Perturbations of spectral measures for Feller operators

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Abstract. In stochastic spectral analysis of selfadjoint Feller operators quantitative error estimates are given for the corresponding spectral measures. Both regular and singular perturbations are considered.

1. The background idea of stochastic spectral analysis

Many spectral properties of selfadjoint operators can be studied indirectly by the spectral behaviour of their semigroups. For Schödinger operators of the form $H_0 + V$, H_0 selfadjoint realization of $-\Delta$, V a Kato–class potential, in $L^2(\mathbb{R}^n)$ a summary of these results can be found by Simon (1982) or by Demuth (1991). The spectral consequences are mainly based on the explicit representation of the integral kernels for the semigroups or resolvents. That means for instance the resolvent $(H_0 + V + z1)^{-1}$ is an integral operator for appropriate regular values, the kernel of which is given by

$$\left[(H_0 + V + z1)^{-1}\right](x,y) = \int_0^\infty d\lambda\; e^{-\lambda z}\; E_x^{y,\lambda}\left\{e^{-\int_0^\lambda V(\omega(s))ds}\right\} , \tag{1}$$

$x, y \in \mathbb{R}^n$. $E_x^{y,\lambda}\{.\}$ denotes the conditional Wiener measure. $\omega(.)$ are the trajectories of the Wiener process.

For Kato–class potentials the kernel of $e^{-t(H_0+V)}$ can be estimated by the free kernel (where $V \equiv 0$) i.e. by the Wiener density function

$$p_W(t,x,y) = (4\pi t)^{-n/2}\; e^{-|x-y|^2/4t} . \tag{2}$$

Therefore the proofs of spectral properties of $H_0 + V$ using the Feynman–Kac–representation in (1) are mainly based on several characteristic features of the Wiener density function p_W. This density function has many "nice" properties. For instance it is symmetric with respect to any coordinate $x_i \in \mathbb{R}$, $i = 1, \ldots, n$. It is decreasing if any $|x_i| \to \infty$, even with the same rate of convergence. It is uniformly bounded by $|x-y|^{-n}$.

But for many spectral theoretical aspects it is not necessary to have or to use the whole variety of these p_W–properties. Therefore the first main task in *Stochastic Spectral Analysis* was to select a set of sufficient (and almost necessary) assumptions on a transition density function p, which are rich enough to provide spectral results and which are poor enough to include an interesting class of operators $K_0 + V$, where K_0 is *not* canonical to the Laplacian.

That was done by Demuth, van Casteren (1989 and 1992). We have established the following basic assumptions on stochastic spectral analysis, shortly denoted as BASSA.

BASSA:

1. *Existence*

   Let $(E, \mathcal{E})$ be a second countable locally compact Hausdorff space E with the Borel field $\mathcal{E}$. Assume a continuous function

   $$p \quad \text{mapping} \quad (0, \infty) \times E \times E \to [0, \infty)$$

   with the properties

   $$\int_A p(t, x, y) dy \leq 1 \text{ for all } t > 0, x \in E, A \subset E \ ,$$

   and

   $$\int_E p(t, x, u) p(s, u, y) du = p(s + t, x, y) \ .$$

2. *Continuity*

   Let $C_\infty(E)$ be the set of continuous functions vanishing at infinity. For any $f \in C_\infty$ and any $x \in E$ we assume

   $$\lim_{t \to 0} \int_E f(y) p(t, x, y) dy = f(x) \ .$$

3. *Symmetry*

   For all $t > 0$, $x, y \in E$ we assume

   $$p(t, x, y) = p(t, y, x) \ .$$

4. *Feller property*

   For any $f \in C_\infty(E)$ we assume

   $$x \to \int_E f(y) p(t, x, y) dy \in C_\infty(E) \ .$$

□

2. Free Feller operators

Definition:
Assuming BASSA the function p corresponds to a semigroup. Its generator is denoted with K_0, i.e.

$$(e^{-tK_0}f)(x) = \int_E f(y)p(t,x,y)dy \ .$$

Because e^{-tK_0} satisfies the Feller property K_0 is called the free Feller operator. (This corresponds to the name "free Schrödinger operator" for the Laplacian). □

Remarks concerning the assumptions.
The density function is one central link between operator theory and stochastic analysis. The existence and the Feller property ensure that the underlying process $(\mathbb{R}_+; \Omega, \mathcal{F}, P_F; \omega(.))$ is a strong Markov process with the Feller property. Together with the continuity assumption it implies that the process has rigth continuous path with left–hand limits. The symmetry condition is equivalent to the selfadjointness of K_0. □

Examples:
The most crucial condition in BASSA is the Feller property. This property has its own interest and is studied seperately in the literature. Let me mention here only two examples.

Davies (1991) studied locally finite Riemannian manifolds where K_0 is formally given by

$$K_0 f = -\frac{1}{\sigma^2} \nabla (\sigma^2 \nabla f) \ .$$

Here $\sigma = \sigma(x)$ are strictly positive measurable functions on E such that $\sigma \in L^\infty_{loc}(E)$ and $\sigma^{-1} \in L^\infty_{loc}(E)$. K_0 is defined correctly via closable Dirichlet forms. e^{-tK_0} is then a positivity preserving strongly continuous semigroup on L^p, $1 \leq p < \infty$. It is an integral operator with a kernel p_D. The Feller property is proved if

$$\lim_{|x|\to\infty} p_D(t,x,y) = 0$$

for all $y \in E$, $t > 0$. This is shown by pointwise estimates for $p_D(t,x,y)$. It is remarkable that the conditions for σ are very general, in particular it is not necessary to have any differentiability of σ.

The next example is given by Jacob (1992) in the present proceedings. In a series of articles he considered pseudo–differential operators defined as extension of an operator $a(x,D)$:

$$(a(x,D)u)(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} a(x,\xi)\hat{u}(\xi)d\xi$$

$u \in C_0^\infty(\mathbb{R}^n)$ for special classes of $a(x,D)$, in particular for $a(x,\xi)$ of the form

$$a(x,\xi) = \sum_{j=1}^{n} b_j(x)|\xi_j|^{2r}, \qquad 0 < r \leq 1$$

with $b_j \in C^\infty(\mathbf{R}^n)$, b_j independent of x_j. Then the corresonding extension generates a Feller semigroup. Further examples by Jacob include the relativistic Hamiltonian, where $a(x,\xi) = (|\xi|^2 + m^2)^{1/2} - m$. □

3. Feller operators

Definition: *Kato-Feller-potentials*
Assume a density function satisfying BASSA. Let V be a real-valued function on E, $V = V_+ - V_-$. V is called a Kato-Feller-potential if

$$\lim_{\tau \to 0} \sup_x \int_0^\tau ds \int_E p(s,x,y)[V_-(y) + \chi_B(y)V_+(y)]dy = 0 \ , \tag{3}$$

where B is a compact subset of E. □

These Kato-Feller-class is optimal for

$$\lim_{\lambda \to \infty} \|(K_0 + \lambda)^{-1} V_-\|_{\infty,\infty} = 0 \ , \tag{4}$$

which determines the relative form bound of V_- with respect to K_0. Then the right-hand side of the generalized Feynman-Kac-formula

$$E_x \left\{ e^{-\int_0^t V(\omega(s))ds} f(\omega(t)) \right\} \tag{5}$$

(here E_x is the exspectation with respect to the Feller measure P_F) yields a strongly continuous semigroup on L^2, its generator is the selfadjoint operator $K_0 \dot{+} V$. $e^{-t(K_0 \dot{+} V)}$ is again an integral operator. The kernel can be estimated by

$$(e^{-t(K_0 \dot{+} V)})(x,y) \leq c \ e^{ct} \ p^{1/2}(t,x,y) \sup_{x,y \in E} p^{1/2}(t,x,y) \tag{6}$$

(see van Casteren (1989)). If $p(t,x,y)$ is uniformly bounded in x and y the last estimate implies the Feller property for the semigroup $e^{-t(K_0 \dot{+} V)}$, too.

Definition: *Feller operator*
The generator of a Feller semigroup is denoted as Feller operator. □

Therefore $K_0 \dot{+} V$ is a Feller operator with a regular perturbation. This denotation corresponds to the name for generators of Schrödinger semigroups.

Singular perturbations can also be included. Let Γ be a subset of E, $|\Gamma| > 0$. With S we denote the penetration time

$$S := \inf\{\tau > 0 : \int_0^\tau 1_\Gamma(\omega(s))ds > 0\} \ . \tag{7}$$

Then we define the absorption semigroup

$$\begin{aligned} & E_x \left\{ e^{-\int_0^t V(\omega(s))ds} f(\omega(t)), \ S > t \right\} \\ =: \ & (U(t)f)(x) \ . \end{aligned} \tag{8}$$

Let $\Sigma = E \setminus \Gamma$, then $U(t) \uparrow L^2(\Sigma)$ is again a Feller semigroup, its generator is denoted with $(K_0\dot{+}V)_\Sigma$. It is a selfadjoint operator in $L^2(\Sigma)$.

4. Spectral measures

The spectral measure plays a fundamental role in characterizing the different parts of the spectrum for selfadjoint operators. For the selfadjoint Feller operators $K_0\dot{+}V$, $K_0\dot{+}W$, considered here, we denote the spectral measures with $E_{K_0\dot{+}V}(\cdot)$, $E_{K_0\dot{+}W}(\cdot)$, respectively. Instead of considering the potential dependence of matrix elements of these spectral measures in weighted L^2-spaces, we study the operator norm of sandwiched spectral measures.

Because we have assumed Kato–Feller–potentials a natural norm is the Kato–Feller–norm, which is defined by

$$\|V\|_{KF} := \sup_{x\in E} \int_0^1 ds \int_E p(s,x,y)|V(y)|dy \ . \tag{9}$$

The objective of the following theorem is to control explicitly the changes of the spectral measure in terms of $\|V-W\|_{KF}$.

Theorem 1: Assume BASSA and two Kato–Feller–potentials V and W. Let φ be a multiplication operator with a nonvanishing continuous function $\varphi : E \to \mathbf{R}_+$ with $\varphi^{-1} \leq 1$. Let $\Delta = (\alpha_1, \alpha_2)$ be an interval on $\mathbf{R}_+$ such that α_1 and α_2 are no eigenvalues of $K_0\dot{+}V$ or $K_0\dot{+}W$, respectly. Assume that

$$\sup_{\substack{\lambda\in\Delta \\ \varepsilon\to(0,1]}} \|\varphi^{-1}(K_0 - \lambda \pm i\varepsilon)^{-1}\varphi^{-1}\| =: d_\Delta < \infty \ . \tag{10}$$

Moreover, suppose a pointwise estimate

$$\int_E p(t,x,y)\varphi^4(y)dy \leq c_0(1+t^m)\varphi^4(x) \tag{11}$$

with some $m \in \mathbf{N}$.

Let us denote positive constants c_V and A_V by

$$E_x\left\{e^{-\int_0^\lambda V(\omega(s))ds}\right\} \leq c_V \ e^{A_V\lambda} \ . \tag{12}$$

Take the Kato–Feller–norms $\|V\varphi^2\|_{KF}$, $\|W\varphi^2\|_{KF}$ small enough, i.e. take for instance

$$\|V\varphi^2\|_{KF} < \frac{1}{12}\,\frac{1}{b_V}\,\frac{1}{c_0^{1/2}}\,\frac{1}{c_{4V}^{1/4}}\,\frac{1}{1+3d_\Delta} \tag{13}$$

with $b_V > \max\{m, \alpha_2, 2A_{4V}\}$. ($\|W\varphi^2\|_{KF}$ correspondingly small).
Then the difference of the spectral measures can be estimated by

$$\|\varphi^{-1}(E_{K_0\dot{+}V}(\Delta) - E_{K_0\dot{+}W}(\Delta))\varphi^{-1}\| \leq c(V,W,\Delta)\|(V-W)\varphi^2\|_{KF} \ . \tag{14}$$

The constant $c(V, W, \Delta)$ depending on V, W, Δ and on the geometry of E can be estimated explicitly. If the condition in (13) is satisfied one has

$$\begin{aligned} c(V,W,\Delta) \;&\leq\; 2|\Delta|\; c_0^{1/2}\; c_{4V}^{1/4}\; c_{4W}^{1/4}\; a_0^{-1} \\ &\quad (1+3a_0\gamma_V^{-1}(d_\Delta+1))(1+3a_0\gamma_W^{-1}(d_\Delta+1)) \;, \end{aligned} \tag{15}$$

with $a_0 > \max\{m, \alpha_2, 2A_{4V}, 2A_{4W}\}$ and with

$$\gamma_V = 1 - 12 b_V c_0^{1/2} c_{4V}^{1/4}(1+3d_\Delta)\|V\varphi^2\|_{KF} \,.$$

□

Remark: Note that V and W are not assumed to be bounded. The condition on $\|W\varphi^2\|_{KF}$, corresponding to (13), could be neglected. It would follow from the condition for $\|V\varphi^2\|_{KF}$ and an analogous result for $(K_0+V-\lambda\pm i0)^{-1}-(K_0+W-\lambda\pm i0)^{-1}$. □

Proof of Theorem 1: The spectral measure of a selfadjoint operator H on a bounded open interval $\Delta = (\alpha_1, \alpha_2)$ where neither α_1 or α_2 is an eigenvalue of H is given by

$$E_H(\Delta) = s - \lim_{\varepsilon\to 0}\,(2\pi i)^{-1}\int_{\alpha_1}^{\alpha_2}[(H-\lambda-i\varepsilon)^{-1}-(H-\lambda+i\varepsilon)^{-1}]d\lambda \,. \tag{16}$$

We set $R_V(\lambda\pm i\varepsilon) := (K_0+V-\lambda\mp i\varepsilon)^{-1}$. Then

$$\begin{aligned} &\|\varphi^{-1}(E_{K_0\dot{+}V}(\Delta)-E_{K_0\dot{+}W}(\Delta))\varphi^{-1}\| \\ \leq\;& \lim_{\varepsilon\to 0}\,(2\pi)^{-1}\int_{\alpha_1}^{\alpha_2} d\lambda\;\{\|\varphi^{-1}[R_V(\lambda+i\varepsilon)-R_W(\lambda+i\varepsilon)]\varphi^{-1}\| \\ +\;& \|\varphi^{-1}[R_V(\lambda-i\varepsilon)-R_W(\lambda-i\varepsilon)]\varphi^{-1}\|\}\,. \end{aligned} \tag{17}$$

The first term in (17) is estimated by:

$$\begin{aligned} &\|\varphi^{-1}[R_V(\lambda+i\varepsilon)-R_W(\lambda+i\varepsilon)]\varphi^{-1}\| \\ \leq\;& (1+|\lambda+i\varepsilon-a|\;\|\varphi^{-1}R_V(\lambda+i\varepsilon)\varphi^{-1}\|) \\ & (1+|\lambda+i\varepsilon-a|\;\|\varphi^{-1}R_W(\lambda+i\varepsilon)\varphi^{-1}\|) \\ & \|\varphi[R_V(-a)-R_W(-a)]\varphi\|\,, \end{aligned} \tag{18}$$

where a is any regular value for $K_0\dot{+}V$ and $K_0\dot{+}W$. The rest of the proof is splitted into several lemmata. The objective is to estimate the terms in (18) uniformly in λ and ε.

Lemma 2: Take the assumptions of Theorem 1. Then

$$\|\varphi[R_V(-a)-R_W(-a)]\varphi\| \leq 4\; c_0^{1/2}\; c_{4V}^{1/4}\; c_{4W}^{1/4}\;\cdot\;\|(V-W)\varphi^2\|_{KF}\,. \tag{19}$$

if $a > \max\{m, 2A_{4V}, 2A_{4W}\}$. □

Proof of Lemma 1: Demuth, van Casteren (1991) have shown that

$$\begin{aligned} \|\varphi\;[R_V(-a)-R_W(-a)]\;\varphi\| \;&\leq\; \|R_0(-a)\;|V-W|\;R_{2V}(-a)\;\varphi^2\|_\infty^{1/2} \\ &\cdot\;\; \|R_0(-a)\;|V-W|\;R_{2W}(-a)\;\varphi^2\|_\infty^{1/2}\,. \end{aligned}$$

The first factor squared is smaller than

$$\sup_x \int_0^\infty d\lambda\; e^{-a\lambda} E_x \left\{ |V(\omega(\lambda)) - W(\omega)(\lambda))|\; [R_{2V}(-a)\varphi^2](\omega(\lambda)) \right\} .$$

But

$$\begin{aligned} (R_{2V}(-a)\varphi^2)(x) &\leq \left[\int_0^\infty d\lambda\; e^{-a\lambda} E_x \left\{ e^{-4\int_0^\lambda V(\omega(s))ds} \right\} \right]^{1/2} \\ &\quad \left[\int_0^\infty d\lambda\; e^{-a\lambda} E_x \{ \varphi^4(\omega(\lambda)) \} \right]^{1/2} \\ &\leq\; c_{4V}^{1/2}\; c_0^{1/2} (a - A_{4V})^{-1/2} \left(\frac{1}{a} + \frac{m!}{a^{m+1}} \right)^{1/2} \varphi^2(x) \\ &\leq\; 2\; c_0^{1/2}\; c_{4V}^{1/2}\; a^{-1}\; \varphi^2(x) , \end{aligned}$$

if $a > m$, $a > 2A_{4V}$.
Therefore

$$\begin{aligned} \|\varphi(R_V(-a) - R_W(-a))\varphi\| &\leq 2\; c_0^{1/2}\; c_{4V}^{1/4}\; c_{4W}^{1/4}\; a^{-1} \| R_0(-a) |V - W| \varphi^2 \|_\infty \\ &\leq\; 2\; c_0^{1/2}\; c_{4V}^{1/4}\; c_{4W}^{1/4}\; a^{-1} \sum_{k=0}^{\infty} e^{-ak}\; \|(V - W)\varphi^2\|_{KF} \end{aligned}$$

which proves Lemma 2. q.e.d.

Corollary 3: Setting $W \equiv 0$ Lemma 2 provides

$$\|\varphi[R_V(-a) - R_0(-a)]\varphi\| \leq 4\; c_0^{1/2}\; c_{4V}^{1/4}\; a^{-1}\; \|V\varphi^2\|_{KF} \tag{20}$$

if $a > \max\{m, 2A_{4V}\}$. □

In the next lemma we estimate the perturbed sandwiched resolvent near the real axis. It is a consequence of Lemma 2.

Lemma 4: Take the assumptions of Theorem 1. Then

$$\|\varphi^{-1} R_V(\lambda + i\varepsilon)\varphi^{-1}\| \leq \gamma_V^{-1} (d_\Delta + \frac{1}{3}) , \tag{21}$$

with

$$\begin{aligned} \gamma_V &:= 1 - 12\; b_V\; c_0^{1/2}\; c_{4V}^{1/4} (1 + 3d_\Delta) \|V\varphi^2\|_{KF} \\ b_V &> \max\{m, \alpha_2, 2A_{4V}\} , \end{aligned}$$

where $\Delta = (\alpha_1, \alpha_2)$ and d_Δ is given in (10). γ_V is greater than zero because of the assumption in (13). □

Proof of Lemma 4: Using again (18) and Corollary 3 we obtain

$$\begin{aligned} \|\varphi^{-1} R_V(\lambda + i\varepsilon)\varphi^{-1}\| \;\leq\; & d_\Delta + (1 + 3b_V d_\Delta)\; 4\; c_0^{1/2}\; c_{4V}^{1/4}\; \|V\varphi^2\|_{KF}\; b_V^{-1} \\ & \cdot\; (1 + 3b_V \|\varphi^{-1} R_V(\lambda + i\varepsilon)\varphi^{-1}\|) . \end{aligned}$$

$\|V\varphi^2\|_{KF}$ is chosen small enough. Then (21) follows obviously. q.e.d.

Rest of the proof of Theorem 1:
From the Lemmata 2 and 4 follows

$$\begin{aligned} & \|\varphi^{-1}[R_V(\lambda+i\varepsilon)-R_W(\lambda+i\varepsilon)]\varphi^{-1}\| \\ \leq\ & (1+|\lambda+i\varepsilon-a_0|\gamma_V^{-1}(d_\Delta+\frac{1}{3})) \\ & (1+|\lambda+i\varepsilon-a_0|\gamma_W^{-1}(d_\Delta+\frac{1}{3})) \\ & 4\ c_0^{1/2}\ c_{4V}^{1/4}\ c_{4W}^{1/4}\ a_0^{-1}\|(V-W)\varphi^2\|_{KF} \end{aligned}$$

with $a_0 > \max\{m, \alpha_2, 2A_{4V}, 2A_{4W}\}$, which implies (14) with the constant in (15). q.e.d.

The next and last objective in this article is to analyse perturbations of the spectral measures for infinitely high potentials. As mentioned in (8)

$$E_x\left\{e^{-\int_0^t V(\omega(s))ds}f(\omega(t)), S>t\right\}$$

establishes a strongly continuous semigroup in $L^2(\Sigma)$, $\Sigma = E\setminus\Gamma$, the generator of which is $(K_0\dot{+}V)_\Sigma$. For the singularity region Γ we assume that the regular points of Γ and the regular points of the interior of Γ form the same set.

On the other hand $e^{-t(K_0\dot{+}V)_\Sigma}$ is the limit of a family of semigroups $e^{-t(K_0+V+\beta U)}$ as $\beta\to\infty$. Here U is an additional positive potential with support Γ. $(K_0\dot{+}V)_\Sigma$ and $K_0\dot{+}V+\beta U$ are selfadjoint operators in $L^2(\Sigma)$ and $L^2(E)$, respectively. Their spectral measures are denoted with $E_\Sigma(.)$ and $E_\beta(.)$, respectively. Because these are operators in different Hilbert spaces we introduce an embedding operator by $(Jf)(x) := \chi_\Gamma(x)f(x)$.

Theorem 5: Assume BASSA and a Kato–Feller–potential V. Let φ be a multiplication operator with a continuous function satisfying $|\varphi^{-1}|\leq 1$,

$$\int_E p(t,x,y)\varphi^8(y)dy \leq c_0\ (1+t^m)\ \varphi^8(x)\ , \tag{22}$$

$m\in\mathbb{N}$, and for arbitrary large R let

$$\sup_{|x|\geq R}\varphi^2(x)\,[E_x\{S<\lambda\}]^{1/2}<\varepsilon \tag{23}$$

where ε is chosen arbitrarily small.

Let $\Delta=(\alpha_1,\alpha_2)$ and assume

$$\sup_{\substack{\lambda\in\Delta\\ \varepsilon\in(0,1]}}\|\varphi^{-1}J^*((K_0)_\Sigma-\lambda\pm i\varepsilon)^{-1}J\varphi^{-1}\| =: d_{\Delta,\Sigma}<\infty\ . \tag{24}$$

For the Kato–Feller–potential we assume, according to Theorem 1 (see (13)),

$$\|V\varphi^2\|_{KF} < \frac{1}{12\ c_0^{1/2}\ c_{4V}^{1/4}}\ \frac{1}{b_V}\ \frac{1}{1+3d_{\Delta,\Sigma}} \tag{25}$$

with $b_V > \max\{m, \alpha_2, 2A_{4V}\}$. And we set

$$\gamma_{V,\Sigma} := 1 - 12\ b_V\ c_0^{1/2}\ c_{4V}^{1/4}\ (1 + 3d_{\Delta,\Sigma})\|V\varphi^2\|_{KF}\ . \tag{26}$$

Then we have the following assertions:

a) If we denote

$$\rho(\beta) := \int_0^\infty d\lambda\ e^{-\lambda} \sup_x \varphi^2(x) \left[E_x\left\{e^{-\beta\int_0^\lambda U(\omega(s))ds}, S < \lambda\right\}\right]^{1/2}\ , \tag{27}$$

then $\rho(\beta)$ tends to zero as $\beta \to \infty$.

b) The difference of the spectral measures can be estimated quantitatively if β is sufficiently large:

$$\begin{aligned} & \|\varphi^{-1}[E_\beta(\Delta) - J^* E_\Sigma(\Delta) J]\varphi^{-1}\| \\ \leq\ & \pi^{-1}|\Delta|\ \gamma_{V,\Sigma}^{-2}\ [1 + 3a_0(d_{\Delta,\Sigma} + 2)]^2 \\ & \|\varphi[(K_0 \dot{+} V + \beta U + a_0)^{-1} - J^*((K_0 \dot{+} V)_\Sigma + a_0)^{-1} J]\varphi\| \end{aligned} \tag{28}$$

$$\begin{aligned} \leq\ & (2\pi)^{-1}|\Delta|\ \gamma_{V,\Sigma}^{-2}\ [1 + 3a_0(d_{\Delta,\Sigma} + 2)]^2 \\ & (c_{2V}^{1/2} + c_0^{1/4}\ c_{4V}^{1/4}\ (1 + m^m) \cdot \rho(\beta) \end{aligned} \tag{29}$$

with $a_0 > \max\{4, \alpha_2, A_{2V}, A_{4V}\}$. □

Remark: The estimation of $\rho(\beta)$ is a difficult problem. One first quantitative estimate of $\rho(\beta)$ in the case $K_0 = -\Delta$ is given Demuth, Jeske, Kirsch (1992). The rate of convergence depends on the size of the boundary $\delta\Gamma$. □

Proof of Theorem 5: As above we set

$$R_{V,\Sigma}(-a) := ((K_0 \dot{+} V)_\Sigma + a)^{-1}\ .$$

For $a > 1$, $a > \alpha_2$ one has to estimate the product

$$[1 + 3a\ \|\varphi^{-1} J^* R_{V,\Sigma}(\lambda \pm i\varepsilon) J\varphi^{-1}\|] \tag{30}$$

$$\cdot\ [1 + 3a\ \|\varphi^{-1} R_{V+\beta U}(\lambda \pm i\varepsilon)\varphi^{-1}\|] \tag{31}$$

$$\cdot\ \|\varphi(R_{V+\beta U}(-a) - J^* R_{V,\Sigma}(-a) J)\varphi\| \tag{32}$$

uniformly in λ and ε.

The factor in (30) corresponds to Lemma 4 with $d_{\Delta,\Sigma}$ indead of d_Δ. The only point is that

$$\|\varphi J^*(R_{V,\Sigma}(-b) - R_{0,\Sigma}(-b)) J\varphi\| \leq \|\varphi(R_V(-b) - R_0(-b))\varphi\|$$

if $b > 1$ and $b > \alpha_2$.

The second factor (31) can be estimated using the fact that $\|\varphi^{-1} R_{V+\beta U}(\lambda \pm i\varepsilon)\varphi^{-1}\|$ converges to $\|\varphi^{-1} J^* R_{V,\Sigma}(\lambda \pm i\varepsilon) J\varphi^{-1}\|$ as $\beta \to \infty$. This convergence will be considered in Lemma 6. Hence for sufficiently large β

$$\|\varphi^{-1} R_{V+\beta U}(\lambda \pm i\varepsilon)\varphi^{-1}\| \leq 1 + \|\varphi^{-1} J^* R_{V,\Sigma}(\lambda \pm i\varepsilon) J\varphi^{-1}\|\ .$$

The main intersting factor is (32). It will be considered separately.

Lemma 6: Under the assumptions of Theorem 5 it holds

$$\begin{aligned} & \|\varphi\ (R_{V+\beta U}(-a_0) - J^* R_\Sigma(-a_0)J)\varphi\| \\ \leq\ & \frac{1}{2}(c_{2V}^{1/2} + c_0^{1/4}\ c_{4V}^{1/4}\ (1+m^m)) \\ & \int_0^\infty d\lambda\ e^{-\lambda} \sup_x \varphi^2(x) \left[E_x\left\{e^{-\beta\int_0^\lambda U(\omega(s))ds}, S<\lambda\right\}\right]^{1/2}\ , \end{aligned} \tag{33}$$

$\alpha_0 > \max\{4, \alpha_2, A_{2V}, A_{4V}\}$, and it tends to zero as $\beta \to \infty$. □

Proof: $\varphi(R_{V+\beta U}(-a) - J^* R_\Sigma(-a)J)\varphi$ is an integral operator with a symmetric kernel. Therefore its norm is smaller than

$$\begin{aligned} & \sup_x \int_0^\infty d\lambda\ e^{-a\lambda}\varphi(x) E_x\left\{e^{-\int_0^\lambda V(\omega(s))ds} e^{-\beta\int_0^\lambda U(\omega(s))ds}\varphi(\omega(\lambda)), S<\lambda\right\} \\ \leq\ & \frac{1}{2}\int_0^\infty d\lambda\ e^{-a\lambda}\sup_x \varphi^2(x)\left[E_x\left\{e^{-2\int_0^\lambda V(\omega(s))ds}\right\}\right]^{1/2} \\ & \cdot\left[E_x\left\{e^{-2\beta\int_0^\lambda U(\omega(s))ds}, S<\lambda\right\}\right]^{1/2} \\ +\ & \frac{1}{2}\int_0^\infty d\lambda\ e^{-a\lambda}\sup_x\left[E_x\left\{e^{-4\int_0^\lambda V(\omega(s))ds}\right\}\right]^{1/4}\cdot\left[E_x\{\varphi^8(\omega(\lambda))\}\right]^{1/4} \\ & \cdot\left[E_x\left\{e^{-2\beta\int_0^\lambda U(\omega(s))ds}, S<\lambda\right\}\right]^{1/2}\ . \end{aligned}$$

For $|x| \geq R$ it is assumed that

$$\int_0^\infty d\lambda\ e^{-\lambda}\sup_{|x|\geq R}\varphi^2(x)[E_x\{S<\lambda\}]^{1/2}$$

is arbitrarily small independently of β. For $|x| \leq R$ notice that

$$\varphi^2(x)\left[E_x\left\{e^{-\beta\int_0^\lambda U(\omega(s))ds}, S<\lambda\right\}\right]^{1/2}$$

is monotoneously decreasing in β for any fixed x, continuous in x, and x is in a compact set. Hence the Theorem of Dini provides the convergence of the integral in (33). The constant factor in (33) follows in a similar way as for regular perturbations mentioned above. q.e.d.

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