# THE KUZ′MINOV — SHVEDOV ADDITION LEMMA IN A QUASI-ABELIAN CATEGORY

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Abstract. We study the question of the validity in a quasi-abelian category of some diagram lemma proved by Kuz′minov and Shvedov in 1994 for abelian groups and used by them as a tool for calculating the reduced $L_p$-cohomology of Riemannian manifolds.

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*Key words and phrases*: strict morphism, quasi-abelian category, homology, pullback.

## Introduction

In [7, Lemma 1], Kuz′minov and Shvedov proved the following assertion as a tool for constructing addition sequences for the reduced $L_p$-cohomology of a Riemannian manifold:

*Suppose that in the commutative diagram*

$$\begin{array}{ccccccc}
A_{01} & \xrightarrow{\alpha_{01}} & A_{02} & & & & \\
\beta_{01}\downarrow & & \beta_{02}\downarrow & & & & \\
A_{11} & \xrightarrow{\alpha_{11}} & A_{12} & \xrightarrow{\alpha_{12}} & A_{13} & \xrightarrow{\alpha_{13}} & A_{14} \\
\beta_{11}\downarrow & & \beta_{12}\downarrow & & \beta_{13}\downarrow & & \\
A_{21} & \xrightarrow{\alpha_{21}} & A_{22} & \xrightarrow{\alpha_{22}} & A_{23} & & \\
\beta_{21}\downarrow & & \beta_{22}\downarrow & & & & \\
A_{31} & \xrightarrow{\alpha_{31}} & A_{32} & & & & \\
\beta_{31}\downarrow & & \beta_{32}\downarrow & & & & \\
A_{41} & \xrightarrow{\alpha_{41}} & A_{42} & & & &
\end{array} \qquad (*)$$

*of abelian groups and homomorphisms the rows and columns are semi-exact, the second row is exact at the term $A_{13}$, the first column is exact at $A_{11}$ and $A_{21}$, the*

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*Key words and phrases.* strict morphism, quasi-abelian category, homology, pullback.

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*second column is exact, $\beta_{13}$ and $\alpha_{31}$ are isomorphisms, $\alpha_{01}$ is an epimorphism, and* $\operatorname{Ker}(\alpha_{13}\beta_{13}^{-1}\alpha_{22}) = \operatorname{Ker}(\beta_{31}\alpha_{31}^{-1}\beta_{22})$. *Put*

$$H_1 = \operatorname{Ker}\alpha_{12}/\operatorname{Im}\alpha_{11}, \quad H_2 = \operatorname{Ker}\alpha_{22}/\operatorname{Im}\alpha_{21}, \qquad H_3 = \operatorname{Ker}\beta_{31}/\operatorname{Im}\beta_{21}.$$

*Then the homomorphisms $\beta_{12}$ and $\alpha_{31}^{-1}\beta_{22}$ induce homomorphisms $H_1 \to H_2$ and $H_2 \to H_3$. The resulting sequence $0 \to H_1 \to H_2 \to H_3 \to 0$ is exact.*

The proof in [7] obviously extends to modules over an arbitrary ring and, thus, by Mitchells's Embedding Theorem [8], the assertion also holds in any abelian category.

There appears the question of the validity of the above lemma in more general additive categories, for example, in the category of topological abelian groups or in various categories of topological vector spaces (Banach, normed, locally convex spaces). A natural framework for this is provided by some class of categories now known under the name of quasi-abelian [12, 13].

In a quasi-abelian category, (*) also induces a semi-exact homology sequence but its exactness relies on the strictness of some morphisms in (*). We find sufficient conditions for the exactness of (*) at particular terms.

In the quasi-abelian categories of topological algebra and functional analysis, strict morphisms admit clear explicit descriptions. For example, in the category $\mathcal{B}an$ of Banach spaces and bounded linear operators, a morphism is strict if and only if it has closed range; a number of important examples can be found in [12].

## 1. Quasi-Abelian Categories

We consider additive categories satisfying the following axiom.

**Axiom 1.** Each morphism has kernel and cokernel.

We denote by $\ker\alpha$ ($\operatorname{coker}\alpha$) an arbitrary kernel (cokernel) of $\alpha$ and by $\operatorname{Ker}\alpha$ ($\operatorname{Coker}\alpha$) the corresponding object; the equality $a = \ker b$ ($a = \operatorname{coker} b$) means that $a$ is a kernel of $b$ ($a$ is a cokernel of $b$).

In a category meeting Axiom 1, every morphism $\alpha$ admits a canonical factorization $\alpha = (\operatorname{im}\alpha)\overline{\alpha}(\operatorname{coim}\alpha)$, where $\operatorname{im}\alpha = \ker\operatorname{coker}\alpha$, $\operatorname{coim}\alpha = \operatorname{coker}\ker\alpha$. A morphism $\alpha$ is called *strict* if $\overline{\alpha}$ is an isomorphism. Below we often use the abbreviation $\tilde{\alpha}$ for $\overline{\alpha}\operatorname{coim}\alpha$.

We use the following notations:

$O_c$ is the class of all strict morphisms;

$M$ is the class of all monomorphisms;

$M_c$ is the class of all strict monomorphisms (= kernels);

$P$ is the class of all epimorphisms;

$P_c$ is the class of all strict epimorphisms (= cokernels).

We write $\alpha \,|\, \beta$ if $\alpha = \ker\beta$ and $\beta = \operatorname{coker}\alpha$.

**Lemma 1.** [1, 6, 10] *The following assertions hold in an additive category meeting Axiom 1*:

(1) $\ker\alpha \in M_c$ *and* $\operatorname{coker}\alpha \in P_c$ *for every* $\alpha$;

(2) $\alpha \in M_c \iff \alpha = \operatorname{im}\alpha$, $\alpha \in P_c \iff \alpha = \operatorname{coim}\alpha$;

(3) *a morphism $\alpha$ is strict if and only if it is representable in the form $\alpha = \alpha_1\alpha_0$ with $\alpha_0 \in P_c$, $\alpha_1 \in M_c$; in every such representation, $\alpha_0 = \operatorname{coim}\alpha$ and $\alpha_1 = \operatorname{im}\alpha$;*

(4) *if a commutative square*

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & D \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow \\ A & \xrightarrow{\beta} & B \end{array} \qquad (1)$$

*is a pullback then* $f \in M \Longrightarrow g \in M$*,* $f \in M_c \Longrightarrow g \in M_c$*, if the square is pushout then* $g \in P \Longrightarrow f \in P$*,* $g \in P_c \Longrightarrow f \in P_c$*.*

An additive category meeting Axiom 1 is abelian if and only if $\overline{\alpha}$ is an isomorphism for every $\alpha$.

**Axiom 2.** For every morphism $\alpha$, $\overline{\alpha}$ is a monomorphism and an epimorphism.

Additive categories with kernels and cokernels satisfying Axioms 1 and 2 are called *P-semi-abelian* or simply *semi-abelian* (in the sense of Palamodov) [9, 11].

**Lemma 2.** [4] *The following hold in a P-semi-abelian category*:

(1) $gf \in M_c \Longrightarrow f \in M_c$*,* $gf \in P_c \Longrightarrow g \in P_c$*;*

(2) *if* $f, g \in M_c$ *and* $fg$ *is defined then* $fg \in M_c$*; if* $f, g \in P_c$ *and* $fg$ *is defined then* $fg \in P_c$*;*

(3) *if* $fg \in O_c$*,* $f \in M$ *then* $g \in O_c$*; if* $fg \in O_c$*,* $g \in P$ *then* $f \in O_c$*.*

An additive category satisfying Axiom 1 is called *quasi-abelian* [2, 12, 13] (*semi-abelian* in the sense of Raĭkov [10], or *almost abelian* [11]) (Jurchescu called such categories *preabelian* in [3]; Yoneda [14] did not assume the existence of kernels and cokernels) if it meets the following

**Axiom 3.** If square (1) is a pullback then $f \in P_c \Longrightarrow g \in P_c$. If (1) is a pushout then $g \in M_c \Longrightarrow f \in M_c$.

As is well known [6, 10, 11, 12], every quasi-abelian category is P-semi-abelian.

A sequence $\dots \xrightarrow{a} B \xrightarrow{b} \dots$ in a quasi-abelian category (or even in a P-semi-abelian category) is said to be *exact at the term* $B$ if $\operatorname{im} a = \ker b$ (or, equivalently, $\operatorname{coker} a = \operatorname{coim} b$). Below we call a sequence *semi-exact* if the composition of its two consecutive morphisms is zero.

By the homology of a sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ at the term $B$ in a quasi-abelian category such that $\psi\varphi = 0$ we mean the cokernel of the natural morphism $r : \operatorname{Im}\varphi \to \operatorname{Ker}\psi$ or, equivalently, the kernel of the natural morphism $q : \operatorname{Coker}\varphi \to \operatorname{Coim}\psi$ (see [5]).

For a commutative square (1), denote by $\hat{g} : \operatorname{Ker}\alpha \to \operatorname{Ker}\beta$ the morphism defined by the condition $g(\ker\alpha) = (\ker\beta)\hat{g}$ and by $\hat{f} : \operatorname{Coker}\alpha \to \operatorname{Coker}\beta$, the morphism defined by the condition $\hat{f}(\operatorname{coker}\alpha) = (\operatorname{coker}\beta)f$.

## 2. The Main Theorem

The main result of the article is formulated as follows:

**Theorem 1.** *Suppose that in the commutative diagram*

$$\begin{array}{ccccccc}
A_{01} & \xrightarrow{\alpha_{01}} & A_{02} & & & & \\
{\scriptstyle\beta_{01}}\downarrow & & {\scriptstyle\beta_{02}}\downarrow & & & & \\
A_{11} & \xrightarrow{\alpha_{11}} & A_{12} & \xrightarrow{\alpha_{12}} & A_{13} & \xrightarrow{\alpha_{13}} & A_{14} \\
{\scriptstyle\beta_{11}}\downarrow & & {\scriptstyle\beta_{12}}\downarrow & & {\scriptstyle\beta_{13}}\downarrow & & \\
A_{21} & \xrightarrow{\alpha_{21}} & A_{22} & \xrightarrow{\alpha_{22}} & A_{23} & & \\
{\scriptstyle\beta_{21}}\downarrow & & {\scriptstyle\beta_{22}}\downarrow & & & & \\
A_{31} & \xrightarrow{\alpha_{31}} & A_{32} & & & & \\
{\scriptstyle\beta_{31}}\downarrow & & {\scriptstyle\beta_{32}}\downarrow & & & & \\
A_{41} & \xrightarrow{\alpha_{41}} & A_{42} & & & &
\end{array} \qquad (2)$$

*in a quasi-abelian category the rows and columns are semi-exact, the second row is exact at the term $A_{13}$, the first column is exact at $A_{11}$ and $A_{21}$, the second column is exact, $\beta_{13}$ and $\alpha_{31}$ are isomorphisms, $\alpha_{01}$ is an epimorphism, and* $\ker(\alpha_{13}\beta_{13}^{-1}\alpha_{22}) = \ker(\beta_{31}\alpha_{31}^{-1}\beta_{22})$*. Denote by $H_1$ the homology of the second row at the term $A_{12}$, by $H_2$, the homology of the third row at the term $A_{22}$, and by $H_3$, the homology of the first column at the term $A_{31}$.*

*Then the morphisms $\beta_{12}$ and $\alpha_{31}^{-1}\beta_{22}$ induce homomorphisms $\varphi : H_1 \to H_2$ and $\psi : H_2 \to H_3$ such that $\psi\varphi = 0$, that is, the sequence*

$$0 \to H_1 \xrightarrow{\varphi} H_2 \xrightarrow{\psi} H_3 \to 0 \qquad (3)$$

*is semi-exact.*

*Moreover, the following sufficient conditions for (3) to be exact at particular terms hold:*

*(a) if in (2) $\alpha_{21}$, $\beta_{11}$, $\beta_{02}$ are strict and $\alpha_{01} \in P_c$ then (3) is exact at $H_1$, i.e., $\varphi \in M$;*

*(b) if in (2) $\beta_{12}$ and $\beta_{21}$ are strict then (3) is exact at $H_2$;*

*(c) if in (2) $\alpha_{22}$, $\alpha_{13}$, $\beta_{32}$, and the composition $\beta_{31}\alpha_{31}^{-1}\beta_{22}$ are strict then (3) is exact at $H_3$, that is, $\psi \in P$.*

*Proof.* The commutative square

$$\begin{array}{ccc}
A_{12} & \xrightarrow{\alpha_{12}} & A_{13} \\
{\scriptstyle\beta_{12}}\downarrow & & {\scriptstyle\beta_{13}}\downarrow \\
A_{22} & \xrightarrow{\alpha_{22}} & A_{23}
\end{array}$$

gives rise to a unique morphism $\hat{\beta}_{12} : \operatorname{Ker}\alpha_{12} \to \operatorname{Ker}\alpha_{22}$ such that $\beta_{12}\ker\alpha_{12} = (\ker\alpha_{22})\hat{\beta}_{12}$ (see the end of Section 1). Similarly, $\beta_{12}$ induces a morphism $\beta'_{12} : \operatorname{Coker}\alpha_{11} \to \operatorname{Coker}\alpha_{21}$ of the cokernels of the rows of the square

$$\begin{array}{ccc}
A_{11} & \xrightarrow{\alpha_{11}} & A_{12} \\
{\scriptstyle\beta_{11}}\downarrow & & {\scriptstyle\beta_{12}}\downarrow \\
A_{21} & \xrightarrow{\alpha_{21}} & A_{22}
\end{array}$$

and, hence, since im = coker ker, we have a morphism $\check{\beta}_{12} : \operatorname{Im}\alpha_{11} \to \operatorname{Im}\alpha_{21}$ making the diagram

$$\begin{array}{ccccc}
\operatorname{Im}\alpha_{11} & \xrightarrow{\operatorname{im}\alpha_{11}} & A_{12} & \xrightarrow{\operatorname{coker}\alpha_{11}} & \operatorname{Coker}\alpha_{11} \\
{\scriptstyle \check{\beta}_{12}}\big\downarrow & & {\scriptstyle \beta_{12}}\big\downarrow & & {\scriptstyle \beta'_{12}}\big\downarrow \\
\operatorname{Im}\alpha_{21} & \xrightarrow{\operatorname{im}\alpha_{21}} & A_{22} & \xrightarrow{\operatorname{coker}\alpha_{21}} & \operatorname{Coker}\alpha_{21}
\end{array}$$

commute. Denote by $\varepsilon_1 : \operatorname{Im}\alpha_{11} \to \operatorname{Ker}\alpha_{12}$ and $\varepsilon_2 : \operatorname{Im}\alpha_{21} \to \operatorname{Ker}\alpha_{22}$ are the natural morphisms such that $\operatorname{im}\alpha_{11} = (\ker\alpha_{12})\varepsilon_1$ and $\operatorname{im}\alpha_{21} = (\ker\alpha_{22})\varepsilon_2$ (by Lemma 2(1), $\varepsilon_1$ and $\varepsilon_2$ are both strict monomorphisms). Then $\hat{\beta}_{12}\varepsilon_1 = \varepsilon_2\check{\beta}_{12}$, $H_1 = \operatorname{coker}\varepsilon_1$, $H_2 = \operatorname{coker}\varepsilon_2$, and we have a natural morphism $\varphi : H_1 \to H_2$ making the diagram

$$\begin{array}{ccccc}
\operatorname{Im}\alpha_{11} & \xrightarrow{\varepsilon_1} & \operatorname{Ker}\alpha_{12} & \xrightarrow{\operatorname{coker}\varepsilon_1} & H_1 \\
{\scriptstyle \check{\beta}_{12}}\big\downarrow & & {\scriptstyle \hat{\beta}_{12}}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow \\
\operatorname{Im}\alpha_{21} & \xrightarrow{\varepsilon_2} & \operatorname{Ker}\alpha_{22} & \xrightarrow{\operatorname{coker}\varepsilon_2} & H_2
\end{array}$$

commute.

Now, since, obviously, $\alpha_{13}\beta_{13}^{-1}\alpha_{22}\ker\alpha_{22} = 0$ and, by hypothesis,

$$\ker(\alpha_{13}\beta_{13}^{-1}\alpha_{22}) = \ker(\beta_{31}\alpha_{31}^{-1}\beta_{22}),$$

we have $\ker\alpha_{22} = \ker(\beta_{31}\alpha_{31}^{-1}\beta_{22})h$ for some morphism $h$. This implies that

$$\beta_{31}\alpha_{31}^{-1}\beta_{22}\ker\alpha_{22} = 0,$$

and thus there is a unique morphism $\mu : \operatorname{Ker}\alpha_{22} \to \operatorname{Ker}\beta_{31}$ with the property

$$\alpha_{31}^{-1}\beta_{22}\ker\alpha_{22} = (\ker\beta_{31})\mu.$$

Moreover, since

$$(\operatorname{coker}\beta_{21})\alpha_{31}^{-1}\beta_{22}\alpha_{21} = (\operatorname{coker}\beta_{21})\beta_{21} = 0,$$

we have a morphism $\pi : \operatorname{Coker}\alpha_{21} \to \operatorname{Coker}\beta_{21}$ such that $\pi\operatorname{coker}\alpha_{21} = (\operatorname{coker}\beta_{21})\alpha_{31}\beta_{22}$ and, thus, a morphism $\tau : \operatorname{Im}\alpha_{21} \to \operatorname{Im}\beta_{21}$ making the diagram

$$\begin{array}{ccccc}
\operatorname{Im}\alpha_{21} & \xrightarrow{\operatorname{im}\alpha_{21}} & A_{22} & \xrightarrow{\operatorname{coker}\alpha_{21}} & \operatorname{Coker}\alpha_{21} \\
{\scriptstyle \tau}\big\downarrow & & {\scriptstyle \alpha_{31}^{-1}\beta_{22}}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\
\operatorname{Im}\beta_{21} & \xrightarrow{\operatorname{im}\alpha_{21}} & A_{31} & \xrightarrow{\operatorname{coker}\beta_{21}} & \operatorname{Coker}\beta_{21}
\end{array}$$

commute.

Denote by $\varepsilon_3$ the morphism $\operatorname{Im}\beta_{21} \to \operatorname{Ker}\beta_{31}$ such that $\operatorname{im}\beta_{21} = (\ker\beta_{31})\varepsilon_3$. Then

$$(\ker\beta_{31})\mu\varepsilon_2 = \alpha_{31}^{-1}\beta_{22}(\ker\alpha_{22})\varepsilon_2 = \alpha_{31}^{-1}\beta_{22}\operatorname{im}\alpha_{21} = (\operatorname{im}\beta_{21})\tau = (\ker\beta_{31})\varepsilon_3\tau.$$

Since $\ker\beta_{31}$ is a monomorphism, this yields the relation $\mu\varepsilon_2 = \varepsilon_3\tau$ and thus there exists a unique morphism $\psi : H_2 \to H_3$ making the diagram

$$\begin{array}{ccccc}
\operatorname{Im}\alpha_{21} & \xrightarrow{\varepsilon_2} & \operatorname{Ker}\alpha_{22} & \xrightarrow{\operatorname{coker}\varepsilon_2} & H_2 \\
{\scriptstyle \tau}\big\downarrow & & {\scriptstyle \mu}\big\downarrow & & {\scriptstyle \psi}\big\downarrow \\
\operatorname{Im}\beta_{21} & \xrightarrow{\varepsilon_3} & \operatorname{Ker}\beta_{31} & \xrightarrow{\operatorname{coker}\varepsilon_3} & H_3
\end{array}$$

commute.

We now prove that $\psi\varphi = 0$.

We have the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Im}\alpha_{11} & \xrightarrow{\varepsilon_1} & \operatorname{Ker}\alpha_{12} & \xrightarrow{\operatorname{coker}\varepsilon_1} & H_1 \\
\check{\beta}_{12}\big\downarrow & & \hat{\beta}_{12}\big\downarrow & & \varphi\big\downarrow \\
\operatorname{Im}\alpha_{21} & \xrightarrow{\varepsilon_2} & \operatorname{Ker}\alpha_{22} & \xrightarrow{\operatorname{coker}\varepsilon_2} & H_2 \\
\tau\big\downarrow & & \mu\big\downarrow & & \psi\big\downarrow \\
\operatorname{Im}\beta_{21} & \xrightarrow{\varepsilon_3} & \operatorname{Ker}\beta_{31} & \xrightarrow{\operatorname{coker}\varepsilon_3} & H_3.
\end{array} \tag{4}$$

Note that

$$(\ker\beta_{31})\mu\hat{\beta}_{12} = \alpha_{31}^{-1}\beta_{22}(\ker\alpha_{22})\hat{\beta}_{12} = \alpha_{31}^{-1}\beta_{22}\beta_{12}\ker\alpha_{12} = 0.$$

Since $\ker\beta_{31} \in M$, this gives $\mu\hat{\beta}_{12} = 0$. Thus,

$$\psi\varphi\operatorname{coker}\varepsilon_1 = \psi(\operatorname{coker}\varepsilon_2)\hat{\beta}_{12} = (\operatorname{coker}\varepsilon_3)\mu\hat{\beta}_{12} = 0.$$

Now, $\operatorname{coker}\varepsilon_1 \in P$, and, hence, $\psi\varphi = 0$.

Thus, we have constructed the semi-exact sequence (3).

Now we consecutively prove assertions (a), (b), and (c).

(a) Suppose that $\alpha_{21}, \beta_{11}, \beta_{02} \in O_c$ and $\alpha_{01} \in P_c$. We need to prove that $\varphi$ is a monomorphism. To this end, take a morphism $x : X \to H_1$ such that $\varphi x = 0$ and show that $x = 0$.

Consider the pullback

$$\begin{array}{ccc}
T & \xrightarrow{t_2} & X \\
t_1\big\downarrow & & x\big\downarrow \\
\operatorname{Ker}\alpha_{12} & \xrightarrow{\operatorname{coker}\varepsilon_1} & H_1.
\end{array}$$

Since $(\operatorname{coker}\varepsilon_2)\hat{\beta}_{12}t_1 = \varphi(\operatorname{coker}\varepsilon_1)t_1 = \varphi x t_2 = 0$, $\varepsilon_1 = \ker\operatorname{coker}\varepsilon_1$, there exists a morphism $u : T \to \operatorname{Im}\alpha_1$ such that $\hat{\beta}_{12}t_1 = \varepsilon_2 u$. Consider the pullback

$$\begin{array}{ccc}
V & \xrightarrow{v_2} & T \\
v_1\big\downarrow & & u\big\downarrow \\
A_{21} & \xrightarrow{\tilde{\alpha}_{21}} & \operatorname{Im}\alpha_{21}.
\end{array}$$

We have

$$\begin{aligned}
\alpha_{31}\beta_{21}v_1 &= \beta_{22}\alpha_{21}v_1 = \beta_{22}\operatorname{im}\alpha_{21}uv_2 \\
&= \beta_{22}(\ker\alpha_{22})\varepsilon_2 uv_2 = \beta_{22}(\ker\alpha_{22})\hat{\beta}_{12}t_1uv_2 = \beta_{22}\beta_{12}(\ker\alpha_{12})t_1uv_2.
\end{aligned}$$

Since $\alpha_{31}$ is an isomorphism, this implies that $\beta_{21}v_1 = 0$. From the exactness of the first column at the term $A_{21}$ it follows that then $v_1 = (\operatorname{im}\beta_{11})w$ for some unique morphism $w$. We can write the following commutative diagram:

$$\begin{array}{ccccc}
C & \xrightarrow{w'} & A_{11} & \xrightarrow{\alpha_{11}} & A_{12} \\
{\scriptstyle\varkappa}\downarrow & & {\scriptstyle\tilde\beta_{11}}\downarrow & & {\scriptstyle\tilde\beta_{12}}\downarrow \\
V & \xrightarrow{w} & \operatorname{Im}\beta_{11} & \xrightarrow{\sigma} & \operatorname{Im}\beta_{12} \\
 & & {\scriptstyle\operatorname{im}\beta_{11}}\downarrow & & {\scriptstyle\operatorname{im}\beta_{12}}\downarrow \\
 & & A_{21} & \xrightarrow{\alpha_{21}} & A_{22},
\end{array}$$

where the left upper square is a pullback and $\sigma : \operatorname{Im}\beta_{11} \to \operatorname{Im}\beta_{12}$ is the natural morphism of the images induced by $\alpha_{21}$.

We infer:

$$\begin{aligned}
\beta_{12}\alpha_{11}w' = \alpha_{21}\beta_{11}w' = \alpha_{21}(\operatorname{im}\beta_{11})w\varkappa = (\operatorname{im}\alpha_{21})\tilde\alpha_{21}v_1\varkappa \\
= (\operatorname{im}\alpha_{21})uv_2\varkappa = (\ker\alpha_{22})\varepsilon_2uv_2\varkappa = (\ker\alpha_{22})\hat\beta_{12}t_1v_2\varkappa.
\end{aligned}$$

Consequently,

$$\beta_{12}((\ker\alpha_{12})t_1v_2\varkappa - \alpha_{11}w') = 0.$$

The exactness of the second column at the term $A_{12}$ yields

$$(\ker\alpha_{12})t_1v_2\varkappa - \alpha_{11}w' = (\operatorname{im}\beta_{02})\gamma \tag{5}$$

for some morphism $\gamma : C \to A_{12}$. Consider the commutative diagram

$$\begin{array}{ccc}
A_{01} & \xrightarrow{\alpha_{01}} & A_{02} \\
{\scriptstyle\tilde\beta_{01}}\downarrow & & {\scriptstyle\tilde\beta_{02}}\downarrow \\
\operatorname{Im}\beta_{01} & \xrightarrow{\check\alpha_{11}} & \operatorname{Im}\beta_{02} \\
{\scriptstyle\operatorname{im}\beta_{01}}\downarrow & & {\scriptstyle\operatorname{im}\beta_{02}}\downarrow \\
A_{11} & \xrightarrow{\alpha_{11}} & A_{12}.
\end{array}$$

Here $\check\alpha_{11} : \operatorname{Im}\beta_{01} \to \operatorname{Im}\beta_{02}$ is the natural morphism of the images induced by $\alpha_{11}$. Consider the pullback

$$\begin{array}{ccc}
P & \xrightarrow{p_2} & C \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle\gamma}\downarrow \\
A_{02} & \xrightarrow{\tilde\beta_{02}} & B_{02}
\end{array}$$

and then the pullback

$$\begin{array}{ccc}
R & \xrightarrow{r_2} & P \\
{\scriptstyle r_1}\downarrow & & {\scriptstyle p_1}\downarrow \\
A_{01} & \xrightarrow{\alpha_{01}} & A_{02}.
\end{array}$$

We obtain from (5):

$$\begin{aligned}
(\ker\alpha_{12})t_1v_2\varkappa p_2r_2 - \alpha_{11}w'p_2r_2 = (\operatorname{im}\beta_{02})\gamma p_2r_2 = (\operatorname{im}\beta_{02})\tilde\beta_{02}p_1r_2 \\
= \beta_{02}p_1r_2 = \beta_{02}\alpha_{01}r_1 = \alpha_{11}\beta_{01}r_1.
\end{aligned}$$

Therefore,

$$(\ker\alpha_{12})t_1v_2\varkappa p_2r_2 = \alpha_{11}(w'p_2r_2 + \beta_{01}r_1) = (\ker\alpha_{12})\varepsilon_1\tilde\alpha_{11}(w'p_2r_2 + \beta_{01}r_1),$$

which, since $\ker\alpha_{12} \in M$, gives the relation

$$t_1 v_2 \varkappa p_2 r_2 = \varepsilon_1 \tilde{\alpha}_{11}(w' p_2 r_2 + \beta_{01} r_1).$$

By Axiom 3, from the relations $\tilde{\alpha}_{21} \in P_c$, $\tilde{\beta}_{11} \in P_c$, $\tilde{\beta}_{02} \in P_c$, $\alpha_{01} \in P_c$, it follows that $v_2 \in P_c$, $\varkappa \in P_c$, $p_2 \in P_c$, $r_2 \in P_c$ respectively. Hence, by Lemma 2(2), $v_2\varkappa p_2 r_2 \in P_c$. Put $a = t_1 v_2 \varkappa p_2 r_2$, $b = \tilde{\alpha}_{11}(w' p_2 r_2 + \beta_{01} r_1)$. We have $\operatorname{im} a = \operatorname{im} t_1 = \varepsilon_1 \operatorname{im} b$. Thus, $t_1 = \varepsilon_1 (\operatorname{im} b)\tilde{t}_1$. Therefore,

$$x t_2 = (\operatorname{coker}\varepsilon_1) t_1 = (\operatorname{coker}\varepsilon_1)\varepsilon_1(\operatorname{im} b)\tilde{t}_1 = 0.$$

Since $t_2 \in P$, this implies that $x = 0$.

(b) Suppose that $\beta_{12}$ and $\beta_{21}$ are strict in (2). Let $x : X \to H_2$ be a morphism with $\psi x = 0$. Demonstrate that $x = (\operatorname{im}\varphi)x'$ for some unique $x'$. We may assume without loss of generality that $x = \operatorname{im} x \in M_c$.

Consider the pullback

$$\begin{array}{ccc} G & \xrightarrow{g_2} & X \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle x} \\ \operatorname{Ker}\alpha_{12} & \xrightarrow{\varphi} & H_2. \end{array}$$

We infer from (4) that

$$0 = \psi x g_2 = \psi(\operatorname{coker}\varepsilon_2) g_1 = (\operatorname{coker}\varepsilon_3)\mu g_1 = 0.$$

Since $\varepsilon_3 = \ker\operatorname{coker}\varepsilon_3$, this implies that $\mu g_1 = \varepsilon_3 g$ for some $g$. Consider now the pullback

$$\begin{array}{ccc} B & \xrightarrow{b_2} & G \\ {\scriptstyle b_1}\downarrow & & \downarrow{\scriptstyle g} \\ A_{21} & \xrightarrow{\tilde{\beta}_{21}} & \operatorname{Im}\beta_{21}. \end{array}$$

Recalling that $(\ker\beta_{31})\mu = \alpha_{31}^{-1}\beta_{22}\ker\alpha_{22}$, we infer

$$\begin{aligned} \beta_{21} b_1 &= (\operatorname{im}\beta_{21})\tilde{\beta}_{21} b_1 = (\operatorname{im}\beta_{21}) g b_2 \\ &= (\ker\beta_{31})\varepsilon_3 g b_2 = (\ker\beta_{31})\mu g_1 b_2 = \alpha_{31}^{-1}\beta_{22}(\ker\alpha_{22}) g_1 b_2. \end{aligned}$$

Consequently,

$$\beta_{22}(\ker\alpha_{22}) g_1 b_2 = \alpha_{31}\beta_{21} b_1 = \beta_{22}\alpha_{21} b_1.$$

But then

$$\beta_{22}((\ker\alpha_{22}) g_1 b_2 - \alpha_{21} b_1) = 0.$$

Hence, by the exactness of the second column at the term $A_{22}$, there exists a unique morphism $\theta : \Theta \to \operatorname{Im}\beta_{12}$ such that $(\ker\alpha_{22}) g_1 b_2 - \alpha_{21} b_1 = (\operatorname{im}\beta_{12})\theta$. Consider the pullback

$$\begin{array}{ccc} \Theta' & \xrightarrow{\theta_1} & \Theta \\ {\scriptstyle \theta'}\downarrow & & \downarrow{\scriptstyle \theta} \\ A_{12} & \xrightarrow{\tilde{\beta}_{12}} & \operatorname{Im}\beta_{12}. \end{array}$$

We infer:

$$\begin{aligned} \beta_{12}\theta' &= (\operatorname{im}\beta_{12})\tilde{\beta}_{12}\theta' = (\operatorname{im}\beta_{12})\theta\theta_1 \\ &= (\ker\alpha_{22}) g_2 b_2 \theta_1 - (\operatorname{im}\alpha_{21})\tilde{\alpha}_{21} b_1 \theta_1 = (\ker\alpha_{22})(g_1 b_2 \theta_1 - \varepsilon_2 \tilde{\alpha}_{21} b_1 \theta_1). \end{aligned}$$

Therefore,

$$\beta_{13}\alpha_{12}\theta' = \alpha_{22}\beta_{12}\theta' = 0.$$

Since $\beta_{13}$ is an isomorphism, this means that $\alpha_{12}\theta' = 0$. Hence, there is a unique morphism $\theta_3 : \Theta' \to \operatorname{Ker}\alpha_{12}$ with $\theta' = (\ker\alpha_{12})\theta_3$. Consequently,

$$\beta_{12}\theta' = \beta_{12}(\ker\alpha_{12})\theta_3 = (\ker\alpha_{22})\hat{\beta}_{12}\theta_3.$$

Thus, we have the equality

$$(\ker\alpha_{22})(g_1 b_2\theta_1 - \varepsilon_2\tilde{\alpha}_{21}b_1\theta_1) = (\ker\alpha_{22})\hat{\beta}_{12}\theta_3.$$

Since $\ker\alpha_{22}$ is a monomorphism, this yields

$$g_1 b_2\theta_1 - \varepsilon_2\tilde{\alpha}_{21}b_1\theta_1 = \hat{\beta}_{12}\theta_3 \tag{6}$$

Apply $\operatorname{coker}\varepsilon_2$ to both sides of (6). We infer:

$$(\operatorname{coker}\varepsilon_2)g_1 b_2\theta_1 = (\operatorname{coker}\varepsilon_2)\hat{\beta}_{12}\theta_3,$$
$$x g_2 b_2\theta_1 = \varphi(\operatorname{coker}\varepsilon_1)\theta_3.$$

By Axiom 3, we have the implications: $\operatorname{coker}\varepsilon_2 \in P_c \Longrightarrow g_2 \in P_c$; $\beta_{21} \in P_c \Longrightarrow b_2 \in P_c$; $\beta_{12} \in P_c \Longrightarrow \theta_1 \in P_c$. By Lemma 2(2), the morphism $c = g_2 b_2\theta_1 \in P_c$. Put $xc = d$, $(\operatorname{coker}\varepsilon_1)\theta_3 = l$, $\tilde{\varphi}(\operatorname{coker}\varepsilon_1)\theta_3 = l'$. Then we have two canonical decompositions of $d$:

$$d = xc = (\operatorname{im}\varphi)(\operatorname{im} l')\overline{l'}\operatorname{coim} l'.$$

Hence,

$$x = (\operatorname{im}\varphi)\operatorname{im} l'. \tag{7}$$

Since $\operatorname{im}\varphi$ is a monomorphism, $\operatorname{im} l'$ is defined by (7) uniquely.

Item (b) is proved.

(c) Pass to the dual category (obviously also quasi-abelian) and consider the dual assertion:

**Lemma 3.** *Suppose that in the commutative diagram*

$$\begin{array}{ccccccc}
 & & & & C_{42} & \xrightarrow{\gamma_{41}} & C_{41}\\
 & & & & {\scriptstyle\delta_{32}}\downarrow & & {\scriptstyle\delta_{31}}\downarrow\\
 & & & & C_{32} & \xrightarrow{\gamma_{31}} & C_{31}\\
 & & & & {\scriptstyle\delta_{22}}\downarrow & & {\scriptstyle\delta_{21}}\downarrow\\
 & & C_{23} & \xrightarrow{\gamma_{22}} & C_{22} & \xrightarrow{\gamma_{21}} & C_{21}\\
 & & {\scriptstyle\delta_{13}}\downarrow & & {\scriptstyle\delta_{12}}\downarrow & & {\scriptstyle\delta_{11}}\downarrow\\
C_{14} & \xrightarrow{\gamma_{13}} & C_{13} & \xrightarrow{\gamma_{12}} & C_{12} & \xrightarrow{\gamma_{11}} & C_{11}\\
 & & & & {\scriptstyle\delta_{02}}\downarrow & & {\scriptstyle\delta_{01}}\downarrow\\
 & & & & C_{02} & \xrightarrow{\gamma_{01}} & C_{01}
\end{array} \tag{8}$$

*in a quasi-abelian category the rows and columns are semi-exact, the penultimate row is exact at the term $C_{13}$, the last column is exact at $C_{11}$ and $C_{21}$, the penultimate column is exact, $\delta_{13}$ and $\gamma_{31}$ are isomorphisms, $\gamma_{01}$ is a monomorphism, and* $\operatorname{coker}(\gamma_{22}\delta_{13}^{-1}\gamma_{13}) = \operatorname{coker}(\delta_{22}\gamma_{31}^{-1}\delta_{31})$. *Denote by $\hat{H}_3$ the homology of the last column at the term $C_{31}$ and by $\hat{H}_2$, the homology of the third row at the term $C_{22}$.*

*Then the morphism* $\delta_{22}\gamma_{31}^{-1}$ *induces a homomorphism* $\hat{\psi} : \hat{H}_3 \to \hat{H}_2$*. If* $\gamma_{22}$*,* $\gamma_{13}$*,* $\delta_{32}$*, and* $\delta_{22}\gamma_{31}^{-1}\delta_{31}$ *are strict in (8) then* $\hat{\psi}$ *is a monomorphism.*

*Proof.* The commutative square

$$\begin{array}{ccc} C_{31} & \xrightarrow{\delta_{22}\gamma_{31}^{-1}} & C_{22} \\ {\scriptstyle\delta_{21}}\downarrow & & {\scriptstyle\gamma_{21}}\downarrow \\ C_{21} & \xrightarrow{\mathrm{id}} & C_{21}. \end{array}$$

induces a natural morphism $\lambda : \operatorname{Ker}\delta_{21} \to \operatorname{Ker}\gamma_{21}$ such that $(\ker\gamma_{21})\lambda = \delta_{22}\gamma_{31}^{-1}\ker\delta_{21}$ and a natural morphism of the cokernels $\omega : \operatorname{Coker}\delta_{21} \to \operatorname{Coker}\gamma_{21}$ such that $\operatorname{coker}\gamma_{21} = \omega\operatorname{coker}\delta_{21}$ giving a natural morphism of the images $\lambda' : \operatorname{Im}\delta_{21} \to \operatorname{Im}\gamma_{21}$ such that $(\operatorname{im}\delta_{21} = (\operatorname{im}\gamma_{21})\lambda'$. Consequently, the morphism $\delta_{22}\gamma_{31}^{-1}$ defines a unique morphism of the homologies $\hat{\psi} : \hat{H}_3 \to \hat{H}_2$ — the morphism of the cokernels of the rows of the square

$$\begin{array}{ccc} \operatorname{Im}\delta_{21} & \xrightarrow{\hat{\varepsilon}_3} & \operatorname{Ker}\delta_{21} \\ {\scriptstyle\lambda'}\downarrow & & {\scriptstyle\lambda}\downarrow \\ \operatorname{Im}\gamma_{21} & \xrightarrow{\hat{\varepsilon}_2} & \operatorname{Ker}\gamma_{21}. \end{array}$$

Here $\hat{\varepsilon}_3 : \operatorname{Im}\delta_{21} \to \operatorname{Ker}\delta_{21}$ and $\hat{\varepsilon}_2 : \operatorname{Im}\gamma_{21} \to \operatorname{Ker}\gamma_{21}$ are the natural inclusions.

Let $x : X \to \hat{H}_3$ be a morphism such that $\hat{\psi}x = 0$. Prove that $x = 0$. Consider the pullback

$$\begin{array}{ccc} Y & \xrightarrow{y_1} & X \\ {\scriptstyle y_2}\downarrow & & {\scriptstyle x}\downarrow \\ \operatorname{Ker}\delta_{21} & \xrightarrow{\operatorname{coker}\varepsilon_3} & \hat{H}_3. \end{array}$$

Since $(\operatorname{coker}\hat{\varepsilon}_2)\lambda y_2 = \hat{\psi}(\operatorname{coker}\hat{\varepsilon}_3)y_2 = \hat{\psi}xy_1 = 0$ and $\varepsilon_2 = \ker\operatorname{coker}\varepsilon_2$, there is a morphism $y : Y \to \operatorname{Im}\gamma_{21}$ with $\lambda y_2 = \hat{\varepsilon}_2 y$. Next, consider the pullback

$$\begin{array}{ccc} V' & \xrightarrow{v_2'} & Y \\ {\scriptstyle v_1'}\downarrow & & {\scriptstyle y}\downarrow \\ C_{23} & \xrightarrow{\tilde{\gamma}_{22}} & \operatorname{Im}\gamma_{22}. \end{array}$$

We have:

$$\begin{aligned} \gamma_{12}\delta_{13}v_1' &= \delta_{12}\gamma_{22}v_1' = \delta_{12}(\operatorname{im}\gamma_{22})\tilde{\gamma}_{22}v_1' = \delta_{12}(\operatorname{im}\gamma_{22})yv_2' \\ &= \delta_{12}(\ker\gamma_{21})\hat{\varepsilon}_2 yv_2' = \delta_{12}(\ker\gamma_{21})\lambda y_2 v_2' = \delta_{12}\delta_{22}\gamma_{31}^{-1}(\ker\delta_{21})y_2v_2' = 0. \end{aligned}$$

The exactness of the penultimate row of (8) at the term $C_{13}$ implies that $\delta_{13}v_1' = (\operatorname{im}\gamma_{13})w'$ for a suitable (unique) morphism $w' : V' \to \operatorname{Im}\gamma_{13}$, i.e., $v_1' = \delta_{13}^{-1}(\operatorname{im}\gamma_{13})w'$.

Consider the pullback

$$\begin{array}{ccc} W' & \xrightarrow{w_1'} & V' \\ {\scriptstyle w_2'}\downarrow & & {\scriptstyle w'}\downarrow \\ C_{14} & \xrightarrow{\tilde{\gamma}_{13}} & \operatorname{Im}\gamma_{13}. \end{array}$$

Put $f=\gamma_{22}\delta_{13}^{-1}\gamma_{13}$, $f_0=\delta_{22}\gamma_{31}^{-1}\delta_{31}$. By hypothesis, $\operatorname{im} f=\operatorname{im} f_0$. We infer

$$\gamma_{22}v_1'w_1'=\gamma_{22}\delta_{13}^{-1}\gamma_{13}w_2'=fw_2'=(\operatorname{im} f)\tilde{f}w_2'=(\operatorname{im} f_0)\tilde{f}w_2'.$$

Consider now the pullback

$$\begin{array}{ccc} Z & \xrightarrow{f_0'} & W' \\ {\scriptstyle w_2''}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{f}w_2'} \\ C_{41} & \xrightarrow{\tilde{f}_0} & \operatorname{Im} f_0. \end{array}$$

We have

$$\delta_{22}\gamma_{31}^{-1}\delta_{31}w_2''=(\operatorname{im} f_0)\tilde{f}w_2'f_0'=\gamma_{22}\delta_{13}^{-1}\gamma_{13}w_2'f_0'.$$

Furthermore,

$$\delta_{22}\gamma_{31}^{-1}(\ker\delta_{21})y_2v_2'=(\ker\gamma_{21})\lambda y_2v_2'=(\ker\gamma_{21})\hat{\varepsilon}_2yv_2'=(\operatorname{im}\gamma_{22})\tilde{\gamma}_{22}v_1'=\gamma_{22}v_1'.$$

Consequently,

$$\begin{aligned}\delta_{22}\gamma_{31}^{-1}(\ker\delta_{21})y_2v_2'w_1'f_0'&=\gamma_{22}v_1'w_1'f_0'=\gamma_{22}\delta_{13}^{-1}(\operatorname{im}\gamma_{13})w'w_1'f_0'\\ &=\gamma_{22}\delta_{13}^{-1}(\operatorname{im}\gamma_{13})\tilde{\gamma}_{13}w_2'f_0'=\gamma_{22}\delta_{13}^{-1}\gamma_{13}w_2'f_0'\\ &=(\operatorname{im} f)\tilde{f}w_2'f_0'=(\operatorname{im} f)\tilde{f}_0w_2''=f_0w_2''=\delta_{22}\gamma_{31}^{-1}\delta_{31}w_2''.\end{aligned}$$

Thus,

$$\delta_{22}\gamma_{31}^{-1}(\ker\delta_{21})y_2v_2'w_1'f_0'=\delta_{22}\gamma_{31}^{-1}\delta_{31}w_2'',$$

that is,

$$\delta_{22}\gamma_{31}^{-1}(\delta_{31}w_2''-(\ker\delta_{21})y_2v_2'w_1'f_0')=0.$$

By the exactness of the penultimate column at the term $C_{32}$, we infer that

$$\gamma_{31}^{-1}(\delta_{31}w_2''-(\ker\delta_{21})y_2v_2'w_1'f_0')=(\operatorname{im}\delta_{32})\zeta$$

for some unique morphism $\zeta: Z'\to\operatorname{Im}\delta_{32}$. Consider the pullback

$$\begin{array}{ccc} K & \xrightarrow{k_2} & C_{42} \\ {\scriptstyle k_1}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\delta}_{32}} \\ Z & \xrightarrow{\zeta} & \operatorname{Im}\delta_{32}. \end{array}$$

Hence,

$$\delta_{31}w_2''k_1-(\ker\delta_{21})y_2v_2'w_1'f_0'k_1=\gamma_{31}\delta_{32}k_2=\delta_{31}\gamma_{41}k_2,$$

or

$$(\ker\delta_{21})y_2v_2'w_1'f_0'k_1=\delta_{31}(w_2'k_1-\gamma_{41}k_2)=(\ker\delta_{21})\hat{\varepsilon}_3\tilde{\delta}_{31}(w_2'k_1-\gamma_{41}k_2),$$

Since $\ker\delta_{21}$ is a monomorphism, this yields

$$y_2v_2'w_1'f_0'k_1=\hat{\varepsilon}_3\tilde{\delta}_{31}(w_2'k_1-\gamma_{41}k_2).$$

By Axiom 3, we have the implications: $\gamma_{22}\in O_c\Longrightarrow v_2'\in P_c$; $\gamma_{13}\in O_c\Longrightarrow w_1'\in P_c$; $\delta_{22}\gamma_{31}^{-1}\delta_{31}\in O_c\Longrightarrow f_0'\in P_c$; $\delta_{32}\in O_c\Longrightarrow k_1\in P_c$. Thus, by Lemma 2(2), $y_2v_2'w_1'f_0'k_1\in P_c$.

Put $a=\hat{\varepsilon}_3\tilde{\delta}_{31}(w_2'k_1-\gamma_{41}k_2)$, $a'=\tilde{\delta}_{31}(w_2'k_1-\gamma_{41}k_2)$. Since $\hat{\varepsilon}_3\in M_c$, then $a=\hat{\varepsilon}_3(\operatorname{im} a)\overline{a}(\operatorname{coim} a)$ and $\operatorname{im} a=\hat{\varepsilon}_3\operatorname{im} a'=\operatorname{im} y_2$. Hence, $y_2=\hat{\varepsilon}_2y_2'$ for a suitable morphism $y_2$. Therefore,

$$xy_1=(\operatorname{coker}\hat{\varepsilon}_3)y_2=(\operatorname{coker}\hat{\varepsilon}_3)\widehat{\varepsilon}_3y_2'=0.$$

Since $y_2 \in P$, we have $x = 0$, q.e.d.

Lemma 3 is proved, and so is the dual assertion to it, (c) of Theorem 1. □

Theorem 1 is proved. □

**Remark.** Assume that all assumptions of Theorem 1 hold. Then (3) is exact at all terms. By analyzing (4), we easily see that:

(i) if $(\operatorname{coker}\varepsilon_2)\hat{\beta}_{12}$ is strict then so is $\varphi$ and, thus, $\varphi = \ker\psi$;

(ii) if $(\operatorname{coker}\varepsilon_3)\mu$ is strict then so is $\psi$ and, hence, $\psi = \operatorname{coker}\varphi$.

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