On irreducibility of tensor products of Yangian modules associated with skew Young diagrams

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Introduction

In this article we continue our study [NT1,NT2] of the finite-dimensional modules over the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ of the general linear Lie algebra $\mathfrak{gl}_N$. The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is a canonical deformation of the enveloping algebra $\mathrm{U}(\mathfrak{gl}_N[u])$ in the class of Hopf algebras [D1]. The unital associative algebra $\mathrm{Y}(\mathfrak{gl}_N)$ has an infinite family of generators $T_{ij}^{(s)}$ where $s=1,2,\ldots$ and $i,j=1,\ldots,N$. In Section 4 we recall the defintion of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$ in terms of the generating series

$$T_{ij}(u)=\delta_{ij}+T_{ij}^{(1)}u^{-1}+T_{ij}^{(2)}u^{-2}+\ldots\,. \tag{0.1}$$

The classification of the irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_N)$-modules has been given by V. Drinfeld [D2] by generalizing the results of the second author [T1,T2]. But the structure of these modules still needs better understanding. For instance, the dimensions of these modules are not explicitly known in general.

There is a distinguished family of irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_N)$-modules [C2,O]. We call these modules elementary. Every elementary $\mathrm{Y}(\mathfrak{gl}_N)$-module is determined by a complex number z, and by two partitions $\lambda=(\lambda_1,\lambda_2,\ldots)$ and $\mu=(\mu_1,\mu_2,\ldots)$ such that $\lambda_i\geqslant\mu_i$ for any $i=1,2,\ldots$. This module depends on λ and μ through the skew Young diagram $\omega=\lambda/\mu$, see Section 1. We denote the corresponding $\mathrm{Y}(\mathfrak{gl}_N)$-module by $V_\omega(z)$, its explicit construction will be given in Section 4. The vector spaces of the modules $V_\omega(z)$ are the same for all values of z, we denote this vector space by V_ω. Suppose that for some positive integer M the numbers of non-zero parts of λ and μ do not exceed $M+N$ and M respectively. Then the dimension of V_ω equals the multiplicity of the irreducible $\mathfrak{gl}_M$-module of highest weight $(\mu_1,\ldots,\mu_M)$ in the restriction to $\mathfrak{gl}_M$ of the the irreducible $\mathfrak{gl}_{M+N}$-module of highest weight $(\lambda_1,\ldots,\lambda_{M+N})$. In particular, the space V_ω is non-zero if and only if $\lambda_i^*-\mu_i^*\leqslant N$ for any index $i=1,2,\ldots$. Here $\lambda^*=(\lambda_1^*,\lambda_2^*,\ldots)$ and $\mu^*=(\mu_1^*,\mu_2^*,\ldots)$ are the partitions conjugate to λ and μ respectively.

For any formal series $f(u)\in\mathbb{C}[[u^{-1}]]$ with the leading term 1, the assignment $T_{ij}(u)\mapsto f(u)\cdot T_{ij}(u)$ determines an automorphism of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$. Let us denote by α_f this automorphism. Further, there is a canonical chain of algebras

$\mathrm{Y}(\mathfrak{gl}_1) \subset \ldots \subset \mathrm{Y}(\mathfrak{gl}_N)$. The elementary modules are distinguished amongst all irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_N)$-modules V by the following fact. Consider the commutative subalgebra in $\mathrm{Y}(\mathfrak{gl}_N)$ generated by the centres of all algebras in that chain. This subalgebra is maximal commutative [C2,NO]. The action of this subalgebra in V is semi-simple if and only if the module V is obtained by pulling back through some automorphism α_f from the tensor product

$$V_{\omega_1}(z_1) \otimes \ldots \otimes V_{\omega_k}(z_k) \tag{0.2}$$

of the elementary $\mathrm{Y}(\mathfrak{gl}_N)$-modules, for some skew Young diagrams $\omega_1, \ldots, \omega_k$ and some complex numbers $z_1, \ldots, z_k$ such that $z_i - z_j \notin \mathbb{Z}$ when $i \neq j$. This fact was conjectured by I. Cherednik, and proved by him in [C2] under certain extra conditions on the module V. In full generality this fact has been proved in [NT1].

Furtermore, if $z_i - z_j \notin \mathbb{Z}$ for all $i \neq j$, the tensor product of $\mathrm{Y}(\mathfrak{gl}_N)$-modules (0.2) is always irreducible. Given the skew Young diagrams $\omega_1, \ldots, \omega_k$ it is natural to ask, for which values of $z_1, \ldots, z_k \in \mathbb{C}$ the $\mathrm{Y}(\mathfrak{gl}_N)$-module (0.2) is reducible. In [NT2] we answered this question in the case when each of the the diagrams $\omega_1, \ldots, \omega_k$ has a rectangular shape. In that particular case, our answer confirmed the following general conjecture. For any indices i and j consider the tensor product $V_{\omega_i}(z_i) \otimes V_{\omega_j}(z_j)$. Consider also the tensor product $V_{\omega_i}(z_i) \,\widetilde{\otimes}\, V_{\omega_j}(z_j)$ obtained via the opposite comultiplication on $\mathrm{Y}(\mathfrak{gl}_N)$. If $z_i - z_j \notin \mathbb{Z}$, these two tensor products are irreducible and equivalent. Hence there is an invertible intertwining operator

$$R_{ij}(z_i, z_j) : V_{\omega_i}(z_i) \otimes V_{\omega_j}(z_j) \longrightarrow V_{\omega_i}(z_i) \,\widetilde{\otimes}\, V_{\omega_j}(z_j),$$

unique up to a non-zero multiplier from $\mathbb{C}$. These multipliers can be chosen so that $R_{ij}(z_i, z_j)$ depends on z_i and z_j as a rational function of $z_i - z_j$, see Section 4. Now for any $z_i - z_j \in \mathbb{C}$ denote by I_{ij} the leading coefficient of the expansion of the rational function $R_{ij}(z_i + u, z_j)$ into Laurent series in u near the origin $u = 0$. This coefficient is always an interwining operator between the two tensor products. If $z_i - z_j \notin \mathbb{Z}$, we simply have $I_{ij} = R_{ij}(z_i, z_j)$. It has been conjectured that the $\mathrm{Y}(\mathfrak{gl}_N)$-module (0.2) is irreducible if and only if for all $i < j$ the operators I_{ij} are invertible; see for instance [CP]. We prove this conjecture in the present article, see Theorem 4.8.

This theorem implies, in particular, that the $\mathrm{Y}(\mathfrak{gl}_N)$-module (0.2) is irreducible if and only if for all $i < j$ the pairwise tensor products $V_{\omega_i}(z_i) \otimes V_{\omega_j}(z_j)$ are irreducible. Given the skew Young diagrams ω_i and ω_j, it would be interesting to describe explicitly the set of all differences $z_i - z_j \in \mathbb{Z}$, such that the module $V_{\omega_i}(z_i) \otimes V_{\omega_j}(z_j)$ is reducible. In the special case when both ω_i and ω_j are usual Young diagrams, this description was recently given in [LNT] for sufficiently large N, and independently in [M] in general.

The *only if* part of Theorem 4.8 is an easy and well known fact. Our proof of the *if* part is based on the following general observation. Let A be any unital associative algebra over $\mathbb{C}$, and let W be any finite-dimensional A-module. Denote by ρ the corresponding homomorphism $\mathrm{A} \to \mathrm{End}(W)$. Suppose that the subalgebra

$$\rho(\mathrm{A}) \otimes \mathrm{End}(W) \subset \mathrm{End}(W) \otimes \mathrm{End}(W) = \mathrm{End}(W \otimes W)$$

contains the flip map $x \otimes y \mapsto y \otimes x$ in $W \otimes W$. Then $\rho(\mathrm{A}) = \mathrm{End}(W)$ and the A-module W is irreducible, see Lemma 4.10. We employ this observation when

$\mathrm{A} = \mathrm{Y}(\mathfrak{gl}_N)$, and W is the tensor product (0.2). Moreover, to prove the irreducibility of (0.2) under the condition that all the operators I_{ij} with $i < j$ are invertible, it suffices to show that I_{ii} is proportional to the flip map in $V_{\omega_i} \otimes V_{\omega_i}$ for each index i. For the details of this argument see Proposition 4.14. We took the idea of this argument from the recent physical literature [KMT, MT]. This argument extends to the quantum affine algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_N)$ in a straightforward way. We work with the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ in this article only to simplify the exposition. Moreover, this argument extends to other quantum affine algebras due to the existence of the universal R-matrix; see [FM, K] for the precise formulation of the problem and known results in the general case.

In this article, we use the explicit realization of the elementary $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_\omega(z)$, introduced in [C2]. Denote $\lambda_1 - \mu_1 + \lambda_2 - \mu_2 + \ldots = n$. Let $n > 0$. Consider the action of the symmetric group S_n in $(\mathbb{C}^N)^{\otimes n}$ by permutations of the n tensor factors. Denote by π_n the corresponding homomorphism $\mathbb{C} \cdot S_n \to \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big)$. Then V_ω is the subspace in $(\mathbb{C}^N)^{\otimes n}$ defined as the image of the operator $\pi_n(F_\omega)$ for a certain element F_ω of the group ring $\mathbb{C} \cdot S_n$. This element is constructed by a limiting process, called the fusion procedure [C1]. Note that in [C1] most of the results were given without proofs. In Section 2 we give proofs of all properties of the element F_ω which we need in this article. These proofs are new, cf. [N].

This choice of the realization of the $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_\omega(z)$ allows us to write down explicitly, for any $z' \in \mathbb{C}$ such that $z - z' \notin \mathbb{Z}$, an invertible intertwining operator

$$R(z,z') : V_\omega(z) \otimes V_\omega(z') \longrightarrow V_\omega(z) \,\widetilde{\otimes}\, V_\omega(z') \,.$$

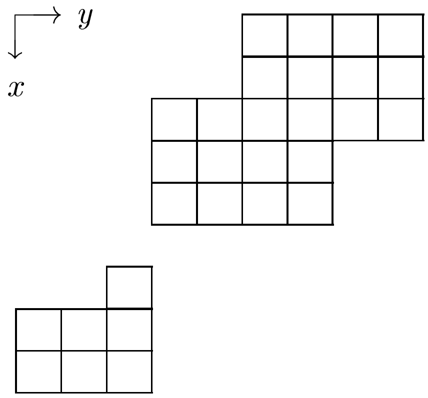
This operator comes with a natural normalization. In partucular, it is a rational function in $z - z'$ with values in $\mathrm{End}(V_\omega \otimes V_\omega)$. Denote by P_ω the flip map in $V_\omega \otimes V_\omega$. We prove that the leading coefficient of the Laurent expansion in u near the origin $u = 0$ of the function $R(z+u, z)$ equals $c(\omega)\, P_\omega\, u^{-d(\omega)}$ for a certain non-zero rational number $c(\omega)$ and a positive integer $d(\omega)$, see Proposition 4.7. The proof of this proposition is based on the results of Section 3. We call $d(\omega)$ the Durfee rank of the skew Young diagram ω. When ω is a usual Young diagram, $d(\omega)$ is the number of boxes on its main diagonal. In Section 1 we study the remarkable combinatorial properties of the numbers $d(\omega)$ for all skew Young diagrams ω.

1. Durfee rank of a skew Young diagram

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be any two partitions. As usual, here the parts are arranged in the non-increasing order: we have $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0$ and $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0$. We will always suppose that $\lambda_i \geqslant \mu_i$ for any $i = 1, 2, \ldots$. Consider the *skew Young diagram* $\omega = \lambda/\mu$. It can be defined as the set of pairs

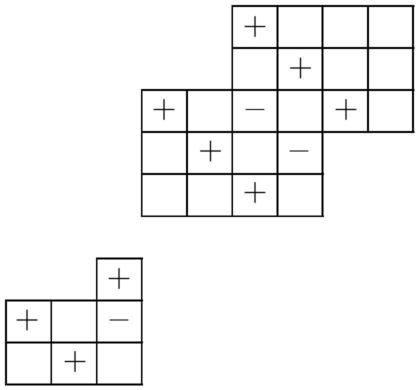
$$\{\, (i,j) \in \mathbb{Z}^2 \mid i \geqslant 1,\ \lambda_i \geqslant j > \mu_i \,\}$$

When $\mu = (0,0,\ldots)$, this is the Young diagram of the partition λ. We will employ the standard graphic representation of Young diagrams on the plane $\mathbb{R}^2$ with the matrix style coordinates (x,y). Here the first coordinate x increases from top to bottom, while the second coordinate y increases from left to right. The element $(i,j) \in \omega$ is represented by the unit box with the bottom right corner at the point $(i,j) \in \mathbb{R}^2$. For example, below we represent the skew Young diagram ω for the partitions $\lambda = (9,9,9,7,7,3,3,3,3,0,0,\ldots)$ and $\mu = (5,5,3,3,3,3,2,0,0,\ldots)$.



We will identify the elements of the set ω with the corresponding unit boxes on $\mathbb{R}^2$.

Let us call an element $(i,j) \in \omega$ *left-convex* if $(i-1,j) \notin \omega$ and $(i,j-1) \notin \omega$. An element $(i,j) \in \omega$ will be called *left-concave* if both $(i-1,j) \in \omega$ and $(i,j-1) \in \omega$, but $(i-1,j-1) \notin \omega$. Each of these two definitions has a natural interpretation in terms of boxes of the Young diagram ω. If $\lambda \neq \mu = (0,0,\ldots)$ then there is only one left-convex element $(1,1) \in \omega$ and no left-concave elements. Now consider those diagonals of ω, which contain a left-convex box. Let d be the total number of boxes on those diagonals. Let d' be the total number of boxes on the diagonals of ω which contain a left-concave box. The difference $d - d'$ will be called the *Durfee rank* of the skew diagram ω and denoted by $d(\omega)$. If $\mu = (0,0,\ldots)$ then $d(\omega) = d$ is the number of boxes on the main diagonal $\{\,(i,i) \in \mathbb{Z}^2 \mid \lambda_i \geqslant i \geqslant 1\,\}$ of the Young diagram of λ, and our definition coincides with the usual one [A, Section 2.3]. Below for our exemplary skew Young diagram ω we mark by the $+$ or $-$ signs the diagonals, which contain a left-convex or a left-concave box respectively.



In this example the Durfee rank $d(\omega)$ equals $9 - 3 = 6$. Our first observation is

Proposition 1.1. *If the diagram ω is not empty then $d(\omega) > 0$.*

Proof. For each $i = 1,2,\ldots$ let d_i be the number of boxes on that diagonal of ω which contains the leftmost box of the i-th row. If the i-th row of ω is empty, we set $d_i = 0$. Let d_i' be the number of boxes on the next diagonal down. Then

$$d(\omega) = d_1 - d_1' + d_2 - d_2' + \ldots \,. \tag{1.1}$$

By the definition of a skew Young diagram we have $d_i - d_i' \geqslant 0$ for every index $i = 1,2,\ldots$. But if i is the index of the last non-empty row of ω, then $d_i = 1$ and $d_i' = 0$ $\square$

Denote by $\ell(\omega)$ the number of non-empty rows in ω. Here is a formula for $d(\omega)$.

Proposition 1.2: $d(\omega) = \ell(\omega) - \#\{\,(i,j)\mid \lambda_j - j = \mu_i - i,\ i<j\,\}$.

Proof. We will keep to the notation from the proof of Proposition 1.1. Further, for each $i = 1,2,\ldots$ put $e_i = 0$ or $e_i = 1$ depending on whether the i-th row of ω is empty or not. By the definition of a skew Young diagram we have the equalities

$$\begin{aligned} d_i &= \#\{\,j \mid \lambda_j - j \geqslant \mu_i - i + 1 \geqslant \mu_j - j + 1\,\} \\ &= \#\{\,j \mid \lambda_j - j \geqslant \mu_i - i + 1,\ i \leqslant j\,\} \\ &= \#\{\,j \mid \lambda_j - j \geqslant \mu_i - i + 1,\ i < j\,\} + e_i\,, \\ d_i' &= \#\{\,j \mid \lambda_j - j \geqslant \mu_i - i \geqslant \mu_j - j + 1\,\} \\ &= \#\{\,j \mid \lambda_j - j \geqslant \mu_i - i,\ i < j\,\}. \end{aligned}$$

Therefore,

$$d_i - d_i' = e_i - \#\{\,j \mid \lambda_j - j = \mu_i - i,\ i<j\,\}.$$

Hence we obtain from (1.1) that

$$d(\omega) = (e_1 + e_2 + \ldots) - \#\{\,(i,j) \mid \lambda_j - j = \mu_i - i,\ i<j\,\}.$$

By observing that $e_1 + e_2 + \ldots = \ell(\omega)$ we now complete the proof $\square$

Let $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots)$ and $\mu^* = (\mu_1^*, \mu_2^*, \ldots)$ be the partitions conjugate to λ and μ respectively. Consider the corresponding skew Young diagram $\omega^* = \lambda^*/\mu^*$. It is obtained from the diagram $\omega \subset \mathbb{Z}^2$ by transposition $(i,j) \mapsto (j,i)$. Evidently, $d(\omega^*) = d(\omega)$. Therefore, Proposition 1.2 can be reformulated as

Proposition 1.3: $d(\omega) = \ell(\omega^*) - \#\{\,(i,j) \mid \lambda_j^* - j = \mu_i^* - i,\ i<j\,\}$.

By our definition, the Durfee rank $d(\omega)$ depends only on the shape of the diagram ω. That is, if a skew Young diagram $\omega^\flat$ is obtained from $\omega \subset \mathbb{Z}^2$ by translation $(i,j) \mapsto (i+k, j+l)$ for a certain $(k,l) \in \mathbb{Z}^2$, then $d(\omega^\flat) = d(\omega)$. Furthermore, suppose that a skew Young diagram $\omega^\natural$ is obtained from ω via the map $(i,j) \mapsto (k-i+1, l-j+1)$. The boxes of $\omega^\natural$ on the plane $\mathbb{R}^2$ are obtained from those of ω via rotation by 180° around the point $(\frac{k}{2}, \frac{l}{2})$. The main result of this section is

Theorem 1.4: $d(\omega^\natural) = d(\omega)$.

Proof. We have the inequalities $\lambda_1 - 1 > \lambda_2 - 2 > \ldots$ and $\mu_1 - 1 > \mu_2 - 2 > \ldots$. So under the condition $\lambda_j - j = \mu_i - i$ in Proposition 1.2 we have the equivalencies

$$\begin{aligned} i<j \;&\Leftrightarrow\; \lambda_i - i > \lambda_j - j \;\Leftrightarrow\; \lambda_i - i > \mu_i - i \;\Leftrightarrow\; \lambda_i > \mu_i\,, \\ i<j \;&\Leftrightarrow\; \mu_i - i > \mu_j - j \;\Leftrightarrow\; \lambda_j - j > \mu_j - j \;\Leftrightarrow\; \lambda_j > \mu_j\,. \end{aligned}$$

Denote by $\mathcal{L}(\omega)$ the set of differences $\mu_i - i$ such that the i-th row of ω is not empty. The set of differences $\lambda_j - j$ such that the j-th row of ω is not empty, will be denoted by $\mathcal{R}(\omega)$. By the above equivalencies, Proposition 1.2 can be restated as

$$d(\omega) = \ell(\omega) - \#\big(\mathcal{L}(\omega) \cap \mathcal{R}(\omega)\big)\,. \tag{1.2}$$

We have $\ell(\omega^\natural)=\ell(\omega)$. We also have $\#\bigl(\mathcal{L}(\omega^\natural)\cap\mathcal{R}(\omega^\natural)\bigr)=\#\bigl(\mathcal{L}(\omega)\cap\mathcal{R}(\omega)\bigr)$ since

$$\mathcal{L}(\omega^\natural)=l-k-\mathcal{R}(\omega)-1\,,\quad \mathcal{R}(\omega^\natural)=l-k-\mathcal{L}(\omega)-1\,,$$

by the defintion of $\omega^\natural$. Thus we obtain Corollary 1.4 from the formula (1.2) $\square$

Let us call an element $(i,j)\in\omega$ *right-convex* if $(i+1,j)\notin\omega$ and $(i,j+1)\notin\omega$. An element $(i,j)\in\omega$ is called *right-concave* if both $(i+1,j)\in\omega$ and $(i,j+1)\in\omega$, but $(i+1,j+1)\notin\omega$. Each of these two definitions has a natural interpretation in terms of boxes of the Young diagram ω. We could define the Durfee rank of ω using the right-convex and right-concave boxes instead of the left-convex and left-concave, respectively. Theorem 1.4 ensures that we would then get the same number $d(\omega)$. Note that although the Young diagram of a non-empty partition always has only one left-convex box and no left-concave boxes, it may have several right-convex and right-concave boxes. So our definition of the Durfee rank of ω is quite natural.

2. Fusion procedure for a skew Young diagram

Take a non-empty skew Young diagram ω. Let n be the total number of boxes in ω. In this section we will introduce a certain element F_ω of the symmetric group ring $\mathbb{C}\cdot S_n$. Under some extra conditions on the diagram ω, this element will be used in Section 4 to determine a family of irredicible modules $V_\omega(z)$ over the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$. Here the parameter z is ranging over the complex field $\mathbb{C}$. For the skew Young diagram $\omega^\flat$ obtained from ω by translation $(i,j)\mapsto(i+k,j+l)$ for a certain $(k,l)\in\mathbb{Z}^2$, we will have $V_{\omega^\flat}(z)=V_\omega(z-k+l)$.

Consider the *column tableau* of shape ω. It is obtained by filling the boxes of the diagram ω with the numbers $1,\ldots,n$ consecutively by columns from left to right, downwards in every column. We will denote this tableau by Ω. Further, for each $p=1,\ldots,n$ put $c_p=j-i$ if the box $(i,j)\in\omega$ is filled with the number p in the column tableau Ω. The difference $j-i$ is called the *content* of the box (i,j) of the diagram ω. Our choice of the tableau Ω provides an ordering of the collection of all contents of ω. Below on the left for the partitions $\lambda=(5,3,3,3,3,0,0,\ldots)$ and $\mu=(3,3,3,2,0,0,\ldots)$ we show the column tableau Ω. On the right we indicate the contents of all boxes of the diagram ω.

$$\begin{array}{ccccc}
 & & & \boxed{8} & \boxed{9}\\
 & & & & \\
 & & \boxed{5} & & \\
\boxed{1} & \boxed{3} & \boxed{6} & & \\
\boxed{2} & \boxed{4} & \boxed{7} & &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & & \boxed{3} & \boxed{4}\\
 & & & & \\
 & & \boxed{0} & & \\
\boxed{-3} & \boxed{-2} & \boxed{-1} & & \\
\boxed{-4} & \boxed{-3} & \boxed{-2} & &
\end{array}$$

Here $n=9$ and the sequence of contents $(c_1,\ldots,c_9)$ is $(\text{-}3,\text{-}4,\text{-}2,\text{-}3,0,\text{-}1,\text{-}2,3,4)$.

For any two distinct numbers $p,q\in\{1,\ldots,n\}$ let (pq) be the transposition in the symmetric group S_n. Consider the rational functions of two complex variables u,v with values in the group ring $\mathbb{C}\cdot S_n$

$$f_{pq}(u,v)=1-\frac{(pq)}{u-v}\,. \tag{2.1}$$

As a direct calculation shows, these rational functions satisfy the equations

$$f_{pq}(u,v)\,f_{pr}(u,w)\,f_{qr}(v,w)=f_{qr}(v,w)\,f_{pr}(u,w)\,f_{pq}(u,v) \tag{2.2}$$

for all pairwise distinct indices p,q,r. Evidently, for all pairwise distinct p,q,s,t

$$f_{pq}(u,v)\,f_{st}(z,w)=f_{st}(z,w)\,f_{pq}(u,v)\,. \tag{2.3}$$

For all distinct p,q we also have

$$f_{pq}(u,v)\,f_{qp}(v,u)=1-\frac{1}{(u-v)^2}\,. \tag{2.4}$$

Our construction of the element $F_\omega\in\mathbb{C}\cdot S_n$ will be based on the following simple observation. Consider the rational function of u,v,w defined by the product at either side of (2.2). The factor $f_{pr}(u,w)$ in (2.2) has a pole at $u=w$. However, we have

Lemma 2.1. *The restriction of the rational function* (2.2) *to the set of* (u,v,w) *such that* $u=v\pm1$, *is regular at* $u=w$.

Proof. Under the condition $u=v\pm1$ the product on the left hand side of (2.2) can be written as

$$\bigl(1\mp(pq)\bigr)\cdot\left(1-\frac{(pr)+(qr)}{v-w}\right).$$

The latter rational function of v,w is manifestly regular at $w=v\pm1$ □

Now introduce n complex variables $z_1,\dots,z_n$. Equip the set of all pairs (p,q) where $1\leqslant p<q\leqslant n$, with the lexicographical ordering. Take the ordered product

$$\prod_{(p,q)}^{\longrightarrow} f_{pq}(c_p+z_p\,,c_q+z_q) \tag{2.5}$$

over this set. Consider (2.5) as a rational function of the variables $z_1,\dots,z_n$ with values in $\mathbb{C}\cdot S_n$. Denote this rational function by $F_\omega(z_1,\dots,z_n)$. Let $\mathcal{D}_\omega$ be the vector subspace in $\mathbb{C}^n$ consisting of all tuples $(z_1,\dots,z_n)$ such that $z_p=z_q$ whenever the numbers p and q appear in the same column of the tableau Ω. The origin $(0,\dots,0)\in\mathbb{C}^n$ belongs to $\mathcal{D}_\omega$. The following statement goes back to [C1].

Proposition 2.2. *The restriction of the rational function* $F_\omega(z_1,\dots,z_n)$ *to the subspace* $\mathcal{D}_\omega\subset\mathbb{C}^n$ *is regular at the point* $(0,\dots,0)$.

Proof. We shall provide an expression for the restriction of $F_\omega(z_1,\dots,z_n)$ to $\mathcal{D}_\omega$ which is manifestly regular at $(0,\dots,0)$. The factor $f_{pq}(c_p+z_p\,,c_q+z_q)$ has a pole at $z_p=z_q$ if and only if the numbers p and q stand on the same diagonal of the tableau Ω. We shall then call the pair (p,q) *singular.*

Let any singular pair (p,q) be fixed. We will write $(p,q)\prec(s,t)$ if the pair (p,q) precedes (s,t) in the lexicographical ordering, that is, if $p<s$ or if $p=s\,,q<t$. In this ordering, the pair immediately before (p,q) is $(p,q-1)$. Moreover, the number $q-1$ stands just above q in the tableau Ω. Therefore, $z_{q-1}=z_q$ on $\mathcal{D}_\omega$. We also have $(p,q)\prec(q-1,q)$, since $p<q$. By (2.2) and (2.3) the product over the pairs (s,t)

$$\prod_{(p,q)\prec(s,t)}^{\longrightarrow} f_{st}(c_s+z_s\,,c_t+z_t)$$

is divisible on the left by the factor $f_{q-1,q}(c_{q-1}+z_{q-1}\,,c_q+z_q)$. Restriction of this factor to $z_{q-1}=z_q$ is just $1-(q-1,q)$ since $c_{q-1}=c_q+1$.

The element $\big(1-(q-1,q)\big)/2\in\mathbb{C}\cdot S_n$ is an idempotent. Now for every singular pair (p,q) let us replace the two adjacent factors in (2.5)

$$f_{p,q-1}(c_p+z_p\,,c_{q-1}+z_{q-1})\,f_{pq}(c_p+z_p\,,c_q+z_q)$$

by

$$f_{p,q-1}(c_p+z_p\,,c_{q-1}+z_{q-1})\,f_{pq}(c_p+z_p\,,c_q+z_q)\,f_{q-1,q}(c_{q-1}+z_{q-1},\,c_q+z_q)/2\,.$$

This does not affect the values of the restriction of (2.5) to $\mathcal{D}_\omega$. But the restriction to $z_{q-1}=z_q$ of the replacement product is regular at $z_p=z_q$ by Lemma 2.1 $\square$

Due to Proposition 2.2, an element $F_\omega\in\mathbb{C}\cdot S_n$ can be now defined as the value at the point $(0,\ldots,0)$ of the restriction to $\mathcal{D}_\omega$ of the function $F_\omega(z_1,\ldots,z_n)$. Note that for $n=1$ we have $F_\omega=1$. For any $n\geqslant 1$ we have the following fact.

Proposition 2.3. *The coefficient of $F_\omega\in\mathbb{C}\cdot S_n$ at the unit element of S_n is 1.*

Proof. For each $r=1,\ldots,n-1$ let $g_r\in S_n$ be the adjacent transposition $(r,r+1)$. Let $g_0\in S_n$ be the element of the maximal length. Let us multiply the ordered product (2.5) by the element g_0 on the right. Using the reduced decomposition

$$g_0=\prod_{(p,q)}^{\longrightarrow} g_{q-p}\,, \tag{2.6}$$

we get the product

$$\prod_{(p,q)}^{\longrightarrow}\left(g_{q-p}-\frac{1}{c_p+z_p-c_q-z_q}\right).$$

It expands as a sum of the elements $g\in S_n$ with the coefficients from the field of rational functions of $z_1,\ldots,z_n$ valued in $\mathbb{C}$. Since the decomposition (2.6) is reduced, the coefficient at g_0 is 1. By the definition of F_ω this implies that the coefficient of $F_\omega\,g_0\in\mathbb{C}\cdot S_n$ at $g_0\in S_n$ is also 1 $\square$

In particular, Proposition 2.3 shows that $F_\omega\neq 0$ for any non-empty diagram ω. Let us now denote by α the involutive antiautomorphism of the group ring $\mathbb{C}\cdot S_n$ defined by $\alpha(g)=g^{-1}$ for every $g\in S_n$.

Proposition 2.4. *The element $F_\omega\in\mathbb{C}\cdot S_n$ is α-invariant.*

Proof. Any element of the group ring $\mathbb{C}\cdot S_n$ of the form $f_{pq}(u,v)$ is α-invariant. Applying the antiautomorphism α to an element of the form (2.5) just reverses the ordering of the factors corresponding to the pairs (p,q). However, the initial ordering can be then restored by using the relations (2.2) and (2.3). Therefore, any value of the function $F_\omega(z_1,\ldots,z_n)$ is α-invariant. So is the element F_ω $\square$

The next property of the element $F_\omega\in\mathbb{C}\cdot S_n$ also easily follows from its definition.

Theorem 2.5. *Suppose the numbers r and $r+1$ stand in the same column of Ω. Then the element $F_\omega \in \mathbb{C}\cdot S_n$ is divisible by $f_{r,r+1}(c_r,c_{r+1}) = 1-(r,r+1)$ on the left and right.*

Proof. By Proposition 2.4 the divisibility of F_ω by the element $1-(r,r+1)$ on the left is equivalent to the divisibility by the same element on the right. Let us prove the divisibility on the left. Using the definition (2.5) along with the relations (2.2) and (2.3), we can write

$$F_\omega(z_1,\dots,z_n) = f_{r,r+1}(c_r+z_r, c_{r+1}+z_{r+1})\, F(z_1,\dots,z_n) \tag{2.7}$$

for some rational function $F(z_1,\dots,z_n)$ valued in $\mathbb{C}\cdot S_n$. By Proposition 2.2, the restriction of the function $F_\omega(z_1,\dots,z_n)$ to $\mathcal{D}_\omega$ is regular at $(0,\dots,0)$. But the restriction to $\mathcal{D}_\omega$ of the factor $f_{r,r+1}(c_r+z_r, c_{r+1}+z_{r+1})$ in (2.7) is $f_{r,r+1}(c_r,c_{r+1})$. In particular, the value at $(0,\dots,0)$ of the restriction of $F_\omega(z_1,\dots,z_n)$ to $\mathcal{D}_\omega$ is divisible on the left by $f_{r,r+1}(c_r,c_{r+1}) = 1-(r,r+1)$ $\square$

Let $l = 1,\dots,n-1$. Denote by σ_l the embedding $\mathbb{C}\cdot S_{n-l} \to \mathbb{C}\cdot S_n$ defined by $\sigma_l : (pq) \mapsto (p+l, q+l)$ for all distinct $p,q = 1,\dots,n-l$. Define a skew Young diagram φ as the shape of the tableau obtained from Ω by removing each of the numbers $1,\dots,l$. Consider the corresponding element $F_\varphi \in \mathbb{C}\cdot S_{n-l}$.

Proposition 2.6. *The element F_ω is divisible by $\sigma_l(F_\varphi)$ on the left and right.*

Proof. Due to Proposition 2.4 the divisibility of F_ω by the element $\sigma_l(F_\varphi)$ on the left is equivalent to the divisibility by the same element on the right. We will actually prove the divisibility on the right. By the definition (2.5) we have

$$F_\omega(z_1,\dots,z_n) = \prod_{(p,q)}^{\longrightarrow} f_{pq}(c_p+z_p, c_q+z_q) \cdot \sigma_l\big(F_\varphi(z_{l+1},\dots,z_n)\big) \tag{2.8}$$

where the pairs (p,q) are ordered lexicigraphically, and precede the pair $(l+1,l+2)$. According to our proof of Proposition 2.2, the product over the pairs (p,q) in (2.8) maybe replaced by a certain rational function of $z_1,\dots,z_n$ with the restriction to $\mathcal{D}_\omega$ regular at $(0,\dots,0)$, without affecting the values of the restriction. Now the divisibility of F_ω by $\sigma_l(F_\varphi)$ on the right follows from the decomposition (2.8) and the definition of the element $F_\varphi \in \mathbb{C}\cdot S_{n-l}$ $\square$

Let $m = 1,\dots,n-1$. Regard the group ring $\mathbb{C}\cdot S_m$ as a subalgebra in $\mathbb{C}\cdot S_n$ using the standard embedding $S_m \to S_n$. Define a skew Young diagram ψ as the shape of the tableau obtained from Ω by removing each of the numbers $m+1,\dots,n$. Consider the corresponding element $F_\psi \in \mathbb{C}\cdot S_m$. The following property of the element F_ω is obtained by combining the arguments of Propositions 2.2 and 2.6.

Proposition 2.7. *The element $F_\omega \in \mathbb{C}\cdot S_n$ is divisible by F_ψ on the left and right.*

Proof. Let us introduce another expression for restriction of $F_\omega(z_1,\dots,z_n)$ to $\mathcal{D}_\omega$ which is again manifestly regular at $(0,\dots,0)$. Reorder the pairs (p,q) in the product (2.5) as follows; this reordering will not affect the values of the product due to (2.2) and (2.3). The pair (s,t) will precede the pair (p,q) if $t<q$ or if $t=q$, $s<p$. Let us now write $(s,t) \prec (p,q)$ referring to this new ordering.

Let any singular pair (p,q) be fixed. In our new ordering, the pair immediately after (p,q) is $(p{+}1,q)$. Moreover, the number $p{+}1$ stands just below p in the tableau Ω. Therefore, $z_p = z_{p+1}$ on $\mathcal{D}_\omega$. We also have $(p,p{+}1) \prec (p,q)$, since $p < q$. By (2.2) and (2.3) the product

$$\prod_{(s,t)\prec(p,q)}^{\longrightarrow} f_{st}(c_s + z_s\,, c_t + z_t)$$

over the pairs (s,t) is divisible on the right by the factor $f_{p,p+1}(c_p + z_p\,, c_{p+1} + z_{p+1})$. Restriction of this factor to $z_p = z_{p+1}$ is just $1-(p,p+1)$ since $c_{p+1} = c_p - 1$.

The element $\bigl(1-(p,p+1)\bigr)/2 \in \mathbb{C}\cdot S_n$ is an idempotent. Now for every singular pair (p,q) let us replace the two adjacent factors in the reordered product (2.5)

$$f_{pq}(c_p + z_p\,, c_q + z_q)\, f_{p+1,q}(c_{p+1} + z_{p+1}\,, c_q + z_q)$$

by

$$f_{p,p+1}(c_p + z_p\,, c_{p+1} + z_{p+1})\, f_{pq}(c_p + z_p\,, c_q + z_q)\, f_{p+1,q}(c_{p+1} + z_{p+1}\,, c_q + z_q)/2\,.$$

This does not affect the values of the restriction of (2.5) to $\mathcal{D}_\omega$. But the restriction to $z_p = z_{p+1}$ of the replacement product is regular at $z_p = z_q$ by Lemma 2.1.

Due to Proposition 2.4 the divisibility of F_ω by the element F_ψ on the left is equivalent to the divisibility by the same element on the right. Let us now prove the divisibility on the left. In our new ordering of the factors in (2.5), we have

$$F_\omega(z_1\,,\ldots,z_n) = F_\psi(z_1\,,\ldots,z_m)\,\cdot \prod_{(p,q)}^{\longrightarrow} f_{pq}(c_p + z_p\,, c_q + z_q) \tag{2.9}$$

where $(p,q) \succ (m{-}1,m)$. By the above argument, in (2.9) the product over the pairs (p,q) may be replaced by a certain rational function of $z_1\,,\ldots,z_n$ with the restriction to $\mathcal{D}_\omega$ regular at $(0,\ldots,0)$, without affecting the values of the restriction. Now the divisibility of F_ω by F_ψ on the left follows from the decomposition (2.9) and the definition of the element $F_\psi \in \mathbb{C}\cdot S_m$ $\square$

We can now give a relatively short proof of the main result of this section, cf. [N].

Theorem 2.8. *Suppose the numbers $p<q$ stand next to each other in the same row of the tableau Ω. Let r be the number at the bottom of the column of Ω containing p. Then the element F_ω is divisible on the left by the product*

$$\prod_{s=p,\ldots,r}^{\longleftarrow} \Bigl(\prod_{t=r+1,\ldots,q}^{\longrightarrow} f_{st}(c_s, c_t) \Bigr)\,. \tag{2.10}$$

Proof. It suffices to prove Theorem 2.8 only in the particular case $q=n$. For $q<n$ the required statement would then follow by Proposition 2.7 applied to $m=q$. Furthermore, it suffices to consider only the case $p=1$. For $p>1$ the required statement would then follow by Proposition 2.6 applied to $l=p-1$.

Assume that $p=1$ and $q=n$. Then the skew Young diagram ω consists of two columns only. Further, every row of ω but one consists of a single box. There is also a row of two boxes, which in the tableau Ω are filled with numbers 1 and n. So there is only one box on every diagonal of ω. The function $F_\omega(z_1, \ldots, z_n)$ is then regular at $(0, \ldots, 0)$ and takes the value

$$\prod_{s=1,\ldots,n-1}^{\longrightarrow}\Big(\prod_{t=s+1,\ldots,n}^{\longrightarrow} f_{st}(c_s,c_t)\Big) = \prod_{s=1,\ldots,r}^{\longleftarrow}\Big(\prod_{t=r+1,\ldots,n}^{\longrightarrow} f_{st}(c_s,c_t)\Big) \times$$
$$\prod_{s=1,\ldots,r-1}^{\longrightarrow}\Big(\prod_{t=s+1,\ldots,r}^{\longrightarrow} f_{st}(c_s,c_t)\Big) \cdot \prod_{s=r+1,\ldots,n}^{\longrightarrow}\Big(\prod_{t=s+1,\ldots,n}^{\longrightarrow} f_{st}(c_s,c_t)\Big);$$

the last equality was obtained by using the relations (2.2) and (2.3). So the element F_ω is indeed divisible on the left by the product (2.10) with $p = 1$ and $q = n$ $\square$

3. Leading term near the origin

Take any two non-empty skew Young diagrams ω and ω'. Let n and n' be the numbers of boxes in ω and ω' respectively. In this section we will introduce a certain rational function $F_{\omega\omega'}(u)$ of one complex variable u with the values in the symmetric group ring $\mathbb{C}\cdot S_{n+n'}$. Let z and z' be any two complex numbers. In Section 4, the leading term of the Laurent expansion of the function $F_{\omega\omega'}(u)$ at the point $u = z - z'$ will determine an intertwining operator of $\mathrm{Y}(\mathfrak{gl}_N)$-modules

$$V_\omega(z) \otimes V_{\omega'}(z') \longrightarrow V_\omega(z) \,\widetilde{\otimes}\, V_{\omega'}(z').$$

Here the tilde refers to the opposite comultiplication on the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$. In the present section we will compute the leading term of the Laurent expansion of $F_{\omega\omega'}(u)$ with $\omega = \omega'$ near the origin $u = 0$. This result will be used in Section 4.

Let us regard $\mathbb{C}\cdot S_n$ as a subalgebra in $\mathbb{C}\cdot S_{n+n'}$ via the standard embedding $S_n \to S_{n+n'}$. In particular, $F_\omega \in \mathbb{C}\cdot S_n$ may be regarded as an element of $\mathbb{C}\cdot S_{n+n'}$. Consider also the embedding $\mathbb{C}\cdot S_{n'} \to \mathbb{C}\cdot S_{n+n'}$ defined by $(pq) \mapsto (p+n, q+n)$ for distinct $p, q = 1, \ldots, n'$. In this section, the image in $\mathbb{C}\cdot S_{n+n'}$ of any element $F \in \mathbb{C}\cdot S_{n'}$ under the latter embedding will be denoted by $F^\vee$.

Let $c_1, \ldots, c_n$ and $c'_1, \ldots, c'_{n'}$ be the sequences of contents corresponding to the skew diagrams ω and ω'. Here we order the contents using the column tableaux. Define the function $F_{\omega\omega'}(u)$ as the ordered product

$$F_\omega \cdot \prod_{p=1,\ldots,n}^{\longrightarrow}\Big(\prod_{q=1,\ldots,n'}^{\longrightarrow} f_{p,q+n}(c_p+u, c'_q)\Big) \cdot F^\vee_{\omega'}. \tag{3.1}$$

Note that the so defined rational function $F_{\omega\omega'}(u)$ may have poles only at $u \in \mathbb{Z}$.

Proposition 3.1. *The function $F_{\omega\omega'}(u)$ can be written as either of the products*

$$\prod_{p=1,\ldots,n}^{\longleftarrow}\Big(\prod_{q=1,\ldots,n'}^{\longrightarrow} f_{p,q+n}(c_p+u, c'_q)\Big) \cdot F_\omega F^\vee_{\omega'}, \tag{3.2}$$

$$F_\omega F^\vee_{\omega'} \cdot \prod_{p=1,\ldots,n}^{\longrightarrow}\Big(\prod_{q=1,\ldots,n'}^{\longleftarrow} f_{p,q+n}(c_p+u, c'_q)\Big). \tag{3.3}$$

Proof. Using the definition (2.5) along with the relations (2.2) and (2.3), we obtain the equality of the rational functions in $z_1, \ldots, z_n$ and u

$$F_\omega(z_1, \ldots, z_n) \cdot \prod_{p=1,\ldots,n}^{\longrightarrow} \Bigl(\prod_{q=1,\ldots,n'}^{\longrightarrow} f_{p,q+n}(c_p + z_p + u, c'_q) \Bigr) =$$

$$\prod_{p=1,\ldots,n}^{\longleftarrow} \Bigl(\prod_{q=1,\ldots,n'}^{\longrightarrow} f_{p,q+n}(c_p + z_p + u, c'_q) \Bigr) \cdot F_\omega(z_1, \ldots, z_n) .$$

By the definition of the element $F_\omega \in \mathbb{C} \cdot S_n$, this equality implies that the products (3.1) and (3.2) are also equal to each other. Similarly, we can prove the equality of the products (3.1) and (3.3) $\square$

Now consider the function $F_{\omega\omega}(u)$. For any diagram $\omega^\flat$ obtained from $\omega \subset \mathbb{Z}^2$ by a translation, we have $F_{\omega^\flat\omega^\flat}(u) = F_{\omega\omega}(u)$. Our preliminary result on $F_{\omega\omega}(u)$ is

Proposition 3.2. *The order of the pole of $F_{\omega\omega}(u)$ at $u=0$ does not exceed $d(\omega)$.*

Proof. We will use the expression for $d(\omega)$ given by Proposition 1.3. Our argument will be similar to that of Proposition 2.2, but more elaborate, cf. [N, Theorem 4.1].

The function $F_{\omega\omega}(u)$ is defined as the ordered product

$$F_\omega \cdot \prod_{p=1,\ldots,n}^{\longrightarrow} \Bigl(\prod_{q=1,\ldots,n}^{\longrightarrow} f_{p,q+n}(c_p + u, c_q) \Bigr) \cdot F_\omega^\vee . \tag{3.4}$$

The factor $f_{p,q+n}(c_p + u, c_q)$ in (3.4) has a pole at $u = 0$ if and only if the numbers p and q stand on the same diagonal of the tableau Ω. This pole is simple. Again, we will then call the pair (p,q) singular. Any singular pair (p,q) belongs to one of the following three types:

(i) the number p is not at the bottom of its column in Ω;
(ii) p is at the bottom of its column in Ω but q is not leftmost in its row;
(iii) p is at the bottom of its column in Ω and q is leftmost in its row.

Observe that the total number of singular pairs of type (iii) is exactly $d(\omega)$. Indeed, let p be at the bottom of the j-th column. Then $c_p = j - \lambda^*_j$. Consider the diagonal of the tableau Ω containing p. Any number q on this diagonal, except maybe for the first number, has in Ω a neighbour on the left. As usual, here we are reading the diagonal from top left to bottom right. Suppose q is the first on the diagonal. The total number of such pairs (p,q) equals the number of the columns, which is $\ell(\omega^*)$. But if q has a neighbour on the left, say in the i-th row, then $i - \mu^*_i \geqslant c_q$. If we had $i - \mu^*_i > c_q$, the number q could not be the first on its diagonal. So we must have $i - \mu^*_i = c_q = c_p = j - \lambda^*_j$. Here we also have $i < j$. Thus the total number of singular pairs (iii) equals

$$\ell(\omega^*) - \#\{\, (i,j) \mid \lambda^*_j - j = \mu^*_i - i,\ i < j \,\} = d(\omega) .$$

We shall prove that when we estimate the order of pole of (3.4) at $u = 0$ from above, the singular pairs (i) and (ii) do not count. Proposition 3.2 will then follow.

I. Using only the relations (2.3), we can rewrite the ordered product (3.4) as

$$F_\omega \cdot \prod_{q=1,\ldots,n}^{\longrightarrow} \Big(\prod_{p=1,\ldots,n}^{\longrightarrow} f_{p,q+n}(c_p+u\,,c_q) \Big) \cdot F_\omega^\vee\,. \tag{3.5}$$

Consider any singular pair (p,q) of type (i). In the ordered product (3.5) the factor $f_{p,q+n}(c_p+u\,,c_q)$ is immediately followed by factor $f_{p+1,q+n}(c_{p+1}+u\,,c_q)$. Note that here $c_p=c_q$. The numbers p and $p+1$ stand in the same column of the tableau Ω. By Theorem 2.5, the element F_ω is divisible on the right by

$$f_{p,p+1}(c_p,c_{p+1}) = 1-(p,p+1) = f_{p,p+1}(c_p+u,c_{p+1}+u)\,.$$

Again using the relations (2.2) and (2.3) we obtain, that the product of all factors in (3.5) preceding $f_{p,q+n}(c_p+u\,,c_q)$ is also divisible by $f_{p,p+1}(c_p+u,c_{p+1}+u)$ on the right. But the product

$$f_{p,p+1}(c_p+u,c_{p+1}+u)\,f_{p,q+n}(c_p+u\,,c_q)\,f_{p+1,q+n}(c_{p+1}+u\,,c_q)$$

has no pole at $u=0$ by Lemma 2.1. Thus when we estimate the order of pole of (3.5) at $u=0$ from above, the singular pairs (i) do not count.

II. Now consider the diagonal of Ω containing the number n. Let $q_1,\ldots,q_d$ be all entries of Ω on that diagonal. We assume that $q_1<\ldots<q_d$ so that $q_d=n$. The singular pairs $(n,q_2),\ldots,(n,q_d)$ are all of type (ii). The singular pair (n,q_1) is of type (ii) or (iii). Depending on this, let the index k range over $1,\ldots,d$ or over $2,\ldots,d$ respectively. For every such k let p_k be the number standing in Ω next to the left of q_k. Of course, for $k>1$ we have $p_k=q_{k-1}+1$. However, to include the possible case $k=1$ in our argument, we will still use the notation p_k.

Let r_k be the number standing in the tableau Ω at the bottom of the column containing p_k. The number at the top of the column containing q_k is then r_k+1. By Theorem 2.8, the element $F_\omega\in\mathbb{C}\cdot S_n$ is divisible on the left by the product

$$\prod_{s=p_k,\ldots,r_k}^{\longleftarrow} \Big(\prod_{t=r_k+1,\ldots,q_k}^{\longrightarrow} f_{st}(c_s,c_t) \Big) = \prod_{t=r_k+1,\ldots,q_k}^{\longrightarrow} \Big(\prod_{s=p_k,\ldots,r_k}^{\longleftarrow} f_{st}(c_s,c_t) \Big)\,. \tag{3.6}$$

For distinct indices k the elements of $\mathbb{C}\cdot S_n$ defined by (3.6) pairwise commute. Let F be the element of $\mathbb{C}\cdot S_n$ obtained by multiplying the elements (3.6) over all k. We can write $F_\omega=FG$ for some $G\in\mathbb{C}\cdot S_n$. We can also write $F_\omega^\vee=F^\vee G^\vee$ in $\mathbb{C}\cdot S_{2n}$.

Put $r=r_d$ for short. Using the relations (2.3) we can rewrite the product (3.5) as

$$F_\omega \cdot \prod_{q=1,\ldots,n}^{\longrightarrow} \Big(\prod_{p=1,\ldots,r}^{\longrightarrow} f_{p,q+n}(c_p+u\,,c_q) \Big) \times \prod_{q=1,\ldots,n}^{\longrightarrow} \Big(\prod_{p=r+1,\ldots,n}^{\longrightarrow} f_{p,q+n}(c_p+u\,,c_q) \Big) \cdot F_\omega^\vee\,. \tag{3.7}$$

Denote by $\mathcal{Q}$ the sequence obtained from the sequence $1,\ldots,n$ by transposing its segments $p_k,\ldots,r_k$ and $r_k{+}1,\ldots,q_k$ for every possible index k. Using the relations (2.2), (2.3) and the decomposition $F^{\vee}_{\omega}=F^{\vee}G^{\vee}$, we can rewrite (3.7) as

$$F_{\omega}\cdot\prod_{q=1,\ldots,n}^{\longrightarrow}\Bigl(\prod_{p=1,\ldots,r}^{\longrightarrow} f_{p,q+n}(c_p+u,c_q)\Bigr)\times \tag{3.8}$$
$$F^{\vee}\cdot\prod_{q\in\mathcal{Q}}^{\longrightarrow}\Bigl(\prod_{p=r+1,\ldots,n}^{\longrightarrow} f_{p,q+n}(c_p+u,c_q)\Bigr)\cdot G^{\vee}.$$

As well as in the product (3.5), here for every singular pair (p,q) of type (i) the factor $f_{p,q+n}(c_p+u,c_q)$ is still followed by factor $f_{p+1,q+n}(c_{p+1}+u,c_q)$.

Let us now take any singular pair (n,q_k) of type (ii). In the second line of the ordered product (3.8), consider the sequence of factors

$$\prod_{q=q_k,\,p_k}^{\longrightarrow}\Bigl(\prod_{p=r+1,\ldots,n}^{\longrightarrow} f_{p,q+n}(c_p+u,c_q)\Bigr). \tag{3.9}$$

Note that $c_p=c_q$ in (3.9) only if $(p,q)=(n,q_k)$. The rightmost factor in the product (3.6) is $f_{p_kq_k}(c_{p_k},c_{q_k})$. Due to (2.2) and (2.3), the product of all factors in (3.8) preceding the sequence (3.9) is divisible by $f_{p_k+n,q_k+n}(c_{p_k},c_{q_k})$ on the right. But the cumulate product

$$f_{p_k+n,q_k+n}(c_{p_k},c_{q_k})\cdot\prod_{q=q_k,\,p_k}^{\longrightarrow}\Bigl(\prod_{p=r+1,\ldots,n}^{\longrightarrow} f_{p,q+n}(c_p+u,c_q)\Bigr)=$$
$$\prod_{q=p_k,\,q_k}^{\longrightarrow}\Bigl(\prod_{p=r+1,\ldots,n-1}^{\longrightarrow} f_{p,q+n}(c_p+u,c_q)\Bigr)\times$$
$$f_{n,p_k+n}(c_n+u,c_{p_k})\,f_{n,q_k+n}(c_n+u,c_{q_k})\,f_{p_k+n,q_k+n}(c_{p_k},c_{q_k})$$

has no pole at $u=0$ due to Lemma 2.1. Thus when we estimate the order of pole of (3.8) at $u=0$ from above, any singular pair (n,q_k) of type (ii) does not count.

III. By rewriting the product (3.7) in its second line as (3.8), we excluded from our count all singular pairs (p,q) of type (ii) with $p=n$. If the skew Young diagram ω consists of only one column, there is no singular pairs of type (ii) and the proof finishes on step I already. Suppose that the diagram ω has more than one column. Now let ψ be the skew Young diagram obtained from ω by removing the last non-empty column. Let m be the total number of boxes in ψ. Due to Proposition 2.7, the element $F^{\vee}_{\omega}$ is divisible by $F^{\vee}_{\psi}$ on the left. But we also have

$$\prod_{q=1,\ldots,n}^{\longrightarrow}\Bigl(\prod_{p=r+1,\ldots,n}^{\longrightarrow} f_{p,q+n}(c_p+u,c_q)\Bigr)\cdot F^{\vee}_{\psi}\;=\;F^{\vee}_{\psi}\times$$
$$\prod_{q=m,\ldots,1,m+1,\ldots,n}^{\longrightarrow}\Bigl(\prod_{p=r+1,\ldots,n}^{\longrightarrow} f_{p,q+n}(c_p+u,c_q)\Bigr),$$

see Proposition 3.1. Hence the entire product in the second line of (3.7) is divisible on the left by $F_\psi^\vee$. Now arguing similarly to step II, we exclude from our count all singular pairs (p,q) of type (ii) with $p=m$. Continuing this argument, we exclude from our count all singular pairs of type (ii). All singular pairs of type (i) can be still excluded from our count, as it was done on step I $\square$

For the skew Young diagram ω, let us introduce the rational function of u

$$h_\omega(u)=\left(\frac{-1}{u}\right)^n\cdot\prod_{1\leqslant p<q\leqslant n}\left(1-\frac{1}{(u-c_p+c_q)^2}\right). \tag{3.10}$$

Recall once again, that here the contents $c_1,\dots,c_n$ of ω are ordered by using the column tableau Ω. Now consider the skew Young diagram ω^* conjugate to ω, and the corresponding rational function $h_{\omega^*}(u)$. In general, we do not have the equality $h_{\omega^*}(u)=(-1)^n\cdot h_\omega(-u)$. But the following proposition is always true.

Proposition 3.3. *The leading term of the Laurent expansion at $u=0$ of the function $h_{\omega^*}(u)$ coincides with that of the function $(-1)^n\cdot h_\omega(-u)$.*

Proof. Consider the sequence of numbers obtained by reading the entries of the column tableau Ω in the natural way, that is by rows downwards, from left to right in every row. This sequence can be also obtained from the sequence $1,\dots,n$ by a certain permutation $g\in S_n$. In general, the permutation g is non-trivial. The sequence of contents corresponding to the diagram ω^* is $-c_{g(1)},\dots,-c_{g(n)}$. So

$$(-1)^n\cdot h_{\omega^*}(-u)=\left(\frac{-1}{u}\right)^n\cdot\prod_{1\leqslant p<q\leqslant n}\left(1-\frac{1}{(u-c_{g(p)}+c_{g(q)})^2}\right).$$

Dividing the right hand side of this equality by that of (3.10), we get the product of

$$\left(1-\frac{1}{(u+c_p-c_q)^2}\right)\cdot\left(1-\frac{1}{(u-c_p+c_q)^2}\right)^{-1} \tag{3.11}$$

over all pairs (p,q) such that $p<q$ but $g^{-1}(p)>g^{-1}(q)$. For any such pair, the number q stands above and to the right of p in the tableau Ω. In particular, then $c_p-c_q\neq\pm1$. But then the factor (3.11) is regular at $u=0$, and takes the value 1 at that point $\square$

The next proposition provides another expression for the function $h_\omega(u)$ with the arbitrary skew Young diagram $\omega=\lambda/\mu$. Consider the infinite product

$$\prod_i\frac{u-\lambda_i+\mu_i}{u}\cdot\prod_{i<j}\frac{(u+\lambda_i-\mu_j-i+j)\,(u+\mu_i-\lambda_j-i+j)}{(u+\lambda_i-\lambda_j-i+j)\,(u+\mu_i-\mu_j-i+j)} \tag{3.12}$$

where $i,j=1,2,\dots$. Here the first fraction equals 1 for all but a finite number of indices i. The second fraction equals 1 for all but a finite number of indices i and j. Hence the product (3.12) determines a rational function of the variable u.

Proposition 3.4. *The rational function* (3.12) *is equal to* $h_{\omega^*}(u)\cdot(1-u)^n$.

Proof. For each $i=1,2,\dots$ denote by $h_i(u)$ the product of the factors

$$1-\frac{1}{(u-c_p+c_q)^2}\;=\;\frac{u-c_p+c_q-1}{u-c_p+c_q}\cdot\frac{u-c_p+c_q+1}{u-c_p+c_q} \tag{3.13}$$

over $p<q$ such that the numbers p and q stand in the i-th column of the tableau Ω. If this column is empty, we put $h_i(u)=1$. For these numbers p and q we have $c_p-c_q=q-p$. The total number of boxes in this column is $\lambda_i^*-\mu_i^*$. So by direct calcultion

$$h_i(u)\;=\;\frac{(u-\lambda_i^*+\mu_i^*)\,u^{\lambda_i^*-\mu_i^*-1}}{(u-1)^{\lambda_i^*-\mu_i^*}}\,.$$

Further, for each i put $a_i=i-\mu_i^*-1$ and $b_i=i-\lambda_i^*$. The numbers a_i and b_i are the contents respectively of the top and the bottom boxes of the i-th column of the diagram ω, if this column is not empty. If it is empty, $\lambda_i^*=\mu_i^*$ and $b_i=a_i+1$.

Now for $i<j$ denote by $h_{ij}(u)$ the product of all those factors (3.13), where the numbers p and q stand respectively in the i-th and the j-th columns of Ω. If at least one of these two columns is empty, we put $h_{ij}(u)=1$. By direct calculation,

$$h_{ij}(u)\;=\;\frac{u-a_i+b_j-1}{u-a_i+a_j}\cdot\frac{u-b_i+a_j+1}{u-b_i+b_j}\,.$$

Note that we also have $\lambda_1^*-\mu_1^*+\lambda_2^*-\mu_2^*+\ldots=n$. So by the definition (3.10)

$$h_\omega(u)\cdot(1-u)^n\;=\;\left(\frac{u-1}{u}\right)^{n}\cdot\prod_i h_i(u)\cdot\prod_{i<j}h_{ij}(u)\;=$$

$$\prod_i\frac{u-\lambda_i^*+\mu_i^*}{u}\cdot\prod_{i<j}\frac{(u+\mu_i^*-\lambda_j^*-i+j)\,(u+\lambda_i^*-\mu_j^*-i+j)}{(u+\mu_i^*-\mu_j^*-i+j)\,(u+\lambda_i^*-\lambda_j^*-i+j)}\,.$$

Replacing here the diagram ω by its conjugate ω^*, we get Proposition 3.4 $\square$

The total number of indices i such that $\lambda_i\neq\mu_i$, is the number $\ell(\omega)$ of non-empty rows in the skew Young diagram ω. For any $i<j$ we also have the inequalities

$$\lambda_i-i>\mu_j-j\,,\quad \lambda_i-i>\lambda_j-j\,,\quad \mu_i-i>\mu_j-j\,.$$

Therefore, the order of the pole of the rational function (3.12) at $u=0$ is

$$\ell(\omega)-\#\{\,(i,j)\mid\lambda_j-j=\mu_i-i,\ i<j\,\}\;=\;d(\omega)\;=\;d(\omega^*)\,,$$

see Proposition 1.2. Now Proposition 3.4 implies that the order of the pole of the function $h_\omega(u)$ at $u=0$ is also $d(\omega)$. Denote by $c(\omega)$ the coefficient at $u^{-d(\omega)}$ in the Laurent expansion of $h_\omega(u)$ at $u=0$. By Proposition 3.3, we have

$$c(\omega^*)\;=\;c(\omega)\cdot(-1)^{n+d(\omega)}\,. \tag{3.14}$$

Here is the main result of this section. We will employ this result in Section 4.

Theorem 3.5. *The leading term of the Laurent expansion at* $u=0$ *of* $F_{\omega\omega}(u)$ *is*

$$c(\omega)\cdot(1,n+1)\,(2,n+2)\,\ldots\,(n,2n)\cdot F_{\omega}\,F_{\omega}^{\vee}\,u^{-d(\omega)}\,. \tag{3.15}$$

Proof. We will use induction on n, the total number of boxes in ω. If $n=1$, then

$$F_{\omega}=1,\quad F_{\omega}^{\vee}=1\quad\text{and}\quad F_{\omega\omega}(u)=1-(12)/u\,.$$

But for $n=1$ we have $h_{\omega}(u)=-1/u$ by the definition (3.10). So the statement of Theorem 3.5 is obvious in this case. From now on we will assume that $n>1$.

Suppose that the diagram ω consists of one column only. Then the diagram ω^{*} consists of only one row. It follows from Proposition 2.3 and Theorems 2.5, 2.7 that then

$$F_{\omega}=\sum_{g\in S_n}\operatorname{sgn}(g)\cdot g\quad\text{and}\quad F_{\omega^{*}}=\sum_{g\in S_n}g$$

where $\operatorname{sgn}(g)=\pm1$ depending on whether the permutation g is even or odd. Now denote by β the involutive automorphism of the group ring $\mathbb{C}\cdot S_{2n}$ defined by $\beta(g)=\operatorname{sgn}(g)\cdot g$ for every $g\in S_{2n}$. Then we have $\beta\big(F_{\omega\omega}(u)\big)=F_{\omega^{*}\omega^{*}}(-u)$ by the definition (3.4). On the other hand, the image of (3.15) under β is

$$(-1)^{n}\,c(\omega)\cdot(1,n+1)\,(2,n+2)\,\ldots\,(n,2n)\cdot F_{\omega^{*}}F_{\omega^{*}}^{\vee}\,u^{-d(\omega)}\,.$$

Using the equality (3.14) we see, that the statements of Theorem 3.5 for ω and ω^{*} are equivalent in the case, when ω consists of one column only. Note that then $d(\omega)=1$, but we did not use the last equality here. We only used $F_{\omega^{*}}=\beta(F_{\omega})$.

From now on we will assume that there is more than one column in the skew Young diagram ω. Let δ be the skew Young diagram consisting of the first non-empty column of ω. Let φ the diagram obtained from ω by removing this column, the diagram φ is also non-empty. Each of the diagrams δ and φ has less than n boxes. We will assume that the statement of Theorem 3.5 is valid for each of the diagrams δ and φ instead of ω.

I. Let l be the number of boxes in the column δ. As well as in Section 2, denote by σ_l the embedding $\mathbb{C}\cdot S_{n-l}\to\mathbb{C}\cdot S_n$ defined by $\sigma_l:(pq)\mapsto(p+l,q+l)$ for all distinct $p,q=1,\ldots,n-l$. Using (3.4) with (2.3), we can write $F_{\omega\omega}(u)$ as

$$F_{\omega}\cdot A(u)\,B(u)\,C(u)\,D(u)\cdot F_{\omega}^{\vee} \tag{3.16}$$

where $A(u),B(u),C(u),D(u)$ are ordered products of the factors $f_{p,q+n}(c_p+u,c_q)$ respectively over

$$p,q=1,\ldots,l;$$
$$p=1,\ldots,l\quad\text{and}\quad q=l+1,\ldots,n;$$
$$p=l+1,\ldots,n\quad\text{and}\quad q=1,\ldots,l;$$
$$p,q=l+1,\ldots,n\,.$$

The pairs (p,q) in each of these four products are ordered lexicographically. By Propositions 2.6 and 2.7, the element $F_{\omega}\in\mathbb{C}\cdot S_n$ is divisible on the left and on the right by $\sigma_l(F_{\varphi})$ and F_{δ}. The element $F_{\delta}/l!\in\mathbb{C}\cdot S_l$ is an idempotent. In general, there maybe no non-zero idempotents in the subset $\mathbb{C}\cdot F_\varphi \subset \mathbb{C}\cdot S_{n-l}$. But there is always an element $I_\varphi \in \mathbb{C}\cdot S_{n-l}$ divisible on the left by F_φ , such that $I_\varphi F_\varphi = F_\varphi$. By Proposition 2.4 applied to F_φ , this follows from the semisimplicity of $\mathbb{C}\cdot S_{n-l}$. Due to Proposition 2.3, the element $J_\varphi = \alpha(I_\varphi)$ is divisible by F_φ on the right, and we have $F_\varphi J_\varphi = F_\varphi$. Using Proposition 3.1, we can now rewrite (3.16) as

$$F_\omega \cdot F_\delta\, A(u)\, F_\delta^\vee \cdot F_\delta\, B(u)\, \sigma_l(I_\varphi)^\vee \cdot \sigma_l(J_\varphi)\, C(u)\, D(u)\, F_\omega^\vee \cdot (l\,!)^{-3}\,. \tag{3.17}$$

II. By the inductive assumption for δ , the factor $F_\delta\, A(u)\, F_\delta^\vee$ in (3.17) has a simple pole at $u=0$, and the corresponding residue coincides with that of

$$h_\delta(u)\cdot(1,n+1)\ldots(l,n+l)\cdot F_\delta\, F_\delta^\vee\,. \tag{3.18}$$

Further, $F_\delta\, B(u)\, \sigma_l(F_\varphi)^\vee$ is conjugate by the involution $(l+1,n+l+1)\ldots(n,2n)$ to

$$F(u) \;=\; F_\delta\cdot \prod_{p=1,\ldots,l}^{\longrightarrow}\Bigl(\;\prod_{q=l+1,\ldots,n}^{\longrightarrow} f_{pq}(\,c_p+u\,,\,c_q)\Bigr)\cdot \sigma_l(F_\varphi)\,. \tag{3.19}$$

The function $F(u)$ is regular at $u=0$, and $F(0)=F_\omega$. This follows by setting $z_1=\ldots=z_l=u$ in Proposition 2.2. In particular, the factor $F_\delta\, B(u)\, \sigma_l(I_\varphi)^\vee$ in (3.17) is regular at $u=0$. We will show that the order of the pole at $u=0$ of the factor $\sigma_l(J_\varphi)\, C(u)\, D(u)\, F_\omega^\vee$ in (3.17) does not exceed $d(\omega)-1$. This will allow us to replace the factor $F_\delta\, A(u)\, F_\delta^\vee$ in (3.17) by the product (3.18), without affecting the coefficient at $u^{-d(\omega)}$ in the Laurent expansion at $u=0$. Independently of Proposition 3.2, this will also imply that the order of the pole at $u=0$ of $F_{\omega\omega}(u)$ does not exceed $d(\omega)$.

It is the argument of Proposition 3.2 that we will use here. In particular, for any numbers p and q standing on the same diagonal of Ω, let us keep calling the pair (p,q) singular. Any singular pair belongs to one of the types (i,ii,iii) as described in the beginning of the proof of Proposition 3.2. Note that the number l stands at the bottom of the first column in the tableau Ω . Consider the product

$$\sigma_l(J_\varphi)\, C(u)\, D(u)\, F_\omega^\vee \;=\; \sigma_l(J_\varphi)\cdot \prod_{p=l+1,\ldots,n}^{\longrightarrow}\Bigl(\;\prod_{q=1,\ldots,n}^{\longrightarrow} f_{p,q+n}(\,c_p+u\,,\,c_q)\Bigr)\cdot F_\omega^\vee\,.$$

Here the factor $f_{p,q+n}(\,c_p+u\,,\,c_q)$ has a pole at $u=0$ if and only if the pair (p,q) is singular. The factor $\sigma_l(J_\varphi)$ is divisible on the right by $\sigma_l(F_\varphi)$. By the argument of Proposition 3.2, when we estimate the order of pole of the above product at $u=0$ from above, the singular pairs of types (i) and (ii) do not count. The total number of the singular pairs (p,q) of type (iii) is $d(\omega)$. For any singular pair (p,q) of type (iii) we have $p>l$, except for the pair $(p,q)=(l,l)$. Hence in the above product, the number of singular pairs (p,q) of type (iii) is exactly $d(\omega)-1$.

III. Thus the coefficient at $u^{-d(\omega)}$ in the Laurent expansion at $u=0$ of the function $F_{\omega\omega}(u)$ coincides with that of the function

$$\begin{gathered} h_\delta(u)\, F_\omega\cdot(1,n+1)\ldots(l,n+l)\cdot F_\delta\, F_\delta^\vee\;\times \\ F_\delta\, B(u)\, \sigma_l(I_\varphi)^\vee\cdot\sigma_l(J_\varphi)\, C(u)\, D(u)\, F_\omega^\vee\cdot(l\,!)^{-3} \;=\; \\ h_\delta(u)\, F_\omega\cdot(1,n+1)\ldots(l,n+l)\cdot F_\delta\, B(u)\, \sigma_l(I_\varphi)^\vee\cdot C(u)\, D(u)\, F_\omega^\vee\cdot(l\,!)^{-1}\,. \end{gathered} \tag{3.20}$$

Without affecting that coefficient, we can also replace in (3.20) the constant factor F_ω by the function $F(-u)$; see (3.19). The function $F(-u)$ can be also written as

$$\sigma_l(F_\varphi)\cdot\prod_{p=1,\ldots,l}^{\longleftarrow}\Bigl(\prod_{q=l+1,\ldots,n}^{\longleftarrow} f_{pq}(c_p\,,c_q+u)\Bigr)\cdot F_\delta\,,$$

see Proposition 3.1. By exchanging the indices p,q and using the commutation relations (2.3), the last product can be rewritten as

$$\sigma_l(F_\varphi)\cdot\prod_{p=l+1,\ldots,n}^{\longleftarrow}\Bigl(\prod_{q=1,\ldots,l}^{\longleftarrow} f_{qp}(c_q\,,c_p+u)\Bigr)\cdot F_\delta\,. \tag{3.21}$$

Let us now denote by $E(u)$ the ordered product of the factors $f_{q+n,p}(c_q\,,c_p+u)$ over

$$p=l+1,\ldots,n\quad\text{and}\quad q=1,\ldots,l\,;$$

the pairs (p,q) being taken in the reversed lexicographical order. Using the relations (2.4) repeatedly, we obtain the equality

$$E(u)\,C(u)=\prod_{p=l+1}^{n}\,\prod_{q=1}^{l}\,\Bigl(1-\frac{1}{(u+c_p-c_q)^2}\Bigr)\,. \tag{3.22}$$

Denote by $h(u)$ the rational function of u at the right hand side of this equality.

The product (3.21) is conjugate by the involution $(1,n+1)\ldots(l,n+l)\in S_{2n}$ to $\sigma_l(F_\varphi)\,E(u)\,F_\delta^\vee$. By the above argument, the coefficient at $u^{-d(\omega)}$ in the Laurent expansion at $u=0$ of the function (3.20) coincides with that of the function

$$\begin{aligned}
&h_\delta(u)\,F(-u)\cdot(1,n+1)\ldots(l,n+l)\cdot F_\delta\,B(u)\,\sigma_l(I_\varphi)^\vee\cdot C(u)\,D(u)\,F_\omega^\vee\cdot(l!)^{-1}\ =\\
&h_\delta(u)\cdot(1,n+1)\ldots(l,n+l)\cdot F_\delta\,B(u)\,\sigma_l(I_\varphi)^\vee\cdot\sigma_l(F_\varphi)\,E(u)\,F_\delta^\vee\,C(u)\,D(u)\,F_\omega^\vee\cdot(l!)^{-1}\\
&=\ h_\delta(u)\,h(u)\cdot(1,n+1)\ldots(l,n+l)\cdot F_\delta\,B(u)\,\sigma_l(I_\varphi)^\vee\cdot\sigma_l(F_\varphi)\,D(u)\,F_\omega^\vee\,;
\end{aligned}$$

here we also used divisibility of the product of $F_\omega^\vee$ and of the product $C(u)\,D(u)\,F_\omega^\vee$ by $F_\delta^\vee$ on the left, along with the equality (3.22).

IV. The element $F_\omega^\vee$ is also divisible on the left by $\sigma_l(F_\varphi)^\vee$. By the inductive assumption for the diagram φ, the leading term in the Laurent expansion at $u=0$ of $\sigma_l(F_\varphi)\,D(u)\,\sigma_l(F_\varphi)^\vee$ coincides with that of the function

$$h_\varphi(u)\cdot(l+1,n+l+1)\ldots(n,2n)\cdot\sigma_l(F_\varphi)\,\sigma_l(F_\varphi)^\vee\,.$$

Therefore, by the result of step III, the coefficient at $u^{-d(\omega)}$ in the Laurent expansion at $u=0$ of the function $F_{\omega\omega}(u)$ coincides with that of the function

$$\begin{aligned}
&h_\delta(u)\,h(u)\,h_\varphi(u)\cdot(1,n+1)\ldots(l,n+l)\ \times\\
&F_\delta\,B(u)\,\sigma_l(I_\varphi)^\vee\cdot(l+1,n+l+1)\ldots(n,2n)\cdot\sigma_l(F_\varphi)\,F_\omega^\vee\ =\\
&h_\delta(u)\,h(u)\,h_\varphi(u)\cdot(1,n+1)\ldots(l,n+l)\cdot(l+1,n+l+1)\ldots(n,2n)\cdot F(u)\,F_\omega^\vee\,;
\end{aligned}$$

see the definition (3.19). But here we have $h_\delta(u)\,h(u)\,h_\varphi(u) = h_\omega(u)$ by (3.10). The statement of Theorem 3.5 for the skew Young diagram ω now follows from the equality $F(0) = F_\omega$. This equality has been already obtained on step II $\square$

We will complete this section with one remark on the ordered product

$$\prod_{p=1,\dots,n}^{\longleftarrow}\Bigl(\prod_{q=1,\dots,n'}^{\longrightarrow} f_{p,q+n}(c_p+u\,,c'_q)\Bigr) \tag{3.23}$$

for two non-empty skew Young diagrams ω and ω'. This product appeared in (3.2). Denote the product (3.23) by $G_{\omega\omega'}(u)$, this is a rational function of u which may have poles only at $u\in\mathbb{Z}$. By Proposition 3.1, we have $F_{\omega\omega'}(u) = G_{\omega\omega'}(u)\,F_\omega\,F^\vee_{\omega'}$. Let us now extend the notation $F^\vee$ used earlier in this section for the elements $G\in\mathbb{C}\cdot S_{n'}$ as follows. For any $G\in\mathbb{C}\cdot S_{n+n'}$ denote by $G^\vee$ the element $g\,G g^{-1}$ conjugate to G by the permutation from $S_{n+n'}$

$$g:(1,\dots,n',n'+1,\dots,n+n')\,\mapsto\,(n+1,\dots,n+n',1,\dots,n)\,.$$

Proposition 3.6. *We have the equality*

$$G_{\omega\omega'}(u)\,G^\vee_{\omega'\omega}(-u)\;=\;\prod_{p=1}^{n}\prod_{q=1}^{n'}\left(1-\frac{1}{(u+c_p-c'_q)^2}\right).$$

Proof. By the definition of the function $G_{\omega'\omega}(u)$, we have

$$\begin{aligned} G^\vee_{\omega'\omega}(-u)\;&=\;\prod_{p=1,\dots,n'}^{\longleftarrow}\Bigl(\prod_{q=1,\dots,n}^{\longrightarrow} f_{p+n,q}(c'_p-u\,,c_q)\Bigr)\\ &=\;\prod_{p=1,\dots,n}^{\longrightarrow}\Bigl(\prod_{q=1,\dots,n'}^{\longleftarrow} f_{q+n,p}(c'_q\,,c_p+u)\Bigr)\,; \end{aligned}$$

here the second equality has been obtained by exchanging the indices p,q and using the relations (2.3). Now the required statement can be derived from the definition (3.23) of the function $G_{\omega\omega'}(u)$ by using the relations (2.4) $\square$

4. Irreducibility of Yangian modules

We will begin this section with recalling the definition of the Yangian of the general linear Lie algebra $\mathfrak{gl}_N$. This is the unital associative algebra $\mathrm{Y}(\mathfrak{gl}_N)$ over $\mathbb{C}$, with the infinite family of generators $T_{ij}^{(s)}$ where $s=1,2,\dots$ and $i,j=1,\dots,N$. The defining relations can be written in terms of the generating series (0.1) as

$$(u-v)\cdot[\,T_{ij}(u)\,,T_{kl}(v)\,]\,=\,T_{kj}(u)\,T_{il}(v)-T_{kj}(v)\,T_{il}(u)\,.$$

The Yangian $\mathrm{Y}(\mathfrak{gl}_N)$ is a Hopf algebra. Coproduct $\Delta:\mathrm{Y}(\mathfrak{gl}_N)\to\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$ is given by

$$\Delta\bigl(T_{ij}(u)\bigr)\,=\,\sum_{k=1}^{N}\,T_{ik}(u)\otimes T_{kj}(u)\,.$$

The opposite coproduct $\widetilde{\Delta}:\mathrm{Y}(\mathfrak{gl}_N)\to\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$ is the composition of the coproduct Δ and subsequent transposition of tensor factors in $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)$. For any two $\mathrm{Y}(\mathfrak{gl}_N)$-modules U and V we will denote by $U\otimes V$ and $U\mathbin{\widetilde{\otimes}}V$ the Yangian modules, where the action of $\mathrm{Y}(\mathfrak{gl}_N)$ on the tensor product of vector spaces is determined via by the coproduct Δ and the opposite coproduct $\widetilde{\Delta}$ respectively.

Our definition implies, that for any $z\in\mathbb{C}$ the assignment $T_{ij}(u)\mapsto T_{ij}(u-z)$ defines an automorphism of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$. Here the formal series in $(u-z)^{-1}$ should be re-expanded in u^{-1}. Denote this automorphism by τ_z.

We will also need a matrix form of the definition of the Hopf algebra $\mathrm{Y}(\mathfrak{gl}_N)$. Let us introduce the following notation. Consider the tensor product of any unital associative algebras $\mathrm{A}_1\otimes\ldots\otimes\mathrm{A}_n$. For $p=1,\ldots,n$ let $\iota_p:\mathrm{A}_p\to\mathrm{A}_1\otimes\ldots\otimes\mathrm{A}_n$ be the embedding as the p-th tensor factor:

$$\iota_p(X)=1^{\otimes(p-1)}\otimes X\otimes 1^{\otimes(n-p)},\qquad X\in\mathrm{A}_p\,.$$

For $X\in\mathrm{A}_p$ and $Y\in\mathrm{A}_q$ where $p\neq q$, let us write $(X\otimes Y)^{(pq)}=\iota_p(X)\,\iota_q(Y)$. For any $Z\in\mathrm{A}_p\otimes\mathrm{A}_q$, determine the element $Z^{(pq)}\in\mathrm{A}_1\otimes\ldots\otimes\mathrm{A}_n$ by linearity. For any vector spaces $U_1,\ldots,U_n$ we identify the algebras $\mathrm{End}(U_1\otimes\ldots\otimes U_n)$ and $\mathrm{End}(U_1)\otimes\ldots\otimes\mathrm{End}(U_n)$.

Now take the vector space $\mathbb{C}^N=U$. For $i,j=1,\ldots,N$ let $E_{ij}\in\mathrm{End}(U)$ be the matrix units. Let $P:x\otimes y\mapsto y\otimes x$ be the flip map in $U\otimes U$. Put $R(u)=u-P$. Identifying $\mathrm{End}(U\otimes U)=\mathrm{End}(U)\otimes\mathrm{End}(U)$, we can write

$$R(u)\,=\,u\,-\sum_{i,j=1}^{N}E_{ij}\otimes E_{ji}\,. \tag{4.1}$$

Combine all series (0.1) into a series $T(u)$ with coefficients in $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(U)$:

$$T(u)=\sum_{i,j=1}^{N}T_{ij}(u)\otimes E_{ji}\,. \tag{4.2}$$

Then the defining relations of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ are equivalent to the relation

$$T^{(01)}(u)\,T^{(02)}(v)\,R^{(12)}(u-v)\;=\;R^{(12)}(u-v)\,T^{(02)}(v)T^{(01)}(u) \tag{4.3}$$

for the series with coefficients in $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(U)\otimes\mathrm{End}(U)$, where we label the tensor factors by the indices $0,1,2$.

Now label the tensor factors in $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(U)$ by 1,2,3 as usual. Put

$$T^{[k]}(u)=T^{(k3)}(u)\,;\qquad k=1,2. \tag{4.4}$$

Then we have the equality

$$(\Delta\otimes\mathrm{id})\big(T(u)\big)=T^{[2]}(u)\,T^{[1]}(u)\,. \tag{4.5}$$

We refer the reader to the survey [MNO] for more details on the definition of the Yangian $\mathrm{Y}(\mathfrak{gl}_N)$, and for description of its basic properties. Note that the series denoted by $T(u)$ in [MNO] differs from our series (4.2) by transposition $E_{ij}\mapsto E_{ji}$ in the second tensor factor.

Let $\omega=\lambda/\mu$ be a non-empty skew Young diagram. Let n be the number of boxes in ω. Take the vector space $U^{\otimes n}$. The symmetric group S_n acts naturally in this vector space, permuting the tensor factors. Denote by π_n the corresponding homomorphism $\mathbb{C}\cdot S_n\to\mathrm{End}(U^{\otimes n})$. In Section 2 we defined a certain element $F_\omega\in\mathbb{C}\cdot S_n$. Let the subspace $V_\omega\subset U^{\otimes n}$ be the image of the operator $\pi_n(F_\omega)$.

Proposition 4.1 : *$V_\omega \neq \{0\}$ if and only if the length of every column of ω does not exceed N.*

The results of the present article do not depend on this proposition. Hence we omit the proof, and refer to [C1,NT2] instead.

Now take two non-empty skew Young diagrams ω and ω' with n and n' boxes respectively. Let $c_1, \ldots, c_n$ and $c'_1, \ldots, c'_{n'}$ be the corresponding sequences of contents, ordered according to the column tableaux. Consider the rational function $G_{\omega\omega'}(u)$ with the values in $\mathbb{C}\cdot S_{n+n'}$, defined by (3.23). For any value of $u \notin \mathbb{Z}$ the operator $\pi_{n+n'}(G_{\omega\omega'}(u))$ is well defined, and by Proposition 3.1 preserves the subspace

$$V_\omega \otimes V_{\omega'} \subset U^{\otimes n} \otimes U^{\otimes n'} = U^{\otimes (n+n')}.$$

Let $R_{\omega\omega'}(u)$ be the value of the restriction of $\pi_{n+n'}(G_{\omega\omega'}(u))$ to $V_\omega \otimes V_{\omega'}$; $R_{\omega\omega'}(u)$ is a rational function of u valued in $\mathrm{End}(V_\omega) \otimes \mathrm{End}(V_{\omega'})$. Suppose that each of the vector spaces V_ω and $V_{\omega'}$ is non-zero, see Proposition 4.1. Then by Proposition 3.6

$$R_{\omega\omega'}(u)\, R^{(21)}_{\omega'\omega}(-u) \;=\; \prod_{p=1}^{n} \prod_{q=1}^{n'} \Bigl(1 - \frac{1}{(u + c_p - c'_q)^2}\Bigr). \tag{4.6}$$

Take one more non-empty skew Young diagram ω''. Let n'' be its number of boxes.

Proposition 4.2. *We have the equality of rational functions of u, v with values in the tensor product* $\mathrm{End}(V_\omega) \otimes \mathrm{End}(V_{\omega'}) \otimes \mathrm{End}(V_{\omega''})$

$$R^{(12)}_{\omega\omega'}(u-v)\, R^{(13)}_{\omega\omega''}(u)\, R^{(23)}_{\omega'\omega''}(v) \;=\; R^{(23)}_{\omega'\omega''}(v)\, R^{(13)}_{\omega\omega''}(u)\, R^{(12)}_{\omega\omega'}(u-v)\,. \tag{4.7}$$

Proof. The relation (4.7) for the functions $R_{\omega\omega'}(u), R_{\omega\omega''}(u), R_{\omega'\omega''}(u)$ follows from the respective relation for the functions $G_{\omega\omega'}(u), G_{\omega\omega''}(u), G_{\omega'\omega''}(u)$. The latter relation is an equality of rational functions of u, v with the values in the algebra $\mathbb{C}\cdot S_{n+n'+n''}$. That equality follows from (2.2) and (2.3), we will not write it here. By applying the homomorphism $\pi_{n+n'+n''}$ to both sides of that equality and taking the restriction to $V_\omega \otimes V_{\omega'} \otimes V_{\omega''} \subset U^{\otimes(n+n'+n'')}$, we obtain (4.7) $\square$

Denote by ε the Young diagram with one box $(1,1)$. Then $V_\varepsilon = U$. Note that $R_{\varepsilon\varepsilon}(u) = R(u)/u$. We will now define a family of modules over the algebra $\mathrm{Y}(\mathfrak{gl}_N)$, we call these modules elementary. Take any $z \in \mathbb{C}$. Consider the rational function $R_{\omega\varepsilon}(z-u)$ of the variable u. This function takes values in $\mathrm{End}(V_\omega) \otimes \mathrm{End}(U)$. This function is regular at $u = \infty$. Moreover, at $u = \infty$ it takes the value 1. Here we keep to the assumption $V_\omega \neq \{0\}$. Consider the Laurent expansion at $u = \infty$

$$R_{\omega\varepsilon}(z-u) \;=\; 1 \;+\; \sum_{i,j=1}^{N} u^{-s}\; E^{(s)}_{ij}(\omega, z) \otimes E_{ji}\,.$$

Here each tensor factor $E^{(s)}_{ij}(\omega, z)$ is a certain element of the algebra $\mathrm{End}(V_\omega)$.

Proposition 4.3. *The assignment for all* $s=1,2,\ldots$ *and* $i,j=1,\ldots,N$

$$T_{ij}^{(s)} \mapsto E_{ij}^{(s)}(\omega,z) \tag{4.8}$$

defines an $\mathrm{Y}(\mathfrak{gl}_N)$*-module structure on the vector space* V_ω *.*

Proof. We have to verify that the operators $E_{ij}^{(s)}(\omega,z)$ satisfy the defining relations of $\mathrm{Y}(\mathfrak{gl}_N)$. Let us rewrite the definition (4.8) in a matrix form. It will be

$$T(u) \mapsto R_{\omega\varepsilon}(z-u)\,. \tag{4.9}$$

In view of the defining relation (4.3), we have to verify the equality

$$R_{\omega\varepsilon}^{(01)}(z-u)\,R_{\omega\varepsilon}^{(02)}(z-v)\,R^{(12)}(u-v) \;=\; R^{(12)}(u-v)\,R_{\omega\varepsilon}^{(02)}(z-v)\,R_{\omega\varepsilon}^{(01)}(z-u)$$

of rational functions in u,v with values in $\mathrm{End}(V_\omega)\otimes\mathrm{End}(U)\otimes\mathrm{End}(U)$, where the tensor factors are labelled by the indices $0,1,2$. Changing these labels respectively to $1,2,3$ and taking into account that $R(u)=u\,R_{\varepsilon\varepsilon}(u)$, we come to verifying

$$R_{\omega\varepsilon}^{(12)}(z-u)\,R_{\omega\varepsilon}^{(13)}(z-v)\,R_{\varepsilon\varepsilon}^{(23)}(u-v) \;=\; R_{\varepsilon\varepsilon}^{(23)}(u-v)\,R_{\omega\varepsilon}^{(13)}(z-v)\,R_{\omega\varepsilon}^{(12)}(z-u)\,.$$

But the latter equality follows from Proposition 4.2 when $\omega'=\omega''=\varepsilon$ ☐

The $\mathrm{Y}(\mathfrak{gl}_N)$-module defined by Proposition 4.3 will be denoted by $V_\omega(z)$, and called an *elementary* module. Note that for any $z\in\mathbb{C}$ the $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_\omega(z)$ is obtained from the module $V_\omega(0)$ by pulling back through the automorphism τ_z of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$.

Let us now give another description of the elementary $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_\omega(z)$. First consider the module $V_\varepsilon(z)$. The vector space of this module is $U=\mathbb{C}^N$, and the action of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ in this space is determined by the assignment $T_{ij}^{(s)}\mapsto z^sE_{ij}$, see (4.1).

Proposition 4.4. *The action of the algebra* $\mathrm{Y}(\mathfrak{gl}_N)$ *in the tensor product of the modules* $V_\varepsilon(c_1+z)\otimes\ldots\otimes V_\varepsilon(c_n+z)$ *preserves the subspace* $V_\omega\subset U^{\otimes n}$*. The restriction of this action to* V_ω *gives exactly the* $\mathrm{Y}(\mathfrak{gl}_N)$*-module* $V_\omega(z)$*.*

Proof. Let us write the definition of the action of the algebra $\mathrm{Y}(\mathfrak{gl}_N)$ in $V_\varepsilon(z)$ in a matrix form. It will be $T(u)\mapsto R_{\varepsilon\varepsilon}(z-u)$. Then due to (4.5) the action of $\mathrm{Y}(\mathfrak{gl}_N)$ in the tensor product $V_\varepsilon(c_1+z)\otimes\ldots\otimes V_\varepsilon(c_n+z)$ is determined by the assignment

$$T(u)\;\mapsto\;\prod_{p=1,\ldots,n}^{\longleftarrow} R_{\varepsilon\varepsilon}^{(p,n+1)}(c_p+z-u)\,.$$

The right hand side of this assignment can be rewritten as

$$\prod_{p=1,\ldots,n}^{\longleftarrow}\pi_{n+1}\bigl(f_{p,n+1}(c_p+z-u,0)\bigr) \;=\; \pi_{n+1}\bigl(G_{\omega\varepsilon}(z-u)\bigr)\,.$$

We have already shown that $\pi_{n+1}\bigl(G_{\omega\varepsilon}(z-u)\bigr)$ preserves the subspace $V_\omega\otimes V_\varepsilon$ in $U^{\otimes n}\otimes U$. This means that the action of $\mathrm{Y}(\mathfrak{gl}_N)$ in $V_\varepsilon(c_1+z)\otimes\ldots\otimes V_\varepsilon(c_n+z)$

preserves the subspace $V_\omega \subset U^{\otimes n}$. Restriction of this action to V_ω is determined exactly by the assignment

$$T(u) \mapsto R_{\omega\varepsilon}(z-u)$$

because of the definition of $R_{\omega\varepsilon}(z-u)$. But the last assignment coincides with the matrix definition of the $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_\omega(z)$ $\square$

For any $z \in \mathbb{C}$ let us expand the rational function $R_{\omega\omega'}(u)$ into the Laurent series at $u = z$. Let $(u-z)^{-a_{\omega\omega'}(z)}\, I_{\omega\omega'}(z)$ be the leading term of this expansion. This defines an integer $a_{\omega\omega'}(z)$ and a non-zero linear operator $I_{\omega\omega'}(z)$ in $V_\omega \otimes V_{\omega'}$. Again, here we keep to the assumption that $V_\omega\,, V_{\omega'} \neq \{0\}$. The rational function $R_{\omega\omega'}(u)$ is regular at $u = z$ for any $z \notin \mathbb{Z}$. Then by (4.6), this function also does not vanish at $u = z$ for any $z \notin \mathbb{Z}$. Thus for any $z \notin \mathbb{Z}$ we have $a_{\omega\omega'}(z) = 0$ and $I_{\omega\omega'}(z) = R_{\omega\omega'}(z)$. Furthermore, the equality (4.6) implies

Proposition 4.5. *For any $z \in \mathbb{C}$ the product $I_{\omega\omega'}(z)\, I^{(21)}_{\omega'\omega}(-z) \in \mathrm{End}\,(V_\omega \otimes V_{\omega'})$ is proportional to the identity map. In particular, the maps $I_{\omega\omega'}(z)$ and $I_{\omega'\omega}(-z)$ are invertible simultaneously.*

The basic property of the linear operator $I_{\omega\omega'}(z)$ is given by the next proposition.

Proposition 4.6. *For any $z, z' \in \mathbb{C}$ the element $I_{\omega\omega'}(z-z') \in \mathrm{End}\,(V_\omega \otimes V_{\omega'})$ is an intertwiner of the Yangian modules $V_\omega(z) \otimes V_{\omega'}(z') \longrightarrow V_\omega(z)\, \widetilde{\otimes}\, V_{\omega'}(z')$.*

Proof. The formula (4.7) implies the following equality of rational functions in u, v:

$$R^{(12)}_{\omega\omega'}(z-v)\, R^{(13)}_{\omega\varepsilon}(z-u)\, R^{(23)}_{\omega'\varepsilon}(v-u) \;=\; R^{(23)}_{\omega'\varepsilon}(v-u)\, R^{(13)}_{\omega\varepsilon}(z-u)\, R^{(12)}_{\omega\omega'}(z-v)\,.$$

Taking the leading terms of the Laurent expansions of both sides at $v = z'$ we get

$$I^{(12)}_{\omega\omega'}(z-z')\, R^{(13)}_{\omega\varepsilon}(z-u)\, R^{(23)}_{\omega'\varepsilon}(z'-u) \;=\; R^{(23)}_{\omega'\varepsilon}(z'-u)\, R^{(13)}_{\omega\varepsilon}(z-u)\, I^{(12)}_{\omega\omega'}(z-z')\,.$$

The latter equality implies Proposition 4.6, see (4.5) and (4.9) $\square$

Recall that the positive integer $d(\omega)$ is the Durfee rank of the non-empty skew Young diagram ω, see Section 1. The non-zero rational number $c(\omega)$ is the value of the function $u^{d(\omega)}\, h_\omega(u)$ at $u = 0$, see (3.10). Denote by P_ω be the flip map in $V_\omega \otimes V_\omega$, so that $P_\omega(x \otimes y) = y \otimes x$ for any $x, y \in V_\omega$. The main result of the previous section, Theorem 3.5, can be now restated as follows.

Proposition 4.7. *We have $a_{\omega\omega}(0) = d(\omega)$ and $I_{\omega\omega}(0) = c(\omega)\, P_\omega$.*

Proof. The rational function $F_{\omega\omega}(u)$ with the values in $\mathbb{C}\cdot S_{2n}$ can be written as $G_{\omega\omega}(u)\, F_\omega\, F^\vee_\omega$. Hence the required statement follows from Theorem 3.5, by the definition of the operator $I_{\omega\omega}(0)$ in $V_\omega \otimes V_\omega$ $\square$

Now take $k \geqslant 1$ non-empty skew Young diagrams $\omega_1, \ldots, \omega_k$. We will assume that each of the vector spaces $V_{\omega_1}, \ldots, V_{\omega_k}$ is not zero, see Proposition 4.1. Take also k arbitrary complex numbers $z_1, \ldots, z_k$. Consider the elementary $\mathrm{Y}(\mathfrak{gl}_N)$-modules $V_{\omega_1}(z_1), \ldots, V_{\omega_k}(z_k)$. The next theorem is the main result of the present paper.

Theorem 4.8. *The $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_{\omega_1}(z_1) \otimes \ldots \otimes V_{\omega_k}(z_k)$ is irreducible if and only if all the intertwiners $I_{\omega_i\omega_j}(z_i - z_j)$ with $1 \leqslant i < j \leqslant k$ are invertible.*

There is an immediate corollary, we state it as a theorem because of its importance.

Theorem 4.9. *The* $\mathrm{Y}(\mathfrak{gl}_N)$*-module* $V_{\omega_1}(z_1)\otimes\ldots\otimes V_{\omega_k}(z_k)$ *is irreducible if and only if all the modules* $V_{\omega_i}(z_i)\otimes V_{\omega_j}(z_j)$ *with* $1\leqslant i<j\leqslant k$ *are irreducible.*

We shall prove Theorem 4.8 in the remainder of this section. However, before giving the proof, let us make a few remarks. Applying Theorem 4.8 when $k=1,2$ we obtain, in particular, the following properties of the elementary $\mathrm{Y}(\mathfrak{gl}_N)$-modules:

(a) the $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_\omega(z)$ is irreducible;
(b) the $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_\omega(z)\otimes V_{\omega'}(z')$ is irreducible provided $z-z'\notin\mathbb{Z}$;
(c) if $V_\omega(z)\otimes V_{\omega'}(z')$ is irreducible, then it is equivalent to $V_\omega(z)\,\widetilde{\otimes}\,V_{\omega'}(z')$.

The properties (a,b,c) are well known, see for instance [C2,CP,NT1]. But we would like to emphasize here, that Theorem 4.8 gives each of them a new independent proof. Using Theorem 4.8 together with Proposition 4.7, we obtain for any $k=2,3,\ldots$ a new remarkable property of the elementary $\mathrm{Y}(\mathfrak{gl}_N)$-modules:

(d) the k-th tensor power of the $\mathrm{Y}(\mathfrak{gl}_N)$-module $V_\omega(z)$ is irreducible.

Conversely, the property (d) for $k=2$ implies that the operator $I_{\omega\omega}(0)$ in the vector space $V_\omega\otimes V_\omega$ is proportional to the flip map P_ω; see Proposition 4.6.

The *only if* part of Theorem 4.8 is an easy and well known statement. For the sake of completeness, with will give the proof of this statement at the end of this section. Our proof of the *if* part is based on the following general observation. Let A be any unital associative algebra over $\mathbb{C}$, and let W be any finite-dimensional A-module. Denote by ρ the corresponding homomorphism $\mathrm{A}\to\mathrm{End}(W)$. Let $P_W\in\mathrm{End}(W\otimes W)$ be the flip map.

Lemma 4.10. *If* $P_W\in\rho(\mathrm{A})\otimes\mathrm{End}(W)$, *then the* A*-module* W *is irreducible.*

Proof. Let $1\in\mathrm{End}(W)$ be the unit element, we have $1\in\rho(\mathrm{A})$ by our assumption. Furthermore, we have $(1\otimes X)\cdot P_W\in\rho(\mathrm{A})\otimes\mathrm{End}(W)$ for any $X\in\mathrm{End}(W)$. Now let $\mathrm{tr}:\,\mathrm{End}(W)\to\mathbb{C}$ be the trace map. Then the subalgebra $\rho(\mathrm{A})\subset\mathrm{End}(W)$ contains the element

$$(\mathrm{id}\otimes\mathrm{tr})\big((1\otimes X)\cdot P_W\big)=X\,.$$

So we actually have $\rho(\mathrm{A})=\mathrm{End}(W)$ $\square$

Now take the algebra $\mathrm{A}=\mathrm{Y}(\mathfrak{gl}_N)$ and its module $W=V_{\omega_1}(z_1)\otimes\ldots\otimes V_{\omega_k}(z_k)$. Under the assumption that all the intertwiners $I_{\omega_i\omega_j}(z_i-z_j)$ with $1\leqslant i<j\leqslant k$ are invertible, we will show that in the above notation,

$$P_W\in\rho\,(\mathrm{Y}(\mathfrak{gl}_N))\otimes\mathrm{End}(W)\,. \tag{4.10}$$

The *if* part of Theorem 4.8 will then follow by Lemma 4.10. We will split the proof of (4.10) into four propositions below.

Let ω be a non-empty skew Young diagram with n boxes. We keep assuming that the subspace $V_\omega\subset U^{\otimes n}$ is non-zero. Let $\mathrm{End}_\omega(U^{\otimes n})\subset\mathrm{End}(U^{\otimes n})$ be the stabilizer of V_ω. Let

$$\chi:\,\mathrm{End}_\omega(U^{\otimes n})\to\mathrm{End}(V_\omega)$$

be the canonical homomorphism: $\chi(X)=X|_{V_\omega}$. Introduce the series $\widehat{T}_\omega(u)$ with coefficients in the tensor product $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(U^{\otimes n})=\mathrm{Y}(\mathfrak{gl}_N)\otimes(\mathrm{End}(U))^{\otimes n}$

$$\widehat{T}_\omega(u)=T^{(01)}(u+c_1)\;\ldots\;T^{(0n)}(u+c_n)\,.$$

Here we labelled the factors in $\mathrm{Y}(\mathfrak{gl}_N)\otimes(\mathrm{End}(U))^{\otimes n}$ by the indices $0,1,\ldots,n$.

Proposition 4.11. *Coefficients of the series* $\widehat{T}_\omega(u)$ *are in* $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}_\omega(U^{\otimes n})$.

Proof. Consider the rational function $F_\omega(z_1,\dots,z_n)$, defined in Section 2 as the product (2.5). The relation (4.3) implies that

$$T^{(01)}(u+c_1+z_1)\ \dots\ T^{(0n)}(u+c_n+z_n)\cdot\pi_n\bigl(F_\omega(z_1,\dots,z_n)\bigr)=$$
$$\pi_n\bigl(F_\omega(z_1,\dots,z_n)\bigr)\cdot T^{(0n)}(u+c_n+z_n)\ \dots\ T^{(01)}(u+c_1+z_1)\,.$$

Hence, by Proposition 2.2

$$\widehat{T}_\omega(u)\,\pi_n(F_\omega)\,=\,\pi_n(F_\omega)\,T^{(0n)}(u+c_n)\,\dots\,T^{(01)}(u+c_1)\quad\square$$

Due to the Proposition 4.11, we can apply the homomorphism $\mathrm{id}\otimes\chi$ to the series $\widehat{T}_\omega(u)$, and obtain in this way a series with the coefficients in $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(V_\omega)$. We will denote the latter series by $T_\omega(u)$.

Henceforth, we will use the following generalization of notation (4.4). Given a vector space V and an element $X\in\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(V)$, we will write

$$X^{[k]}=X^{(k3)}\in\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(V)\,,\qquad k=1,2\,.$$

Proposition 4.12: $\Delta\bigl(T_\omega(u)\bigr)=T_\omega^{[2]}(u)\,T_\omega^{[1]}(u)$.

Proof. The formula (4.5) implies that $\Delta\bigl(\widehat{T}_\omega(u)\bigr)=\widehat{T}_\omega^{[2]}(u)\,\widehat{T}_\omega^{[1]}(u)$. By applying the homomorphism $\mathrm{id}\otimes\mathrm{id}\otimes\chi$ to the last equality, the proposition follows $\square$

Let $\gamma:\mathrm{Y}(\mathfrak{gl}_N)\to\mathrm{End}(V_\omega)$ be the defining homomorphism of the module $V_\omega(z)$. Take one more skew Young diagram ω', we will again assume that $V_{\omega'}\neq\{0\}$.

Proposition 4.13: $(\gamma\otimes\mathrm{id})\bigl(T_{\omega'}(u)\bigr)=R_{\omega\omega'}(z-u)$.

Proof. Let n' be the numbers of boxes in the diagram ω'. Consider the canonical homomorphism

$$\chi':\ \mathrm{End}_{\omega'}(U^{\otimes n'})\to\mathrm{End}(V_{\omega'})\,.$$

It is straighforward to verify that each of $(\gamma\otimes\mathrm{id})\bigl(T_{\omega'}(u)\bigr)$ and $R_{\omega\omega'}(z-u)$ equals

$$\bigl((\,\chi\otimes\chi')\circ\pi_{n+n'}\bigr)\bigl(G_{\omega\omega'}(z-u)\bigr)\quad\square$$

Now consider the formal power series in u^{-1}

$$T_W(u)\,=\,T^{(01)}_{\omega_1}(u+z_1)\ \dots\ T^{(0k)}_{\omega_k}(u+z_k)\in\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(W)\,,$$

here we identify $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(W)$ with $\mathrm{Y}(\mathfrak{gl}_N)\otimes\mathrm{End}(V_{\omega_1})\otimes\ldots\otimes\mathrm{End}(V_{\omega_k})$, and label the tensor factors by $0,1,\dots,k$. The next proposition completes the proof of the *if* part of Theorem 4.8.

Proposition 4.14. *The image* $(\rho\otimes\mathrm{id})\big(T_W(u)\big)$ *is a rational function in* u *with values in*

$$\rho(\mathrm{Y}(\mathfrak{gl}_N))\otimes\mathrm{End}\,(W)\subset\mathrm{End}\,(W)\otimes\mathrm{End}\,(W)\,.$$

The leading coefficient in its Laurent expansion at $u=0$ *is proportional to* P_W .

Proof. It follows from Proposition 4.12 that $\Delta\big(T_W(u)\big)=T_W^{[2]}(u)\,T_W^{[1]}(u)$. Using this relation and Proposition 4.13, we get

$$(\rho\otimes\mathrm{id})\big(T_W(u)\big)\;=\prod_{j=1,\dots,k}^{\longrightarrow}\Bigg(\prod_{i=1,\dots,k}^{\longleftarrow}R^{(i,j+k)}_{\omega_i\omega_j}(z_i-z_j-u)\Bigg)\,.\tag{4.11}$$

Here we regard $\mathrm{End}\,(W)\otimes\mathrm{End}\,(W)$ as the $2k$-fold tensor product

$$\mathrm{End}\,(V_{\omega_1})\otimes\ldots\otimes\mathrm{End}\,(V_{\omega_k})\otimes\mathrm{End}\,(V_{\omega_1})\otimes\ldots\otimes\mathrm{End}\,(V_{\omega_k})\,.\tag{4.12}$$

The right hand side of the equality (4.11) is a rational function in u by definition. Denote

$$a\;=\sum_{i,j=1}^{k}a_{\,\omega_i\omega_j}(z_i-z_j)\,.$$

Then the product of the leading terms of the Laurent expansions at $u=0$ of all factors in the product (4.11) equals $(-u)^{-a}$ multiplied by

$$\prod_{j=1,\dots,k}^{\longrightarrow}\Bigg(\prod_{i=1,\dots,k}^{\longleftarrow}I^{(i,j+k)}_{\omega_i\omega_j}(z_i-z_j)\Bigg)\,.\tag{4.13}$$

By Proposition 4.7, any factor $I^{(i,i+k)}_{\omega_i\omega_j}(0)$ corresponding to $i=j$, equals $P^{(i,i+k)}_{\omega_i}$ up to a non-zero multiplier. Using this result, we can rewrite (4.13) as

$$\prod_{i=1}^{k}P^{(i,i+k)}_{\omega_i}\cdot\prod_{1\leqslant i<j\leqslant k}\Big(I^{(j+k,i)}_{\omega_j\omega_i}(z_j-z_i)\,I^{(i,j+k)}_{\omega_i\omega_j}(z_i-z_j)\Big)\tag{4.14}$$

up to a nonzero multiplier. Here all factors $P^{(i,i+k)}_{\omega_i}$ pairwise commute. By Proposition 4.5, each of the factors in (4.14) corresponding to $1\leqslant i<j\leqslant k$, is proportional to the identity map in (4.12). In particular, these factors also pairwise commute. Moreover, each of these factors is non-zero, because the operators $I_{\omega_i\omega_j}(z_i-z_j)$ with $1\leqslant i<j\leqslant k$ are invertible by our assumption. So the entire product (4.13) is non-zero and proportional to

$$\prod_{i=1}^{k}P^{(i,i+k)}_{\omega_i}=P_W\,.$$

Thus the leading term in the Laurent expansion of the function $(\rho\otimes\mathrm{id})\big(T_W(u)\big)$ near the origin $u=0$ equals $(-u)^{-a}P_W$, up to a non-zero factor from $\mathbb{C}$ $\square$

We will complete this section with the proof of the *only if* part of Theorem 4.8. Consider the $\mathrm{Y}(\mathfrak{gl}_N)$-module $W=V_{\omega_1}(z_1)\otimes\ldots\otimes V_{\omega_k}(z_k)$. Suppose there is a non-invertible intertwiner $I_{\omega_i\omega_j}(z_i-z_j)$ with $1\leqslant i<j\leqslant k$. Assume that the pair (i,j) is lexicographically minimal with this non-invertibility property. The operator

$$I^{(ij)}_{\omega_i\omega_j}(z_i-z_j)\,\ldots\,I^{(i,i+1)}_{\omega_i\omega_{i+1}}(z_i-z_{i+1})$$

is non-zero and non-invertible. By Proposition 4.6, this is an intertwiner of Yangian modules

$$W\;\longrightarrow\;V_{\omega_1}(z_1)\otimes\ldots\otimes V_{\omega_{i-1}}(z_{i-1})\,\otimes$$
$$\big(V_{\omega_i}(z_i)\,\widetilde{\otimes}\,\big(V_{\omega_{i+1}}(z_{i+1})\otimes\ldots\otimes V_{\omega_j}(z_j)\big)\big)\,\otimes\,V_{\omega_{j+1}}(z_{j+1})\otimes\ldots\otimes V_{\omega_k}(z_k)\,.$$

So the $\mathrm{Y}(\mathfrak{gl}_N)$-module W is reducible. This proves the *only if* part of Theorem 4.8.

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