Minimizing p-Harmonic Maps

into Spheres

by

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MPI/87-13

0. Introduction

Consider two Riemannian manifolds M^m and $N^n \subset \mathbb{R}^d$, where M is compact, possibly with boundary, and $m \geq 3$. A map $f : M \longrightarrow N$ is harmonic if it is stationary for Dirichlet's integral ("energy")

$$E_2(f) = \int_M |\nabla f|^2 d\mathrm{Vol},$$

where $|\nabla f|^2 = \sum_{i=1}^{d} \sum_{\alpha,\beta=1}^{m} \gamma^{\alpha\beta} \frac{\partial f_i}{\partial x_\alpha} \frac{\partial f_i}{\partial x_\beta}$, and where $\gamma_{\alpha\beta}(x) = (\gamma^{\alpha\beta}(x))^{-1}$ represents the metric of M. In a fundamental paper ([SU1]), Schoen and Uhlenbeck showed that near any singularity, a minimizing harmonic map $f : M^m \longrightarrow N^n$ converges strongly to a minimizing tangent map $u : \mathbb{R}^m \longrightarrow N^n$, which is harmonic and homogeneous of degree zero. The investigation of minimizing tangent maps $u : B^m \longrightarrow N^n$ is therefore an important aspect of current research into minimizing harmonic maps.

We restrict our attention in this paper to the case $N = S^n$, the unit sphere in $\mathbb{R}^{n+1}$. Even in this case, surprisingly few examples are known of maps $u : B^m \longrightarrow S^n$, homogeneous of degree zero, which minimize energy for given Dirichlet boundary conditions. The first nonconstant example was given by Jäger and Kaul in 1983, who proved that the map $u_0 : B^m \longrightarrow S^m$ defined by $u_0(x) = (x/|x|, 0)$ minimizes energy if $m \geq 7$ ([JK]; see also [SU3]). Recently, Brézis, Coron and Lieb have shown that the map $u_0(x) = x/|x|$ from B^3 to S^2 minimizes E_2 ([BCL]). A proof

was communicated to us by Lin that $u_0(x) = x/|x|$ from B^m to S^{m-1} has minimum energy, for all m ([L]). In a related result, Hélein has shown that $E_2(u) \geq E_2(u_0) + \alpha\, E_2(u-u_0)$ for some $\alpha > 0$, provided that $n = m-1$ and $m \geq 9$ ([H]).
In contrast, it is shown in [SU3] that any minimizing tangent map $u : B^m \longrightarrow S^n$ is constant if $m \leq d(n)$, where $d(3) := 3$ and $d(n) := 1 + \min\{n/2, 5\}$ otherwise.

A natural generalization of the functional E_2 is the p-energy

$$E_p(u) = \int_{B^m} |\nabla u|^p dx ,$$

which is finite if and only if u belongs to the Sobolev class $W^{1,p}(B^m, S^n) := \{u \in W^{1,p}(B^m, \mathbb{R}^{n+1}) : |u| = 1 \text{ a.e.}\}$. Mappings which are stationary for E_p are called p-<u>harmonic</u> maps. Note that regularity theorems analogous to results for $p = 2$ in [SU1] have not yet been proved for general p (uniform ellipticity is lost). One may well expect, however, that minimizing tangent maps will play a role similar to their role in the theory for $p = 2$.

One result of the present paper concerns the homogeneous mapping $u_0 : B^m \longrightarrow S^n$ defined by $u_0(y,z) = y/|y|$, where $y \in \mathbb{R}^{n+1}$ and $z \in \mathbb{R}^{m-n-1}$. We have

<u>Theorem 2.4</u>. <u>If</u> $p \leq n \leq m-1$, <u>then</u> $E_p(u_0) \leq E_p(u)$ <u>for any</u> $u \in W^{1,p}(B^m, S^n)$ <u>with</u> $u = u_0$ <u>on</u> ∂B^m.

If $p = n = m-1$, then this result may be proved by the methods of [BCL]. If $p = 2$ and $n = m-1$, then this is exactly Lin's result. Our proof was discovered later than Lin's and independently, and is of a quite different nature.

Two interesting examples of mappings from B^{2n} to S^n are provided by the homogeneous extension

$$u_0(x) = H(x/|x|)$$

of the Hopf maps $H : S^{2n-1} \longrightarrow S^n$ related to the multiplication of complex numbers $(n = 2)$ and the quaternions $(n = 4)$. We shall prove that both are minimizing maps for E_2 (Theorems 5.1 and 6.1).

Using similar techniques, we shall prove a sharp lower bound

$$E_n(u) \geq n^{n/2} \text{ Volume } (S^n)$$

for $u \in W^{1,n}(B^{n+1}, S^n)$ such that $u(-x) = -u(x)$ for all $x \in \partial B^{n+1}$ (Theorem 4.1).

Finally, we give a theorem with general hypotheses on a mapping $u_0 : B^m \longrightarrow S^n$ which allow us to conclude that u_0 minimizes E_p for its boundary data. The hypotheses are similar to the conditions for a harmonic morphism (compare p. 123 of [B]).

We would like to point out that the results of the present paper do not include a classification of all minimizing tangent maps into S^n . For example, up to an orthogonal motion, $u_0(x) = x/|x|$ is the only known example of a minimizing tangent map from B^4 to S^3 ; it is not known whether any others exist. It was proved in [BCL] that $u_0(x) = x/|x|$ is the unique minimizing tangent map from B^3 to S^2 modulo $\mathbb{O}(3)$.

One idea in our proof is to bound the p-energy of a map $v : B^m \longrightarrow S^p$ from below by a coarea formula. The usefulness of the coarea formula in the context of the functional E_p for mappings into a p-dimensional manifold was made clear in the paper of Almgren, Browder and Lieb [ABL]. An analogous framework of ideas had been constructed in [BCL] for the case $p = n = m-1$.

A new idea, which plays a central role in our proof, is to estimate the p-energy of a map $u : B^m \longrightarrow S^n$ by averaging a related functional of the composition of u with all nearest-point projections π_Y of S^n onto its totally geodesic p-spheres (Lemma 2.2). This averaging method is simplest in the classical case $p = 2$: the energy of any map $u : B^m \longrightarrow S^n$ is a constant times the average of $E_2(\pi_Y \circ u)$ over all 3-planes Y in $\mathbb{R}^{n+1}$. Here $\pi_Y : S^n \longrightarrow Y \cap S^n$ maps $s \in S^n$ to the nearest point in the 2-sphere $Y \cap S^n$ (Lemma 1.2).

An important technical tool in our proof is a new approximation result for mappings into the p-sphere of class $W^{1,p}$ (Theorem 3.2), which is based on methods of Hardt-Lin and of Bethuel-Zheng. Note that smooth mappings are not dense ([SU2], p. 267 for $p = 2$). However, we construct a dense class $\mathcal{R}$ of mappings whose singularities form submanifolds of codimension $p + 1$, with simple structure near the singularities. Of course, the slicing theorems of Federer ([F], 4.3.1), which are relevant to the coarea formula, are valid only for Lipschitz-continuous mappings; in effect, the singular set of a mapping of class R contributes to the boundary of each slice. This difficulty is overcome by considering the difference of the slices at two distinct points in S^p ; the difference is a current having no boundary in the interior of the domain.

The authors would like to thank Martin Guest and Uwe Abresch for useful references to the literature. They gratefully acknowledge the hospitality of the Max-Planck-Institut für Mathematik, where this work was carried out.

1. Projection to lower-dimensional spheres (p = 2)

Consider $n \leqq m-1$ and an integer p, $1 \leqq p \leqq n$. For this section and the following one, we define boundary data $g : \partial B^m \longrightarrow S^n$ by

$$g(y,z) = y/|y| ,$$

where $y \in \mathbb{R}^{n+1}$ and $z \in \mathbb{R}^{m-n-1}$. The class of admissible mappings is

$$\mathcal{E}_p(g) = \{u \in W^{1,p}(B^m,S^n) : u = g \text{ on } \partial B^m\} .$$

The homogeneous extension of g is $u_0(y,z) = y/|y|$, which is singular on $\{0\} \times \mathbb{R}^{m-n-1} \subset \mathbb{R}^m$. Note that $E_p(u_0)$ is the integral of $|y|^{-p}$, which is finite since $p < n+1$. This shows that $u_0 \in \mathcal{E}_p(g)$, and the admissible class is not empty.

In this section, we shall consider only the case $p = 2$, which is simpler than the general case (compare the averaging Lemmas 1.2 and 2.2). Our result is

Theorem 1.1. $E_2(u_0) \leqq E_2(u)$ for any $u \in \mathcal{E}_2(g)$.

Given a 3-plane $Y \subset \mathbb{R}^{n+1}$, we define $\pi_Y : S^n \longrightarrow S^n \cap Y$ by $\pi_Y(u) = u'/|u'|$, where u' is the orthogonal projection of u onto Y. The singular set of π_Y is the $(n-3)$-sphere $S^n \cap Y^\perp$.

Lemma 1.2. There is a constant $c = c(n)$ such that for any $u \in W^{1,2}(B^m, S^n)$,

$$(1.1) \quad c\, E_2(u) = \int_{Y \in G_3(\mathbb{R}^{n+1})} E_2(\pi_Y \circ u)\, dG(Y) \ .$$

Here dG is the bi-invariant volume form on the Grassmann manifold $G_3(\mathbb{R}^{n+1})$.

Proof. For any tangent vector V to S^n , we have

$$(1.2) \quad c|V|^2 = \int_{Y \in G_3(\mathbb{R}^{n+1})} |D\pi_Y(V)|^2 dG(Y) \ ,$$

since $\mathbb{O}(n+1)$ acts transitively on the unit tangent vectors to S^n and leaves $dG(Y)$ invariant on $G_3(\mathbb{R}^{n+1})$. Note that π_Y is singular along a totally geodesic $(n-3)$-sphere of S^n , and $|D\pi_Y(V)| \leq C|V|/r$, where r is the distance to the singular set; therefore, the integral in equation (1.2) is finite. Since $|\nabla u|^2 = \sum_{\alpha=1}^{m} \left|\frac{\partial u}{\partial x_\alpha}\right|^2$, this formula applied to $V = \frac{\partial u}{\partial x_\alpha}$ yields

$$c|\nabla u|^2 = \int_{Y \in G_3(\mathbb{R}^{n+1})} |\nabla(\pi_Y \circ u)|^2 \, dG(Y) \ .$$

We integrate both sides over B^m to obtain (1.1) by Fubini's theorem.

q.e.d.

Corollary 1.3. Let $v_0 : B^m \longrightarrow S^2$ be defined by $v_0(x,y) = \frac{x}{|x|}$, where $x \in \mathbb{R}^3$, $y \in \mathbb{R}^{m-3}$. If $E_2(v) \geq E_2(v_0)$ for every

$v \in W^{1,2}(B^m,S^2)$ *with* $v = v_0$ *on* ∂B , *then* $E_2(u) \geq E_2(u_0)$ *for every* $u \in W^{1,2}(B^m,S^n)$ *with* $u = u_0$ *on* ∂B .

Proof. Note that $\pi_Y \circ u_0 = v_0$ after performing an appropriate rotation in $\mathbb{R}^m$. Using Lemma 1.2,

$$c\, E_2(u) = \int_{G_3(\mathbb{R}^{n+1})} E_2(\pi_Y \circ u)\, dG(Y) \geq$$

$$\int_{G_3(\mathbb{R}^{n+1})} E_2(\pi_Y \circ u_0)\, dG(Y) = c\, E_2(u_0) .$$

q.e.d.

The coarea formula has the serious weakness that it gives a lower bound for energy E_2 only for mappings to a manifold of dimension $n = 2$. The above corollary bypasses this weakness in the case of mappings to the n-sphere.

Lemma 1.4. (*Coarea formula*, $p = 2$). *If* $v \in C^{0,1}(\Omega,S^2)$ *for* Ω *open in* B^m , *then*

$$\int_\Omega |\nabla v|^2 dx \geq 2 \int_{S^2} \mathcal{H}^{m-2}(v^{-1}(s))\, dA_{S^2}(s)$$

where $\mathcal{H}^{m-2}$ *denotes* $(m-2)$-*dimensional Hausdorff measure*.

Proof. See [F, 3.2.22], with the observation that $|\nabla v|^2 \geq 2\, J(v)$, where $J(v)$ is the determinant of ∇v restricted to the 2-dimensional space orthogonal to $v^{-1}(s)$, and s is any regular value of v .

q.e.d.

In order to use Lemma 1.4, which is only valid for Lipschitz mappings, we need to approximate $W^{1,2}(B^m,S^2)$ by mappings having precisely controlled singularities (recall that Lipschitz functions are not dense for $m \geq 3$: see [SU2], p. 267). Let $\mathcal{R}$ be the class of mappings $v \in W^{1,2}(B^m,S^2)$ such that

(1.3) $v = v_0$ on a neighborhood of ∂B^m (whose size may depend on v) and on a neighborhood of the singular set $\Delta = \{0\} \times \mathbb{R}^{m-3}$ of v_0 ;

(1.4) $v \in C^\infty(B^m \setminus (\Delta \cup \Sigma))$ for some Lipschitz (m-3)-dimensional manifold $\Sigma \subset\subset B^m \setminus \Delta$ $(\partial\Sigma = \phi)$; and

(1.5) for a.e. $s \in S^2$, $v^{-1}(s) \cup v^{-1}(-s) \cup \Sigma \cup \Delta$ is a Lipschitz (m-2)-dimensional manifold with boundary $\subset \partial B^m$.

<u>Approximation Theorem 1.5</u>. <u>If</u> $v_0 \in \mathcal{R}$, <u>then</u> $\mathcal{R}$ <u>is</u> <u>dense</u> <u>in</u> $\mathcal{E}_2(v_0) = \{v \in W^{1,2}(B^m,S^2) : v = v_0$ <u>on</u> $\partial B^m\}$.

We defer the proof of Theorem 1.5 to section 3.

<u>Proof of Theorem 1.1</u>. According to Corollary 1.3 and the Approximation Theorem 1.5, we need only to show that for $v \in \mathcal{R}$, $E_2(v) \geq E_2(v_0)$. We use the coarea formula of Lemma 1.4, with $\Omega = B^m \setminus (\Sigma \cup \Delta)$:

$$(1.6)\quad \int_\Omega |\nabla v|^2 dx \geq 2 \int_{S^2} \mathcal{H}^{m-2}(v^{-1}(s) \cap \Omega)\, dA_{S^2}(s) \ .$$

Since $\mathcal{H}^{m-2}(\Sigma \cup \Delta) = 0$, we have $\mathcal{H}^{m-2}(v^{-1}(s) \cap \Omega) = \mathcal{H}^{m-2}(v^{-1}(s))$. Note that the antipodal map from S^2 to S^2 defined by $s \longmapsto -s$ preserves the volume form $dA_{S^2}(s)$, so that the right-hand side of (1.6) equals

$$\int_{S^2} \mathcal{H}^{m-2}(v^{-1}(s) \cup v^{-1}(-s))\, dA_{S^2}(s) \ .$$

It follows from conditions (1.5) and (1.3) that for almost all s, $v^{-1}(s) \cup v^{-1}(-s) \cup \Sigma \cup \Delta$ is a regular manifold with boundary a totally geodesic sphere of dimension $m-3$. In particular, it has $(m-2)$-dimensional measure $\geq \mathcal{H}^{m-2}(B^{m-2}) =: \alpha_{m-2}$. Thus $\int_{B^m} |\nabla v|^2 dx \geq \alpha_{m-2}\, \mathcal{H}^2(S^2) = 4\pi\, \alpha_{m-2}$.

Meanwhile, $|\nabla v_0(x,y)|^2 = \dfrac{2}{|x|^2}$, so that

$$E_2(v_0) = \int_{B^m} \frac{2}{|x|^2}\, dx\, dy = 4\pi\, \alpha_{m-3} \int_0^1 2(1-r^2)^{\frac{m-3}{2}}\, dr$$

$$= 4\pi\, \alpha_{m-2} \ .$$

q.e.d.

2. Projection to lower-dimensional spheres (general p)

We may now turn our attention to the case of a general integer exponent $1 \leq p \leq n$. Somewhat surprisingly, the counterpart of Lemma 1.2 fails: if the constant c is defined so that

$$c\, E_p(u_0) = \int_{Y \in G_{p+1}(\mathbb{R}^{n+1})} E_p(\pi_Y \circ u_0)\, dG(Y) ,$$

then it is not true that

$$c\, E_p(u) \geq \int_{Y \in G_{p+1}(\mathbb{R}^{n+1})} E_p(\pi_Y \circ u)\, dG(Y) .$$

In order to carry out our program, we will instead compute the average of squares of the Jacobian determinants of $\pi_Y \circ u$ (see Lemma 2.2 below).

Given a linear transformation $L : \mathbb{R}^m \longrightarrow \mathbb{R}^n$, where $m \geq n$, we may write $L = Q_1 \Lambda Q_2$, where $Q_1 \in \mathbb{O}(n)$, $Q_2 \in \mathbb{O}(m)$ and Λ has the form

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n & 0 & \dots & 0 \end{pmatrix} ,$$

with $\lambda_j \geq \lambda_{j+1} \geq 0$. (For example, $L = \nabla u(x)$.) In fact, $\lambda_1^2, \dots, \lambda_n^2$ are the eigenvalues of the positive semi-definite symmetric operator

LL^T on $\mathbb{R}^n$. We shall refer to $\lambda_1,\ldots,\lambda_n$ as the <u>singular values</u> of L. Observe that λ_k may be given a variational interpretation:

(2.1) $\lambda_k = \max_{Z \in G_k(\mathbb{R}^m)} \min \{|Lx| : x \in Z, |x| = 1\}$.

For any p-plane $Z \in G_p(\mathbb{R}^n)$, let π_Z be the orthogonal projection of $\mathbb{R}^n$ onto Z, and write $\mu_1(Z),\ldots,\mu_p(Z)$ for the singular values of $\pi_Z \circ L : \mathbb{R}^m \longrightarrow Z \cong \mathbb{R}^p$.

Recall the definition of the elementary symmetric functions $\sigma_1,\ldots,\sigma_n$ of $(\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n$:

$$\sigma_k(\alpha_1,\ldots,\alpha_n) := \sum_{i_1<i_2<\ldots<i_k} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} .$$

We have the following formula:

<u>Lemma 2.1</u>. <u>Given any linear transformation</u> $L : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ <u>with singular values</u> $\lambda_1,\ldots,\lambda_n$, <u>the average</u>

$$⨍_{G_p(\mathbb{R}^n)} [\mu_1(Z)\ldots\mu_p(Z)]^2 dG(Z) = \binom{n}{p}^{-1} \sigma_p(\lambda_1^2,\ldots,\lambda_n^2) .$$

<u>Proof</u>. Without loss of generality, we may assume $L = \Lambda$. Write $M = \Lambda\Lambda^T$, a symmetric linear operator on $\mathbb{R}^n$ with eigenvalues $\alpha_1 = \lambda_1^2,\ldots,\alpha_n = \lambda_n^2$. For any $Z \in G_p(\mathbb{R}^n)$, $\mu_1(Z)^2\ldots\mu_p(Z)^2 = \det(\pi_Z M \pi_Z^T)$, which is a homogeneous polynomial of degree p in $\alpha_1,\ldots,\alpha_n$. Define

$$f(\alpha_1,\ldots,\alpha_n) = ⨍_{G_p(\mathbb{R}^n)} \det(\pi_Z M \pi_Z^T) dG(Z) ;$$

we need only show that

$$f(\alpha_1,\ldots,\alpha_n) = \binom{n}{p}^{-1} \sigma_p(\alpha_1,\ldots,\alpha_n) =: \binom{n}{p}^{-1} \sigma_p$$

for any $\alpha_1,\ldots,\alpha_n \in \mathbb{R}$. Note that f is a homogeneous polynomial of degree p . Further, $f(\alpha_1,\ldots,\alpha_n)$ is symmetric under permutations of $(\alpha_1,\ldots,\alpha_n)$, since a permutation corresponds to the isometry of $\mathbb{R}^n$ which permutes the eigenvectors, leaving $dG(Z)$ unchanged.

According to the fundamental theorem on symmetric functions (see e.g. [Md], p. 13), any symmetric polynomial $f(\alpha_1,\ldots,\alpha_n)$ is equal to a polynomial $P_0(\sigma_1,\ldots,\sigma_n)$, with real coefficients; moreover, $\sigma_1,\ldots,\sigma_n$ are algebraically independent. In our case $f(\alpha_1,\ldots,\alpha_n)$ is homogeneous of degree $p \leq n$, and therefore, for some $\gamma \in \mathbb{R}$,

$$P_0(\sigma_1,\ldots,\sigma_n) = \gamma\sigma_p + P_1(\sigma_1,\ldots,\sigma_{p-1}) .$$

Consider the special case $\alpha_p = \ldots = \alpha_n = 0$: in this case $\sigma_i(\alpha_1,\ldots,\alpha_{p-1}, 0,\ldots,0) = \tilde{\sigma}_i(\alpha_1,\ldots,\alpha_{p-1})$, the elementary symmetric function in $p-1$ variables. On the other hand, for each $Z \in G_p(\mathbb{R}^n)$, $\det(\pi_Z M \pi_Z^T) = 0$ since M has rank $\leq p-1$, and hence $f(\alpha_1,\ldots,\alpha_n) = P_0(\sigma_1,\ldots,\sigma_n) = 0$. Clearly, $\sigma_p = 0$ as well. Therefore, for any $(\alpha_1,\ldots,\alpha_{p-1}) \in \mathbb{R}^{p-1}$, $P_1(\tilde{\sigma}_1,\ldots,\tilde{\sigma}_{p-1}) = 0$. But according to the fundamental theorem on symmetric functions, $\tilde{\sigma}_1,\ldots,\tilde{\sigma}_{p-1}$ are algebraically independent in $p-1$ variables, so that the polynomial P_1 is itself zero. This shows that

$$f(\alpha_1,\ldots,\alpha_n) = \gamma\sigma_p(\alpha_1,\ldots,\alpha_n) \ .$$

Finally, we may evaluate the constant γ by choosing $M = id$, which implies $\mu_1(Z) = \ldots = \mu_p(Z) = 1$ for each Z in $G_p(\mathbb{R}^n)$. Since $\sigma_p(1,\ldots,1) = \binom{n}{p}$ we have $\gamma = \binom{n}{p}^{-1}$ as claimed.

q.e.d.

Define, for $v : B^m \longrightarrow S^p$ and for each x in B^m, $J(v)(x) := \lambda_1(x)\ldots\lambda_p(x)$, the product of the p singular values of $L = \nabla v(x) : \mathbb{R}^m \longrightarrow T_{v(x)}S^p$. We recall that $u : B^m \longrightarrow S^n$ is said to be <u>horizontally conformal</u> (see <u>e.g.</u>[B]) if for almost all x in B^m, the singular values of $\nabla u(x) : \mathbb{R}^m \longrightarrow T_{u(x)}S^n$ are equal.

We have the following averaging result:

<u>Lemma 2.2</u>. <u>For</u> $n \geq p$, <u>there is a constant</u> $c = c(n,p)$ <u>such that for any</u> $u \in W^{1,p}(B^m,S^n)$

$$(2.2) \quad c\, E_p(u) \geq ⨍_{G_{p+1}(\mathbb{R}^{n+1})} \int_{B^m} J(\pi_Y \circ u)\,dx\, dG(Y) \ .$$

<u>Moreover, equality holds if</u> u <u>is horizontally conformal</u>.

<u>Proof</u>. We first observe that

$$(2.3) \quad \sigma_p(\alpha_1,\ldots,\alpha_n) \leq \binom{n}{p} \left(\frac{1}{n}\sum_{i=1}^{n}\alpha_i\right)^p$$

for $\alpha_i \geq 0$, with equality if $\alpha_1 = \ldots = \alpha_n$. The case $p = n-1$ was given in inequality (8.5) of [BCL] and A.1.3 of [ABL]. We prove inequality (2.3) by induction on n ; for $n = p$ it is the well-known arithmetic-geometric inequality. By reordering, we may assume $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$; by homogeneity, we may assume $\alpha_1 + \ldots + \alpha_n = 1$. Now consider $(\alpha_1,\ldots,\alpha_n)$ which maximizes σ_p . If $\alpha_n = 0$, we use the induction hypothesis: $\sigma_p(\alpha_1,\ldots,\alpha_{n-1},0) \leq \binom{n-1}{p} (n-1)^{-p} < \binom{n}{p} n^{-p}$. If $\alpha_n > 0$, then by the method of Lagrange, there is $\beta \in \mathbb{R}$ with

$$\beta = \frac{\partial}{\partial\alpha_i} \sigma_p(\alpha_1,\ldots,\alpha_n) = \sigma_{p-1}(\alpha_1,\ldots,\alpha_{i-1}, \alpha_{i+1},\ldots,\alpha_n) =: \Theta_i$$

for all $1 \leq i \leq n$. But $\Theta_{i+1} \geq \Theta_i$, and equality implies that $\alpha_i = \alpha_{i+1}$; inequality (2.3) follows.

Applying inequality (2.3) to the eigenvalues $\alpha_1,\ldots,\alpha_n$ of $\nabla u(x)(\nabla u(x))^T$ for some $x \in B^m$, we have

$$(2.4) \quad |\nabla u(x)|^{2p} \geq n^p \binom{n}{p}^{-1} \sigma_p(\alpha_1,\ldots,\alpha_n) =$$

$$= n^p \fint_{G_p(T_{u(x)}S^n)} \det(\pi_Z \nabla u(x) \nabla u(x)^T \pi_Z^T)\, dG(Z)$$

by Lemma 2.1. An application of the Cauchy-Schwartz inequality yields

$$(2.5) \quad |\nabla u(x)|^p \geq n^{p/2} \fint_{G_p(T_{u(x)}S^n)} [\det(\pi_Z \nabla u(x) \nabla u(x)^T \pi_Z^T)]^{1/2}\, dG(Z) \ .$$

Note that equality holds in the inequalities (2.4) and (2.5) if u is horizontally conformal at x.

Let Y be a $(p+1)$-plane in $\mathbb{R}^{n+1}$, and write Z for the p-plane in $T_{u(x)}S^n$ parallel to the subspace of Y orthogonal to $\pi_Y(u(x))$. Then up to a parallel translation in $\mathbb{R}^{n+1}$, we have

$$\nabla(\pi_Y \circ u)(x) = \frac{\pi_Z \circ \nabla u(x)}{\cos d(u(x),Y)}$$

where $d(u_1,Y)$ is the distance in S^n from $u_1 \in S^n$ to $Y \cap S^n$. Thus

$$(2.6) \quad J(\pi_Y \circ u)(x) = \frac{[\det(\pi_Z \nabla u(x) \nabla u(x)^T \pi_Z^T)]^{1/2}}{\cos^p d(u(x),Y)} .$$

Integrating formula (2.6) over $Y \in G_{p+1}(\mathbb{R}^{n+1})$, we find that

$$(2.7) \quad ⨍_{G_{p+1}(\mathbb{R}^{n+1})} J(\pi_Y \circ u)(x)\, dG(Y) =$$

$$= c' ⨍_{G_p(T_{u(x)}S^n)} [\det(\pi_Z \nabla u(x) \nabla u(x)^T \pi_Z^T)]^{1/2} \, dG(Z) ,$$

where $c' = c'(n,p)$ is independent of x and u. Finally, using inequality (2.5) and equation (2.7), and integrating over $x \in B^m$, we find the inequality (2.2) with $c = n^{-p/2} c'$.

q.e.d.

<u>Lemma 2.3</u>. (<u>Coarea formula, general</u> p). <u>If</u> $v \in C^{0,1}(\Omega, S^p)$ <u>for an open set</u> $\Omega \subset B^m$, <u>then</u>

$$\int_{\Omega} J(v)\,dx = \int_{S^p} \mathcal{H}^{m-p}(v^{-1}(s))\,dA_{S^p}(s) \quad .$$

Proof. See [F, 3.2.22].

Let $g : S^{m-1} \longrightarrow S^n$ and $u_0 : B^m \longrightarrow S^n$ be as in Theorem 1.1.

Theorem 2.4. *For any* $1 \leq p \leq n$, $E_p(u_0) \leq E_p(u)$ *for all* $u \in \mathcal{E}_p(g)$.

Proof. According to Lemma 2.2, we have

$$\text{(2.8)} \quad c\, E_p(u) \geq ⨍_{G_{p+1}(\mathbb{R}^{n+1})} \int_{B^m} J(\pi_Y \circ u)\,dx\; dG(Y) \quad .$$

Note that for almost all $Y \in G_{p+1}(\mathbb{R}^{n+1})$, the map $v = \pi_Y \circ u \in W^{1,p}(B^m, S^p)$. Write $v_0 = \pi_Y \circ u_0$. We shall show that for all such $Y \in G_{p+1}(\mathbb{R}^{n+1})$,

$$\text{(2.9)} \quad \int_{B^m} J(v)\,dx \geq \int_{B^m} J(v_0)\,dx \quad .$$

According to Approximation Theorem 3.2, it is enough to prove inequality (2.9) for v in the class $\mathcal{R}$ of mappings with controlled singularities, since $J(v)$ is dominated by $|\nabla v|^p$. As before, choose $\Omega = B^m \setminus (\Sigma \cup \Delta)$, where $\Sigma \cup \Delta$ is the singular set of v , as in the definition of $\mathcal{R}$; and apply the coarea formula of Lemma 2.3. This yields

$$(2.10) \quad \int_{\Omega} J(v)\,dx = \int_{S^p} \mathcal{H}^{m-p}(v^{-1}(s))\,dA_{S^p} =$$

$$= \frac{1}{2} \int_{S^p} \mathcal{H}^{m-p}(M(s))\,dA_{S^p} ,$$

where $M(s) := v^{-1}(s) \cup v^{-1}(-s) \cup \Sigma \cup \Delta$ is a regular, oriented Lipschitz manifold having boundary a totally geodesic sphere of dimension $m-p-1$ in ∂B^m , for a.a. $s \in S^p$. In particular,

$$(2.11) \quad \mathcal{H}^{m-p}(M(s)) \geq \mathcal{H}^{m-p}(B^{m-p}) \; .$$

Note that equality holds in (2.11) for

$$M_0(s) = v_0^{-1}(s) \cup v_0^{-1}(-s) \cup \Delta \; .$$

Inequality (2.9) now follows from equation (2.10) for v and for v_0 .

With u replaced by u_0 , we attain equality in (2.8), according to Lemma 2.2, since u_0 is horizontally conformal. Therefore

$$c\, E_p(u) \geq c\, E_p(u_0) \; .$$

q.e.d.

3. Approximation and slicing

We saw in sections 1 and 2 that it may be useful to approximate mappings in $W^{1,p}(B^m, S^p)$ by mappings with controlled singularities, and particularly, with regular slices $u^{-1}(s)$ for almost all $s \in S^p$. Consider boundary data $g \in W^{1,p}(\partial B^m, S^p)$, such that g is C^∞ except on a Lipschitz submanifold $\Gamma \subset \partial B^m$ of dimension at most $m-p-2$. Write $u_0 \in W^{1,p}(B^m, S^p)$ for the homogeneous mapping $u_0(x) := g(\frac{x}{|x|})$. Observe that u_0 is singular on the cone $\Delta := \{tx : 0 \leq t \leq 1, x \in \Gamma\}$. We define R to be the class of mappings $u \in W^{1,p}(B^m, S^p)$ such that

(3.1) $u = u_0$ on a neighborhood of $\partial B^m \cup \Delta$;

(3.2) u is locally Lipschitz on $B^m \setminus (\Delta \cup \Sigma)$, for some Lipschitz submanifold $\Sigma \subset\subset B^m \setminus \Delta$ $(\partial \Sigma = \phi)$, of dimension $m-p-1$; and

(3.3) for a.a. $s \in S^p$, $u^{-1}(s) \cup u^{-1}(-s) \cup \Sigma \cup \Delta$ is a regular, oriented $(m-p)$-dimensional Lipschitz submanifold of B^m, having boundary only in ∂B^m.

Remark 3.1. The conditions (3.3) and (3.1) are both possible only if the restriction of g to a small p-sphere linking Γ in ∂B^m is one-to-one. In the present paper, this condition is always satisfied; the general case requires methods of geometric measure theory.

Theorem 3.2. *If* $u_0 \in R$ *, then* R *is dense in*

$$E_p(g) = \{u \in W^{1,p}(B^m, S^p) : u = u_0 \text{ on } \partial B^m\} .$$

Proof. We follow ideas of Bethuel and Zheng [BZ]. Consider $u \in E_p(g)$: we wish to find $u_k \in R$, $u_k \longrightarrow u$ in $W^{1,p}(B^m, S^p)$. First observe that by radially homogeneous extension beyond ∂B and rescaling we may assume that $u = u_0$ on $\{x \in \mathbb{R}^m : |x| \geq 1-\varepsilon\}$. We form $w_k = u * \rho_k$ for some compactly supported mollifier $\rho : \mathbb{R}^m \longrightarrow [0,\infty)$, where $\rho_k(x) := k^m \rho(kx)$. Note that $w_k \longrightarrow u$ in $W^{1,p}(B^m, \bar{B}^{p+1})$. Since Δ has p-capacity zero, we may find a sequence $v_k' \in W^{1,p}(B^m, \bar{B}^{p+1})$ such that each $v_k' = u_0$ on a neighborhood (of size depending on k) of Δ , $v_k' \in C^\infty(B^m \setminus \Delta, \mathbb{R}^{p+1})$ and $v_k' \longrightarrow u$ in $W^{1,p}(B^m, \bar{B}^{p+1})$ and a.e.. Let $\eta \in C^\infty(\mathbb{R}^m, \mathbb{R})$ have support in $B^m_{1-\varepsilon/2}$, such that $\eta(x) \equiv 1$ for $|x| \leq 1-\varepsilon$. Define

$$v_k = \eta \, v_k' + (1-\eta) u_0 ;$$

then $v_k = u_0$ on a neighborhood of $\Delta \cup \partial B^m$, and $v_k \longrightarrow u$ in $W^{1,p}(B^m, \bar{B}^{p+1})$ and a.e.. Let

$$\Omega_k := \{x \in B^m : |v_k(x)| < \tfrac{1}{2}\} ;$$

then $\operatorname{mes}(\Omega_k) \longrightarrow 0$, and hence

$$\int_{\Omega_k} |\nabla v_k|^p \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty .$$

Consider a regular value ("center") $a \in B^{p+1}_{1/4}$; let $\Sigma(a,k) = \{x \in B^m : v_k(x) = a\}$. Note that since $|v_k| = |u_0| = 1$ on a neighborhood of $\partial B^m \cup \Delta$, $\Sigma(a,k)$ lies in a compact subset of $B^m \setminus \Delta$. Since $v_k \in C^\infty(B^m \setminus \Delta, \bar{B}^{p+1})$, $\Sigma(a,k)$ is a regular submanifold of dimension $m-p-1$. Define $q_a : \bar{B}^{p+1}_{1/2} \longrightarrow \partial B^{p+1}_{1/2}$ so that x lies in the line segment from a to $q_a(x)$; then $q_a(x) = x$ for $x \in \partial B^{p+1}_{1/2}$. Extend q_a to $\bar{B}^{p+1}$ by defining $q_a(x) = x$ when $|x| \geq 1/2$. We note that

$$|\nabla q_a(x)| \leq C/|x-a| ,$$

where C is independent of a and x . As in the paper of Hardt and Lin ([HL]; see also [HKL], p. 556), we apply Fubini's theorem to show that for every $\varphi \in W^{1,p}(\Omega_k, \bar{B}^{p+1})$,

$$⨍_{B^m_{1/4}} \int_{\Omega_k} |\nabla(q_a \circ \varphi)|^p \, dx \, da \leq C' \int_{\Omega_k} |\nabla \varphi|^p \, dx$$

where C' is independent of φ, a and k . It follows that for a in a subset of $B^m_{1/4}$ of positive measure

$$(3.4) \quad \int_{\Omega_k} |\nabla(q_a \circ v_k)|^p \, dx \leq C' \int_{\Omega_k} |\nabla v_k|^p \, dx ,$$

which tends to zero.

We define $u_k(x) := \dfrac{q_a(v_k(x))}{|q_a(v_k(x))|}$. From inequality (3.4), we see that $u_k \longrightarrow u$ in $W^{1,p}(B^m, S^p)$. Note that u_k satisfies condition (3.1), and satisfies condition (3.2) for almost all $a \in B^m_{1/4}$.

In order to check condition (3.3), we first observe that for almost all lines $a + \mathbb{R}b$ in $\mathbb{R}^{p+1}$, $v_k^{-1}(a + \mathbb{R}b) \setminus \Delta$ is locally a smooth submanifold of $B^m \setminus \Delta$, as follows from Sard's theorem. Note that for $s \in S^p$,
$M(s,a) := u_k^{-1}(s) \cup u_k^{-1}(-s) \cup \Sigma = v_k^{-1}(q_a^{-1}(\mathbb{R}s))$, while $q_a^{-1}(\mathbb{R}s)$ is the union of the four line segments

$$[s, \tfrac{1}{2} s] \cup [\tfrac{1}{2} s, a] \cup [a, -\tfrac{1}{2} s] \cup [-\tfrac{1}{2} s, -s] \ .$$

Observe that $q_a^{-1}(\mathbb{R}s)$ is a Lipschitz 1-manifold with boundary $\{s, -s\}$. In particular, for almost all $a \in B^m_{1/4}$ and $s \in S^p$, $M(s,a)$ is locally a Lipschitz (m-p)-dimensional manifold in $B^m \setminus \Delta$. In a neighborhood of $\partial B^m \cup \Delta$, we have $u_k = u_0$; but since $u_0 \in \mathcal{R}$ by hypothesis, $M(s,a) \cup \Delta$ is also a Lipschitz manifold near $\partial B \cup \Delta$, hence everywhere in $\bar{B}^m$. Finally, $M(s,a) \cup \Delta$ is composed of four smooth manifolds-with-boundary, of which two meet at the smooth manifolds $\Sigma = v_k^{-1}(a)$, at $v_k^{-1}(\frac{1}{2} s)$ and at $v_k^{-1}(-\frac{1}{2} s)$; so that its boundary is $(\Delta \cup u_0^{-1}(s) \cup u_0^{-1}(-s)) \cap \partial B^m$.

q.e.d.

4. Odd boundary data

In this section, we consider smooth boundary data $g : S^{m-1} \longrightarrow S^{m-1}$ satisfying the hypothesis

$$(4.1) \quad g(-x) = -g(x)$$

for all $x \in S^{m-1}$, with $p = m-1$. This includes the specific case of Theorem 1.1 with $n = p = m-1$. Let $u_0 : B^m \longrightarrow S^{m-1}$ be defined by $u_0(x) = \frac{x}{|x|}$. We have the following lower bound for any such g:

Theorem 4.1. *For any* $u \in W^{1,m-1}(B^m, S^{m-1})$ *with* $u = g$ *on* ∂B, $E_{m-1}(u) \geq E_{m-1}(u_0)$.

Remark 4.2. This result settles a conjecture of Brézis-Coron-Lieb [BCL, Remark 7.3]; they proved this theorem under the additional hypothesis that the Jacobian $J(g) \geq 0$ and g has degree 1.

Proof. According to Approximation Theorem 3.2 (see also Theorem 4 of [BZ]), we may assume u belongs to the class $\mathcal{R}$ of $W^{1,m-1}$ mappings with controlled singularities. In particular, u is locally Lipschitz continuous on $B^m \setminus \Sigma$, where Σ is a finite set. Further, for almost every $s \in S^{m-1}$, $M(s) := u^{-1}(s) \cup u^{-1}(-s) \cup \Sigma$ is a Lipschitz 1-manifold with boundary $g^{-1}(s) \cup g^{-1}(-s)$. Considered as a one-dimensional integral current, $\partial M = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} a_{i+k}$, where $\{a_1,\ldots,a_{2k}\} = g^{-1}(s) \cup g^{-1}(-s)$ and a point of $g^{-1}(\pm s)$ is included in the list $\{a_1,\ldots,a_k\}$ provided $\pm J(g)(a_i) > 0$, otherwise in the list $\{a_{k+1},\ldots,a_{2k}\}$. Note that hypothesis (4.1) implies that $J(g)(-a_i) = J(g)(a_i)$. By reordering $\{a_{k+1},\ldots,a_{2k}\}$, we may assume that $a_{i+k} = -a_i$. According to the well-known theorem of Borsuk and Ulam, g has odd degree. Since $s \in S^{m-1}$ is a regular value of g , the number of points in $g^{-1}(s)$ has the same parity as the degree of g . That is, k is odd.

Now each connected component of $M(s)$ has boundary equal to the zero-dimensional integral current $a_i - a_{k+j}$ for some $1 \leq i$, $j \leq k$; write $j = \sigma(i)$, and note that σ is a permutation of $\{1,\ldots,k\}$. Clearly, therefore, $M(s)$ has length

$$\mathcal{H}^1(M(s)) \geq \sum_{i=1}^{k} |a_i - a_{k+\sigma(i)}| = \sum_{i=1}^{k} |a_i + a_{\sigma(i)}| \ .$$

Since k is odd, we have $\mathcal{H}^1(M(s)) \geq 2$ by Lemma 4.3 below.

From the coarea formula (Lemma 2.3) along with inequality (2.4) with $p = n = m-1$, we have

$$\int_{B^m} |\nabla u|^{m-1} dx \geq (m-1)^{(m-1)/2} \int_{S^{m-1}} \mathcal{H}^1(u^{-1}(s))\, dA_{S^{m-1}}(s)$$

$$= \frac{1}{2}\ (m-1)^{(m-1)/2} \int_{S^{m-1}} \mathcal{H}^1(M(s))\, dA_{S^{m-1}}(s)$$

$$\geq (m-1)^{(m-1)/2}\ m\, \alpha_m = E_{m-1}(u_0) \ .$$

q.e.d.

Lemma 4.3. Consider a set $\{a_1,\ldots,a_k\}$ *of points (not necessarily distinct) satisfying* $|a_i| \geq 1$. *If* k *is odd , then for any permutation* σ *of* $\{1,2,\ldots,k\}$, *the sum*

$$S := \sum_{i=1}^{k} |a_i + a_{\sigma(i)}| \geq 2 .$$

Remark 4.4. Note that any even value of k allows counter-examples.

Proof. If $\sigma(j) = j$ for some $1 \leq j \leq k$, then the term $|a_j + a_{\sigma(j)}| = 2|a_j| \geq 2$, and the conclusion follows. If $k = 1$, then $\sigma(1) = 1$, and the conclusion again follows. Thus we may proceed by induction, with the assumption that $\sigma(j) \neq j$, $1 \leq j \leq k$.

Since $\sigma(k) \neq k$, we may reorder $\{a_1,\ldots,a_k\}$ so that $\sigma(k) = k-1$. Then a_k appears only in the two terms $|a_k + a_{k-1}|$ and $|a_j + a_k|$, where $\sigma(j) = k$. If $j = k-1$, then we may discard these two terms to form the sum

$$\sum_{i=1}^{k-2} |a_i + a_{\sigma(i)}| \geq 2$$

by the induction hypothesis, since the restriction of σ is a permutation of $\{1,\ldots,k-2\}$. If $j \neq k-1$, then a_{k-1} appears in one additional term $|a_{k-1} + a_i|$, where $i = \sigma(k-1)$. By the triangle inequality,

$$|a_k + a_{k-1}| + |a_j + a_k| + |a_{k-1} + a_i| \geq$$

$$\geq |a_{k-1} - a_j| + |a_{k-1} + a_i| \geq |a_j + a_i| \ .$$

Now define the permutation $\tilde{\sigma}$ on $\{1,\dots,k-2\}$ so that $\tilde{\sigma}(j) = i$, and otherwise $\tilde{\sigma} = \sigma$. Then the corresponding sum $\tilde{S} \leq S$. But $\tilde{S} \geq 2$ by the induction hypothesis.

q.e.d.

5. The complex Hopf map

The Hopf map $H : S^3 \longrightarrow S^2$ is defined by the restriction to $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ of

$$H(z,w) = (|z|^2 - |w|^2, 2z\bar{w}) \in \mathbb{R} \times \mathbb{C} = \mathbb{R}^3 .$$

Let $u_0 : B^4 \longrightarrow S^2$ be its homogeneous extension of degree zero:

$$u_0(x) = H(\frac{x}{|x|}) .$$

Note that $u_0(z,w)$ is the stereographic projection of $z/w \in \mathbb{C} \cup \{\infty\}$.

Theorem 5.1. For any $u \in E_2(H) = \{u \in W^{1,2}(B^4, S^2) : u = H$ on $\partial B^4\}$, $E_2(u) \geq E_2(u_0) = 8\pi^2$.

Proof. According to Theorem 3.2, we may assume u belongs to the class R of $W^{1,2}$ maps with controlled singularities. In particular, u is locally Lipschitz on $B^4 \setminus \Sigma$, where Σ is a one-dimensional Lipschitz manifold without boundary. Further, for almost every $s \in S^2$,

$$M(s) := u^{-1}(s) \cup u^{-1}(-s) \cup \Sigma$$

is a Lipschitz 2-manifold with boundary $H^{-1}(s) \cup H^{-1}(-s)$.

The energy-minimizing property of u_0 will be proved by first showing that the cone $M_0(s)$ over $H^{-1}(s) \cup H^{-1}(-s)$ has smallest area. In fact, $M_0(s)$ is the union of the disks of radius 1 in the 2-dimensional planes

$$\{z = \xi w\} \quad \text{and} \quad \{z = -w/\bar{\xi}\}$$

where $\xi \in \mathbb{C} \cup \{\infty\}$ corresponds under stereographic projection to $s \in S^2$. Now any vector $(\xi w_1, w_1)$ in the first plane is orthogonal to any vector $(-w_2/\bar{\xi}, w_2)$ in the second plane, which implies that $M_0(s)$ is a complex-analytic variety for some orthogonal complex structure (which depends on s) for $\mathbb{R}^4$. In particular, $M_0(s)$ has minimum area among all surfaces in B^4 having boundary $H^{-1}(s) \cup H^{-1}(-s)$ (including unorientable surfaces: see [M], Corollary 6). Specifically,

$$\mathcal{H}^2(M(s)) \geq \mathcal{H}^2(M_0(s)) = 2\pi .$$

It now follows from Lemma 1.4 that

$$E_2(u) \geq 2 \int_{S^2} \mathcal{H}^2(u^{-1}(s))\, dA_{S^2}(s) =$$

$$= \int_{S^2} \mathcal{H}^2(M(s))\, dA_{S^2}(s) \geq 8\pi^2 .$$

Meanwhile $|\nabla u_0(x)|^2 = 8/|x|^2$, so that

$$E_2(u_0) = 4\mathcal{H}^3(S^3) = 8\pi^2 .$$

q.e.d.

6. The quaternionic Hopf map

Quaternionic multiplication in $\mathbb{R}^4$ is defined via an orthonormal basis $\{1,i,j,k\}$ with the properties $i^2 = j^2 = k^2 = ijk = -1$, forming a skew field $\mathbb{H}$. The Hopf map $H : S^7 \longrightarrow S^4$ is defined by identifying $\mathbb{R}^8$ as $\mathbb{H} \times \mathbb{H}$ and setting

$$H(q_1,q_2) = (|q_1|^2-|q_2|^2,\ 2q_1\bar{q}_2) \in \mathbb{R} \times \mathbb{H} = \mathbb{R}^5 .$$

Let $u_0 : B^8 \longrightarrow S^4$ be its homogeneous extension of degree zero:

$$u_0(x) = H(\frac{x}{|x|}) .$$

Note that $u_0(q_1, q_2)$ is the stereographic projection of $q_1 q_2^{-1} \in \mathbb{H} \cup \{\infty\}$.

Theorem 6.1. *For any* $u \in E_2(H) = \{u \in W^{1,2}(B^8, S^4) : u = H$ *on* $\partial B^8\}$, *we have* $E_2(u) \geq E_2(u_0)$.

Remark 6.2. The map u_0 also minimizes E_4, as may be proved by direct analogy with the proof of Theorem 5.1, and with the proof of Corollary 6 of [M]. The case $p = 2$, however, requires averaging over projections $\pi_Y : S^4 \longrightarrow S^2$, and is more interesting.

<u>Proof</u>. For any $Y \in G_3(\mathbb{R}^5)$, write $\pi_Y : S^4 \longrightarrow S^2$ for the nearest-point projection. According to Lemma 1.2, we have

$$(6.1) \quad c\, E_2(u) = \fint_{Y \in G_3(\mathbb{R}^5)} E_2(\pi_Y \circ u)\, dG(Y) .$$

Write $v = \pi_Y \circ u : B^8 \longrightarrow S^2$, and $v_0 = \pi_Y \circ u_0$. We need to show that $E_2(v) \geq E_2(v_0)$ for any $v \in W^{1,2}(B^8, S^2)$ with $v = \pi_Y \circ H$ on ∂B^8. It suffices to prove this for $v \in R$, according to Theorem 3.2. Applying Lemma 1.4 and regrouping s with $-s$ as before, we have

$$(6.2) \quad E_2(v) \geq \int_{S^2} H^6(v^{-1}(s) \cup v^{-1}(-s))\, dA_{S^2} .$$

Now since $v \in R$,

$$M(s) := v^{-1}(s) \cup v^{-1}(-s) \cup \Sigma \cup \Delta$$

is a 6-dimensional oriented Lipschitz submanifold of B^8, with $\partial M(s) = (\pi_Y \circ H)^{-1}(s) - (\pi_Y \circ H)^{-1}(-s)$ as integral currents with the natural slice orientations ([F], 4.3.1). The cone over $\partial M(s)$ is

$$M_0(s) := v_0^{-1}(s) \cup v_0^{-1}(-s) \cup \Delta .$$

We need to show that

$$(6.3) \quad H^6(M_0(s)) \leq H^6(M(s)) .$$

For each $Y \in G_3(\mathbb{R}^5)$ and each $s \in S^2 = Y \cap S^4$, observe that $\pi_Y^{-1}(s) \cup \pi_Y^{-1}(-s) = S^4 \cap Z$, where Z is the 3-plane spanned by s and the orthogonal complement of Y. According to Lemma 6.3 below, it is enough to verify inequality (6.3) for the special case where Y is spanned by $(1,0)$, $(0,j)$ and $(0,k)$, and where $s = (1,0)$. In this case, Z is spanned by $(1,0)$, $(0,1)$ and $(0,i)$. Write the point $(q_1,q_2) \in \mathbb{R}^8$ in terms of complex variables z_1, w_1, z_2, w_2 by defining $q_\alpha = z_\alpha + w_\alpha j$. Then

$$q_1\bar{q}_2 = z_1\bar{z}_2 + w_1\bar{w}_2 + (z_2w_1 - z_1w_2)j ,$$

so that $M_0(s)$ is given by

$$\{(z_1,w_1,z_2,w_2) \in B^8 : g(z_1,w_1,z_2,w_2) := z_2w_1 - z_1w_2 = 0\} .$$

Note that the orientation induced on $M_0(s) = g^{-1}(0)$ by $g : B^8 \longrightarrow \mathbb{C}$ from the appropriate orientations on $\mathbb{C}$ and $\mathbb{R}^8$ is consistent with the orientation given in the Approximation Theorem 3.2. For the (standard) complex structure on $\mathbb{R}^8$ given by

$$J(z_1,w_1,z_2,w_2) = (iz_1,iw_1,iz_2,iw_2),$$

$M_0(s)$ is a complex variety, and inequality (6.3) follows, since $\partial M_0(s) = \partial M(s)$ as integral currents ([F], pp. 435 and 652).

On the other hand, $u_0 : B^8 \longrightarrow S^4$ is horizontally conformal ([B], Theorem 7.1.1 and Examples 7.2.1, 8.2.1) and therefore $v_0 : B^8 \longrightarrow S^2$ is horizontally conformal for any choice of Y . It follows from Lemma 1.4 that equality holds in (6.2) when v is replaced by v_0 . Finally, using inequality (6.3), inequality (6.2) for v and equation (6.1) for u_0 and for u , we conclude that

$$c\ E_2(u) \geq c\ E_2(u_0)\ .$$

q.e.d.

The following lemma is known, since it is an immediate consequence of the fact that the Hopf map: $S^7 \longrightarrow S^4$ induces the isomorphism of the symplectic group $Sp(2)$ of quaternionic 2×2 matrices in $SO(8)$ with the oriented double cover of $SO(5)$ (which fact may be proved in analogous fashion to p. 38 of [A]). Since the literature may be unfamiliar to many, we prefer to present a direct proof.

<u>Lemma 6.3</u>. <u>Given</u> $Z_0, Z_1 \in G_3(\mathbb{R}^5)$, <u>there exist rotations</u> $R \in SO(5)$ <u>and</u> $Q \in SO(8)$ <u>such that</u> $R(Z_1) = Z_0$ <u>and</u> $H(Q(q_1,q_2)) = R(H(q_1,q_2))$ <u>for all</u> $q_1, q_2 \in \mathbb{H}$.

<u>Proof</u>. Without loss of generality, we may assume $Z_0 \subset \mathbb{R}^5 = \mathbb{R} \times \mathbb{H}$ is spanned by (1,0), (0,1) and (0,i) . According to a theorem of Cayley ([C], p. 71), any $R_1 \in SO(4)$ may be written in terms of quaternionic multiplication as $R_1(q) = q_1 q q_2$ for some

$q_1, q_2 \in \mathbb{H}$ of norm one. This corresponds to $Q_1 \in SO(8)$ given by $Q_1(p,q) = (q_1 p, \bar{q}_2 q)$. Consider R_1 to be in $SO(5)$ by $R_1(t,q) = (t, R_1(q))$; then $H \circ Q_1 = R_1 \circ H$ (recall that $\overline{pq} = \bar{q}\bar{p}$). By choosing R_1 appropriately, we may achieve $Z_2 = R_1(Z_1)$ so that $(0,1)$ and $(0,i)$ are in Z_2 . Next, let $R_2 \in SO(5)$ be the rotation which fixes $(0,i)$, $(0,j)$ and $(0,k)$, while $R_2(1,0) = (\cos 2\Theta, \sin 2\Theta)$ and $R_2(0,1) = (-\sin 2\Theta, \cos 2\Theta)$. This corresponds to $Q_2 \in SO(8)$ defined by $Q_2(p,q) = ((\cos\Theta)p - (\sin\Theta)q, (\sin\Theta)p + (\cos\Theta)q)$: namely, $H \circ Q_2 = R_2 \circ H$. For two choices of Θ , we find $Z_0 = R_2(Z_2)$.

q.e.d.

We would like to conclude our paper with a theorem of more general character, whose proof is analogous to the proofs of Theorems 1.1, 2.4, 5.1 and 6.1.

Consider $u_0 \in W^{1,p}(B^m, S^n)$ for some integer p , $1 \leq p \leq n$. For each $Y \in G_{p+1}(\mathbb{R}^{n+1})$, let $v_0 = \pi_Y \circ u_0$. We require that

(6.4) $v_0 \in C^{0,1}(B^m \setminus \Delta, S^p)$ for some Lipschitz $(m-p-1)$-submanifold $\Delta \subset B^m$ with $\partial\Delta \subset \partial B^m$;

(6.5) there exists a measurable and measure-preserving map $h : S^p \longrightarrow S^p$ such that the difference of slices $M_0(s) := v_0^{-1}(s) - v_0^{-1}(h(s))$ defines an $(m-p)$-dimensional integral current of smallest mass for its boundary; and

(6.6) v_0 is horizontally conformal a.e. in B^m .

Theorem 6.4. *Suppose that for almost all* $Y \in G_{p+1}(\mathbb{R}^{n+1})$, *hypotheses* (6.4), (6.5) *and* (6.6) *hold. Then* $E_p(u_0) \le E_p(u)$ *for all* $u \in W^{1,p}(B^m, S^n)$ *with* $u = u_0$ *on* ∂B^m .

References

[ABL] Almgren, F., Browder, W., Lieb, E.: Co-area, liquid crystals, and minimal surfaces. Preprint.

[A] Atiyah, M.: Geometry of Yang-Mills Fields (1978 Fermi Lectures). Pisa: Scuola Normale Superiore, 1979.

[B] Baird, P.: Harmonic maps with symmetry, harmonic morphisms and deformations of metrics. Boston-London-Melbourne: Pitman 1983.

[BZ] Bethuel, F., Zheng, X.: Sur la densité des fonctions régulières entre deux variétés dans des espaces de Sobolev, C.R. Acad. Sci. Paris 303, I, 447-449 (1986) and Density of smooth functions between two manifolds in Sobolev spaces, preprint.

[BCL] Brézis, H., Coron, J.-M., Lieb, E.H.: Harmonic maps with defects. Commun. Math. Phys. 107, 649-705 (1986).

[C] Coxeter, H.S.M.: Regular Complex Polytopes. Cambridge: Cambridge University Press 1974.

[F] Federer, H.: Geometric Measure Theory. Berlin - Heidelberg-New York: Springer 1969.

[HKL] Hardt, R., Kinderlehrer, D., Lin, F.-H.: Existence and partial regularity of static liquid crystal configurations. Commun. Math. Phys. 105, 547-570 (1986).

[HL] Hardt, R., Lin, F.-H.: Mappings that minimize the p th power of the gradient. In preparation.

[H] Hélein, : In preparation.

[JK] Jäger, W., Kaul, H.: Rotationally symmetric harmonic maps from a ball into a sphere and the regularity problem for weak solutions of elliptic systems. J. Reine Angew. Math. 343, 146-161 (1983).

[L] Lin, F.-H.: private communication.

[Md] Macdonald, I.G.: Symmetric Functions and Hall Polynomials. Cambridge: Clarendon Press 1979.

[M] Morgan, F.: On the singular structure of two-dimensional area minimizing surfaces in $\mathbb{R}^n$. Math. Ann. 261, 101-110 (1982).

[SU1] Schoen, R., Uhlenbeck, K.: A regularity theory for harmonic maps. J. Diff. Geom. 17, 307-335 (1982).

[SU2] Schoen, R., Uhlenbeck, K.: Boundary regularity and the Dirichlet problem for harmonic maps. J. Differ. Geom. 18, 253-268 (1983).

[SU3] Schoen, R., Uhlenbeck, K.: Regularity of minimizing harmonic maps into the sphere. Invent. Math. 78, 89-100 (1984).