Flatness of equivariant modules

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Introduction

Flatness underlies the existence of homogeneous spaces for group schemes of finite type over a field. There are substantial difficulties in developing the theory of homogeneous spaces for arbitrary Hopf algebras. This paper provides further evidence that a nice extension of the classical theory is likely to be possible for the class of noncommutative algebras which are finitely generated modules over their centers.

Let H be a Hopf algebra over a field k. We will consider algebras A on which H either acts or coacts in a way compatible with the multiplication in A (see [13], [21]). If A is a left H-module algebra, then ${}_H\mathcal{M}_A$ and ${}_{A\#H}\mathcal{M}$ will stand for the categories of H-equivariant right and left A-modules, respectively. When A is a right H-comodule algebra, the analogous categories $\mathcal{M}_A^H$ and ${}_A\mathcal{M}^H$ of H-coequivariant A-modules or (H, A)-Hopf modules were introduced by Takeuchi [22] and Doi [3]. An object of one of these categories will be called *A-finite* if it is finitely generated as an A-module.

This paper is concerned with the A-flatness of H-(co)equivariant A-modules. All main results are obtained under the assumption that A is H-semiprime, noetherian and module-finite over its center. When A is an H-module algebra, two important ingredients of the proofs come from a previous work [19]. One of those is the H-orbit equivalence relation $\sim_H$ on the prime spectrum $\operatorname{Spec} A$. We call the $\sim_H$-equivalence class of a prime P of A the *H-orbit* of P in $\operatorname{Spec} A$ and denote it by $\operatorname{Eq}_H(P)$. It was also established in [19] that A has a quasi-Frobenius classical quotient ring $Q(A)$. This enables us first to prove the flatness of a generic localization of an equivariant A-module and then to apply a kind of translations — twisting operations on modules. Although this technique is far less obvious than in the case of group actions, it nevertheless works efficiently.

If A is an H-comodule algebra, A may be regarded as a module algebra over the finite dual H° of H (see [13], [21]). Therefore all notions and results for module algebras can be reformulated in the context of comodule algebras. In particular, the H°-orbits in $\operatorname{Spec} A$ are defined.

An important class of H-comodule algebras consists of the right coideal subalgebras of H. Such a subalgebra $A \subset H$ is characterized by the property that $\Delta(A) \subset A \otimes H$ where $\Delta : H \to H \otimes H$ is the comultiplication. The coideal subalgebras provide a way to think about quasiaffine homogeneous spaces in purely

Supported by the RFBR grant 07-01-00222

algebraic terms. The next result shows that in the module-finite case we do enjoy the expected properties.

We say that H is *of finite type over k* if H is module-finite over a finitely generated central subalgebra. For each ring R we always consider the Zariski topology on the prime spectrum $\operatorname{Spec} R$ and on its subset $\operatorname{Max} R$ consisting of the maximal ideals of R. Denote by $\mathcal{M}_R$ and ${}_R\mathcal{M}$ the categories of right and left R-modules.

Theorem 0.1. *Let H be a Hopf algebra of finite type over k, and let A be a noetherian right coideal subalgebra module-finite over its center. Then H is flat in $\mathcal{M}_A$ and in ${}_A\mathcal{M}$. There is an open dense H°-orbit in $\operatorname{Max} A$ consisting precisely of those $P \in \operatorname{Max} A$ for which there exists $\mathfrak{P} \in \operatorname{Max} H$ satisfying $A \cap \mathfrak{P} \subset P$. The following conditions are equivalent*:

(i) *H is faithfully flat in $\mathcal{M}_A$,*

(ii) *H is faithfully flat in ${}_A\mathcal{M}$,*

(iii) *A is H-simple, that is, A has no H-costable ideals other than 0 and A itself.*

(iv) *the set $\operatorname{Max} A$ is a single H°-orbit.*

As was shown by Masuoka and Wigner [10], for an arbitrary Hopf algebra with a bijective antipode (ii) is equivalent to several other conditions, e.g. to the condition that H is a projective generator in ${}_A\mathcal{M}$. The importance of (ii) became clear in the work of Takeuchi [22]. This condition implies that $\mathcal{M}_A^H$ is equivalent to the category of comodules over a factor coalgebra of H, which generalizes a description of the category of G-linearized quasicoherent sheaves on an affine homogeneous G-space where G is the group scheme corresponding to a finitely generated commutative Hopf algebra (see, e.g., [7]). If in Theorem 0.1 A is a Hopf subalgebra, then (i) and (ii) hold by [18]. The methods of [18] were insufficient to prove the flatness of H over coideal subalgebras.

Theorem 0.1 presents a special case of more general results. An H-(co)module algebra A is called *H-semiprime* if A has no nonzero nilpotent H-(co)stable ideals. Moreover, A is *H-prime* if $A \neq 0$ and $IJ \neq 0$ for any two nonzero H-(co)stable ideals I, J of A. Thus we use these notions for comodule algebras as well as for module algebras. An H-(co)stable ideal I of A is called H-semiprime or H-prime if so is the H-(co)module algebra A/I.

Theorem 0.2. *Let H be a Hopf algebra of finite type over k and $\varphi : A \to B$ a homomorphism of noetherian H-comodule algebras module-finite over their centers. Suppose that A is H-semiprime and $\varphi(I)B = B$ for each H-costable ideal I of A which contains a regular element of A. Then*:

(i) $\operatorname{Tor}_i^A(M, W) = 0$ *when $i > 0$, $M \in \mathcal{M}_A^H$ and $W \in {}_B\mathcal{M}$.*

(ii) $\operatorname{Tor}_i^A(V, N) = 0$ *when $i > 0$, $N \in {}_A\mathcal{M}^H$ and $V \in \mathcal{M}_B$.*

(iii) *Each object of $\mathcal{M}_B^H$ is flat in $\mathcal{M}_A$ and each object of ${}_B\mathcal{M}^H$ is flat in ${}_A\mathcal{M}$.*

Given a homomorphism of H-(co)module algebras $A \to B$, we say that B is *right of finite type over A* if B has a finite number of central elements $z_1, \ldots, z_n$ such that B is right module-finite over its subring $A'[z_1, \ldots, z_n]$ where A' denotes the image of A in B; "*left of finite type*" is defined in terms of left module-finiteness. Generic freeness [11, Th. 24.1] plays an important role in commutative algebra since it allows one to prove the openness of flat loci. A well-known generalization concerns noncommutative algebras over a commutative integral domain [12, Ch. 9].

In Theorem 0.3 generic freeness is established for an extension of comodule algebras. A weaker restriction on the Hopf algebra is sufficient here. Recall that H is *residually finite dimensional* [13] if its ideals of finite codimension have zero intersection.

Theorem 0.3. *Let H be a residually finite dimensional Hopf algebra with the property that H° has a bijective antipode. Let A be a noetherian H-prime H-comodule algebra module-finite over its center, and B an H-comodule algebra right* (*resp. left*) *of finite type over A. Denote by l the greatest common divisor of the lengths of simple factor rings of the classical quotient ring $Q(A)$. Then for each B-finite object $M \in \mathcal{M}_B^H$* (*resp.* $N \in {}_B\mathcal{M}^H$) *there exists a central regular element $s \in A$ such that $M^l \otimes_A A[s^{-1}]$* (*resp.* $A[s^{-1}] \otimes_A N^l$) *is a free $A[s^{-1}]$-module.*

Here M^l and N^l denote the direct sums of l copies of M and N, respectively. The assumption about the antipode of H° is actually not needed in the $\mathcal{M}_B^H$-part of this theorem. Versions of Theorems 0.2, 0.3 for module algebras are presented in Theorems 4.6, 5.3. For $P \in \operatorname{Max} A$ we denote by P_{H° the largest H-costable (same as H°-stable) ideal of A contained in P.

Theorem 0.4. *Let H be a Hopf algebra of finite type over k, and let A be a noetherian H-comodule algebra module-finite over its center. If there exists $P \in \operatorname{Max} A$ such that $\dim A/P < \infty$ and $P_{H^\circ} = 0$, then*:

(i) *A has a smallest nonzero H-semiprime H-costable ideal J.*

(ii) *The set $\{P' \in \operatorname{Max} A \mid J \not\subset P'\}$ is an open dense H°-orbit in $\operatorname{Max} A$.*

(iii) *$\bigcap_{n>0} J^n = 0$ unless $J = A$.*

It is now possible to generalize a classical fact according to which all orbits of a rational action of an algebraic group on an algebraic variety are locally closed:

Corollary 0.5. *Let H be a Hopf algebra of finite type over k, and let A be a noetherian H-comodule algebra module-finite over its center. For each maximal ideal P of finite codimension in A there exists an H-costable ideal I of A such that*

$$\operatorname{Eq}_{H^\circ}(P) = \{P' \in \operatorname{Max} A \mid P_{H^\circ} \subset P' \text{ and } I \not\subset P'\}.$$

Thus $\operatorname{Eq}_{H^\circ}(P)$ is locally closed in $\operatorname{Max} A$.

In the paper we allow k to be an arbitrary commutative ring. By default algebras and coalgebras are over k. When k is the base ring for $\otimes$ or Hom, it is not indicated explicitly. We denote by Δ, ε, S the comultiplication, the counit and the antipode of H. In sections 1–5 where module algebras are considered the Hopf algebra H is assumed to be the union of a directed family $\mathcal{F}$ of subcoalgebras such that each $C \in \mathcal{F}$ is a finitely generated projective k-module. When k is a field this assumption is clearly satisfied for any Hopf algebra. In section 6 we switch to comodule algebras, and H is assumed to have a family of ideals satisfying several conditions. All module algebras are assumed to be left module algebras, while comodule algebras are right comodule algebras.

We will freely use several properties known for a ring R module-finite over a central subring Z. By a noncommutative version of the Eakin-Nagata Theorem [11, Th. 3.7] R is either right or left noetherian if and only if Z is noetherian, in which case R is two-sided noetherian. A similar conclusion is valid for artinian conditions [8, Th. 3.100]. Each simple right R-module V is annihilated by a maximal ideal of Z; therefore the annihilator of V in R is a maximal ideal of R such that $R/\operatorname{Ann}(V)$ is a simple finite dimensional algebra over a field.

1. Preliminaries: the orbit equivalences and quotient rings

We adopt the conventions about k, H, $\mathcal{F}$ stated in the introduction. Let H^{bop} denote H with the opposite multiplication and comultiplication. So H^{bop} is a Hopf algebra with the same antipode S as for H, and $\mathcal{F}$ is a family of subcoalgebras of H^{bop}. Let A be an H-module algebra. So A has a left H-module structure such that $h(ab) = \sum (h_{(1)}a)(h_{(2)}b)$ for all $h \in H$ and $a, b \in A$. We may view S as a homomorphism of Hopf algebras $H^{\mathrm{bop}} \to H$. In particular, A is an H^{bop}-module algebra with respect to the action $(h, a) \mapsto S(h)a$ where $h \in H$ and $a \in A$. This allows us to make use of the H^{bop}-versions of results proved for H-module algebras.

Given a subcoalgebra C of H and an ideal I of A, the ideal I_C of A is defined by the rule

$$I_C = \{a \in A \mid Ca \subset I\}.$$

In particular, I_H is the largest H-stable ideal of A contained in I. Since H is the union of subcoalgebras in $\mathcal{F}$, we have $I_H = \bigcap_{C \in \mathcal{F}} I_C$. Let C^{cop} denote the opposite to C coalgebra. For each algebra B we consider $\mathrm{Hom}(C, B)$ and $\mathrm{Hom}(C^{\mathrm{cop}}, B)$ equipped with the convolution multiplications. There are algebra homomorphisms

$$\tau : A \to \mathrm{Hom}(C, A/I), \qquad \tau' : A \to \mathrm{Hom}(C^{\mathrm{cop}}, A/I)$$

defined by $\tau(a)(c) = ca + I$ and $\tau'(a)(c) = S(c)a + I$ for $a \in A$ and $c \in C$. Clearly $I_C = \mathrm{Ker}\, \tau$ and $I_{S(C)} = \mathrm{Ker}\, \tau'$, so that τ and τ' induce injective homomorphisms

$$A/I_C \hookrightarrow \mathrm{Hom}(C, A/I), \qquad A/I_{S(C)} \hookrightarrow \mathrm{Hom}(C^{\mathrm{cop}}, A/I). \qquad (*)$$

Denote by $\mathrm{Spec}_f\, A$ the set of those prime ideals P of A for which there exists no infinite strictly ascending chain $P_0 \subset P_1 \subset \cdots$ in $\mathrm{Spec}\, A$ starting at $P_0 = P$. For instance, $\mathrm{Spec}_f\, A = \mathrm{Spec}\, A$ when A is either left or right noetherian.

Theorem 1.1. *Let A be an H-module algebra module-finite over its center. There are equivalence relations $\sim_H$ and $\sim_{H^{\mathrm{bop}}}$ on $\mathrm{Spec}_f\, A$ such that for $P, P' \in \mathrm{Spec}_f\, A$ one has $P \sim_H P'$ (resp. $P \sim_{H^{\mathrm{bop}}} P'$) if and only if P' is a prime minimal over P_C (resp. over $P_{S(C)}$) for some $C \in \mathcal{F}$.*

The statement concerning $\sim_H$ is just [19, Th. 0.1]. The second relation is obtained by regarding A as an H^{bop}-module algebra. Denote by $\mathrm{Eq}_H(P)$ the $\sim_H$-equivalence class of $P \in \mathrm{Spec}_f\, A$. Similarly, $\mathrm{Eq}_{H^{\mathrm{bop}}}(P)$ will denote the $\sim_{H^{\mathrm{bop}}}$-equivalence class. If $P' \in \mathrm{Eq}_H(P)$, then $P_H \subset P'$ and, by symmetry, $P'_H \subset P$, so that $P_H = P'_H$.

The *coheight* of $P \in \mathrm{Spec}\, A$ is the ordinal-valued classical Krull dimension of the factor ring A/P. It is defined by transfinite induction. One has $\mathrm{coheight}\, P = 0$ precisely for maximal ideals of A. Given any ordinal $\alpha > 0$, one sets $\mathrm{coheight}\, P = \alpha$ if $\mathrm{coheight}\, P' < \alpha$ for each prime P' of A properly containing P and if $\mathrm{coheight}\, P \neq \beta$ for each ordinal $\beta < \alpha$. The coheight is well-defined for all primes in $\mathrm{Spec}_f\, A$, e.g. by [6, Prop. 14.1]. The statement below is taken from [19, Prop. 4.10]:

Lemma 1.2. *If two primes $P, P' \in \mathrm{Spec}_f\, A$ are either $\sim_H$ or $\sim_{H^{\mathrm{bop}}}$-equivalent then* $\mathrm{coheight}\, P = \mathrm{coheight}\, P'$.

In particular, $\mathrm{Eq}_H(P) \subset \mathrm{Max}\, A$ when $P \in \mathrm{Max}\, A$; moreover, the ring A/P_C is artinian for each $C \in \mathcal{F}$ by [18, Lemma 3.4].

Corollary 1.3. *If $S(H) = H$, then $\sim_H$ coincides with $\sim_{H^{\mathrm{bop}}}$.*

Proof. For each $C \in \mathcal{F}$ there exists $D \in \mathcal{F}$ such that $S(C) \subset D$. Then $P_D \subset P_{S(C)}$ for any $P \in \operatorname{Spec}_f A$. If P' is a prime of A minimal over $P_{S(C)}$, then $P'' \subset P'$ for some prime P'' minimal over P_D. By [19, Lemma 4.1] $P', P'' \in \operatorname{Spec}_f A$, whence we have $P \sim_{H^{\mathrm{bop}}} P'$ and $P \sim_H P''$. By Lemma 1.2 P, P', P'' have equal coheights. This is only possible when $P' = P''$. This shows that $P \sim_{H^{\mathrm{bop}}} P'$ implies $P \sim_H P'$.

When S is surjective, for each $C \in \mathcal{F}$ there exists $D \in \mathcal{F}$ such that $C \subset S(D)$. Comparison of the coheights shows that the primes minimal over P_C are minimal over $P_{S(D)}$. In this case $P \sim_H P'$ implies $P \sim_{H^{\mathrm{bop}}} P'$. □

The right (resp. left) Ore localization of a ring R at a right (resp. left) denominator subset $\Sigma \subset R$ will be denoted by $R\Sigma^{-1}$ (resp. $\Sigma^{-1}R$) [12, Ch. 2]. Given a right (resp. left) R-module V, the Σ-torsion submodule of V consists of all elements of V annihilated by some element of Σ. This submodule coincides with the kernel of the canonical map $V \mapsto V \otimes_R R\Sigma^{-1}$ (resp. $V \mapsto \Sigma^{-1}R \otimes_R V$). The R-module V is called Σ-torsionfree if it has zero Σ-torsion submodule.

In the special case when Σ is the set of all regular elements, i.e. nonzerodivisors, of R the ring $R\Sigma^{-1}$ will be denoted by $Q(R)$; this ring exists when the set of regular elements is a right denominator set. Any overring of R isomorphic to $Q(R)$ as an overring is called a *classical right quotient ring* of R. Left quotient rings are defined by symmetry. An overring is a *classical quotient ring* if it satisfies both the right hand and left hand conditions. The fact below is well known in the ring theory. We omit the proof.

Lemma 1.4. *Suppose that a ring R has a right artinian classical right quotient ring $Q(R)$. Then the maximal ideals of $Q(R)$ are precisely the ideals $P \cdot Q(R)$ for minimal primes P of R, and one has $Q(R)/\big(P \cdot Q(R)\big) \cong Q(R/P)$ for any such P.*

If Σ is any right denominator set consisting of regular elements of R, then the canonical map $R \to R\Sigma^{-1}$ is injective. If, moreover, the ring $R\Sigma^{-1}$ is right artinian, then it is a classical right quotient ring of R, so that $Q(R) \cong R\Sigma^{-1}$. Indeed, in this case every regular element of R has zero right annihilator in $R\Sigma^{-1}$, and therefore is invertible in $R\Sigma^{-1}$ by [12, Prop. 3.1.1]. In this paper we will be concerned with the situation when Σ is contained in the center of R, in which case $\Sigma^{-1}R \cong R\Sigma^{-1}$. In particular, if Z is a central subring of R such that $Q(R) \cong R \otimes_Z Q(Z)$, then $Q(R) \cong R\Sigma^{-1}$ where Σ is the set of regular elements of Z; since R embeds in $Q(R)$, all elements of Σ are regular in R. Note also that the map $Q(Z) \to Q(R)$ is injective since $Q(Z)$ is a flat Z-algebra.

Suppose that A is module-finite over a central subring Z. Let $P \in \operatorname{Spec} A$ and $\mathfrak{p} = Z \cap P$. Since P is prime, all nonzero elements of $Z/\mathfrak{p}$ are regular in A/P. Hence $\mathfrak{p} \in \operatorname{Spec} Z$ and A/P embeds into the finite dimensional algebra $A/P \otimes_{Z/\mathfrak{p}} \kappa(\mathfrak{p})$ over the field of fractions $\kappa(\mathfrak{p}) = Q(Z/\mathfrak{p})$. It follows that A/P has an artinian classical quotient ring $Q(A/P) \cong A/P \otimes_{Z/\mathfrak{p}} \kappa(\mathfrak{p})$.

Lemma 1.5. *Assume A to be module-finite over its center. If $P \in \operatorname{Spec}_f A$ and $C \in \mathcal{F}$, then A/P_C and $A/P_{S(C)}$ have artinian classical quotient rings, respectively,*

$$A/P_C \otimes_Z Q\big(Z/(Z \cap P_C)\big) \quad \text{and} \quad A/P_{S(C)} \otimes_Z Q\big(Z/(Z \cap P_{S(C)})\big).$$

The injections $A/P_C \to \operatorname{Hom}(C, A/P)$ and $A/P_{S(C)} \to \operatorname{Hom}(C^{\mathrm{cop}}, A/P)$ from $(*)$ extend to injective ring homomorphisms

$$Q(A/P_C) \to \mathrm{Hom}\big(C, Q(A/P)\big) \quad \text{and} \quad Q(A/P_{S(C)}) \to \mathrm{Hom}\big(C^{\mathrm{cop}}, Q(A/P)\big).$$

This is proved in [19, Lemma 4.1]. The injectivity of the homomorphisms is clear since each ideal of $Q(R)$ is generated by elements in R. The next two results are [19, Th. 0.3] and [20, Th. 2.2]:

Theorem 1.6. *Any noetherian H-semiprime H-module algebra A module-finite over its center Z has a quasi-Frobenius classical quotient ring $Q(A) \cong A \otimes_Z Q(Z)$.*

Theorem 1.7. *Suppose that A has a right artinian classical right quotient ring Q. Then the H-module structure on A has a unique extension to Q with respect to which Q becomes an H-module algebra.*

An H-module algebra Q is called *H-simple* if $Q \neq 0$ and Q has no H-stable ideals other than 0 and Q itself. An H-module algebra is *H-semisimple* if it is a finite direct product of H-simple H-module algebras.

Lemma 1.8. *Under the hypotheses of Theorem* 1.7 *Q is H-semisimple when A is H-semiprime, and Q is H-simple when A is H-prime.*

Proof. If I is any H-stable ideal of Q, then $I \cap A$ is an H-stable ideal of A and $I = (I \cap A)Q$. It follows that the H-semiprimeness and the H-primeness pass from A to Q. Now the conclusion follows from [20, Lemma 4.2]. □

Lemma 1.9. *Let $\varphi : A \to B$ be a homomorphism of H-module algebras. Suppose A has a right artinian classical quotient ring Q and B is H-semiprime left noetherian. Denote by Σ the set of all regular elements of A. If $\varphi(I)B = B$ for each $S(H)$-stable ideal I of A such that $I \cap \Sigma \neq \varnothing$, then all elements in $\varphi(\Sigma)$ are regular in B.*

Proof. Denote by K and L, respectively, the kernels of the maps $\iota : B \to Q \otimes_A B$ and $\iota' : B \to B \otimes_A Q$ defined by the rules $b \mapsto 1 \otimes b$ and $b \mapsto b \otimes 1$. Since Q is isomorphic to $\Sigma^{-1}A$ and to $A\Sigma^{-1}$, we have

$$K = \{b \in B \mid \varphi(s)b = 0 \text{ for some } s \in \Sigma\},$$
$$L = \{b \in B \mid b\varphi(s) = 0 \text{ for some } s \in \Sigma\}.$$

The set $I = \{a \in A \mid L\varphi(a) = 0\}$ is an ideal of A since $L\varphi(A) \subset L$. The right annihilator of any finite subset of L contains an element in $\varphi(\Sigma)$. Since L is a finitely generated left ideal of B, we get $I \cap \Sigma \neq \varnothing$. By Theorem 1.7 Q carries an H-module structure. As is checked straightforwardly, the kernel of the canonical surjection $B \otimes Q \to B \otimes_A Q$ is H-stable. Hence the H-module structure passes to $B \otimes_A Q$ and ι' is an H-linear map. It follows that L is H-stable. Since

$$b\varphi\big((Sh)a\big) = \sum_{(h)} (Sh_{(1)})\big((h_{(2)}b) \cdot \varphi(a)\big) = 0$$

for all $b \in L$, $a \in I$ and $h \in H$, the ideal I is $S(H)$-stable. By the hypothesis $1 \in \varphi(I)B$. Since $L\varphi(I)B = 0$, we conclude that $L = 0$.

This shows that $\varphi(s)$ has zero left annihilator, and therefore $B\varphi(s)$ is an essential left ideal of B, for each $s \in \Sigma$. Recall that the left singular ideal $\mathfrak{S}(B)$ of B consists of all elements of B whose left annihilator in B is an essential left ideal. We see that $K \subset \mathfrak{S}(B)$. Since B is left noetherian, $\mathfrak{S}(B)$ is nilpotent [12, Lemma 2.3.4].

Similarly as in case of ι', the map ι is H-linear and K is H-stable. Thus BK is a nilpotent H-stable ideal of B, and the H-semiprimeness of B yields $K = 0$. In other words, $\varphi(s)$ also has zero right annihilator for each $s \in \Sigma$. □

Lemma 1.10. *Let A be an H-module algebra module-finite over its center Z. For each $P \in \operatorname{Spec} A$ denote by $\mathfrak{I}_P$ the intersection of the ideals $P' \in \operatorname{Spec} A$ such that $P_C \subset P'$ for some $C \in \mathcal{F}$. Given $C \in \mathcal{F}$, there exists an integer $n > 0$ depending on C but not on P such that $\mathfrak{I}_P^n \subset P_C$.*

Proof. Let $g > 0$ be any integer such that A is g-generated as a Z-module. Denote $r = \max\{\dim_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}) \otimes C \mid \mathfrak{q} \in \operatorname{Spec} k\}$ where $\kappa(\mathfrak{q}) = Q(k/\mathfrak{q})$. We will show that the conclusion holds with $n = rg$. Put $\mathfrak{p} = Z \cap P$ and $\mathfrak{q} = k \cap P$. The quotient ring $Q(A/P) \cong A/P \otimes_{Z/\mathfrak{p}} \kappa(\mathfrak{p})$ is an algebra of dimension at most g over the field $\kappa(\mathfrak{p}) = Q(Z/\mathfrak{p})$. Note that $\kappa(\mathfrak{q})$ is a subfield of $\kappa(\mathfrak{p})$. The algebra A/P_C embeds into $\operatorname{Hom}(C, A/P)$, and the latter is a subalgebra of

$$\operatorname{Hom}\bigl(C, Q(A/P)\bigr) \cong Q(A/P) \otimes C^* \cong Q(A/P) \otimes_{\kappa(\mathfrak{q})} (\kappa(\mathfrak{q}) \otimes C^*)$$

The k-projectivity of C shows also that $\dim_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}) \otimes C^* = \dim_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}) \otimes C \le r$. Hence $Q(A/P)\otimes C^*$ is an algebra of dimension at most n over $\kappa(\mathfrak{p})$. As is well known, any multiplicatively closed nil subset in such an algebra is nilpotent with nilpotency index at most n [9]. In particular, the prime radical N of A/P_C satisfies $N^n = 0$ since N is nil [12, Th. 0.2.6]. By definition $\mathfrak{I}_P$ is contained in each $P' \in \operatorname{Spec} A$ with $P_C \subset P'$. Hence $(\mathfrak{I}_P + P_C)/P_C \subset N$, and it follows that $\mathfrak{I}_P^n \subset P_C$. □

We say that a subset of $\operatorname{Spec} A$ is *H-stable* if it is a union of H-orbits.

Lemma 1.11. *Let A be a noetherian H-module algebra module-finite over its center.*

(i) *Any H-stable subset $X \subset \operatorname{Spec} A$ has an H-stable closure $\overline{X}$ in* $\operatorname{Spec} A$.

(ii) *The set $U' = \bigcup_{P\in U} \operatorname{Eq}_H(P)$ is open in* $\operatorname{Spec} A$ *for any open subset $U \subset \operatorname{Spec} A$.*

Proof. (i) We have $\overline{X} = \{P \in \operatorname{Spec} A \mid J \subset P\}$ where J denotes the intersection of all ideals $P \in X$. If $P \in X$, then $J \subset P'$ for each $P' \in \operatorname{Eq}_H(P)$ since X is H-stable, and therefore $J \subset \mathfrak{I}_P$. Let $C \in \mathcal{F}$, and let n be as in Lemma 1.10. We have

$$J^n \subset \bigcap_{P\in X} \mathfrak{I}_P^n \subset \bigcap_{P\in X} P_C = J_C.$$

If $P, P' \in \operatorname{Spec} A$ satisfy $J \subset P$ and $P_C \subset P'$, then $J^n \subset P'$ since $J_C \subset P_C$, and therefore $J \subset P'$. This shows that $\operatorname{Eq}_H(P) \subset \overline{X}$ for each $P \in \overline{X}$.

(ii) Consider the complements X and X' of U and U' in $\operatorname{Spec} A$. Then X' is the largest H-stable subset of X. The closure of X' in $\operatorname{Spec} A$ is H-stable by (i) and is contained in X since X is closed. It follows that X' is closed. □

We say that an H-orbit $\operatorname{Eq}_H(P)$ has the *density property* if its closure in $\operatorname{Spec} A$ contains all primes P' of A such that $P_H \subset P'$. Clearly

$$\overline{\operatorname{Eq}_H(P)} = \{P' \in \operatorname{Spec} A \mid \mathfrak{I}_P \subset P'\}.$$

Therefore $\operatorname{Eq}_H(P)$ has the density property if and only if $\mathfrak{I}_P/P_H$ is contained in the prime radical of the factor algebra A/P_H (e.g. this holds when $\mathfrak{I}_P/P_H$ is nilpotent).

If k is a field and H is pointed irreducible, then P_C contains a power of P for each $C \in \mathcal{F}$ (cf. [2, Lemma 2.2] and [14, Th. 3.7]), which shows that $\mathrm{Eq}_H(P) = \{P\}$. It is easy to construct examples in which P_H is a prime ideal of A properly contained in P, so that the density property may not be fulfilled.

As a positive example suppose that there exists an integer $r > 0$ such that H is the sum of subcoalgebras $C \in \mathcal{F}_r$ where

$$\mathcal{F}_r = \{C \in \mathcal{F} \mid \dim_{\kappa(\mathfrak{q})} \kappa(\mathfrak{q}) \otimes C \leq r \text{ for all } \mathfrak{q} \in \operatorname{Spec} k\}.$$

Let $n = rg$ where g is as in Lemma 1.10. Since $P_H = \bigcap_{C \in \mathcal{F}_r} P_C$ and $\mathfrak{I}_P^n \subset P_C$ for each $C \in \mathcal{F}_r$, we get $\mathfrak{I}_P^n \subset P_H$. In this case the density property holds for each $P \in \operatorname{Spec}_f A$. Another result of this kind will be presented in Corollary 6.4.

2. Preliminary results on freeness

Let A be an H-module algebra. The compatibility of the two module structures on objects $M \in {}_H\mathcal{M}_A$ and $N \in {}_{A\#H}\mathcal{M}$ is expressed as

$$h(va) = \sum_{(h)} (h_{(1)}v)(h_{(2)}a), \qquad h(aw) = \sum_{(h)} (h_{(1)}a)(h_{(2)}w)$$

for $h \in H$, $a \in A$, $v \in M$ and $w \in N$. The opposite algebra A^{op} is an H^{cop}-module algebra and ${}_{A\#H}\mathcal{M} \approx {}_{H^{\mathrm{cop}}}\mathcal{M}_{A^{\mathrm{op}}}$, but H^{cop} is only a bialgebra unless the antipode S is bijective. When $A \to B$ is a homomorphism of H-module algebras, there are well-defined H-module structures on $M \otimes_A B$ and $B \otimes_A N$ which give rise to functors

$$? \otimes_A B : {}_H\mathcal{M}_A \to {}_H\mathcal{M}_B, \qquad B \otimes_A ? : {}_{A\#H}\mathcal{M} \to {}_{B\#H}\mathcal{M}.$$

An object of either ${}_H\mathcal{M}_A$ or ${}_{A\#H}\mathcal{M}$ will be called *locally A-finite* if it is a directed union of A-finite subobjects. The functors above take A-finite objects to B-finite objects and locally A-finite objects to locally B-finite objects.

In this section we will be concerned with the freeness of R-modules $M^l \otimes_A R$ and $R \otimes_A N^l$ for certain integers l and ring homomorphisms $A \to R$. We will need several extensions of earlier results. There is a slight asymmetry of formulations for ${}_H\mathcal{M}_A$ and ${}_{A\#H}\mathcal{M}$ which disappears when $S(H) = H$. Only one statement in each right-left hand pair will be proved.

A ring R is said to be *weakly finite* if all one-sided invertible $n \times n$-matrices with entries in R are invertible on both sides. In other words, for each integer $n > 0$ every generating set for the free right R-module R^n containing exactly n elements is a basis for R^n, and the same holds for free left modules. Recall the definition of the injections $(*)$ from section 1. The ${}_H\mathcal{M}_A$-part of the next result in the special case when $Q = A/I_C$ and $R = A/I$ is given in [18, Lemma 1.3].

Lemma 2.1. *Let $n \geq 0$ be an integer, let $C \in \mathcal{F}$, and let I be an ideal of A. Suppose that there is a commutative diagram*

$$\begin{array}{ccc} A/I_C & \xrightarrow{(*)} & \mathrm{Hom}(C, A/I) \\ \downarrow & & \downarrow \\ Q & \longrightarrow & \mathrm{Hom}(C, R) \end{array} \qquad \left(\textit{resp.} \quad \begin{array}{ccc} A/I_{S(C)} & \xrightarrow{(*)} & \mathrm{Hom}(C^{\mathrm{cop}}, A/I) \\ \downarrow & & \downarrow \\ Q & \longrightarrow & \mathrm{Hom}(C^{\mathrm{cop}}, R) \end{array} \right)$$

where Q is an algebra, R is a weakly finite algebra, all maps are homomorphisms of algebras, the map on the right is obtained by functoriality from a homomorphism $A/I \to R$, and the map at the bottom is injective.

Let $M \in {}_H\mathcal{M}_A$ (resp. $N \in {}_{A\#H}\mathcal{M}$). If Q is a right (resp. left) Ore localization of A/I_C (resp. $A/I_{S(C)}$), the R-module $M \otimes_A R$ (resp. $R \otimes_A N$) is free of rank n, and the Q-module $M \otimes_A Q$ (resp. $Q \otimes_A N$) is n-generated, then the latter is free of rank n.

Proof. We will prove the ${}_{A\#H}\mathcal{M}$-part. By the hypothesis $Q = \Sigma^{-1}(A/I_{S(C)})$ where Σ is a left denominator subset of $A/I_{S(C)}$. Let $T = \operatorname{Hom}\big(C^{\mathrm{cop}}, R\big)$, so that the multiplication in T is the twist convolution

$$(\xi \times \eta)(c) = \sum_{(c)} \xi(c_{(2)})\eta(c_{(1)}), \qquad \xi, \eta \in T \text{ and } c \in C,$$

(see [4]). For each left R-module W we define a left action of T on the k-linear maps $\eta : C \to W$ using the same formula. This allows us to view $\operatorname{Hom}(C^{\mathrm{cop}}, ?)$ as an additive functor ${}_R\mathcal{M} \rightsquigarrow {}_TM$. In particular, $F = \operatorname{Hom}\big(C^{\mathrm{cop}}, R \otimes_A N\big)$ is a free left T-module of rank n.

For each $a \in A$ and $v \in N$ define $\tilde{a} \in T$ and $\tilde{v}, \hat{v} \in F$ by the formulas

$$\tilde{a}(c) = \text{the image of } S(c)a \text{ in } R, \qquad \tilde{v}(c) = 1 \otimes S(c)v, \qquad \hat{v}(c) = 1 \otimes \varepsilon(c)v.$$

In particular, $\tilde{a}$ is the image of a under the composite of ring homomorphisms

$$\varphi : A \to A/I_{S(C)} \to Q \hookrightarrow \operatorname{Hom}(C^{\mathrm{cop}}, R) = T$$

Given $a \in A$ and $v \in N$, we have $S(c)(av) = \sum_{(c)} S(c_{(2)})a \cdot S(c_{(1)})v$ for all $c \in C$, whence $\widetilde{av} = \tilde{a} \times \tilde{v}$. In other words, the assignment $v \mapsto \tilde{v}$ extends to a T-linear map $T \otimes_A N \to F$ where T is regarded as a ring extension of A by means of φ.

If $\eta \in F$, then $\eta(C)$ is a finitely generated k-submodule of $R \otimes_A N$. There exist a finite number of elements $w_1, \ldots, w_r \in N$ such that $\eta(C) \subset \sum R \otimes w_j$. Since C is k-projective, we can find $\xi_1, \ldots, \xi_r \in T$ such that $\eta(c) = \sum \xi_j(c) \otimes w_j$ for all $c \in C$, i.e. $\eta = \sum \xi_j \times \hat{w}_j$. This shows that the T-module F is generated by $\{\hat{w} \mid w \in N\}$.

Given $w \in N$, the k-submodule $Cw \subset N$ is finitely generated. Let $w_1, \ldots, w_r$ be any elements of N such that $\sum kw_j = Cw$. Since C is k-projective, there exist $\zeta_1, \ldots, \zeta_r \in \operatorname{Hom}(C, k)$ such that $cw = \sum_{j=1}^r \zeta_j(c)w_j$ for all $c \in C$. Note that

$$\begin{aligned} \hat{w}(c) = \sum_{(c)} 1 \otimes S(c_{(1)})c_{(2)}w &= \sum_{(c)} 1 \otimes S(c_{(1)}) \sum_{j=1}^r \zeta_j(c_{(2)})w_j \\ &= \sum_{j=1}^r \sum_{(c)} \zeta_j(c_{(2)}) 1 \otimes S(c_{(1)})w_j. \end{aligned}$$

In other words, $\hat{w} = \sum_{j=1}^r \zeta'_j \times \tilde{w}_j$ where $\zeta'_j \in R$ denotes the composite of ζ_j with the canonical map $k \to R$. It follows that F is also generated by $\{\tilde{w} \mid w \in N\}$.

Each element of $Q \otimes_A N$ can be written as $t^{-1} \otimes v$ for some $t \in \Sigma$ and $v \in N$. By the hypothesis the Q-module $Q \otimes_A N$ has a set of n generators which we may assume to be $1 \otimes v_1, \dots, 1 \otimes v_n$ for some $v_1, \dots, v_n \in N$. Since $Q \otimes_A (N/\sum Av_i) = 0$, for each $w \in N$ there exists $s \in A$ such that $s + I_{S(C)} \in \Sigma$ and $sw \in \sum Av_i + I_{S(C)}N$. Let $sw = \sum a_i v_i + u$ for some $a_1, \dots, a_n \in A$ and $u \in I_{S(C)}N$. Note that

$$S(c)u \in \sum_{(c)} S(c_{(2)})I_{S(C)} \cdot S(c_{(1)})N \subset IN \subset \mathrm{Ker}(N \to R \otimes_A N),$$

and therefore $\tilde{u}(c) = 1 \otimes S(c)u = 0$ for all $c \in C$. In other words, $\tilde{u} = 0$. It follows that $\tilde{s} \times \tilde{w} = \sum \tilde{a}_i \times \tilde{v}_i$. Since $s + I_{S(C)}$ is invertible in Q, its image $\tilde{s}$ is invertible in T. It follows that $\tilde{w}$ lies in the T-linear span of $\tilde{v}_1, \dots, \tilde{v}_n$.

We conclude that $\tilde{v}_1, \dots, \tilde{v}_n$ is a set of n generators for the T-module F. The ring $T \cong R \otimes (C^{\mathrm{cop}})^*$ is a finitely generated projective module over R. Since R is a weakly finite ring, so too is T, e.g. by [17, Lemma 2.1]. Hence $\tilde{v}_1, \dots, \tilde{v}_n$ are in fact a basis for F over T.

Suppose that $x_1, \dots, x_n \in Q$ are any elements such that $\sum_{i=1}^n x_i \otimes v_i = 0$ in $Q \otimes_A N$. Then $\sum_{i=1}^n \xi_i \otimes v_i = 0$ in $T \otimes_A N$, and therefore $\sum_{i=1}^n \xi_i \tilde{v}_i = 0$ in F, where ξ_i denotes the image of x_i in T. It follows that $\xi_i = 0$, and then $x_i = 0$, for each $i = 1, \dots, n$. Hence $1 \otimes v_1, \dots, 1 \otimes v_n$ are a basis for $Q \otimes_A N$ over Q. $\square$

If $P \in \operatorname{Spec} A$ is such that A/P has a simple artinian classical quotient ring, then we put

$$r_P(M) = \frac{\operatorname{length} M \otimes_A Q(A/P)}{\operatorname{length} Q(A/P)}, \qquad r_P(N) = \frac{\operatorname{length} Q(A/P) \otimes_A N}{\operatorname{length} Q(A/P)}$$

for each $M \in {}_H\mathcal{M}_A$ and $N \in {}_{A\#H}\mathcal{M}$ where length stands for the composition series length of $Q(A/P)$-modules. In particular, this notation will be used when P is a maximal ideal of A with an artinian factor ring A/P; in this case $Q(A/P) = A/P$.

Theorem 2.2. *Suppose that A is semilocal. Put $l = \gcd\{\operatorname{length} A/P \mid P \in \operatorname{Max} A\}$. Let $M \in {}_H\mathcal{M}_A$ and $N \in {}_{A\#H}\mathcal{M}$ be locally A-finite objects.*

(i) *If A is H-simple, then M^l is a free A-module.*

(ii) *If A is $S(H)$-simple, then N^l is a free A-module.*

Proof. Part (i) is proved in [17, Th. 7.6] and [19, Prop. 1.5]. Essentially the same proof will be repeated for (ii). By [17, Lemma 2.7] (or Lemma 2.3 below) the proof is reduced to the case of A-finite objects. So we assume N to be A-finite.

Choose $P \in \operatorname{Max} A$ with the maximum value of $r_P(N)$. Let $r_P(N) = n/d$ for some integers $n \ge 0$ and $d > 0$. Then $r_{P'}(N^d) = r_{P'}(N)d \le n$ for all $P' \in \operatorname{Max} A$, and the equality holds for $P' = P$. Since A/P' is a simple artinian ring, the A/P'-module $N^d/P'N^d$ is n-generated; this module is free of rank n when $P' = P$. It follows from Nakayama's Lemma that N^d is an n-generated A-module. Let $v_1, \dots, v_n$ be any generating set for this module. All semilocal rings, and so all factor rings of A, are weakly finite. Applying Lemma 2.1 with $I = P$, $Q = A/P_{S(C)}$ and $R = A/P$, we deduce that $N^d/P_{S(C)}N^d$ is a free $A/P_{S(C)}$-module of rank n. Since this module is generated by the cosets of $v_1, \dots, v_n$, the latter are a basis for this module. Assuming that $\sum x_i v_i = 0$ for some $x_1, \dots, x_n \in A$, we get $x_i \in \bigcap_{C \in \mathcal{F}} P_{S(C)} = P_{S(H)}$ for each i. Clearly $P_{S(H)}$ is an $S(H)$-stable ideal of A contained in P. Hence $P_{S(H)} = 0$, and so $v_1, \dots, v_n$ are a basis for the A-module N^d. The freeness of N^d implies that $r_P(N) = r_{P'}(N)$, and therefore $r_P(N) \cdot \operatorname{length}(A/P') \in \mathbb{Z}$, for each $P' \in \operatorname{Max} A$. Then $r_P(N)l \in \mathbb{Z}$, and we may take $d = l$. $\square$

In the next lemma H may be assumed to be an arbitrary bialgebra. By replacing H, A with $H^{\mathrm{cop}}, A^{\mathrm{op}}$ the result can be applied to objects of ${}_{A\#H}\mathcal{M}$.

Lemma 2.3. *Let $l > 0$ be an integer and $A \to R$ a homomorphism into a semilocal ring R. If $M^l \otimes_A R$ is a free R-module for each A-finite object $M \in {}_H\mathcal{M}_A$, then the same is true for each locally A-finite M.*

Proof. Our intention is to apply Zorn's Lemma as in the proof of [17, Th. 1.2]. Given an object $M \in {}_H\mathcal{M}_A$ and its subobject M', denote by $K_{M'M}$ and $T_{M'M}$, respectively, the kernel and the image of the canonical map $M' \otimes_A R \to M \otimes_A R$. So $T_{M'M}$ is a submodule of the R-module $M \otimes_A R$, and

$$(M \otimes_A R)/T_{M'M} \cong M/M' \otimes_A R$$

by the right exactness of tensor products. If M is A-finite, then $M^l \otimes_A R$ is a free R-module, and so too is $(M/M')^l \otimes_A R$ since M/M' is also A-finite. In this case

$$M^l \otimes_A R \cong T^l_{M'M} \oplus \big((M/M')^l \otimes_A R\big);$$

since free direct summands over a semilocal ring cancel out, we deduce that $T^l_{M'M}$ is a free R-module. If both M and M' are A-finite, then $K^l_{M'M}$ is also a free R-module since $(M')^l \otimes_A R$ is free and there is a split exact sequence of R-modules

$$0 \to K_{M'M} \to M' \otimes_A R \to T_{M'M} \to 0.$$

Suppose that M is an arbitrary locally A-finite object. Since tensor products commute with inductive direct limits, there is an isomorphism $M \otimes_A R \cong \varinjlim M'' \otimes_A R$ where M'' runs over the A-finite subobjects of M. Hence for each A-finite subobject $M' \subset M$ we have $K_{M'M} = \bigcup K_{M'M''}$ where M'' runs over the A-finite subobjects of M containing M'. As we have seen, each $K_{M'M''}$ here is a direct summand of the R-module $M' \otimes_A R$; moreover, $K^l_{M'M''}$ is a free R-module whose rank is bounded by that of $(M')^l \otimes_A R$. It follows that the family $\{K_{M'M''}\}$ has a largest element. In other words, there exists M'' such that $K_{M'M''} = K_{M'M}$, and then $T_{M'M''} \cong T_{M'M}$. We conclude that $T^l_{M'M}$ is a free R-module for an A-finite M'.

If M', M'' are arbitrary subobjects of M such that $M' \subset M''$, then there is an inclusion $T_{M'M} \subset T_{M''M}$, and $T_{M''M}/T_{M'M}$ may be identified with the image of the canonical map $M''/M' \otimes_A R \to M/M' \otimes_A R$. The previous paragraph with M/M' replacing M shows that $(T_{M''M}/T_{M'M})^l$ is a free R-module whenever M''/M' is A-finite. This implies that $T^l_{M''M} \cong T^l_{M'M} \oplus (T_{M''M}/T_{M'M})^l$. Hence $T^l_{M''M}$ is a free R-module whenever $T^l_{M'M}$ is free. Moreover, each basis for $T^l_{M'M}$ extends to a basis for $T^l_{M''M}$.

Consider the set Ω of all pairs (M', X) where M' is a subobject of M and X is a basis for the R-module $T^l_{M'M}$. For two pairs in Ω set $(M', X) \le (M'', Y)$ if and only if $M' \subset M''$ and $X \subset Y$. By Zorn's Lemma Ω has a maximal element. When $M' \ne M$, there exists an A-finite subobject $F \subset M$ such that $F \not\subset M'$. In this case $M' + F$ is a subobject of M properly containing M', and $(M' + F)/M'$ is A-finite; in view of the observation made in the previous paragraph M' cannot occur as the first component of a maximal element of Ω. Hence any maximal element of Ω is (M, X) where X is a basis for the R-module $M^l \otimes_A R$. $\square$

Proposition 2.4. *Suppose that A is module-finite over its center. Denote by l the length of the simple artinian ring $Q(A/P)$ where $P \in \operatorname{Spec}_f A$. Let $M \in {}_H\mathcal{M}_A$ and $N \in {}_{A\#H}\mathcal{M}$ be locally A-finite objects. Then:*

(i) $M^l \otimes_A Q(A/P_C)$ *is a free* $Q(A/P_C)$*-module for each* $C \in \mathcal{F}$.

(ii) *If* M *is* A*-finite, then* $r_{P'}(M) = r_P(M)$ *for each* $P' \in \mathrm{Eq}_H(P)$.

(iii) $Q(A/P_{S(C)}) \otimes_A N^l$ *is a free* $Q(A/P_{S(C)})$*-module for each* $C \in \mathcal{F}$.

(iv) *If* N *is* A*-finite, then* $r_{P'}(N) = r_P(N)$ *for each* $P' \in \mathrm{Eq}_{H^{\mathrm{bop}}}(P)$.

Proof. By Lemma 1.5 the ring $Q(A/P_C)$ is artinian. Hence Lemma 2.3 allows us to assume in (i) that M is A-finite. Let $g > 0$ be an integer such that A is a generated as a module over its center by g elements. For each $P' \in \operatorname{Spec} A$ the ring $Q(A/P')$ is an algebra of dimension at most g over a field, whence the length of $Q(A/P')$ does not exceed g, and so $r_{P'}(M)$ is a fraction whose denominator is bounded by g and numerator is bounded by the number of A-module generators for M. It follows that there exists finitely many possible values of $r_{P'}(M)$. Let

$$m = \max\{r_{P'}(M) \mid P' \in \mathrm{Eq}_H(P)\}.$$

To prove (ii) we may replace P with any $P' \in \mathrm{Eq}_H(P)$. Furthermore, if (ii) holds, then $r_{P'}(M) = m$ for each $P' \in \mathrm{Eq}_H(P)$. So it suffices to check that (i) and (ii) are true when $r_P(M) = m$.

The rational number $r_P(M)$ is $\frac{1}{l}$ times an integer, i.e. $n = r_P(M^l)$ is an integer. A computation of lengths shows that $M^l \otimes_A Q(A/P)$ is a free $Q(A/P)$-module of rank n. By Lemma 1.4 each simple factor ring of $Q(A/P_C)$ is isomorphic with $Q(A/P')$ where P' is a prime of A minimal over P_C; since $P' \sim_H P$, we have $r_{P'}(M^l) \le n$, and therefore the $Q(A/P')$-module $M^l \otimes_A Q(A/P')$ is n-generated. It follows from Nakayama's Lemma that the $Q(A/P_C)$-module $M^l \otimes_A Q(A/P_C)$ is n-generated. In view of Lemma 1.5 we meet the hypotheses of Lemma 2.1 with $I = P$, $Q = Q(A/P_C)$, $R = Q(A/P)$ and with M^l in place of M. This verifies (i).

Each $P' \in \mathrm{Eq}_H(P)$ is a prime minimal over P_C for some $C \in \mathcal{F}$. Since $Q(A/P')$ is a factor ring of $Q(A/P_C)$, the $Q(A/P')$-module $M^l \otimes_A Q(A/P')$ is free of rank n, and it follows that $r_{P'}(M) = n$. Now (ii) is proved. Parts (iii) and (iv) are similar. $\square$

Proposition 2.5. *Let* $M \in {}_H\mathcal{M}_A$ *be an* A*-finite object. Suppose that* A *is module-finite over its center and there is* $P \in \operatorname{Max} A$ *such that* $P_H = 0$ *and* $r_{P'}(M) \le r_P(M)$ *for all* $P' \in \operatorname{Max} A$. *Then* M *is projective in* $\mathcal{M}_A$ *and* $r_{P'}(M) = r_P(M)$ *for all* P'.

This is proved in [18, Cor. 5.5].

Lemma 2.6. *Let* R *be a ring,* M *a finitely generated right* R*-module,* Σ *a multiplicatively closed set consisting of central regular elements of* R. *Denote* $Q = R\Sigma^{-1}$. *If* $M \otimes_R Q$ *is a projective (resp. free)* Q*-module, then there exists* $s \in \Sigma$ *such that* $M \otimes_R R[s^{-1}]$ *is a projective (resp. free)* $R[s^{-1}]$*-module.*

Proof. Consider any exact sequence $0 \to K \to F \to M \to 0$ in $\mathcal{M}_R$ where F is a finitely generated free R-module. Applying the exact functor $? \otimes_R Q$, we obtain an exact sequence of Q-modules which has to split. Thus the identity transformation of $K \otimes_R Q$ extends to a Q-linear map $\varphi : F \otimes_R Q \to K \otimes_R Q$. We have $\varphi(F \otimes 1) \subset K \otimes s^{-1}$ for some $s \in \Sigma$. Since F is Σ-torsionfree, we may identify $F \otimes_R R[s^{-1}]$ and $K \otimes_R R[s^{-1}]$ with their images in $F \otimes_R Q$ and $K \otimes_R Q$, respectively. Thus φ induces an $R[s^{-1}]$-linear map $F \otimes_R R[s^{-1}] \to K \otimes_R R[s^{-1}]$ which provides a splitting for the exact sequence of $R[s^{-1}]$-modules

$$0 \to K \otimes_R R[s^{-1}] \to F \otimes_R R[s^{-1}] \to M \otimes_R R[s^{-1}] \to 0.$$

The projectivity of $M \otimes_R R[s^{-1}]$ is immediate.

Suppose that $M \otimes_R Q$ is free. We can find finitely many elements $v_1, \ldots, v_m \in M$ such that $v_1 \otimes 1, \ldots, v_m \otimes 1$ are a basis for $M \otimes_R Q$ over Q. Let $\varphi : R^m \to M$ be the R-linear map sending the standard free generators of R^m to $v_1, \ldots, v_m$. Then $\varphi \otimes_R Q$ is an isomorphism, whence $(\operatorname{Ker} \varphi) \otimes_R Q = 0$ and $(\operatorname{Coker} \varphi) \otimes_R Q = 0$. The first equality shows that $\operatorname{Ker} \varphi = 0$ since R^m is Σ-torsionfree. The second equality shows that $Mt \subset \operatorname{Im} \varphi$ for some $t \in \Sigma$ since M is finitely generated. This means that $\varphi \otimes_R R[t^{-1}]$ is an isomorphism of $R[t^{-1}]$-modules. Hence $M \otimes_R R[t^{-1}]$ is free. $\square$

Lemma 2.7. *Let Σ be a multiplicatively closed set consisting of central regular elements of A. Suppose that the ring $Q = A\Sigma^{-1}$ is right artinian. Denote by l the greatest common divisor of the lengths of simple factor rings of Q. Given an A-finite object $M \in {}_H\mathcal{M}_A$ or $N \in {}_{A\#H}\mathcal{M}$, there exists $s \in \Sigma$ such that*

(i) $M \otimes_A A[s^{-1}]$ *is a projective $A[s^{-1}]$-module when A is H-semiprime,*

(ii) $M^l \otimes_A A[s^{-1}]$ *is a free $A[s^{-1}]$-module when A is H-prime,*

(iii) $A[s^{-1}] \otimes_A N$ *is a projective $A[s^{-1}]$-module when A is $S(H)$-semiprime,*

(iv) $A[s^{-1}] \otimes_A N^l$ *is a free $A[s^{-1}]$-module when A is $S(H)$-prime.*

Proof. Since Q is right artinian, we have $Q \cong Q(A)$. The H-module structure extends from A to Q by Theorem 1.7, and $M \otimes_A Q$ is a Q-finite object of ${}_H\mathcal{M}_Q$. If A is H-prime, then Q is H-simple by Lemma 1.8, whence $M^l \otimes_A Q$ is a free Q-module by Theorem 2.2. If A is H-semiprime, then Q is a direct product of finitely many H-simple H-module algebras, say $Q_1, \ldots, Q_r$. In this case $M \otimes_A Q \cong \prod_{i=1}^r M_i$ where $M_i \in {}_H\mathcal{M}_{Q_i}$; since for each i there exists an integer $l_i > 0$ such that $M_i^{l_i}$ is a free Q_i-module, we deduce that $M \otimes_A Q$ is projective in $\mathcal{M}_Q$. An application of Lemma 2.6 yields (i) and (ii). In (iii) and (iv) we can proceed similarly since Q is $S(H)$-(semi)simple when A is $S(H)$-(semi)prime. $\square$

Lemma 2.8. *If s is chosen as in Lemma 2.7 then $\operatorname{Tor}_i^A\big(M, Q(A/P)\big) = 0$ (resp. $\operatorname{Tor}_i^A\big(Q(A/P), N\big) = 0$) for each $i > 0$ and each $P \in \operatorname{Spec} A$ with $s \notin P$ provided that the classical right quotient ring $Q(A/P)$ exists.*

Proof. Since $A[s^{-1}]$ is ${}_A\mathcal{M}$-flat, we have by [1]

$$\operatorname{Tor}_i^A(M, W) \cong \operatorname{Tor}_i^{A[s^{-1}]}(M \otimes_A A[s^{-1}], W) = 0$$

for any left $A[s^{-1}]$-module W and $i > 0$. Since A/P is a prime ring, its central element $s + P$ has to be regular in A/P whenever $s \notin P$. In this case we may take $W = Q(A/P)$ since the canonical ring homomorphism $A \to Q(A/P)$ factors through $A[s^{-1}]$. $\square$

3. Twisting of modules

Let A be an H-module algebra and U a right H-comodule with the comodule structure map $U \to U \otimes H$ written as $u \mapsto \sum_{(u)} u_{(0)} \otimes u_{(1)}$. For $V \in \mathcal{M}_A$ we define right A-module structures on $U \otimes V$ and $\operatorname{Hom}(U, V)$ by the formulas

$$(u \otimes v)a = \sum_{(u)} u_{(0)} \otimes v\big((Su_{(1)})a\big), \qquad (\eta a)(u) = \sum_{(u)} \eta(u_{(0)})(u_{(1)}a).$$

where $a \in A$, $u \in U$, $v \in V$ and $\eta \in \mathrm{Hom}(U, V)$. For $W \in {}_A\mathcal{M}$ we define left A-module structures on $W \otimes U$ and $\mathrm{Hom}(U, W)$ by the formulas

$$a(w \otimes u) = \sum_{(u)} (u_{(1)}a)w \otimes u_{(0)}, \qquad (a\zeta)(u) = \sum_{(u)} \big((Su_{(1)})a\big)\zeta(u_{(0)}).$$

where $a \in A$, $u \in U$, $w \in W$ and $\zeta \in \mathrm{Hom}(U, W)$.

Lemma 3.1. *Given $V, V' \in \mathcal{M}_A$ and $W, W' \in {}_A\mathcal{M}$, there are isomorphisms*

(i) $\mathrm{Hom}_A(U \otimes V, V') \cong \mathrm{Hom}_A\big(V, \mathrm{Hom}(U, V')\big)$,

(ii) $\mathrm{Hom}_A(W \otimes U, W') \cong \mathrm{Hom}_A\big(W, \mathrm{Hom}(U, W')\big)$,

(iii) $(U \otimes V) \otimes_A W \cong V \otimes_A (W \otimes U)$,

(iv) $\mathrm{Tor}_i^A(U \otimes V, W) \cong \mathrm{Tor}_i^A(V, W \otimes U)$ *for all i provided U is k-flat,*

(v) $\mathrm{Ext}_A^i(U \otimes V, V') \cong \mathrm{Ext}_A^i\big(V, \mathrm{Hom}(U, V')\big)$ *for all i provided U is k-projective,*

(vi) $\mathrm{Ext}_A^i(W \otimes U, W') \cong \mathrm{Ext}_A^i\big(W, \mathrm{Hom}(U, W')\big)$ *for all i provided U is k-projective.*

Proof. (i) This is proved in [20, Lemma 1.1].

(ii) Under the canonical bijection $\mathrm{Hom}(W \otimes U, W') \cong \mathrm{Hom}\big(W, \mathrm{Hom}(U, W')\big)$ the A-module homomorphisms $W \to \mathrm{Hom}(U, W')$ correspond precisely to the k-linear maps $\varphi : W \otimes U \to W'$ such that

$$\varphi(aw \otimes u) = \sum_{(u)} \big((Su_{(1)})a\big)\varphi(w \otimes u_{(0)})$$

for all $w \in W$, $u \in U$ and $a \in A$. If φ satisfies this identity then

$$\varphi\Big(\sum_{(u)} (u_{(1)}a)w \otimes u_{(0)}\Big) = \sum_{(u)} \big((Su_{(1)})u_{(2)}a\big)\varphi(w \otimes u_{(0)}) = a\varphi(w \otimes u),$$

which is the identity defining the A-module homomorphisms $W \otimes U \to W'$. Going in the opposite direction we see that the two identities are equivalent.

(iii) Clearly $(U \otimes V) \otimes_A W \cong (U \otimes V \otimes W)/K$ where K is the k-linear span of elements

$$\sum_{(u)} u_{(0)} \otimes v\big((Su_{(1)})a\big) \otimes w - u \otimes v \otimes aw$$

with $u \in U$, $v \in V$, $w \in W$ and $a \in A$. Similarly $V \otimes_A (W \otimes U) \cong (V \otimes W \otimes U)/L$ where L is the k-linear span of elements

$$va \otimes w \otimes u - \sum_{(u)} v \otimes (u_{(1)}a)w \otimes u_{(0)}.$$

Explicit calculations show that K corresponds to L under the canonical k-linear bijection $U \otimes V \otimes W \cong V \otimes W \otimes U$:

$$\sum_{(u)} v\big((Su_{(1)})a\big) \otimes w \otimes u_{(0)} \equiv \sum_{(u)} v \otimes \big(u_{(1)}(Su_{(2)})a\big)w \otimes u_{(0)} \equiv v \otimes aw \otimes u \pmod{L},$$

$$\sum_{(u)} u_{(0)} \otimes v \otimes (u_{(1)}a)w \equiv \sum_{(u)} u_{(0)} \otimes v\big((Su_{(1)})u_{(2)}a\big) \otimes w \equiv u \otimes va \otimes w \pmod{K}.$$

(iv) If F is a flat right A-module, then so too is $U \otimes F$ since (iii) implies that the functor $(U \otimes F) \otimes_A ? \cong F \otimes_A (? \otimes U)$ is exact. Moreover, $U \otimes F_\bullet \to U \otimes V \to 0$ is a flat resolution of $U \otimes V$ in $\mathcal{M}_A$ whenever $F_\bullet \to V \to 0$ is a flat resolution of V. The conclusion is obtained by computing the homology of the complex $(U \otimes F_\bullet) \otimes_A W \cong F_\bullet \otimes_A (W \otimes U)$.

(v) As follows from (i), tensoring with U preserves $\mathcal{M}_A$-projectivity. If $F_\bullet$ is a projective resolution of V then $U \otimes F_\bullet$ is a projective resolution of $U \otimes V$. It remains to use the isomorphism of complexes $\mathrm{Hom}_A(U \otimes F_\bullet, V') \cong \mathrm{Hom}_A\big(F_\bullet, \mathrm{Hom}(U, V')\big)$.

(vi) This is similar to (v). $\square$

Denote by U_{triv} the H-comodule which has the same underlying k-module as U but the trivial coaction of H. The A-module structures in $U_{\mathrm{triv}} \otimes V$ and $W \otimes U_{\mathrm{triv}}$ are given by

$$(u \otimes v)a = u \otimes va, \qquad a(w \otimes u) = aw \otimes u.$$

Lemma 3.2. *For each $M \in {}_H\mathcal{M}_A$ and each $N \in {}_{A\#H}\mathcal{M}$ there are isomorphisms $U \otimes M \cong U_{\mathrm{triv}} \otimes M$ in $\mathcal{M}_A$ and $N \otimes U \cong N \otimes U_{\mathrm{triv}}$ in ${}_A\mathcal{M}$.*

Proof. The first isomorphism is established in [20, Lemma 1.2]. Note that the k-linear transformation Φ of $N \otimes U$ defined by the rule $w \otimes u \mapsto \sum_{(u)} u_{(1)}w \otimes u_{(0)}$ has the inverse given by $w \otimes u \mapsto \sum_{(u)} S(u_{(1)})w \otimes u_{(0)}$. Moreover, Φ is an ${}_A\mathcal{M}$-isomorphism $N \otimes U_{\mathrm{triv}} \to N \otimes U$ since

$$\Phi(aw \otimes u) = \sum_{(u)} (u_{(1)}a)(u_{(2)}w) \otimes u_{(0)} = \sum_{(u)} a \cdot (u_{(1)}w \otimes u_{(0)})$$

for all $w \in N$, $u \in U$ and $a \in A$. $\square$

Lemma 3.3. *Suppose that the underlying k-module of U is finitely generated projectve. Let $U^* = \mathrm{Hom}(U, k)$ and $C = \sum_{\eta \in U^*} f_\eta(U)$ where $f_\eta : U \to H$ is defined by the rule $u \mapsto \sum_{(u)} \eta(u_{(0)})u_{(1)}$. Denote by I the annihilator of $V \in \mathcal{M}_A$ and by J the annihilator of $W \in {}_A\mathcal{M}$. Then:*

(i) *$U \otimes V \cong \mathrm{Hom}(U^*, V)$ in $\mathcal{M}_A$ and $\mathrm{Hom}(U, W) \cong W \otimes U^*$ in ${}_A\mathcal{M}$.*

(ii) *$U \otimes V$ and $\mathrm{Hom}(U, V)$ have annihilators, respectively, $I_{S(C)}$ and I_C in A.*

(iii) *$W \otimes U$ and $\mathrm{Hom}(U, W)$ have annihilators, respectively, J_C and $J_{S(C)}$ in A.*

Proof. (i) The isomorphisms here are just the canonical k-linear bijections; it is straightforward to check that they are indeed A-linear. The right H-comodule structure $\eta \mapsto \eta_{(0)} \otimes \eta_{(1)}$ on U^* satisfies the identity $\sum_{(\eta)} \eta_{(0)}(u)\eta_{(1)} = \sum_{(u)} \eta(u_{(0)})Su_{(1)}$.

(ii) Let $a \in A$. Given $u \in U$ and $v \in V$, we have $(u \otimes v)a = 0$ if and only if $\sum_{(u)} v\big((Su_{(1)})a\big)\eta(u_{(0)}) = 0$ for all $\eta \in U^*$. Hence a annihilates $U \otimes V$ if and only if $\sum_{(u)} \eta(u_{(0)})(Su_{(1)})a \in I$ for all u and η, i.e. $S(C)a \subset I$.

Given $v \in V$ and $\eta \in U^*$, let $\eta_v \in \mathrm{Hom}(U, V)$ correspond to $v \otimes \eta \in V \otimes U^*$ under the canonical k-linear bijection. We have $(\eta_v a)(u) = \sum_{(u)} v\eta(u_{(0)})(u_{(1)}a)$ for

$u \in U$. Hence a annihilates $\mathrm{Hom}(U, V)$ if and only if $\sum_{(u)} \eta(u_{(0)})u_{(1)}a \in I$ for all u and η, i.e. $Ca \subset I$.

(iii) This is similar to (ii). $\square$

If C is a subcoalgebra of H regarded as a right H-comodule with respect to the comultipication Δ, then for $U = C$ the subcoalgebra $\sum f_\eta(U)$ defined in Lemma 3.3 coincides with the original C. Indeed, $f_\varepsilon : C \to H$ is the identity map where ε is the counit of C and $f_\eta(C) \subset C$ for each $\eta \in C^*$.

If I is any ideal of A such that A/I has a classical quotient ring $Q(A/I)$, then the categories of right and left $Q(A/I)$-modules may be identified with full subcategories of $\mathcal{M}_A$ and ${}_A\mathcal{M}$, respectively.

Lemma 3.4. *Suppose that A is module-finite over its center. Let $P \in \mathrm{Spec}_f\, A$ and $C \in \mathcal{F}$. If V is a right $Q(A/P)$-module, W a left $Q(A/P)$-module and U a right C-comodule then*:

(i) *$U \otimes V$ is a $Q(A/P_{S(C)})$-module and $\mathrm{Hom}(U, V)$ is a $Q(A/P_C)$-module,*

(ii) *$W \otimes U$ is a $Q(A/P_C)$-module and $\mathrm{Hom}(U, W)$ is a $Q(A/P_{S(C)})$-module.*

Proof. There is a right $\mathrm{Hom}\big(C^{\mathrm{cop}}, Q(A/P)\big)$-module structure on $U \otimes V$ defined by the rule

$$(u \otimes v)\xi = \sum_{(u)} u_{(0)} \otimes v\xi(u_{(1)})$$

where $u \in U$, $v \in V$ and $\xi \in \mathrm{Hom}\big(C^{\mathrm{cop}}, Q(A/P)\big)$. By Lemma 1.5 we have a commutative diagram

$$\begin{array}{ccc}
A/P_{S(C)} & \longrightarrow & \mathrm{Hom}\big(C^{\mathrm{cop}}, A/P\big) \\
\downarrow & & \downarrow \\
Q(A/P_{S(C)}) & \longrightarrow & \mathrm{Hom}\big(C^{\mathrm{cop}}, Q(A/P)\big)
\end{array}$$

in which all maps are ring homomorphisms and the map on the top is induced by the map $\tau' : A \to \mathrm{Hom}\big(C^{\mathrm{cop}}, A/P\big)$ from section 1. Note that $(u \otimes v)a = (u \otimes v)\tau'(a)$ for all $u \in U$, $v \in V$ and $a \in A$. Thus we obtain a right $Q(A/P_{S(C)})$-module structure on $U \otimes V$ whose pullback to A is the A-module structure originally defined. Similarly, $Q(A/P_C)$ embeds into $\mathrm{Hom}\big(C, Q(A/P)\big)$, and the latter ring operates on $\mathrm{Hom}(U, V)$ by the rule

$$(\eta\xi)(u) = \sum_{(u)} \eta(u_{(0)})\xi(u_{(1)})$$

where $\eta \in \mathrm{Hom}(U, V)$, $\xi \in \mathrm{Hom}\big(C, Q(A/P)\big)$ and $u \in U$. In part (ii) we can use the left $\mathrm{Hom}\big(C, Q(A/P)\big)$-module structure on $W \otimes U$ and the left $\mathrm{Hom}\big(C^{\mathrm{cop}}, Q(A/P)\big)$-module structure on $\mathrm{Hom}(U, W)$ defined by the rules

$$\xi(w \otimes u) = \sum_{(u)} \xi(u_{(1)})w \otimes u_{(0)}, \qquad (\xi\zeta)(u) = \sum_{(u)} \xi(u_{(1)})\zeta(u_{(0)}).$$

$\square$

4. Vanishing of Tor

We assume that $S(H)=H$ throughout the whole section. Under this assumption the H^{bop}-orbits coincide with the H-orbits by Corollary 1.3. This will be used in the following lemma.

Lemma 4.1. *Let $A\to B$ be a homomorphism of H-module algebras. Suppose that B is module-finite over a central subring Z. Let $P\in\operatorname{Spec}_f B$ and $\mathfrak{p}=P\cap Z$. Let $\mathcal{I}$ be either the set of all positive integers or $\{i\in\mathbb{Z}\mid 0<i\le d\}$ for some fixed $d\ge 1$. Given a locally A-finite object $M\in{}_H\mathcal{M}_A$ (resp. $N\in{}_{A\#H}\mathcal{M}$), the equality*

$$\operatorname{Tor}_i^A\big(M,\,Q(B/P')\big)=0\qquad\big(\textit{resp. }\operatorname{Tor}_i^A\big(Q(B/P'),N\big)=0\big)$$

holds for all $i\in\mathcal{I}$ and all P' in the H-orbit of P provided that this is true for those P' that satisfy $P'\cap Z=\mathfrak{p}$.

Proof. We will give a proof for ${}_H\mathcal{M}_A$. Let W' be a simple left $Q(B/P')$-module for some $P'\in\operatorname{Eq}_H(P)$. Any left $Q(B/P')$-module is a direct sum of copies of W'. So we have to show that $\operatorname{Tor}_i^A(M,W')=0$ for all $i\in\mathcal{I}$. Since $P\sim_{H^{\mathrm{bop}}}P'$, there exists $C\in\mathcal{F}$ such that P is a prime minimal over $P'_{S(C)}$. There is a commutative diagram

$$\begin{array}{ccccc}
Z & \longrightarrow & K=Q\big(Z/(Z\cap P'_{S(C)})\big) & \longrightarrow & K/\mathfrak{p}K\\
\downarrow & & \downarrow & & \downarrow\\
B & \longrightarrow & T=Q(B/P'_{S(C)}) & \longrightarrow & T/PT
\end{array}$$

In view of Lemma 1.5 we have $T\cong B/P'_{S(C)}\otimes_{\overline{Z}}K$ where $\overline{Z}=Z/(Z\cap P'_{S(C)})$. Hence $T/PT\cong B/P\otimes_{\overline{Z}}K$. Since K is a flat $\overline{Z}$-algebra and the map $Z/\mathfrak{p}\to B/P$ is injective, the map $K/\mathfrak{p}K\to T/PT$ is also injective. Similarly $K\to T$ is injective.

The simple artinian ring $T/PT\cong Q(B/P)$ (see Lemma 1.4) is module-finite over its central subring $K/\mathfrak{p}K$. It follows that $K/\mathfrak{p}K$ is a field, i.e. $\mathfrak{p}K$ is a maximal ideal of K. If J denotes the Jacobson radical of T and I the intersection of those maximal ideals of T that do not contain $\mathfrak{p}K$, then $\mathfrak{p}I\subset J$, while $I\not\subset J$. Hence $\mathfrak{p}^nI^n=0$ for sufficiently large integer n, while $I^n\ne 0$. By Lemmas 3.3, 3.4 $G=\operatorname{Hom}(C,W')$ is a faithful left T-module. We get $I^nG\ne 0$ and $\mathfrak{p}^nI^nG=0$. Thus $\mathfrak{p}$ annihilates a nonzero T-submodule of G.

Since T is an artinian ring, G contains a simple T-submodule W'' annihilated by $\mathfrak{p}$. By Lemma 1.4 W'' is a $Q(B/P'')$-module for a prime P'' of B minimal over $P'_{S(C)}$. Since $P''\sim_{H^{\mathrm{bop}}}P'$, we have $P''\in\operatorname{Eq}_H(P)$. Moreover, $K\cap P''T=\mathfrak{p}K$, and therefore $Z\cap P''$ coincides with the preimage of $\mathfrak{p}K$ in Z, that is with $\mathfrak{p}$. By the hypothesis $\operatorname{Tor}_i^A(M,W'')=0$ for all $i\in\mathcal{I}$. Lemmas 3.1, 3.2 applied with $U=C$ show that

$$\operatorname{Tor}_i^A(M,W''\otimes U)\cong\operatorname{Tor}_i^A(U\otimes M,W'')\cong\operatorname{Tor}_i^A(U_{\mathrm{triv}}\otimes M,W'').$$

Since U is k-projective, U is a direct summand of a free k-module. Hence $U_{\mathrm{triv}}\otimes M$ is an $\mathcal{M}_A$-direct summand of a direct sum of copies of M, and the above isomorphisms yield $\operatorname{Tor}_i^A(M,W''\otimes U)=0$ for $i\in\mathcal{I}$.

By Lemma 3.1 the inclusion $W''\to G$ corresponds to a nonzero B-linear map $\xi: W''\otimes U\to W'$. The image of ξ is a nonzero B/P'-submodule of W'. Since W'

is a torsionfree B/P'-module, the annihilator of $\xi(W''\otimes U)$ in B coincides with P'. By Lemma 3.4 $W''\otimes U$ is a left $Q(B/P''_C)$-module. In particular, P''_C annihilates $W''\otimes U$, and therefore $P''_C\subset P'$. Then $P'''\subset P'$ for some $P'''\in\operatorname{Spec}B$ minimal over P''_C. We have $P''\sim_H P'''$. By Lemma 1.2 the three primes P',P'',P''' have equal coheights. This forces $P'=P'''$. In other words, P' is minimal over P''_C, whence $Q(B/P')$ is a factor ring of $Q(B/P''_C)$ by Lemma 1.4. Now we may view ξ as a homomorphism of left $Q(B/P''_C)$-modules, and the simplicity of W' ensures that ξ is surjective. Denoting $L=\operatorname{Ker}\xi$, we get

$$\operatorname{Tor}_1^A(M,W')\cong\operatorname{Ker}\bigl(M\otimes_A L\to M\otimes_A(W''\otimes U)\bigr),$$

$$\operatorname{Tor}_{i+1}^A(M,W')\cong\operatorname{Tor}_i^A(M,L)\qquad\text{when } i>0,\ i+1\in\mathcal{I}.$$

Note that $M\otimes_A B$ is a locally B-finite object of ${}_H\mathcal{M}_B$. Therefore

$$M\otimes_A Q(B/P''_C)\cong(M\otimes_A B)\otimes_B Q(B/P''_C)$$

is a projective $Q(B/P''_C)$-module by Proposition 2.4. Hence the functor $M\otimes_A?$ is exact on the category of left $Q(B/P''_C)$-modules. We deduce that $\operatorname{Tor}_1^A(M,W')=0$.

Suppose that $i>0$ is an integer such that for each $\mathfrak{P}\in\operatorname{Eq}_H(P)$ and each left $Q(B/\mathfrak{P})$-module X we have $\operatorname{Tor}_i^A(M,X)=0$. Each $Q(B/P''_C)$-module has a finite chain of submodules such that each factor is a $Q(B/\mathfrak{P})$-module for some prime $\mathfrak{P}$ of B minimal over P''_C; each $\mathfrak{P}$ appearing here is in the H-orbit of P. In particular, this applies to the $Q(B/P''_C)$-module L. We then deduce that $\operatorname{Tor}_i^A(M,L)=0$, and therefore $\operatorname{Tor}_{i+1}^A(M,W')=0$ when $i+1\in\mathcal{I}$. The required property of Tor_i^A has been checked already for $i=1$. Induction on i completes the proof. □

For each $W\in{}_A\mathcal{M}$ and an integer i the functor $\operatorname{Tor}_i^A(?,W)$ commutes with inductive direct limits [1]. Thus, when $\operatorname{Tor}_i^A(M,W)=0$ for all A-finite $M\in{}_H\mathcal{M}_A$, the equality holds also for all locally A-finite objects of ${}_H\mathcal{M}_A$. This provides a reduction to the case of A-finite objects in Proposition 4.2 and Theorem 4.6.

Proposition 4.2. *Suppose that A is a noetherian H-semiprime H-module algebra module-finite over its center Z. Let $P\in\operatorname{Spec}A$. If the closure of $\operatorname{Eq}_H(P)$ in $\operatorname{Spec}A$ contains at least one minimal prime of A then*

$$\operatorname{Tor}_i^A\bigl(M,\,Q(A/P)\bigr)=0\qquad\bigl(\textit{resp.}\ \operatorname{Tor}_i^A\bigl(Q(A/P),N\bigr)=0\bigr)$$

for all $i>0$ and all locally A-finite objects $M\in{}_H\mathcal{M}_A$ (resp. $N\in{}_{A\#H}\mathcal{M}$).

Proof. It suffices to consider the case when M is A-finite. By Theorem 1.6 A has an artinian classical quotient ring $Q(A)\cong A\Sigma^{-1}$ where Σ is the set of all central regular elements of A. By Lemma 2.8 there exists $s\in\Sigma$ such that $\operatorname{Tor}_i^A\bigl(M,Q(A/P')\bigr)=0$ for all $i>0$ and all $P'\in D_s$ where $D_s=\{P'\in\operatorname{Spec}A\mid s\notin P'\}$. The subset D_s is open in $\operatorname{Spec}A$ and contains all minimal primes of A since s is regular modulo each of those, e.g. by [6, Lemmas 7.4, 11.8]. It follows that $D_s\cap\operatorname{Eq}_H(P)\ne\varnothing$. Choose any $P'\in\operatorname{Eq}_H(P)$ with $s\notin P'$. If $P''\in\operatorname{Spec}A$ satisfies $P''\cap Z=P'\cap Z$ then $s\notin P''$; since $P''\in D_s$, we have $\operatorname{Tor}_i^A\bigl(M,Q(A/P'')\bigr)=0$ for $i>0$. Lemma 4.1 applied with $B=A$ and the identity map $A\to B$ shows that $\operatorname{Tor}_i^A\bigl(M,Q(A/\mathfrak{P})\bigr)=0$ for all $i>0$ and all $\mathfrak{P}\in\operatorname{Eq}_H(P')$. In particular, this holds for $\mathfrak{P}=P$. □

Lemma 4.3. *Let $A \to B$ be a ring homomorphism where the ring B is module-finite over a central noetherian subring Z. Suppose that $N \in {}_A\mathcal{M}$ admits a resolution $F_\bullet \to N \to 0$ by finitely generated flat A-modules (e.g., this holds when A is left noetherian and N is finitely generated). Let $\mathcal{I}$ be either the set of all positive integers or $\{i \in \mathbb{Z} \mid 0 < i \le d\}$ for some fixed $d \ge 1$. If $\operatorname{Tor}_i^A(V, N) = 0$ for all $i \in \mathcal{I}$ and all simple $V \in \mathcal{M}_B$, then this is true for arbitrary $V \in \mathcal{M}_B$.*

Proof. Since the functors $\operatorname{Tor}_i^A(?, N)$ commute with inductive direct limits, it suffices to prove the conclusion for finitely generated V.

The hypothesis implies that $\operatorname{Tor}_i^A(V, N) = 0$ for all $i \in \mathcal{I}$ whenever V has finite length in $\mathcal{M}_B$. Suppose that V is any finitely generated right B-module. Then V is also a finitely generated Z-module, whence so are all components of the complex $V \otimes_A F_\bullet$ and its homology groups $\operatorname{Tor}_i^A(V, N)$, $i \ge 0$. If $\mathfrak{p}$ is any maximal ideal of Z and $n > 0$ is an integer, then $B/B\mathfrak{p}^n$ is a module-finite algebra over $Z/\mathfrak{p}^n$; since the latter ring is artinian, so too is $B/B\mathfrak{p}^n$. Then $V/V\mathfrak{p}^n$ is a B-module of finite length, whence $\operatorname{Tor}_i^A(V/V\mathfrak{p}^n, N) = 0$ for all $i \in \mathcal{I}$. Taking $n = 1$, we deduce that the inclusion $V\mathfrak{p} \to V$ induces a surjection

$$\xi_\mathfrak{p} : \operatorname{Tor}_i^A(V\mathfrak{p}, N) \to \operatorname{Tor}_i^A(V, N).$$

Suppose that $0 \to T \to X \to Y \to 0$ is an exact sequence of finitely generated right B-modules. We first prove that the exactness is preserved after tensoring with N over A, and therefore the map $\operatorname{Tor}_1^A(X, N) \to \operatorname{Tor}_1^A(Y, N)$ induced by $X \to Y$ is surjective. Denote $K = \operatorname{Ker}(T \otimes_A N \to X \otimes_A N)$. For each $\mathfrak{p} \in \operatorname{Max} Z$ and each integer $n > 0$ we have an exact sequence

$$0 \to T/(T \cap X\mathfrak{p}^n) \to X/X\mathfrak{p}^n \to Y/Y\mathfrak{p}^n \to 0.$$

Since $\operatorname{Tor}_1^A(Y/Y\mathfrak{p}^n, N) = 0$, the map $T/(T \cap X\mathfrak{p}^n) \otimes_A N \to X/X\mathfrak{p}^n \otimes_A N$ is injective. It follows that K is contained in the kernel of $T \otimes_A N \to T/(T \cap X\mathfrak{p}^n) \otimes_A N$, that is, in the image of $(T \cap X\mathfrak{p}^n) \otimes_A N \to T \otimes_A N$. By the Artin-Rees Lemma for each integer $m > 0$ there exists $n > 0$ such that $T \cap X\mathfrak{p}^n \subset T\mathfrak{p}^m$, whence K is contained in the image of $T\mathfrak{p}^m \otimes_A N \to T \otimes_A N$. It follows that $K \subset \bigcap_{m>0} \mathfrak{p}^m (T \otimes_A N)$. Since $T \otimes_A N$ is a finitely generated Z-module, K is annihilated by an element in $Z \setminus \mathfrak{p}$ [11, Th. 8.9]. Since this is valid for each $\mathfrak{p}$, we conclude that $K = 0$, as claimed.

Suppose that $i \in \mathcal{I}$ has the property that each epimorphism $X \to Y$ of finitely generated right B-modules induces a surjection $\operatorname{Tor}_i^A(X, N) \to \operatorname{Tor}_i^A(Y, N)$. If $\mathfrak{p}$ is any maximal ideal of Z generated by elements $p_1, \ldots, p_l$, then $V\mathfrak{p} = \sum Vp_j$. We obtain an epimorphism $V^l \to V\mathfrak{p}$ in $\mathcal{M}_B$ defining it on the jth copy of V by the rule $v \mapsto vp_j$. The induced map

$$\eta_\mathfrak{p} : \operatorname{Tor}_i^A(V^l, N) \to \operatorname{Tor}_i^A(V\mathfrak{p}, N)$$

is surjective by the assumption. Then $\xi_\mathfrak{p} \circ \eta_\mathfrak{p}$ is also surjective. Note that $\operatorname{Tor}_i^A(V^l, N)$ is a direct sum of l copies of $\operatorname{Tor}_i^A(V, N)$, and the restriction of $\xi_\mathfrak{p} \circ \eta_\mathfrak{p}$ to the jth summand is given by the action of p_j on $\operatorname{Tor}_i^A(V, N)$. It follows that

$$\operatorname{Tor}_i^A(V, N) = \mathfrak{p} \cdot \operatorname{Tor}_i^A(V, N),$$

According to Nakayama's Lemma $\operatorname{Tor}_i^A(V, N)$ is annihilated by an element in $Z \setminus \mathfrak{p}$. Since this is valid for each $\mathfrak{p}$, we get $\operatorname{Tor}_i^A(V, N) = 0$. In particular, $\operatorname{Tor}_i^A(T, N) = 0$

when T is the kernel of an $\mathcal{M}_B$-epimorphism $X \to Y$ with a finitely generated X, which shows that $\operatorname{Tor}^A_{i+1}(X, N) \to \operatorname{Tor}^A_{i+1}(Y, N)$ is surjective. We may now proceed by induction on i. □

Lemma 4.4. *Let A be a ring, M a right A-module, P an ideal of A, and i an integer. Suppose that A/P has a classical right quotient ring. Then $\operatorname{Tor}^A_i\big(M, Q(A/P)\big) = 0$ if and only if $\operatorname{Tor}^A_i(M, A/P)$ is a torsion right A/P-module.*

Proof. Let $F_\bullet \to M \to 0$ be any flat resolution of M. We have an isomorphism of complexes $F_\bullet \otimes_A Q(A/P) \cong (F_\bullet \otimes_A A/P) \otimes_{A/P} Q(A/P)$. Since $Q(A/P)$ is a flat left A/P-module, the ith homology group is

$$\operatorname{Tor}^A_i\big(M, Q(A/P)\big) \cong \operatorname{Tor}^A_i(M, A/P) \otimes_{A/P} Q(A/P),$$

and the conclusion is immediate. □

Lemma 4.5. *Let A be a noetherian ring module-finite over its center, and M a right A-module. Suppose that M is noetherina as a module over the ring $E = \operatorname{End}_A M$. If $\operatorname{Tor}^A_1\big(M, Q(A/P)\big) = 0$ for all $P \in \operatorname{Spec} A$, then M is flat in $\mathcal{M}_A$.*

Proof. Suppose that M is not flat. Then $\operatorname{Tor}^A_1(M, ?)$ is not identically zero. Let P be an ideal of A maximal with respect to the property that P annihilates a finitely generated left A-module W such that $\operatorname{Tor}^A_1(M, W) \ne 0$. If $IJ \subset P$ for two ideals I, J of A, then $W' = JW$ is a submodule of W annihilated by I such that W/W' is annihilated by J; since the equalities $\operatorname{Tor}^A_1(M, W') = 0$ and $\operatorname{Tor}^A_1(M, W/W') = 0$ cannot hold simultaneously, we have either $I \subset P$ or $J \subset P$ by the maximality condition. Thus $P \in \operatorname{Spec} A$.

Recall that $Q(A/P) \cong \Sigma^{-1}(A/P)$ where Σ is the set of central regular elements of A/P. Each finitely generated torsion left A/P-module T is annihilated by an element in Σ. This means that the annihilator of T in A properly contains P, and therefore $\operatorname{Tor}^A_1(M, T) = 0$. In particular, this is valid when T is the torsion submodule of the A/P-module W. Hence $\operatorname{Tor}^A_1(M, W)$ embeds in $\operatorname{Tor}^A_1(M, W/T)$. Replacing W with W/T, we retain $\operatorname{Tor}^A_1(M, W) \ne 0$. So we may assume that W is a torsionfree A/P-module. Then W embeds in $Q(A/P) \otimes_A W$. Since $Q(A/P)$ is a simple artinian ring, we may also assume that $Q(A/P) \otimes_A W$ is a free $Q(A/P)$-module, replacing W with W^l for some integer $l > 0$. Choosing any basis $e_1, \ldots, e_n$ for this module, we can find $s \in \Sigma$ such that $se_i \in W$ for each $i = 1, \ldots, n$. Then $se_1, \ldots, se_n$ generate a free A/P-submoduke $F \subset W$ such that W/F is a torsion A/P-module. This implies that $\operatorname{Tor}^A_1(M, W/F) = 0$, and therefore $\operatorname{Tor}^A_1(M, F)$ is mapped onto $\operatorname{Tor}^A_1(M, W)$. We conclude that $\operatorname{Tor}^A_1(M, F) \ne 0$, i.e. $\operatorname{Tor}^A_1(M, A/P) \ne 0$.

However, $\operatorname{Tor}^A_1(M, A/P)$ is a torsion A/P-module by Lemma 4.4. We may regard $\operatorname{Tor}^A_1(M, A/P)$ as an $E, A/P$-bimodule. There is an exact sequence of left E-modules $0 \to \operatorname{Tor}^A_1(M, A/P) \to M \otimes_A P \to M$. Since P is a finitely generated left ideal of A, the E-module $M \otimes_A P$ is noetherian. Hence $\operatorname{Tor}^A_1(M, A/P)$ is finitely generated over E, and therefore there exists $s \in \Sigma$ annihilating $\operatorname{Tor}^A_1(M, A/P)$. The multiplication by s gives rise to an exact sequence $0 \to A/P \to A/P \to A/I \to 0$ in ${}_A\mathcal{M}$ where I is an ideal of A properly containing P. It follows from the induced exact sequence

$$\operatorname{Tor}^A_1(M, A/P) \to \operatorname{Tor}^A_1(M, A/P) \to \operatorname{Tor}^A_1(M, A/I)$$

that $\operatorname{Tor}^A_1(M, A/P) = \operatorname{Tor}^A_1(M, A/P) \cdot s = 0$ since $\operatorname{Tor}^A_1(M, A/I) = 0$ by the choice of P. We have arrived at a contradiction. □

The density property for H-orbits was introduced at the end of section 1. I don't know whether the result below remains valid when this property is not imposed.

Theorem 4.6. *Let $\varphi : A \to B$ be a homomorphism of noetherian H-module algebras module-finite over their centers. Suppose that the H-orbits in* $\operatorname{Spec} A$ *and* $\operatorname{Spec} B$ *have the density property. Suppose also that A is H-semiprime and $\varphi(I)B = B$ for each H-stable ideal I of A which contains a regular element of A. Then*:

(i) $\operatorname{Tor}_i^A(M, W) = 0$ *when $i > 0$, $M \in {}_H\mathcal{M}_A$ is locally A-finite and $W \in {}_B\mathcal{M}$.*

(ii) $\operatorname{Tor}_i^A(V, N) = 0$ *when $i > 0$, $N \in {}_{A\#H}\mathcal{M}$ is locally A-finite and $V \in \mathcal{M}_B$.*

(iii) *Each B-finite and ${}_H\mathcal{M}_A$-locally A-finite object $M \in {}_H\mathcal{M}_B$ is flat in $\mathcal{M}_A$.*

(iv) *Each B-finite and ${}_{A\#H}\mathcal{M}$-locally A-finite object $N \in {}_{B\#H}\mathcal{M}$ is flat in ${}_A\mathcal{M}$.*

Proof. (ii) It suffices to consider the case when N is A-finite. By Lemma 2.8 there exists a central regular element $s \in A$ such that $\operatorname{Tor}_i^A\big(Q(A/P), N\big) = 0$ for all $i > 0$ and all $P \in \operatorname{Spec} A$ such that $s \notin P$.

By Lemma 4.3 it suffices to prove the conclusion for each simple $V \in \mathcal{M}_B$. The annihilator $\mathfrak{M}$ of V in B is a maximal ideal. Replacing B with $B/\mathfrak{M}_H$, we may assume that $\mathfrak{M}_H = 0$ and, in particular, that B is H-prime. The density property ensures then that $\operatorname{Eq}_H(\mathfrak{M})$ is dense in $\operatorname{Spec} B$. So $\operatorname{Eq}_H(\mathfrak{M})$ intersects each nonempty open subset of $\operatorname{Spec} B$.

Denote by Z the center of B, by Z' the center of the subring $B' = \varphi(A)Z$ of B. Clearly $\varphi(s) \in Z'$. Put $\mathfrak{a} = Z \cap \varphi(s)Z'$. By Lemma 1.9 $\varphi(s)$ is a regular element of B, and therefore $\varphi(s)$ cannot lie in any minimal prime of Z'. Since Z' is an integral extension of Z, for each $\mathfrak{q} \in \operatorname{Spec} Z$ with $\mathfrak{a} \subset \mathfrak{q}$ there exists $\mathfrak{q}' \in \operatorname{Spec} Z'$ such that $\varphi(s) \in \mathfrak{q}'$ and $Z \cap \mathfrak{q}' = \mathfrak{q}$. Since the extension $Z \subset Z'$ satisfies the incomparability and $\mathfrak{q}'$ is not a minimal prime, neither is $\mathfrak{q}$. In particular, $\mathfrak{a}$ is not nil. Choose any nonnilpotent element $u \in \mathfrak{a}$. Then u is not contained in the prime radical of B, and therefore the open subset $D_u = \{\mathfrak{P} \in \operatorname{Spec} B \mid u \notin \mathfrak{P}\}$ of $\operatorname{Spec} B$ is nonempty.

Suppose that $\mathfrak{P}$ is any maximal ideal of B such that $u \notin \mathfrak{P}$. Then $B/\mathfrak{P}$ is a finite dimensional algebra over the field $Z/\mathfrak{p}$ where $\mathfrak{p} = Z \cap \mathfrak{P} \in \operatorname{Max} Z$. In particular, $B/\mathfrak{P}$ has finite length as a right B'-module. Let $V_0 \subset V_1 \subset \cdots \subset V_r$ be any composition series of this B'-module. For $j = 1, \ldots, r$ denote by P_j the annihilator of V_j/V_{j-1} in A. We have $P_j \in \operatorname{Spec} A$ since B' is a centralizing extension of $\varphi(A)$. Since $u \in Z \smallsetminus \mathfrak{p}$, we have $V_j u = V_j$; in particular, u does not annihilate V_j/V_{j-1}. By construction $u \in \varphi(s)Z'$, whence $\varphi(s)$ does not annihilate V_j/V_{j-1} either, i.e. $s \notin P_j$. Let $T_j \subset V_j/V_{j-1}$ denote the subset consisting of elements annihilated by a regular element of A/P_j. Since T_j is stable under the action of both A and Z, it is a B'-submodule. Since T_j is finitely generated over Z, there exists a regular element of A/P_j annihilating the whole T_j. But V_j/V_{j-1} is a faithful A/P_j-module and a simple B'-module. It follows that $T_j \ne V_j/V_{j-1}$, and then $T_j = 0$. This means that each regular element of A/P_j operates as a nonsingular linear transformation on the finite dimensional vector space V_j/V_{j-1} over $Z/\mathfrak{p}$. Hence V_j/V_{j-1} is a $Q(A/P_j)$-module. The choice of s at the very beginning of the proof ensures that $\operatorname{Tor}_i^A(V_j/V_{j-1}, N) = 0$ when $i > 0$. As this is valid for each j, we conclude that $\operatorname{Tor}_i^A(B/\mathfrak{P}, N) = 0$ for all $i > 0$.

Now pick any $\mathfrak{P} \in \operatorname{Eq}_H(\mathfrak{M}) \cap D_u$. We have $u \notin \mathfrak{P}$. Since $\mathfrak{M}$ is a maximal ideal of B, so too is $\mathfrak{P}$ by Lemma 1.2. If $\mathfrak{P}'$ is any maximal ideal of B such that $Z \cap \mathfrak{P}' = Z \cap \mathfrak{P}$, then $u \notin \mathfrak{P}'$, and therefore $\mathrm{Tor}_i^A(B/\mathfrak{P}', N) = 0$ for all $i > 0$. Since $\mathfrak{M} \in \mathrm{Eq}_H(\mathfrak{P})$, Lemma 4.1 yields $\mathrm{Tor}_i^A(B/\mathfrak{M}, N) = 0$, which is equivalent to $\mathrm{Tor}_i^A(V, N) = 0$.

(iii) Given $P \in \operatorname{Spec} A$, the exact sequence $0 \to P/P_H \to A/P_H \to A/P \to 0$ in ${}_A\mathcal{M}$ gives rise to an exact sequence

$$\mathrm{Tor}_1^A(M, A/P_H) \to \mathrm{Tor}_1^A(M, A/P) \to M \otimes_A P/P_H \to M \otimes_A A/P_H.$$

We may regard A/P_H as an A-finite object of ${}_{A\#H}\mathcal{M}$. By (ii) $\mathrm{Tor}_1^A(M, A/P_H) = 0$. The functor $M \otimes_A ?$ on the category of left A/P_H-modules coincides with the functor $M/MP_H \otimes_{A/P_H} ?$. Hence there is also an exact sequence connecting Tor_i^{A/P_H} which yields an isomorphism

$$\begin{aligned}\mathrm{Tor}_1^{A/P_H}(M/MP_H,\, A/P) &\cong \operatorname{Ker}(M \otimes_A P/P_H \to M \otimes_A A/P_H)\\ &\cong \mathrm{Tor}_1^A(M, A/P).\end{aligned}$$

Now M/MP_H is a locally A/P_H-finite object of ${}_H\mathcal{M}_{A/P_H}$. The H-orbit of P/P_H is dense in $\operatorname{Spec} A/P_H$ by the hypothesis. So Proposition 4.2 yields

$$\mathrm{Tor}_1^{A/P_H}\big(M/MP_H,\, Q(A/P)\big) = 0.$$

It follows from Lemma 4.4 with A/P_H in place of A that $\mathrm{Tor}_1^A(M, A/P)$ is a torsion A/P-module, whence $\mathrm{Tor}_1^A\big(M, Q(A/P)\big) = 0$ again by Lemma 4.4. Note that M is a noetherian module over the center Z of B. Since M is a Z, A-bimodule, we may apply Lemma 4.5 to complete the proof.

Parts (i), (iv) are proved similarly. □

Remarks. The density property for H-orbits in $\operatorname{Spec} B$ was actually used only for H-orbits of maximal ideals of B. The equalities $\varphi(I)B = B$ in the hypothesis can be replaced with the equalities $B\varphi(I) = B$, which makes use of a similar alteration in Lemma 1.9. Since inductive direct limits of flat modules are flat, both (iii) and (iv) hold, more generally, for locally B-finite objects.

5. Generic freeness

Let R be a ring, $\Sigma \subset R$ a multiplicatively closed subset consisting of central regular elements. For each ideal I of R denote by $(R/I)\Sigma^{-1}$ the Ore localization of R/I at the image of Σ in R/I. If $s \in \Sigma$, then $(R/I)[s^{-1}]$ will denote the Ore localization of R/I at the multiplicatively closed set $\{s^m + I \mid m \in \mathbb{Z},\ m \geq 0\}$.

Lemma 5.1. *Suppose $I_1, \ldots, I_n$ are ideals of R such that their intersection is nilpotent and $(I_i + I_j) \cap \Sigma \neq \varnothing$ for each pair of indices $i \neq j$. If M is a right R-module such that $M \otimes_R R\Sigma^{-1}$ is a free $R\Sigma^{-1}$-module and for each $i = 1, \ldots, n$ there exists $s_i \in \Sigma$ such that $M \otimes_R (R/I_i)[s_i^{-1}]$ is a free $(R/I_i)[s_i^{-1}]$-module then $M \otimes_R R[s^{-1}]$ is a free $R[s^{-1}]$-module for a suitable $s \in \Sigma$.*

Proof. Multiplying $s_1, \ldots, s_n$ by appropriately chosen elements in Σ, we may assume that $s_1, \ldots, s_n$ are equal and lie in each of the ideals $I_i + I_j$ with $i \neq j$. For any $s \in \Sigma$ we may pass from R to $R[s^{-1}]$, replacing $I_1, \ldots, I_n$ with their extensions in the latter ring and M with $M \otimes_R R[s^{-1}]$. Such an adjustment with $s = s_1$ makes the ideals $I_1, \ldots, I_n$ pairwise comaximal and M/MI_i a free R/I_i-module for each i.

Furthermore, nothing will change if we remove all ideals I_i equal to R. If $s \in I_i \cap \Sigma$, then $I_i \cdot R[s^{-1}] = R[s^{-1}]$. We may assume therefore that $I_i \cap \Sigma = \varnothing$ for all i.

By the Chinese Remainder Theorem $R/J \cong \prod R/I_i$ and $M/MJ \cong \prod M/MI_i$ where $J = \bigcap I_i$. If N is a submodule of M such that $N + MI_i = M$ for each i, then $N + MJ = M$, and therefore $N = M$ by Nakayama's Lemma.

Put $Q = R\Sigma^{-1}$ and $Q_i = (R/I_i)\Sigma^{-1}$. Note that $Q_i \neq 0$ since otherwise I_i would contain an element of Σ. The isomorphisms

$$M/MI_i \otimes_{R/I_i} Q_i \cong M \otimes_R Q_i \cong (M \otimes_R Q) \otimes_Q Q_i$$

show that the Q_i-module $M \otimes_R Q_i$ has two bases, one of the same cardinality as a basis for the R/I_i-module M/MI_i, the other of the same cardinality as a basis for the Q-module $M \otimes_R Q$.

Suppose that $M \otimes_R Q$ is not finitely generated. Neither is then $M \otimes_R Q_i$. The cardinality of a basis is an invariant of a free module unless the module is finitely generated [8, Cor. 1.2]. We deduce that any basis for M/MI_i has the same cardinality as a basis for $M \otimes_R Q$. In particular, this cardinality is independent of i. It follows that M/MJ is a free R/J-module. We can find an R-module homomorphism $\varphi : F \to M$ such that F is a free R-module and $\varphi \otimes_R R/J$ is an isomorphism. Then $\varphi \otimes_R Q_i$ is an isomorphism for each i since the ring extension $R \to Q_i$ factors through R/J. By Nakayama's Lemma φ is surjective. Hence $\psi = \varphi \otimes_R Q$ is also surjective. The latter has to be a split epimorphism of Q-modules by the freeness of $M \otimes_R Q$. Then

$$(\operatorname{Ker} \psi) \otimes_Q Q_i = \operatorname{Ker}(\psi \otimes_Q Q_i) = \operatorname{Ker}(\varphi \otimes_R Q_i) = 0$$

for each i. Since $Q_i \cong Q/I_iQ$ and $I_1Q, \dots, I_nQ$ are pairwise comaximal ideals of Q with a nilpotent intersection, it follows from Nakayama's Lemma that $\operatorname{Ker} \psi = 0$. The map $F \to F \otimes_R Q$ given by the assignment $x \mapsto x \otimes 1$ is injective since F is Σ-torsionfree. If $x \in \operatorname{Ker} \varphi$ then $x \otimes 1 \in \operatorname{Ker} \psi$. This shows that φ is injective. Thus φ is an isomorphism, and M is free.

Consider now the other case when $M \otimes_R Q$ is finitely generated. Then so too are $M \otimes_R Q_i$ and M/MI_i for each i. By Nakayama's Lemma M is finitely generated, so that we may apply Lemma 2.6. □

The following lemma is well known in the special case when A is a commutative domain.

Lemma 5.2. *Let A be a right noetherian ring and $A \to B$ a ring homomorphism such that $B = A'[z_1, \dots, z_n]$ where A' denotes the image of A in B and $z_1, \dots, z_n$ are central elements of B. Suppose that the ring $Q = A\Sigma^{-1}$ is simple artinian where Σ is a multiplicatively closed set consisting of central regular elements of A. If M is a finitely generated right B-module such that $M \otimes_A Q$ is a free Q-module, then $M \otimes_A A[s^{-1}]$ is a free $A[s^{-1}]$-module for a suitable $s \in \Sigma$.*

Proof. We proceed by induction on n. If $n = 0$ then M is finitely generated over A; in this case Lemma 2.6 applies. Suppose that $n > 0$ and the conclusion of the lemma holds for modules over the subring $C = A'[z_1, \dots, z_{n-1}]$ of B. Note that a Q-module is free if it is either not finitely generated or finitely generated of length divisible by $l = \operatorname{length} Q$. In particular $V^l \otimes_A Q$ is a free Q-module for any finitely generated right C-module V; by the induction hypothesis there exists $s \in \Sigma$, depending on V, such that $V^l \otimes_A A[s^{-1}]$ is a free $A[s^{-1}]$-module.

Define an ascending chain of finitely generated C-submodules $0 \subset M_0 \subset M_1 \subset \cdots$ of M as follows. Let M_0 be the C-linear span of a chosen finite generating set for the B-module M and $M_i = M_{i-1} + M_{i-1}z_n$ when $i > 0$. The action of z_n^i on M induces a C-module epimorphism $M_0 \to M_i/M_{i-1}$. Hence $M_i/M_{i-1} \cong M_0/N_i$ for some C-submodule $N_i \subset M_0$. Clearly $N_0 \subset N_1 \subset \cdots$. Since C is right noetherian, the finitely generated C-module M_0 is noetherian. Hence there exists an integer r such that $N_i = N_r$, and therefore $M_i/M_{i-1} \cong M_r/M_{r-1}$, for all $i > r$.

Choose any $s \in \Sigma$ such that both $M_r^l \otimes_A A[s^{-1}]$ and $(M_r/M_{r-1})^l \otimes_A A[s^{-1}]$ are free $A[s^{-1}]$-modules. For $i > r$ the exact sequence of $A[s^{-1}]$-modules

$$0 \to M_{i-1} \otimes_A A[s^{-1}] \to M_i \otimes_A A[s^{-1}] \to M_i/M_{i-1} \otimes_A A[s^{-1}] \to 0$$

splits by the projectivity of the last term. Since $B = C[z_n]$, we have $M = \bigcup M_i$, whence

$$M \otimes_A A[s^{-1}] \cong G \oplus F$$

where $G = M_r \otimes_A A[s^{-1}]$ and $F = \bigoplus_{i>r} M_i/M_{i-1} \otimes_A A[s^{-1}]$. Since F is isomorphic to the direct sum of a countable set of copies of $(M_r/M_{r-1})^l \otimes_A A[s^{-1}]$, it is a free $A[s^{-1}]$-module. If $M_r \otimes_A Q$ is a free Q-module, then we may assume, by the induction hypothesis, that G is a free $A[s^{-1}]$-module; in this case $G \oplus F$ is free. Suppose that $M_r \otimes_A Q$ is not free. Then $M_r \otimes_A Q$ has to be finitely generated over Q. Furthermore, $M_r \otimes_A Q$ has to be a proper submodule of $M \otimes_A Q$, which implies that $F \neq 0$, and then F cannot be finitely generated. Since G^l is a finitely generated free $A[s^{-1}]$-module, the direct sum $G^{(\mathbb{N})}$ of a countable set of copies of G is a countably generated free $A[s^{-1}]$-module. Hence $G^{(\mathbb{N})}$ is a direct summand of F. Since $G \oplus G^{(\mathbb{N})} \cong G^{(\mathbb{N})}$, we get $G \oplus F \cong F$. It follows that $G \oplus F$ is a free $A[s^{-1}]$-module in any case. □

Theorem 5.3. *Let $A \to B$ be a homomorphism of H-module algebras. Suppose Σ is a multiplicatively closed set consisting of central regular elements of A such that the ring $Q = A\Sigma^{-1}$ is right artinian. Put $l = \gcd\{\text{length}\, Q/P \mid P \in \operatorname{Max} Q\}$.*

(i) *If A is H-prime and right noetherian, B is right of finite type over A, then for each B-finite and ${}_H\mathcal{M}_A$-locally A-finite object $M \in {}_H\mathcal{M}_B$ there exists $s \in \Sigma$ such that $M^l \otimes_A A[s^{-1}]$ is a free $A[s^{-1}]$-module.*

(ii) *If A is $S(H)$-prime and left noetherian, B is left of finite type over A, then for each B-finite and ${}_{A\#H}\mathcal{M}$-locally A-finite object $N \in {}_{B\#H}\mathcal{M}$ there exists $s \in \Sigma$ such that $A[s^{-1}] \otimes_A N^l$ is a free $A[s^{-1}]$-module.*

Proof. We will prove (i); part (ii) is similar. Since $Q \cong Q(A)$, by Lemma 1.8 Q is an H-simple H-module algebra. Furthermore, $M \otimes_A Q$ is a locally Q-finite object of ${}_H\mathcal{M}_Q$. By Theorem 2.2 $M^l \otimes_A Q$ is a free Q-module.

Let $K_1, \ldots, K_p$ be all maximal ideals of Q. Denote $I_i = K_i \cap A$ and $A_i = A/I_i$. Then $K_i = I_iQ$, and therefore $A_i\Sigma^{-1} \cong Q/K_i$ is a simple artinian ring for each i. Since $(I_i + I_j)Q = K_i + K_j = Q$, we have $(I_i + I_j) \cap \Sigma \neq \varnothing$ for $i \neq j$. Furthermore, $\bigcap I_i$ is contained in the Jacobson radical $\bigcap K_i$ of Q. The latter is nilpotent since Q is right artinian.

Let $z_1, \ldots, z_n$ be central elements of B such that B is right module-finite over its subring $B' = A[z_1, \ldots, z_n]$. Then $M_i = M/MI_i$ is a finitely generated right module over $B_i = B'/B'I_i$ such that $M_i^l \otimes_{A_i} Q/K_i \cong M^l \otimes_A Q/K_i$ is a free Q/K_i-module. Lemma 5.2 applied to A_i, B_i, M_i^l shows that

$$M_i^l \otimes_{A_i} A_i[s_i^{-1}] \cong M^l \otimes_A A_i[s_i^{-1}]$$

is a free $A_i[s_i^{-1}]$-module for some $s_i \in \Sigma$. Lemma 5.1 completes the proof. $\square$

Lemma 5.4. *Let $\varphi : A \to B$ be an injective homomorphism of H-module algebras module-finite over their centers. Suppose that A is noetherian and H-prime, B is right of finite type over A and locally A-finite as an object of ${}_H\mathcal{M}_A$. Then there exists an H-stable dense open subset $U \subset \operatorname{Spec} A$ such that each $P \in U$ satisfies $\varphi^{-1}(BP) = P$ and $P + \varphi^{-1}(\mathfrak{P}) \neq A$ for some $\mathfrak{P} \in \operatorname{Max} B$; so $\varphi^{-1}(\mathfrak{P}) \subset P$ when $P \in \operatorname{Max} A$.*

Proof. Let Σ denote the set of central regular elements of A. By Theorem 1.6 $A\Sigma^{-1}$ is right artinian. By Theorem 5.3 there exists $s \in \Sigma$ such that the $A[s^{-1}]$-module $F = B^l \otimes_A A[s^{-1}]$ is free. Note that $F \neq 0$ since s is not nilpotent.

The open subsets $D_a = \{P \in \operatorname{Spec} A \mid a \notin P\}$ with $a \in A$ give a base for the topology on $\operatorname{Spec} A$. Since s is central in A, a prime P contains sa if and only if either $s \in P$ or $a \in P$. In other words, $D_s \cap D_a = D_{sa}$. If $D_a \neq \varnothing$, then $a \notin P$ for some minimal prime of A; since the regular element s is contained in none of the minimal primes, we get $sa \notin P$, and therefore $D_{sa} \neq \varnothing$. Thus D_s is dense in $\operatorname{Spec} A$. By Lemma 1.11 the H-stable subset $U = \bigcup_{s \notin P} \operatorname{Eq}_H(P)$ is also open and dense in $\operatorname{Spec} A$.

Let $P \in \operatorname{Spec} A$. The ideal $I = \varphi^{-1}(BP)$ of A satisfies $P \subset I$ and $BI = BP$. Hence $FI = FP$, and the freeness of F ensures that I and P extend to the same ideal of $A[s^{-1}]$, that is, for each $x \in I$ there exists an integer $n > 0$ such that $xs^n \in P$. When $s \notin P$, the primeness of P entails $I = P$.

For $P \in \operatorname{Spec} A$ the equality $\varphi^{-1}(BP) = P$ holds if and only if the map $\varphi \otimes_A A/P$ is injective; since $A/P \hookrightarrow Q(A/P)$ is a flat ring extension, this is equivalent to the injectivity of $\varphi \otimes_A Q(A/P)$. Thus it follows from the exact sequence

$$\operatorname{Tor}_1^A\bigl(B,\, Q(A/P)\bigr) \to \operatorname{Tor}_1^A\bigl(B/A,\, Q(A/P)\bigr) \to Q(A/P) \to B \otimes_A Q(A/P)$$

that $\varphi^{-1}(BP) = P$ whenever $\operatorname{Tor}_1^A\bigl(B/A,\, Q(A/P)\bigr) = 0$. The last equality does hold for each P with $s \notin P$ since in this case

$$\operatorname{Tor}_1^A\bigl(B,\, Q(A/P)\bigr) \cong \operatorname{Tor}_1^{A[s^{-1}]}\bigl(B \otimes_A A[s^{-1}],\, Q(A/P)\bigr) = 0$$

by the freeness of F, and we have checked already that $\varphi^{-1}(BP) = P$, i.e. the map $\varphi \otimes_A Q(A/P)$ is injective. Since B/A is a locally A-finite object of ${}_H\mathcal{M}_A$, Lemma 4.1 shows that $\operatorname{Tor}_1^A\bigl(B/A,\, Q(A/P)\bigr) = 0$ for all $P \in U$.

We conclude that $\varphi^{-1}(BP) = P$ for each $P \in U$. In particular $BP \neq B$ for such a P. Choose any maximal left ideal L of B containing BP. The annihilator $\mathfrak{P}$ of the simple left B-module B/L is a maximal ideal of B. Since the coset $1 + L \in B/L$ is annihilated by $\varphi(P) + \mathfrak{P}$, the conclusion is clear. $\square$

6. Results for comodule algebras

Let H be a Hopf algebra over a commutative ring k. For $\mathfrak{q} \in \operatorname{Spec} k$ we denote by $\kappa(\mathfrak{q})$ the field of fractions $Q(k/\mathfrak{q})$ and regard $H \otimes \kappa(\mathfrak{q})$ as a Hopf algebra over $\kappa(\mathfrak{q})$.

We assume in this section that H is k-flat and H has a family $\mathcal{F}$ of ideals satisfying the following conditions:

(1) for each $K \in \mathcal{F}$ the algebra H/K is finitely generated projective in $\mathcal{M}_k$,

(2) for every $K, L \in \mathcal{F}$ there exists $J \in \mathcal{F}$ such that $J \subset K \cap L$,

(3) for every $K, L \in \mathcal{F}$ there exists $J \in \mathcal{F}$ such that $\Delta(J) \subset K \otimes H + H \otimes L$,

(4) for each $K \in \mathcal{F}$ there exists $J \in \mathcal{F}$ such that $S(J) \subset K$,

(5) for each $K \in \mathcal{F}$ there exists $J \in \mathcal{F}$ such that $S(H) + J = H$ and $S^{-1}(J) \subset K$,

(6) for each $\mathfrak{q} \in \operatorname{Spec} k$ the ideals $K \otimes \kappa(\mathfrak{q})$ with $K \in \mathcal{F}$ have zero intersecction and form a cofinal subset in the set of all ideals of finite codimension in $H \otimes \kappa(\mathfrak{q})$.

The $\mathcal{F}$-*dual* of H is defined to be $H^\circ = \bigcup_{K \in \mathcal{F}} K^\perp$ where

$$K^\perp = \{f \in \operatorname{Hom}(H, k) \mid f(K) = 0\} \cong \operatorname{Hom}(H/K, k).$$

By (1) $K^\perp$ is a coalgebra, finitely generated projective in $\mathcal{M}_k$. In view of (2) the family $\mathcal{F}^\perp = \{K^\perp \mid K \in \mathcal{F}\}$ is directed by inclusion. For $K, L \in \mathcal{F}$ with $K \subset L$ the inclusion $L^\perp \to K^\perp$ is a homomorphism of coalgebras dual to the canonical homomorphism of algebras $H/K \to H/L$. This provides H° with a coalgebra structure. Condition (3) implies that H° is a subalgebra of the convolution algebra $\operatorname{Hom}(H, k)$. If J, K are as in (4), then $f \circ S \in J^\perp$ for each $f \in K^\perp$. Hence the assignment $f \mapsto f \circ S$ defines a k-linear transformation S° of H°. It is easily verified that H° is a Hopf algebra with antipode S°. The family of subcoalgebras $\mathcal{F}^\perp$ of H° satisfies the condition required in sections 1–5.

If J, K are as in (5), then each $f \in K^\perp$ can be written as $g \circ S$ for some $g \in J^\perp$ since S induces a k-linear bijection $H/S^{-1}(J) \to H/J$. This entails $S^\circ(H^\circ) = H^\circ$.

If k is a field, then (1)–(4) are satisfied for the family $\mathcal{F}$ of all ideals of finite codimension in H. Condition (6) means that H is residually finite dimensional. Suppose that H is of finite type over k. Then H is module-finite over its center Z_H and Z_H is a finitely generated k-algebra. By [5, Cor. 2] (see also [15]) there exists an integer $e > 0$ such that

$$\bigcap_{\mathfrak{m} \in \operatorname{Max} Z_H} \mathfrak{m}^e H = 0. \tag{$*$}$$

For each $\mathfrak{m}$ the field $Z_H/\mathfrak{m}$ is a finite extension of k by Hilbert's Nullstellensatz. Hence the artinian algebra $Z_H/\mathfrak{m}^e$ is finite dimensional, and therefore $\mathfrak{m}^e H$ is an ideal of finite codimension in H. In particular, H is residually finite dimensional. Since H is a PI algebra, the antipode S is bijective by [16, Cor. 2]. Therefore (5) is also fulfilled.

We assume further that A is an H-comodule algebra with the comodule structure map $\rho : A \to A \otimes H$. Recall that ρ is required to be an algebra homomorphism. For an ideal P of A we consider the composite homomorphism

$$\rho_P : A \xrightarrow{\rho} A \otimes H \xrightarrow{\text{can.}} A/P \otimes H.$$

We may regard A as a left H°-module algebra with respect to the module structure given by $fa = \sum_{(a)} f(a_{(1)}) a_{(0)}$ for $f \in H^\circ$ and $a \in A$. If $C = K^\perp$ for $K \in \mathcal{F}$, then $\operatorname{Hom}(C, A/P) \cong A/P \otimes H/K$ in view of (1). Since the map $\tau : A \to \operatorname{Hom}(C, A/P)$ from section 1 coincides with the composite of ρ_P and the projection to $A/P \otimes H/K$, it is clear that

$$P_C = \rho_P^{-1}(A/P \otimes K).$$

Any H-costable ideal of A is obviously H°-stable. When k is a field, the converse is true. For an arbitrary k there is a somewhat weaker conclusion:

Lemma 6.1. (i) *For each $P \in \operatorname{Spec} A$ we have $P_{H^\circ} = \operatorname{Ker} \rho_P$. In particular, P_{H° is the largest H-costable ideal of A contained in P.*

(ii) *Suppose that A is right (or left) noetherian. Then an ideal I of A is H-costable if and only if I is H°-stable. In order that A be H-(semi)prime as a comodule algebra, it is necessary and sufficient that A be H°-(semi)prime as a module algebra.*

Proof. Denote by $\mathcal{X}$ the class of right A-modules V satisfying $\bigcap_{K\in\mathcal{F}}(V \otimes K) = 0$ (since K is a k-module direct summand of H by (1), we may identify $V \otimes K$ with a direct summand of $V \otimes H$). First we will check that $\mathcal{X}$ is closed under extensions. Suppose that $V \in \mathcal{M}_A$ has a submodule V' such that both V' and V/V' are in $\mathcal{X}$. Then we must have

$$\bigcap_{K\in\mathcal{F}} (V \otimes K) \subset \operatorname{Ker}(V \otimes H \to V/V' \otimes H) = V' \otimes H.$$

Furthermore, $(V \otimes K) \cap (V' \otimes H) = V' \otimes K$ since the map $V' \otimes H/K \to V \otimes H/K$ is injective. The above inclusion then yields $\bigcap_{K\in\mathcal{F}}(V \otimes K) \subset \bigcap_{K\in\mathcal{F}}(V' \otimes K) = 0$. Thus $V \in \mathcal{X}$, as claimed.

Suppose now that V is any fully faithful right A/P-module for some $P \in \operatorname{Spec} A$, so that each nonzero submodule of V is a faithful A/P-module. Let $\mathfrak{q} \in \operatorname{Spec} k$ denote the contraction of P. Then V is a torsionfree $k/\mathfrak{q}$-module, whence so too is $V \otimes H \cong V \otimes_{k/\mathfrak{q}} H/\mathfrak{q}H$ since $H/\mathfrak{q}H$ is a flat $k/\mathfrak{q}$-module. It follows that $V \otimes H$ embeds in

$$V \otimes H \otimes \kappa(\mathfrak{q}) \cong \big(V \otimes \kappa(\mathfrak{q})\big) \otimes_{\kappa(\mathfrak{q})} \big(H \otimes \kappa(\mathfrak{q})\big).$$

The image of $\bigcap_{K\in\mathcal{F}}(V \otimes K)$ is contained in $\bigcap_{K\in\mathcal{F}}\big(V \otimes \kappa(\mathfrak{q})\big) \otimes_{\kappa(\mathfrak{q})} \big(K \otimes \kappa(\mathfrak{q})\big)$, which is zero by (6). Hence $V \in \mathcal{X}$.

We conclude that $\mathcal{X}$ contains every right A-module which has a finite chain of submodules such that each factor is a fully faithful A/P-module for some prime of A. When A is right noetherian, each finitely generated right A-module has this property and therefore belongs to $\mathcal{X}$.

Now take $V = A/I$. The equality $\bigcap_{K\in\mathcal{F}}(A/I \otimes K) = 0$ implies that

$$I_{H^\circ} = \bigcap_{C\in\mathcal{F}^\perp} I_C = \rho_I^{-1}\Big(\bigcap_{K\in\mathcal{F}} (A/I \otimes K)\Big) = \operatorname{Ker} \rho_I = \rho^{-1}(I \otimes H).$$

Moreover, the inclusion $(\rho \otimes \mathrm{id})\rho(I_{H^\circ}) = (\mathrm{id} \otimes \Delta)\rho(I_{H^\circ}) \subset I \otimes H \otimes H$ shows that

$$\rho(I_{H^\circ}) \subset \operatorname{Ker}\big(\rho_I \otimes \mathrm{id} : A \otimes H \to A/I \otimes H \otimes H\big) = I_{H^\circ} \otimes H.$$

Hence I_{H° is an H-costable ideal of A. If I is H°-stable, then $I_{H^\circ} = I$, and (ii) is proved. When $I = P$, the noetherian hypothesis is not needed, and we obtain (i). □

Lemma 6.2. *If k is a field, then the contraction map $\operatorname{Spec}(A \otimes B) \to \operatorname{Spec} A$ is surjective for any two algebras A and B.*

Proof. We may identify A with the subalgebra $A \otimes 1$ of $A \otimes B$. Let $P \in \operatorname{Spec} A$. Then $P \otimes B$ is an ideal of $A \otimes B$ such that $(P \otimes B) \cap A = P$. By Zorn's Lemma $A \otimes B$ has an ideal $\mathfrak{P}$ maximal with respect to the property $\mathfrak{P} \cap A = P$. A standard check shows that $\mathfrak{P}$ is prime. $\square$

Lemma 6.3. *Suppose that H is of finite type over k and A is module-finite over its center. For $P \in \operatorname{Spec} A$ denote by $\mathfrak{I}_P$ the intersection of the ideals $P' \in \operatorname{Spec} A$ such that $P_C \subset P'$ for some $C \in \mathcal{F}^\perp$. There exists an integer $n > 0$ satisfying $\mathfrak{I}_P^n \subset P_{H^\circ}$. When k is a field, this integer depends only on H and A, but not on P.*

Proof. We will reduce the proof to the case when k is a field. In this case condition (6) allows us to assume that $\mathcal{F}$ is the family of all ideals of finite codimension in H.

Step 1. Suppose that k is an algebraically closed field. Denote by Z_H the center of H, by Z_A the center of A. Let $e, g, h > 0$ be integers such that e satisfies $(*)$, A is a Z_A-linear span of g elements, and H is a Z_H-linear span of h elements. We will show that the conclusion holds with $n = egh$. The quotient ring $Q(A/P)$ is an anlgebra of dimension at most g over the field $\kappa(\mathfrak{p}) = Q(Z_A/\mathfrak{p})$ where $\mathfrak{p} = Z_A \cap P$ (see section 1). Let $C = (\mathfrak{m}H)^\perp$ where $\mathfrak{m} \in \operatorname{Max} Z_H$. Note that $H/\mathfrak{m}H$ is an algebra of dimension at most h over the field $\kappa(\mathfrak{m}) = Z_H/\mathfrak{m}$. Hence $C \in \mathcal{F}^\perp$ and the ring $Q(A/P) \otimes H/\mathfrak{m}H$ has a basis over its central subring $R = \kappa(\mathfrak{p}) \otimes \kappa(\mathfrak{m})$ consisting of at most gh elements. Since k is algebraically closed, R is a domain. Denoting $F = Q(R)$, we obtain embeddings of rings

$$A/P_C \hookrightarrow A/P \otimes H/\mathfrak{m}H \hookrightarrow Q(A/P) \otimes H/\mathfrak{m}H \hookrightarrow \big(Q(A/P) \otimes H/\mathfrak{m}H\big) \otimes_R F$$

where the latter ring is an algebra of dimension at most gh over the field F. Any multiplicatively closed nil subset in this algebra is nilpotent with nilpotency index at most gh. In particular, the prime radical N of A/P_C satisfies $N^{gh} = 0$. Since $(\mathfrak{I}_P + P_C)/P_C \subset N$, we get $\mathfrak{I}_P^{gh} \subset P_C$ (cf. the proof of Lemma 1.10), which yields

$$\rho_P(\mathfrak{I}_P^{gh}) \subset A/P \otimes \mathfrak{m}H.$$

Since this inclusion is valid for each $\mathfrak{m}$, we get

$$\rho_P(\mathfrak{I}_P^n) \subset \rho_P(\mathfrak{I}_P^{gh})^e \subset \bigcap_{\mathfrak{m} \in \operatorname{Max} Z_H} \big(A/P \otimes \mathfrak{m}^e H\big) = 0.$$

Thus $\mathfrak{I}_P^n \subset \operatorname{Ker} \rho_P = P_{H^\circ}$.

Step 2. Suppose that k is an arbitrary field. Let $\bar{k}$ denote the algebraic closure of k. Extending the field, we obtain a Hopf algebra $H \otimes \bar{k}$ of finite type over $\bar{k}$ and an $H \otimes \bar{k}$-module algebra $A \otimes \bar{k}$ module-finite over its center. Let n be the integer given by Step 1 for the pair $H \otimes \bar{k}$, $A \otimes \bar{k}$.

All maximal ideals of $H \otimes \bar{k}$ contract to maximal ideals of Z_H since in the chain of ring extensions $Z_H \hookrightarrow Z_H \otimes \bar{k} \hookrightarrow H \otimes \bar{k}$ the first is integral and the second is module-finite. Each ideal K of finite codimension in $H \otimes \bar{k}$ contains a finite product of maximal ideals. Therefore K contains a finite product $\mathfrak{a}$ of maximal ideals of Z_H, and then $K \supset L \otimes \bar{k}$ where $L = \mathfrak{a}H$ is an ideal of finite codimension in H. In other words, the ideals $L \otimes \bar{k}$ with $L \in \mathcal{F}$ form a cofinal subset in the set of all ideals of finite codimension in $H \otimes \bar{k}$ directed by inverse inclusion. It follows that the finite dual of the Hopf algebra $H \otimes \bar{k}$ is isomorphic with $H^\circ \otimes \bar{k}$.

We may identify A with the k-subalgebra $A \otimes 1$ of $A \otimes \bar{k}$. By Lemma 6.2 there exists $\mathfrak{P} \in \operatorname{Spec} A \otimes \bar{k}$ satisfying $A \cap \mathfrak{P} = P$. Since $A \otimes \bar{k}$ is a centralizing extension of A, we have $A \cap \mathfrak{P}' \in \operatorname{Spec} A$ for each $\mathfrak{P}' \in \operatorname{Spec} A \otimes \bar{k}$. Suppose that $\mathfrak{P}_D \subset \mathfrak{P}'$ for some finite dimensional subcoalgebra D of $H^\circ \otimes \bar{k}$. We have $D \subset C \otimes \bar{k}$ for some $C \in \mathcal{F}^\perp$; clearly $P_C \subset \mathfrak{P}_D$. It follows that $P_C \subset P'$ where $P' = A \cap \mathfrak{P}'$, and therefore $\mathfrak{I}_P \subset P' \subset \mathfrak{P}'$. Taking the intersection over all such primes $\mathfrak{P}'$, we get $\mathfrak{I}_P \subset \mathfrak{I}_\mathfrak{P}$. Hence

$$\mathfrak{I}_P^n \subset A \cap \mathfrak{I}_\mathfrak{P}^n \subset A \cap \mathfrak{P}_{H^\circ \otimes \bar{k}} = P_{H^\circ}.$$

where the last equality is clear since $\mathfrak{P}_{H^\circ \otimes \bar{k}}$ is the largest H°-stable ideal of $A \otimes \bar{k}$ contained in $\mathfrak{P}$.

Step 3. Consider now the general case. Let n be the integer given by Step 2 for the pair $H \otimes \kappa(\mathfrak{q})$, $A \otimes \kappa(\mathfrak{q})$ over the base field $\kappa(\mathfrak{q})$ where $\mathfrak{q} \in \operatorname{Spec} k$ is the contraction of P to k. Condition (6) ensures that the finite dual of the Hopf algebra $H \otimes \kappa(\mathfrak{q})$ is isomorphic with $H^\circ \otimes \kappa(\mathfrak{q})$. Clearly the canonical homomorphism $\varphi : A \to A \otimes \kappa(\mathfrak{q})$ commutes with the action of H°. The prime ideal P of A extends to a prime ideal $\mathfrak{P}$ of $A \otimes \kappa(\mathfrak{q})$ which satisfies $\varphi^{-1}(\mathfrak{P}) = P$. We have $\varphi^{-1}(\mathfrak{P}') \in \operatorname{Spec} A$ for each $\mathfrak{P}' \in \operatorname{Spec} A \otimes \kappa(\mathfrak{q})$. It is checked as in Step 2 that $\varphi(\mathfrak{I}_P) \subset \mathfrak{I}_\mathfrak{P}$, whence

$$\mathfrak{I}_P^n \subset \varphi^{-1}(\mathfrak{I}_\mathfrak{P}^n) \subset \varphi^{-1}(\mathfrak{P}_{H^\circ \otimes \kappa(\mathfrak{q})}) = P_{H^\circ}.$$

□

Corollary 6.4. *If A and H are as in Lemma* 6.3, *then all H°-orbits in* $\operatorname{Spec}_f A$ *have the density property.*

Proof. This follows from the nilpotency of $\mathfrak{I}_P / P_{H^\circ}$ (see section 1). □

Corollary 6.5. *Let A and H be as in Lemma* 6.3. *Suppose also that A is noetherian. Then for each H°-stable closed subset $X \subset \operatorname{Spec} A$ there exists an H-costable ideal I of A such that $X = \{P \in \operatorname{Spec} A \mid I \subset P\}$.*

Proof. Denote by J the intersection of all ideals $P \in X$. The closedness of X means that $X = \{P \in \operatorname{Spec} A \mid J \subset P\}$. Since A is noetherian, there are finitely many primes $P_1, \ldots, P_r$ minimal over J; these are the minimal elements of X. Since X is H°-stable, we have $\operatorname{Eq}_{H^\circ}(P_i) \subset X$ for each i. Therefore $J = \bigcap P_i = \bigcap \mathfrak{I}_{P_i}$. By Lemma 6.3 there exists an integer $n > 0$ such that $\mathfrak{I}_{P_i}^n \subset (P_i)_{H^\circ}$ for each i. The H-costable ideal $I = \bigcap (P_i)_{H^\circ}$ satisfies $J^n \subset I \subset J$, whence I and J are contained in the same primes of A. □

Each object $M \in \mathcal{M}_A^H$ is endowed with structures of a right A-module and a right H-comodule such that

$$\sum (ma)_{(0)} \otimes (ma)_{(1)} = \sum m_{(0)} a_{(0)} \otimes m_{(1)} a_{(1)}$$

for all $m \in M$ and $a \in A$. The formula $fm = \sum m_{(0)} f(m_{(1)})$ where $f \in H^\circ$ and $m \in M$ defines a left H°-module structure which makes M an object of ${}_{H^\circ}\mathcal{M}_A$.

Suppose A is right noetherian. If $V \subset M$ is any finitely generated k-submodule, then $H^\circ V$ is contained in a finitely generated k-submodule of M by the explicit formula defining the action of H°; therefore $H^\circ V A$ is an A-finite ${}_{H^\circ}\mathcal{M}_A$-subobject of M. It follows that all objects of $\mathcal{M}_A^H$ are locally A-finite objects of ${}_{H^\circ}\mathcal{M}_A$.

Similarly, if A is left noetherian then all objects of ${}_A\mathcal{M}^H$ are locally A-finite objects of ${}_{A\#H^\circ}\mathcal{M}$.

Passing to the Hopf algebra H°, we can now apply all results from sections 4 and 5 to the H-coequivariant modules. In particular, Theorem 0.2 is a translation of Theorem 4.6. This theorem is valid over an arbitrary base ring k provided H satisfies all aassumptions made in this section. Theorem 5.3 yields

Theorem 6.6. *Let B be an H-comodule algebra right (resp. left) of finite type over a right (resp. left) noetherian H-prime H-comodule algebra A. Suppose that Σ is a multiplicatively closed set consisting of central regular elements of A such that the ring $Q = A\Sigma^{-1}$ is right artinian. Denote $l = \gcd\{\operatorname{length} Q/P \mid P \in \operatorname{Max} Q\}$. Then for each B-finite object $M \in \mathcal{M}_B^H$ (resp. $N \in {}_B\mathcal{M}^H$) there exists $s \in \Sigma$ such that $M^l \otimes_A A[s^{-1}]$ (resp. $A[s^{-1}] \otimes_A N^l$) is a free $A[s^{-1}]$-module.*

Note that for the $\mathcal{M}_B^H$-part of Theorem 6.6 the equality $S^\circ(H^\circ) = H^\circ$ is not needed, and therefore condition (5) may be omitted. In view of Theorem 1.6 we can apply Theorem 6.6 in the case when A is module-finite over its center. When k is a field, we obtain Theorem 0.3.

Finally we will prove the remaining results stated in the introduction where k is assumed to be a field. We may also assume that $\mathcal{F}$ contains all ideals of finite codimension in H.

Proof of Theorem 0.4. By Corollary 6.4 $\operatorname{Eq}_{H^\circ}(P)$ is dense in $\operatorname{Spec} A$. The assumption $P_{H^\circ} = 0$ also implies that A is an H°-prime H°-module algebra. We may regard $B = A/P \otimes H$ as an H-comodule algebra with respect to the comodule structure map $\mathrm{id} \otimes \Delta : A/P \otimes H \to A/P \otimes H \otimes H$. The map $\rho_P : A \to B$ is a homomorphism of H-comodule algebras, and therefore a homomorphism of H°-module algebras. By Lemma 6.1 ρ_P is injective.

The H°-orbit $\operatorname{Eq}_{H^\circ}(P)$ consists of ideals $P' \in \operatorname{Max} A$ such that $P_C \subset P'$ for some $C \in \mathcal{F}^\perp$, i.e. $\rho_P^{-1}(A/P \otimes K) \subset P'$ for some $K \in \mathcal{F}$. Since $A/P \otimes K$ is an ideal of finite codimension in B, it contains a product of maximal ideals. Hence each $P' \in \operatorname{Eq}_{H^\circ}(P)$ contains $\rho_P^{-1}(\mathfrak{P})$ for some $\mathfrak{P} \in \operatorname{Max} B$. Conversely, suppose that $\mathfrak{P}$ is any maximal ideal of B. Then $K = \{h \in H \mid A/P \otimes h \subset \mathfrak{P}\}$ is an ideal of H such that $A/P \otimes K \subset \mathfrak{P}$. Since $A/P \otimes H$ is an algebra of finite type over k, the codimension of $\mathfrak{P}$ in B is finite. Hence for each $a \in A/P$ the subspace $\{h \in H \mid a \otimes h \in \mathfrak{P}\}$ has finite codimension in H. Since $\dim A/P < \infty$, we conclude that $\dim H/K < \infty$, i.e. $K \in \mathcal{F}$. It follows that

$$\operatorname{Eq}_{H^\circ}(P) = \{P' \in \operatorname{Max} A \mid \rho_P^{-1}(\mathfrak{P}) \subset P' \text{ for some } \mathfrak{P} \in \operatorname{Max} B\}.$$

We may apply Lemma 5.4 with $\varphi = \rho_P$. Let $U \subset \operatorname{Spec} A$ be the H°-stable open subset given by that lemma. We have $U \cap \operatorname{Max} A \subset \operatorname{Eq}_{H^\circ}(P)$. The density property ensures that $U \cap \operatorname{Eq}_{H^\circ}(P) \neq \varnothing$. As $\operatorname{Eq}_{H^\circ}(P)$ has no nonempty proper H°-stable subsets, it holds $U \cap \operatorname{Max} A = \operatorname{Eq}_{H^\circ}(P)$.

For each $P' \in U$ there exists $\mathfrak{P} \in \operatorname{Max} B$ such that $P' + \rho_P^{-1}(\mathfrak{P})$ is a proper ideal of A. This ideal is contained in some maximal ideal P''. Then $P'' \in \operatorname{Eq}_{H^\circ}(P)$ since $\rho_P^{-1}(\mathfrak{P}) \subset P''$. It follows that $P''_{H^\circ} = P_{H^\circ} = 0$, and therefore $P'_{H^\circ} = 0$. This shows that P' contains no nonzero H-costable ideals of A.

The subset $X = \operatorname{Spec} A \setminus U$ is H°-stable and closed in $\operatorname{Spec} A$. Corollary 6.5 provides us with an H-costable ideal J of A such that $X = \{P' \in \operatorname{Spec} A \mid J \subset P'\}$.

Note that $J \subset P'_{H^\circ}$ for each $P' \in X$. Each P'_{H° is an H-prime H-costable ideal of A. Replacing J with $\bigcap_{P' \in X} P'_{H^\circ}$, we may assume that J is H-semiprime. Let I be any nonzero H-costable ideal of A. Since A is noetherian, there are finitely many primes $P_1, \dots, P_r$ minimal over I, and a suitable product of those primes is contained in I. We have $I \subset (P_i)_{H^\circ}$, and so $(P_i)_{H^\circ} \neq 0$, for each i. This forces $P_i \in X$, so that $J \subset P_i$, for each i. Then $J^m \subset I$ for sufficiently large integer $m > 0$. If I is H-semiprime then $J \subset I$. Thus J is a smallest nonzero H-semiprime ideal of A.

Now let $I = \bigcap_{n>0} J^n$. This ideal is H°-stable, hence H-costable. Suppose $I \neq 0$. Then $J^m \subset I$ for sufficiently large $m > 0$ as have been observed above. This means that $I = J^m$ and $IJ = I$. We may regard I as an A-finite object of ${}_{H^\circ}\mathcal{M}_A$. If $P' \in \operatorname{Max} A \smallsetminus \operatorname{Eq}_{H^\circ}(P)$, then $P' \in X$, i.e. $J \subset P'$. In this case $IP' = I$, whence $r_{P'}(I) = 0$. If $P' \in \operatorname{Eq}_{H^\circ}(P)$ then $r_{P'}(I) = r_P(I)$ by Proposition 2.4. It follows from Proposition 2.5 that the rank function $P' \mapsto r_{P'}(I)$ is constant on $\operatorname{Max} A$. On the other hand, I has a simple factor module in $\mathcal{M}_A$, and the latter is annihilated by some $P' \in \operatorname{Max} A$, so that $IP' \neq I$. Hence $r_{P'}(I) > 0$ for all $P' \in \operatorname{Max} A$, and therefore $\operatorname{Eq}_{H^\circ}(P) = \operatorname{Max} A$. It follows that J is contained in none of the maximal ideals of A, whence $J = A$. □

Proof of Corollary 0.5. Replacing A with A/P_{H°, we may assume that $P_{H^\circ} = 0$. Then Theorem 0.4 applies. □

Proof of Theorem 0.1. If I is any H-costable ideal of A, then IH is an $\mathcal{M}_H^H$-subobject of H; by the structure theorem for objects of $\mathcal{M}_H^H$ [21, Th. 4.1.1] we have $IH = H$ whenevser $I \neq 0$. Hence $I_1 I_2 H = H$ for any two nonzero H-costable ideals of A, which shows that A is an H-prime H-comodule algebra. Thus the hypotheses of Theorem 0.2 are satisfied when we take $B = H$ and $\varphi : A \to B$ the inclusion. Since H may be regarded as an object of $\mathcal{M}_H^H$ and an object of ${}_H\mathcal{M}^H$, the flatness of H over A is immediate.

Denote $A^+ = H^+ \cap A$ where $H^+ = \operatorname{Ker} \varepsilon$ is the augmentation ideal of H. So A^+ is a maximal ideal of A with $A/A^+ \cong k$. Since $A^+H \subset H^+ \neq H$, none of the nonzero H-costable ideals of A can be contained in A^+, i.e. $A^+_{H^\circ} = 0$. Thus the hypotheses of Theorem 0.4 are also satisfied, and we deduce that $\operatorname{Eq}_{H^\circ}(A^+)$ is an open dense orbit in $\operatorname{Max} A$. Moreover, the proof of Theorem 0.4 shows that $\operatorname{Eq}_{H^\circ}(A^+)$ consists precisely of those $P \in \operatorname{Max} A$ for which there exists $\mathfrak{P} \in \operatorname{Max} H$ satisfying $\mathfrak{P} \cap A \subset P$ (note that ρ_{A^+} is nothing else but the inclusion $A \to H$).

The proof of Theorem 0.4 shows also that $\operatorname{Eq}_{H^\circ}(A^+) = U \cap \operatorname{Max} A$ where U is the subset of $\operatorname{Spec} A$ defined in Lemma 5.4. Clearly $HP \neq H$ for each $P \in U$. Conversely, if $P \in \operatorname{Max} A$ satsifies $HP \neq H$, then there exists $\mathfrak{P} \in \operatorname{Max} H$ such that $\mathfrak{P} \cap A \subset P$ (see the proof of Lemma 5.4), and therefore $P \in \operatorname{Eq}_{H^\circ}(A^+)$. Hence (iv) amounts to the condition that $HP \neq H$ for all $P \in \operatorname{Max} A$. For H to be faithfully flat in $\mathcal{M}_A$, it is necessary and sufficient that $H \otimes_A W \neq 0$ for each simple $W \in {}_A\mathcal{M}$. The annihilator of such a W is a maximal ideal P of A; since A/P is simple artinian, we have $H \otimes_A W \neq 0$ if and only if $H \otimes_A A/P \neq 0$, i.e. $HP \neq P$. It follows that (i)⇔(iv). Since the antipode of H is bijective, the bialgebra H^{op} is a Hopf algebra. Hence (i)⇔(iv) holds also for the pair $H^{\mathrm{op}}, A^{\mathrm{op}}$ in place of H, A. It is immediate from the definition that the $(H^{\mathrm{op}})^\circ$-orbit equivalence relation on $\operatorname{Spec} A^{\mathrm{op}} = \operatorname{Spec} A$ coincides with $\sim_H$. Therefore (ii)⇔(iv).

If $P \in \operatorname{Eq}_{H^\circ}(A^+)$ then $P_{H^\circ} = 0$, i.e. P contains no nonzero H-costable ideals of A. Since each proper H-costable ideal of A is contained in a maximal ideal of

A, we get (iv)$\Rightarrow$(iii). By Theorem 0.4 $\mathrm{Eq}_{H^\circ}(A^+) = \{P \in \operatorname{Max} A \mid J \not\subset P\}$ where J is a nonzero H-costable ideal of A. If A is H-simple, then $J = A$, and therefore $\mathrm{Eq}_{H^\circ}(A^+) = \operatorname{Max} A$. Hence (iii)$\Rightarrow$(iv). □

Remark. As we have seen, the open dense orbit in $\operatorname{Max} A$ can be characterized as

$$\mathrm{Eq}_{H^\circ}(A^+) = \{P \in \operatorname{Max} A \mid HP \neq H\} = \{P \in \operatorname{Max} A \mid PH \neq H\}.$$

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