphism

$$\widehat{ch}:\hat{K}_0(X)\otimes\mathbb{Q}\to\widehat{CH}(\mathcal{X})_{\mathbb{Q}}.$$

Gillet and Soule posed the following conjecture in [GS1]:

Conjecture (Arithmetic Grothendieck-Riemann-Roch): The folowing diagram commutes

$$\begin{array}{ccc}\hat{K}_0(X) & \xrightarrow{\pi_!} & \hat{K}_0(Y)\\ \downarrow{\scriptstyle \widehat{ch}\cdot Td^A(t)} & & \downarrow{\scriptstyle \widehat{ch}}\\ \widehat{CH}(\mathcal{X})_{\mathbb{Q}} & \xrightarrow{\pi_*} & \widehat{CH}(\mathcal{Y})_{\mathbb{Q}}\end{array}$$

Td^A is here a particular complicated characteristic class (see [GS1] and [K1]). Up to now, one knows by the work of Bismut, Lebeau, Gillet and Soulé that this conjecture is true for the restriction to $\widehat{CH}^1(\mathcal{Y})_{\mathbb{Q}}$:

Theorem (Gillet,Soule [GS2]):For $\alpha\in\hat{K}_0(X)$

$$\hat{c}_1(\pi_!\,\alpha)=\pi_*(\widehat{ch}(\alpha)Td^A(TZ,h^{TZ}))^{(1)}$$

Let $\delta(\pi,E,h,g^{TZ}):=\widehat{ch}(\pi_!\,\alpha)-\pi_*(\widehat{ch}(\alpha)Td^A(TZ,g^{TZ}))$ for $\alpha=[(E,h,\eta)]$, E as in Theorem 1,2,3, be the possible error of a Grothendieck-Riemann-Roch theorem. Then we find the following reformulation of the theorems 1 and 2:

Theorem 1': The error term lives in the image of the map $a:H(Y)\to\widehat{CH}(\mathcal{Y})_{\mathbb{Q}}$,

$$\delta \in a(H(Y)).$$

Theorem 2': The error term is independent of the metrics h, g^{TX}.

The Theorem 3 describes the composition rule for the direct image in degree 1.

IV) Analytic torsion forms for torus fibrations

Firbrations where the fibres are tori give a particularly nice setting for the torsion form construction. It turns out in this case that one may construct the torsion forms even when the fibration is not kählerian. This is a very strong generalization. We construct the torus fibrations in the following way:
Let $\pi: F \to B$ be a holomorphic n-dimensional hermitian vector bundle over a complex manifold B and let Λ be a lattice generating the underlying real vector bundle $F_{\mathbb{R}}$. Assume that local sections of the lattice are holomorphic sections of F. Then the fibration $F/\Lambda \to B$ is a holomorphic torus fibration. We construct the torsion forms $T(F/\Lambda, g^F)$ only for the trivial line bundle O on this fibration. In this situation,

$$(*) \qquad R\pi_* O = \Lambda \bar{F}^*$$

as complex vector bundles, but not as hermitian holomorphic vector bundles. Via (*), F^* is equipped with an exotic holomorphic structure. We assume the volume of F/Λ to be constant equal to 1, so that (*) is an isometry. Using the classical formula

$$ch(\Lambda \bar{F}^*) = \frac{c_n}{Td}(\bar{F}),$$

one notices that the following theorem is the analogue of theorem 1:

Theorem 4 ([K2]): Let $\bar{F}$ be equipped with the holomorphic structure constructed above. Then

$$\frac{\partial\bar{\partial}}{2\pi i} T(F/\Lambda, g^F) = \frac{c_n}{Td}(\bar{F}, g^F).$$

This result is a generalization of some result of Atiyah on 2-tori [A]. The analogue of Theorem 2 is an easy consequence of the axiomatic definition of Bott-Chern classes:

Corollary 5 ([K2]): Let g^F, g'^F be two hermitian metrics on F. Then

$$T(F/\Lambda, g'^F) - T(F/\Lambda, g^F) = Td^{-1}(\bar{F}, g^F, g'^F) c_n(\bar{E}, g^F) + Td^{-1}(\bar{F}, g'^F) c_n(\bar{F}, g^F, g'^F). \mod \partial, \bar{\partial}.$$

Theorem 4 is proven by explicitly double trangressing the Euler class of F using Epstein zeta functions. Sullivan proved that this class is zero in rational cohomology [S]. Bismut and Cheeger transgressed the Euler class in real cohomology when investigating eta invariants of SL(2n,Z) vector bundles [BC]. The case considered here is a bit more sophisticated because not only the metric but also the complex structure is not nessecarily compatible with the flat structure.
The result gives in particular interesting forms on moduli spaces of 2n-tori.

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Titel: Symplectic Donaldson Type Invariants
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In the early 80's, there are two important events in mathematics. One was the spectacular success of Donaldson gauge theory, which produced the differentiable invariants for smooth 4-manifold. Another was the introduction of Gromov theory to symplectic geometry, which is producing symplectic invariants. In the early 80's, Gromov observed that much of the theory of holomorphic curves can be carried out on an almost complex manifold with a symplectic structure. In his historic paper [G], Gromov established the basic properties of the moduli space of pseudo-holomorphic curves. Using this newly found theory, he proved many interesting theorems. A remarkable feature of Gromov's symplectic theory is its strong resemblance to Donaldson gauge theory. Let's just list two instances of this. (i) The noncompactness for both cases are Uhlenbeck bubbling off. (ii) Both theories study the bordism class of a finitely dimensional moduli space inside an infinite dimensional manifold (gauge equivalence class of connections or $Map(\Sigma, V)$), where Σ is a Riemann surface and V is a symplectic manifold. Perhaps the most famous examples is Floer's homology theory for both gauge theory and symplectic geometry. To further explore the power of Gromov theory, it is natural to establish a Donaldson type invariant. Actually, various simple forms of Donaldson type invariants have already been used by Gromov and McDuff to prove many important theorems in symplectic geometry. [G], [M1], [M3], [M5].

Another motivation for Donaldson Type Gromov invariants is from rather different sources. There is a famous "Mirror Symmetry" Phenomenon in mathematical physics and algebraic geometry relating the number of rational curves in a Calabi-Yau 3-fold to the variation of Hodge structures on its mirror, which is another Calabi-Yau 3-fold [Mr]. The core of this mirror symmetry phenomeno is Witten's topological σ-model invariant called k-point corelation function, which is based on the number of rational curves [Wi]. Although mirror symmetry apparently concerns the complex structures; like the variation of Hodge structures, physicists predict the k-point corelation function should only depend on the symplectic structure. Until now, its mathematical foundation still remains to be established. This is one of the goals of this talk.

Here, we introduce two Donaldson Type Gromov Invariants Φ and $\hat{\Phi}$. Intuitively, one can think that Φ is "counting the number of holomorphic curves". Technically speaking, there is also a question of the multiplicity involved. This cause some difficulty in formulating the invariant Φ. The following is a rough definition of Φ. We will explain more detail in the talk or reader can find it in [R].

Recall that a symplectic manifold (V, ω) is semi-positive if $\omega(A) \le 0$ for any A in the image of Hurewicz homomorphism $H: \pi_2(V) \to H_2(V, \mathbb{Z})$ and $3-n \le c_1(V)(A) < 0$. In particular, if $\dim V = 4$ or 6, (V, ω) is always semi-positive.

<u>Theorem A</u>: Let (V, ω) be a semi-positive symplectic manifold. Let $A \in H_2(V, \mathbb{Z})$ with $c_1(V)A > 0$. Choose a generic tamed almost complex structure J. For any $\alpha_1, \cdots, \alpha_K \in H_*(V, \mathbb{Z})$ such that $\deg \alpha_i \le 2n-2$ and $2(n-1)k - \sum_i \deg \alpha_i = 2(c_1(A)+n)-6$. Choose a geometric cycle representing α_i (still denoted by α_i). We can define an integer $\Phi_{(A,J,\omega)}(\alpha_1, \cdots \alpha_K)$ as follows:

(1) There are only finitely many unparameterized J-spheres in $\mathcal{M}'_{(A,J)}$ intersecting $\alpha_1, \cdots \alpha_K$

(2) To each such J-sphere, we can associate a multiplicity $m(f)$

Then, we define $\Phi_{(A,J,\omega)}(\alpha_1, \cdots \alpha_K) = \sum m(f)$

$\Phi_{(A,J,\omega)}(\alpha_1, \cdots, \alpha_K)$ is independent of J, hence an invariant of (ω, A), denoted by $\Phi_{(A,\omega)}$. Furthermore, if ω_t is a path of semi-positive symplectic structures, then

$\Phi_{(A,\omega_0)} = \Phi_{(A,\omega_1)}$.

On the other hand, k-point corelation function $\hat{\Phi}$ is "counting the number of holomorphic maps $f: S^2 \to V$ with marked points. There is also a technical issue of multiple cover maps. We have to perturbe the $\bar{\partial}_J$-equation to $\bar{\partial}_J f = g$. Following is a brief definition of $\hat{\Phi}$.

Theorem B: Let (V, ω) be a semi-positive symplectic manifold. Let $A \in H_2(V, \mathbb{Z})$ with $c_1(V)(A) \geq 0$. Fix a set of distinct points (marked points) $x_1, \dots x_k$. Choose generic J and g. For any $\alpha_1, \dots \alpha_k \in H_*(V, \mathbb{Z})$, with $\sum_i (2n - \deg \alpha_i) = 2(c_1(A) + n)$, there are only finite many perturbed holomorphic maps $f_i: S^2 \to V$ with $f(x_i) \in Im(\alpha_i)$. We can define k-point corelation function $\hat{\Phi}_{(A,J,g,\omega)}(\alpha_1, \dots \alpha_k; x_1, \dots x_k)$ to be the number of such f_i.

This number is independent of the choice of $x_1, \dots x_k$, J, g and the representative of α_i. We denote it by $\hat{\Phi}_{(A,\omega)}(\alpha_1, \dots \alpha_k)$. Furthermore, if ω_t is a path of semi-positive symplectic structures, then $\hat{\Phi}_{(A,\omega_0)} = \hat{\Phi}_{(A,\omega_1)}$.

Note that we only require $c_1(V)(A) \geq 0$ here, including the case of Calabi-Yau manifolds.

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Titel: Subvarieties of moduli spaces.

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We consider moduli spaces of algebraic curves and of abelian varieties. These have the following properties: Let $k = \bar{k}$ be an algebraically closed field, then:

- $\mathcal{M}_g(k) = \{\cong_k \text{cl. } C \mid C \text{ is complete non singular irred. alg. curve genus } g\}$
- $\mathcal{M}_g^{\sim}(k) = \{\cong_k \text{cl } C \mid C \text{ stable, irred, genus } g, \ \& \ \mathrm{Jac}(C) \text{ is an AV}\}$ "nice curve"
- $A_g(k) = \{\cong_k (X, \lambda) \mid X \text{ is AV, } \dim X = g, \ \lambda : X \xrightarrow{\sim} X^t \text{ polar.}\}$
- $\overline{\mathcal{M}_g}$: Deligne-Mumford compactification, $T_g = \overline{j(\mathcal{M}_g)} \subset A_g$

  A_g^* : Satake (= minimal) compactification.

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(1) Question: fix g, fix k, let $Z \subset \mathcal{M}_g \otimes k$, Z complete, irred. $/k$ what is a (sharp) bound for $\dim(Z)$?

(1bis) $Z \subset A_g \otimes k$, (sharp) bound for $\dim(Z)$?

(2) Th (Diaz, see [D1]): $\mathrm{char}(k) = 0$, or $= p \gg 0$; Z complete, $g \geq 3$
$Z \hookrightarrow \mathcal{M}_g \otimes k \Rightarrow \boxed{\dim Z \leq g-2}$

(3) $\mathrm{char}(k) = p > 0$, $A_g \otimes k \supset V_0 := \{(X, \lambda) \mid X[p](\bar{k}) = 0\}$
Th ([K], [N.O], [O1]) $\dim V_0 = \frac{1}{2} g(g-1)$ & V_0 is complete.

(4) Some examples: • [D2] $g \geq 3$, $\forall P \in \mathcal{M}_g$, $\exists P \in Z \subset \mathcal{M}_g$ $\dim Z = 1$. (also see [O1], p.95).

- (use A_g^*): $\forall g$, $\exists Z \hookrightarrow A_g \otimes k$, Z complete, $\dim Z = g-1$.
- [D2] $\forall d$, $\exists g \ (\sim \frac{4}{3} 4^d)$ & $Z \hookrightarrow \mathcal{M}_g$, Z complete, $\dim Z = d$.

(5) See [M], and see [F]: Faber computed Chow rings (rational equivalence, $\mathbb{Q}$-coefficients) for several moduli spaces, see [F], page 92: $(Y=) \ \xi$ is a certain class in $A^3(\mathcal{M}_3 \otimes \mathbb{C})$ &:

Question: $\exists ?$ $Z \subset \mathcal{M}_3^{\sim}$ (or $Z \subset A_3$) such that $[Z] = \xi$?

(6) Let us accept as a working hypothesis (??):
$A^{\cdot}(\mathcal{A}_g\otimes\overline{\mathbb{Q}},\mathbb{Q}) \xrightarrow{\sim} A^{\cdot}(\mathcal{A}_g\otimes\overline{\mathbb{F}_p},\mathbb{Q})$
(true for rational equivalence? otherwise take numerical equivalence, or cohomology, or ⋯).

(7) Conjecture (Manin, oral communication in 1988):
$\xi\in A^3(\mathcal{A}_3\otimes\overline{\mathbb{Q}})$ as in (5) has the property:

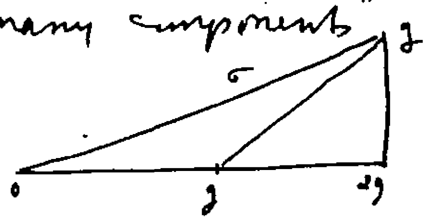
$$\forall p, \exists c_p\in\mathbb{Q}, \quad \boxed{\xi = c_p\cdot[V_0]\in A^3(\mathcal{A}_3\otimes\overline{\mathbb{F}_p})} \quad !$$

(for $g=3$, and $p>7$ I showed (6) to be correct, so this makes sense)

(8) Th (G. van der Geer, unpublished): $[V_f]=(p-1)^{g\cdot f}\lambda_{g-f}$.
(λ_i : i-th Chern class of the universal cotangent bundle)
(this proves (7), !): $g=3$, $[V_0]=(p-1)^3\lambda_3\in A^3(\mathcal{A}_3\otimes\overline{\mathbb{F}_p},\mathbb{Q})$.

(9) Here $\mathcal{A}_g\otimes\mathbb{F}_p\supset V_f=\{(X,\lambda)\mid X[p](k)\cong(\mathbb{Z}/p)^f\}$. This is a special case of the stratification by Newton polygons, see [O2]; $\forall$ Newton polygon α, $\dim(W_\alpha)$ is known,
$\mathcal{A}_g\otimes\mathbb{F}_p\supset W_\alpha=\{(X,\lambda)\mid NP \text{ of } X \text{ does not lie below } \alpha\}$.

(10) Th (Conj. by Tadao Oda & FO, T. Katsura & FO, see [L.O]):
$\dim W_\sigma = \left[\frac{g^2}{4}\right]$ & W_σ has "many components"
(a certain class number)
(here σ : supersingular, all Newton-slopes $=\frac{1}{2}$).



(8 bis) Conj. (GvdGeer): $\exists\, c_{p,\alpha}$: $[W_\alpha]=c_{p,\alpha}\cdot \mathrm{Pol}_\alpha\in A^{\cdot}(\mathcal{A}_g\otimes\mathbb{F}_p)$,
where "Pol_α" is a concrete polynomial in the λ_i's, $c_{p,\alpha}=?\in\mathbb{Q}$.

(11) Conj : $\forall p$, $\forall$ Newton polygon α, $\alpha\neq\sigma$, W_α is irreducible.
Th(11.1) (E.O) : $g\geq 2$, $\forall p$, V_{g-1} is irreducible.
Th(11.2) : $g=3$, $\forall p$, $V_0\subset\mathcal{A}_3\otimes\mathbb{F}_p$ is irreducible.

(12) Question What is $\mathcal{J}_g\cap W_\alpha\subset\mathcal{A}_g\otimes\mathbb{F}_p$, i.e. curves with Newton polygon not below α? I have no guess. Note that $g\geq 9 \Rightarrow \frac{1}{2}g(g+1)-\left[\frac{g^2}{4}\right]>3g-3$,

But for large g there exist supersingular curves of genus g, hence the stratification by Newton polygons is not "transversal" to the Torelli locus $\mathcal{T}_g \subset \mathcal{A}_g$.

(13) Exa (see Texel conference:) $g = 3$, let $\mathcal{B} \subset \mathcal{A}_3 \otimes \mathbb{F}_p$ be the closure of the hyperelliptic locus, $\dim \mathcal{B} = 5$, let $\mathcal{S} = W_\sigma$ be the supersingular locus, $\dim \mathcal{S} = 2$. Computations in [K] suggest that $\mathcal{B} \cap \mathcal{S}$ could have 2-dimensional components, however:

Th ($g=3$, char $k = p$) Every component of $\mathcal{B} \cap \mathcal{S} \subset \mathcal{A}_3 \otimes \mathbb{F}_p$ has dim $= 1$.

Now we come back to $Z \subset \mathcal{A}_g \otimes k$, Z complete ($g \geq 3$):

(14) (14.1) Conj: $\dim Z \leq \frac{1}{2} g(g-1) = \frac{1}{2} g(g+1) - g =: m$

(14.2) Note (3): $V_0 \subset \mathcal{A}_g \otimes \mathbb{F}_p$, $\dim (V_0 = Z) = m$,

(14.3) Conj: $Z \subset \mathcal{A}_g \otimes \mathbb{C} \Rightarrow \dim Z < m$. (!)

(14.4) ? sharp bound for $\dim Z$ if char$(k) = 0$?

(14.5) ? sharp bound for $\dim Z$, with $Z \subset \mathcal{M}_g \otimes \mathbb{C}$ or $Z \subset \mathcal{M}_g^{\sim} \otimes \mathbb{C}$?

(15) Let us try: ($g \geq 3$)

Z complete, $Z \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$, $\dim Z = m \overset{?}{\Rightarrow} [Z] \in \mathbb{Z} \cdot [V_0] \in A^{\cdot}(\mathcal{A}_g)$

If this is true, (14.3) would follow ...

Acknowledgments:

- to Yu. I. Manin for suggestions, especially see Conjecture 7,
- to G. van der Geer & C. Faber, for stimulating discussions,
- to coauthors Tadao Oda, P. Norman, T. Katsura, T. Ekedahl, and K.-Z. Li for common passions and ideas.

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Hope & last conjecture: ∃ A. ⊂ Bonn, 1995 … ?!