Integral Formulas for Affine Surfaces and Rigidity Theorems of Cohn-Vossen Type

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In this note we establish some integral formulas for surfaces in affine space which are analogous to those for surfaces in Euclidean space. As applications we provide proofs for the classical theorems of Blaschke that characterize ellipsoids among the ovaloids. Then we proceed to prove some results on the rigidity of compact affine surfaces provided with equiaffine transversal fields.

Emphasis in the note is more on exposition than on new, original results. The rigidity theorem of Cohn-Vossen type (Theorem 4) we establish is in weaker form than what is given in [S], but our approach, based on the integral formulas, is elementary and easily approachable. It is comparable to Herglotz's proof of the original Cohn-Vossen theorem for surfaces in Euclidean space (see [C]).

In Section 1 we recall the basic facts about affine surfaces and derive integral formulas. In Section 2, we prove two characterization theorems (Theorems 1, 2), due to Blaschke, for ellipsoids. There are perhaps several different proofs which are scattered in [B] and in some references contained in [B]. In Section 3 we prepare a few lemmas that are used in Section 4 in order to prove several congruence results, Theorems 3, 4 and 5. As mentioned already, Theorem 4 is our affine version of the rigidity theorem of Cohn-Vossen; the case of Blaschke surfaces (Corollary to Theorem 4) was first presented at a seminar at KU, Louvain, in June 1990.

1. Integral formulas

Let M^2 be a connected, orientable, differentiable 2-manifold. Let $f : M^2 \to R^3$ be an immersion into the affine 3-space equipped with a globally defined transversal vector field ξ along f. For all tangent vector fields X, Y we can write

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{1}$$

where D denotes the usual flat connection in R^3. This formula defines a torsion-free affine connection ∇, said to be induced by (f, ξ) and a bilinear symmetric tensor h, called the fundamental form for (f, ξ). The rank of h does not depend on the choice of transversal vector field. If the rank is n, we say that f is nondegenerate.

We restrict our attention to transversal vector fields ξ with the property that $D_X\xi$ is tangential to $f_*(M^2)$; they are called equiaffine transversal vector fields. If we set

$$\theta(X,Y) = \det\,(f_*X, f_*Y, \xi), \tag{2}$$

for all tangent vectors X, Y to M^2, θ is a volume element on M^2 which is ∇-parallel. We may also write

$$D_X\xi = -f_*(SX), \tag{3}$$

thus defining a (1,1)-tensor field S, called the shape operator for (f, ξ).

Remark 1. For the fundamental equations of Gauss, Codazzi and Ricci, see [N-P]. The Ricci tensor for ∇ is given by Ric $(Y, Z) = h((\mathrm{tr}SI - S)Y, Z)$. For surfaces, if Ric is nondegenerate, then S is nonsingular and h is nondegenerate. The converse also holds.

The classical Blaschke theory goes like this. Assume $f : M^2 \to R^3$ is nondegenerate. Then we can find a unique equiaffine transversal vector field ξ (up to sign), called the affine normal, such that the form θ defined by (2) coincides with the volume element of the nondegenerate metric h. When we choose an affine normal ξ, the corresponding structures ∇, h and S are called, respectively, the Blaschke connection on M^2, the affine fundamental form (or affine metric) and the affine shape operator. Or we may simply refer to $f : M^2 \to R^3$ as a Blaschke immersion or a Blaschke surface. We call $K = \det S$ and $H = \mathrm{tr}S/2$ the affine Gaussian curvature and the affine mean curvature, respectively.

Suppose that M^2 is compact and $f : M^2 \to R^3$ is a nondegenerate imbedding with an equiaffine transversal vector field ξ (not necessarily an affine normal). It is known that $f(M^2)$ is a convex surface which is the boundary of a convex body. By changing ξ to $-\xi$, if necessary, we may assume that h is positive-definite and ξ is inward. We call M^2 an ovaloid (in the sense of affine differential geometry).

It goes without saying that all the definitions go over to the case of an immersion $M^n \to R^{n+1}$. But we are concerned with the case $n = 2$ in this paper.

Now assume that f and $\bar{f}$ are two immersions: $M^2 \to R^3$, where M^2 is compact. Assume that we have equiaffine transversal vector fields $\xi, \bar{\xi}$ for $f, \bar{f}$ such that $\nabla = \overline{\nabla}, \theta = \bar{\theta}$. We shall derive a mixed integral formula. We define a 1-form α by

$$\alpha(X) = \theta(Z, \bar{S}X), \tag{4}$$

where $\bar{S}$ is the shape operator for $\bar{f}$ and Z denotes the tangential component of the position vector $f(x)$:

$$f(x) = \rho\xi_x + f_*Z_x. \tag{5}$$

Obviously, we can define a similar form, say, $\bar{\alpha}$ by interchanging the roles of f and $\bar{f}$:

$$\bar{\alpha}(X) = \theta(\bar{Z},\ SX), \tag{$\bar{4}$}$$

where S is the shape operator for f and

(5) $$\bar{f}(x) = \bar{\rho}\bar{\xi} + \bar{f}_*\bar{Z}_x.$$

We compute $d\alpha$. From $\alpha(Y) = \theta(Z, \bar{S}Y)$ we obtain

(6) $$X\alpha(Y) = (\nabla_X\theta)(Z, \bar{S}Y) + \theta(\nabla_X Z, \bar{S}Y) + \theta(Z, (\nabla_X\bar{S})Y) + \theta(Z, \bar{S}(\nabla_X Y)).$$

Here we have $\nabla_X\theta = 0$. In order to take care of $\nabla_X Z$, we consider the conormal map $v : M^2 \to R_3$: for each $x \in M^2$, v_x is the covector in the dual space R_3 such that $v_x(\xi_x) = 1$ and $v_x(f_*X) = 0$ for all $X \in T_x(M^2)$. Then in (5), we have $\rho = v(f(x))$. We define the affine distance function $\rho : M^2 \to R$ by $\rho(x) = v(f(x)), x \in M^2$. Thus

(5′) $$f(x) = \rho\xi_x + f_*Z_x.$$

Differentiating this equation relative to X we get

(7) $$\nabla_X Z = \rho SX + X \quad \text{and} \quad h(X, Z) = -X\rho.$$

We can rewrite (6) as follows:

$$X\alpha(Y) = \rho\theta(SX, \bar{S}Y) + \theta(X, \bar{S}Y) + \theta(Z, (\nabla_X\bar{S})Y) + \theta(Z, \bar{S}(\nabla_X Y)).$$

Taking $X\alpha(Y) - Y\alpha(X) - \alpha([X, Y])$ and using the Codazzi equation $(\nabla_Y\bar{S})(X) = (\nabla_X\bar{S})(Y)$ we obtain

(8) $$(d\alpha)(X, Y) = \rho[\theta(SX, \bar{S}Y) + \theta(\bar{S}X, SY) + \theta(X, \bar{S}Y) + \theta(\bar{S}X, Y)].$$

Let $\{X_1, X_2\}$ be a basis in $T_x(M^2)$ such that $\theta(X_1, X_2) = 1$ and write

$$SX_j = \sum_{i=1}^{2} S^i_j X_i \quad \text{and} \quad \bar{S}X_j = \sum_{i=1}^{2} \bar{S}^i_j X_i.$$

From (8) we compute

$$\begin{aligned}(d\alpha)(X_1, X_2) &= \rho[S^1_1\bar{S}^2_2 - S^2_1\bar{S}^1_2 + \bar{S}^1_1 S^2_2 - \bar{S}^2_1 S^1_2] + \mathrm{tr}\bar{S} \\ &= -2\rho < S, \bar{S} > + \mathrm{tr}\bar{S},\end{aligned}$$

where $<\ ,\ >$ denotes the inner product of signature $(-, -, +, +)$ in the space of all endomorphisms of $T_x(M^2)$ defined by

$$< A, B >= [\mathrm{tr}(AB) - \mathrm{tr}A \cdot \mathrm{tr}B]/2.$$

We also have

$$-2 < A, B >= \det A + \det B - \det(A - B).$$

It finally follows that

$$d\alpha = [\rho(\det S + \det\bar{S} - \det(\bar{S} - S)) + \mathrm{tr}\bar{S}]\theta. \tag{9}$$

Next we define a 1-form β by $\beta(X) = \theta(Z, X)$, where X is any tangent vector and Z is as in (5). It is quite simple to get

$$d\beta = [2 + \rho\mathrm{tr}S]\theta. \tag{10}$$

From (9) we get

$$\int_{M^2} [2\rho\det S + \rho(\det\bar{S} - \det S)) - \rho\det(\bar{S} - S) + \mathrm{tr}\bar{S}]\theta = 0. \tag{I}$$

In particular, taking $\bar{f} = f$ we get

$$\int_{M^2} [2\rho\det S + \mathrm{tr}S]\theta = 0. \tag{II}$$

From (I) and (II) we obtain

$$\int_{M^2} (\mathrm{tr}\bar{S} - \mathrm{tr}S + \rho(\det\bar{S} - \det S)]\theta = \int_{M^2} \rho\det(\bar{S} - S))\theta. \tag{III}$$

From (10) we get

$$\int_{M^2} (2 + \rho\mathrm{tr}S)\theta = 0. \tag{IV}$$

2. Characterizations of ellipsoids

We shall now prove two characterization theorems for ellipsoids among the ovaloids with Blaschke structures.

Theorem 1. *If the affine Gaussian curvature K of an ovaloid $f : M^2 \to R^3$ is constant, then K is positive and $f(M^2)$ is an ellipsoid.*

Theorem 2. *If the affine mean curvature H of an ovaloid $f : M^2 \to R^3$ is constant, then $f(M^2)$ is an ellipsoid.*

Remark 2. The original result by Blaschke for Theorem 1 assumes that K is a positive constant, see [B, p.248]. Theorem 2 is stated in [B, p.201] and proved in the reference cited there. To prove Theorems 1 and 2, it is sufficient to show that, for an ovaloid $f : M^2 \to R^3$ with an equiaffine transversal vector field, the assumption that $\det S =$ constant or $\mathrm{tr} S =$ constant implies $S = \rho I$, where ρ is a nonzero constant. Then if f is further a Blaschke imbedding, it follows that $f(M^2)$ is an ellipsoid by another theorem of Blaschke [B, p.212].

Proof of Theorem 1. We may assume that h is positive-definite. By Lemma 3.1 in [O-V], it follows that $\det S$ is a positive constant. We may assume $K = 1$, without loss of generality. Let k_1, k_2 be two eigenvalues of S so that $k_1 k_2 = 1$. Hence

$H = (k_1 + k_2)/2 = (k_1 + 1/k_1)/2 \geq 1$, and the equality holds if and only if $k_1 = k_2$. Now since M^2 is convex, we may choose a point, say, o in the interior of the convex body bounded by $f(M^2)$ so that the position vector from o to $f(x)$ for each point x of M^2 is transversal to $f(M^2)$. Then the affine distance ρ from o is negative-valued on M^2. From (IV) and (II) we obtain

$$\int_{M^2} -\rho\theta = \int_{M^2} H\theta \geq \int_{M^2} \theta = \int_{M^2} -\rho H\theta$$

and hence

$$\int_{M^2} \rho(1 - H)\theta \leq 0.$$

But $\rho < 0$ and $1 - H \leq 0$. Hence the integral above is ≥ 0. It follows that the integral must be 0 and hence $H = 1$. We get $k_1 = k_2$.

Proof of Theorem 2. We may assume $1 = H = (k_1 + k_2)/2$. Then

$$K = k_1 k_2 = k_1(2 - k_1) = 2k_1 - (k_1)^2 \leq 1,$$

and the equality holds if and only if $k_1 = 1$. By (IV) and (II) we get

$$\int_{M^2} -\rho\theta = \int_{M^2} \theta = \int_{M^2} -\rho K\theta,$$

which implies $\int_{M^2} \rho(1 - K)\theta = 0$. Since $\rho < 0$ as before and $1 - K \geq 0$, it follows that $K = 1$ and hence $k_1 = k_2 = 1$, as desired..

3. Lemmas

The following lemma is essentially in [SL].

Lemma 1. *Let $f : (M^2, \nabla) \to R^3$ be a Blaschke surface. Then the affine Gaussian curvature $K = \det S$ is equal to $\epsilon \det R(X_1, X_2)$, where $\epsilon = \pm 1$ depending on whether h is definite or not, and $\{X_1, X_2\}$ is a unimodular basis (i.e. $\theta(X_1, X_2) = 1$), $\det R(X_1, X_2)$ being independent of the choice of such $\{X_1, X_2\}$.*

Proof. We include a proof for the sake of completeness. Let h be the fundamental form for f and let $\{X_1, X_2\}$ be an orthonormal basis in $T_x(M^2)$ relative to $h : h(X_1, X_1) = 1, h(X_1, X_2) = 0, h(X_2, X_2) = \epsilon$, where $\epsilon = \pm 1$. From the Gauss equation we have $R(X_1, X_2)X_1 = -SX_2 = -S_2^1 X_1 - S_2^2 X_2$ and $R(X_1, X_2)X_2 = \epsilon SX_1 = \epsilon(S_1^1 X_1 + S_1^2 X_2)$; by computing det $R(X_1, X_2)$ we find it to be $\epsilon\,\mathrm{det}S = \epsilon K$. Now observe furthermore that $R(X_1, X_2)$ is independent of the choice of X_1, X_2 such that $\theta(X_1, X_2) = 1$. Thus K is determined by (∇, θ) and the signature of h.

Lemma 2. *Let $f : M^n \to R^{n+1}$ be a hypersurface with a transversal vector field ξ. If $\alpha, \beta \in R - \{0\}, \tilde{f} = \alpha f, \tilde{\xi} = \beta\xi$, and if $\tilde{\nabla}, \tilde{h}, \tilde{S}$, and $\tilde{\theta}$ are the induced connection, the fundamental form, the shape operator and the volume element corresponding to $(\tilde{f}, \tilde{\xi})$, then we have*

$$\tilde{\nabla} = \nabla, \quad \tilde{h} = \frac{\alpha}{\beta}h, \quad \tilde{S} = \frac{\beta}{\alpha}S, \quad \tilde{\theta} = \alpha^n \beta\theta.$$

In particular, if $\alpha = \beta$, then

$$\tilde{\nabla} = \nabla, \quad \tilde{h} = h, \quad \tilde{S} = S, \quad \tilde{\theta} = \alpha^{n+1}\theta.$$

Lemma 3. *Let $g, \bar{g}$ be positive-definite scalar products on a 2-dimensional vector space V and $A, \bar{A}$ be endomorphisms, both positive-definite (or both negative-definite) and symmetric relative to g and $\bar{g}$, respectively. Assume that $g, \bar{g}, A, \bar{A}$ are related by*

$$g(Y, Z)AX - g(X, Z)AY = \bar{g}(Y, Z)\bar{A}X - \bar{g}(X, Z)\bar{A}Y \tag{11}$$

for every $X, Y, Z \in V$. If $detA = det\bar{A}$ or $trA = tr\bar{A}$, then $det(\bar{A} - A) \leq 0$, and the equality holds if and only if $A = \bar{A}$.

Proof. Fix an orientation on V. Let $\{e_1, e_2\}$ be a positively oriented g-orthonormal basis of V diagonalizing A and $\{\bar{e_1}, \bar{e_2}\}$ a positively oriented $\bar{g}$-orthogonal basis diagonalizing $\bar{A}$. Then

$$Ae_1 = \lambda_1 e_1, \quad Ae_2 = \lambda_2 e_2; \qquad \bar{A}\bar{e}_1 = \mu e_1, \quad \bar{A}\bar{e}_2 = \mu_2 \bar{e}_2$$

for some $\rho_1, \rho_2, \mu_1, \mu_2$. We may assume that $\lambda_1 \geq \lambda_2$ and $\mu_1 \geq \mu_2$.

Let $\bar{e}_1 = ae_1 + be_2, \quad \bar{e}_2 = ce_1 + de_2$. Also consider the matrices

$$L = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad B = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad C = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

By a straightforward computation we obtain

$$\det(A - \bar{A}) = \det(C - LBL^{-1}) = (\lambda_1 - \mu_1)(\lambda_2 - \mu_2) + (\beta/l)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2), \tag{12}$$

where $l = \det L > 0$ and $\beta = bc$.

From (11) we can compute

$$\begin{aligned}\lambda_2 e_2 = g(e_1, e_1)Ae_2 &= \bar{g}(e_1, e_1)\bar{A}e_2 - \bar{g}(e_1, e_2)\bar{A}e_1 \\ &= (1/l^3)\{\bar{g}(d\bar{e}_1 - b\bar{e}_2, d\bar{e}_1 - b\bar{e}_2)\bar{A}(-c\bar{e}_1 + a\bar{e}_2) \\ &\quad -\bar{g}(d\bar{e}_1 - b\bar{e}_2, -c\bar{e}_l + a\bar{e}_2)\bar{A}(d\bar{e}_1 - b\bar{e}_2)\} \\ &= (1/l^2)[(dc\mu_2 + ba\mu_1)e_1 + (b^2\mu_1 + d^2\mu_2)e_2].\end{aligned}$$

Hence

$$dc\mu_2 + ba\mu_1 = 0 \tag{13}$$

and

$$\lambda_2 = (1/l^2)(b^2\mu_1 + d^2\mu_2). \tag{14}$$

Assume that $c \neq 0$. Then by (13) we have $d\mu_2 = -ba\mu_1/c$. By substituting this formula into (14) and using the equality $l = ad - bc$ we get

$$\lambda_2 = -(b/lc)\mu_1. \tag{15}$$

Since A and $\bar{A}$ are both positive-definite or both negative-definite, we have by (15) $b/c < 0$, i.e. $\beta = bc < 0$. Of course, if $c = 0$, then $\beta = 0$. Hence we conclude $\beta \leq 0$. Since $\det A = \det\bar{A}$ or $\mathrm{tr}A = \mathrm{tr}\bar{A}$, we have $(\lambda_1 - \mu_1)(\lambda_2 - \mu_2) \leq 0$. Of course, we have $(1/l)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2) \geq 0$. Using formula (12) we see that $\det(\bar{A} - A) \leq 0$.

Now suppose we have the equality in this formula. Then

$$(\lambda_1 - \mu_1)(\lambda_2 - \mu_2) = 0, \quad \text{i.e.} \quad \lambda_1 = \mu_1 \quad \text{or} \quad \lambda_2 = \mu_2,$$

and

$$(\beta/l)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2) = 0.$$

Since $\lambda_1\lambda_2 = \mu_1\mu_2$ or $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$, we get

$$\lambda_1 = \mu_1 \quad \text{and} \quad \lambda_2 = \mu_2. \tag{16}$$

If $\beta < 0$, then $\mu_1 = \mu_2$ or $\lambda_1 = \lambda_2$. By (16) we have $\mu_1 = \mu_2 = \lambda_1 = \lambda_2$, which clearly implies the equality $A = \bar{A}$. If $\beta = 0$, then by (13) and by the fact that $l \neq 0$ we get $b = c = 0$. Then $LBL^{-1} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ and by (16) we get $A = \bar{A}$.

Lemma 4. *Let $g, \bar{g}$ be positive-definite scalar products on a 2-dimensional vector space V and $A, \bar{A}$ isomorphisms of V symmetric relative to g and $\bar{g}$, respectively. Assume that*

$g, \bar{g}, A, \bar{A}$ are related by (11). If $\det A = \det \bar{A}$ and $trA = tr\bar{A} = 0$, then $\det(\bar{A} - A) \geq 0$; the equality holds if and only if $A = \bar{A}$.

Proof. Using the same notation as in the proof of Lemma 3, we have

$$B = C = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$$

and by (12)

$$\det(\bar{A} - A) = 4(\beta/l)\lambda^2. \tag{17}$$

Using (15) we get $\beta > 0$, i.e. $\det(\bar{A} - A) \geq 0$.

Now suppose $\det(\bar{A} - A) = 0$. By (17) we get $\beta = 0$. Then by (13) and by the fact $l = da - bc \neq 0$, we obtain $b = c = 0$. Hence $LBL^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ and, consequently, $A = \bar{A}$.

4. Congruence theorems

We shall first prove the following local congruence theorem.

Theorem 3. *Let $f, \bar{f} : M^2 \to R^3$ be Blaschke surfaces with definite affine metrics $h, \bar{h}$. Assume that the Blaschke connections induced by f and $\bar{f}$ are equal and have nondegenerate Ricci tensors.*

(i) If the affine shape operators S and $\bar{S}$ are both positive-definite (or both negative-definite), then $\det(\bar{S} - S) \leq 0$; the equality holds if and only if f and $\bar{f}$ are affinely congruent.

(ii) If f and $\bar{f}$ are affine minimal, then $\det(\bar{S} - S) \geq 0$; the equality holds if and only if f and $\bar{f}$ are affinely congruent.

Proof. First remark that S and $\bar{S}$ are nonsingular, because the Ricci tensor is non-degenerate, and that $\det S = \det\bar{S}$ by virtue of Lemma 1. To prove (i), we note that the Gauss equation implies an equation of the form (11), where $g = h, \bar{g} = \bar{h}, A = S$, and $\bar{A} = \bar{S}$. By Lemma 3, we get $\det(\bar{S} - S) \leq 0$. If the equality holds, then again by Lemma 3 we conclude that $S = \bar{S}$. By the Gauss equation, we get $h = \bar{h}$. It now follows that f and $\bar{f}$ are affinely congruent. Similarly, Lemma 4 proves (ii).

We are now in a position to prove the following main theorem.

Theorem 4. *Let M^2 be a connected, compact, orientable 2-manifold and let $f, \bar{f} : M^2 \to R^3$ be two nondegenerate imbeddings with equiaffine transversal vector fields $\xi, \bar{\xi}$. Assume that $\det S$ for f is nowhere 0. If the induced connections coincide and if $\det S = \det \bar{S}$ at every point, then f and $\bar{f}$ are affinely congruent.*

Proof. The fundamental forms $h, \bar{h}$ for $f, \bar{f}$ are definite. We can assume that they are positive-definite; namely, if, for instance, h is negative-definite, then we can replace f by $-f$. We may also assume that the induced volume elements θ and $\bar{\theta}$ coincide, because $\nabla\theta = \nabla\bar{\theta} = 0$ implies that $\bar{\theta} = c\theta$ with some constant c and it is sufficient, by virtue of Lemma 2, to multiply f and ξ by $c^{1/3}$.

As before, we choose a point o in the interior of the body bounded by $f(M)$ so that the affine distance function ρ for (f, ξ) from o is negative. There is a point where the form $B(X,Y) = h(SX,Y)$ is positive-definite (see [O-V], Lemma 3.1). At that point S is positive-definite. By assumption, $\det S$ is never 0 and hence S is positive-definite on M. Because of the same reason, the shape operator $\bar{S}$ is positive-definite on M. Now we can apply Lemma 3 and get $\rho \det(\bar{S} - S) \geq 0$ on M. From the integral formula (III):

$$\int (\mathrm{tr}\bar{S} - \mathrm{tr}S)\theta = \int \rho \det(\bar{S} - S)\theta,$$

we obtain

$$\int \mathrm{tr}\bar{S}\theta - \int \mathrm{tr}S\theta \geq 0.$$

By interchanging S and $\bar{S}$ we obtain the reversed inequality, and consequently $\det(\bar{S} - S) = 0$ on M^2. By Lemma 3 we have $S = \bar{S}$. The rest of the proof is similar to that of Theoem 3.

The following is the rigidity theorem for Blalschke imbeddings.

Corollary. *Let $f : M^2 \to R^3$ be an ovaloid such that the affine Gaussian curvature K vanishes nowhere. If a Blaschke imbedding $\bar{f} : M^2 \to R^3$ has the same induced connection as that of f, then $\bar{f}$ is affinely congruent to f.*

Proof. This follows from Theorem 4 and Lemma 1.

Remark 3. In Theorem 4 and its Corollary, we can replace the assumption that $\det S$ is nowhere 0 by the assumption that the Ricci tensor is nondegenerate. See Remark 1.

We shall now prove

Theorem 5. *Let M^2 be a connected, compact orientable 2-manifold and let $f, \bar{f}$: M^2 be a nondegenerate imbedding and a nondegenerate immersion, respectively, with equiaffine transversal vector fields $\xi, \bar{\xi}$. Assume that they have the same induced connection ∇ with nondegenerate Ricci tensor. If $trS = tr\bar{S}$ and $detS \leq det\bar{S}$, then f and $\bar{f}$ are affinely congurent.*

Proof. Since the forms $B(X,Y) = h(SX,Y)$ for (f,ξ) and $\bar{B}(X,Y) = \bar{h}(\bar{S}X,Y)$ for $(\bar{f},\bar{\xi})$ are positive-definite on M^2 and the forms h and $\bar{h}$ are definite, we see that S and $\bar{S}$ are definite. Since $\mathrm{tr}S = \mathrm{tr}\bar{S}$, it follows that S and $\bar{S}$ are both positive-definite or both negative-definite. It also means that h and $\bar{h}$ are both positive-definite or both negative-definite. We may assume that they are both positive-definite. By choosing a point o in the interior of the body bounded by $f(M^2)$ we obtain a negative-valued distance function ρ for (f,ξ) relative to o. By Lemma 3, we have $\det(\bar{S}-S)\leq 0$. Hence we get

$$\int_{M^2}\rho\det(\bar{S}-S)\geq 0.$$

On the other hand, since ρ is negative and $\det\bar{S}-\det S\geq 0$, we have

$$\int_{M^2}\rho\,(\det\bar{S}-\det S)\leq 0.$$

Using the integral formula (III) we obtain

$$\int_{M^2}\rho\det(\bar{S}-S)=0,$$

and, consequently, $\det(\bar{S}-S)=0$ on M^2. By Lemma 3, we get $S=\bar{S}$. Then we get $h=\bar{h}$ as before.

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