The Cohen-Macaulay type of points in generic position

by

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§ 1. Introduction

Let $X = \{P_1,\dots,P_s\}$ be a set of s distinct points in $\mathbb{P}^n_k$, k an algebraically closed field, and let I be the defining ideal of X in the polynomial ring $R = k[X_0,\dots,X_n]$.

We denote by A the homogeneous coordinate ring of X , $A = R/I = \bigoplus_{t=0}^{\infty} A_t$. We say, following Geramita and Orecchia [G-O₁], that the points $P_1,\dots,P_s$ are in generic position if the Hilbert function $H_A(t) := \dim_k A_t$ satisfies

$$H_A(t) = \min\left\{s, \binom{n+t}{n}\right\} \quad \text{for all} \quad t \geq 0 .$$

It is well-known that A is a Cohen-Macaulay one-dimensional graded k-algebra whose Cohen-Macaulay type $\tau(A)$ is defined as the k-dimension of the socle of an artinian reduction of A .

Let d be the least integer such that $s \leq \binom{d+n}{n}$ and let $L \in R_1$ be a non zero divisor on A . If $B = \bigoplus_{i=0}^{\infty} B_i$ denote the graded artinian ring $B := A/LA$, then the socle of B is the k-vector space $(I,L) : R_1/(I,L)$ which we denote by $s(B)$. Since $H_B(t) = 0$ for all $t > d$, we have $\partial(B) = \dim_k s(B)_d + \dim_k s(B)_{d-1}$. But $s(B)_d = B_d$, hence we need to compute $\dim_k s(B)_{d-1}$.

Since $s(B)_{d-1}$ is the kernel of the linear transformation

$$\varphi : B_{d-1} \longrightarrow \operatorname{Hom}_k(B_1, B_d)$$

which is induced by the multiplication of B , it is clear that

$$\dim_k s(B)_{d-1} \geq \dim_k B_{d-1} - (\dim_k B_1 \cdot \dim_k B_d) .$$

The Cohen-Macaulay type conjecture made by L.G. Roberts in [R] is that for a general set of points in generic position in $\mathbb{P}^n$, we have

$$\dim_k s(B)_{d-1} = \max\{0, \dim_k B_{d-1} - n \cdot \dim_k B_d\} .$$

This conjecture was verified in $\mathbb{P}^2$ in [G-M] , and, when $n > 2$, for special values of s in [R], [G-G-R] and [G-O_2] .

In this paper we verify the conjecture in its wide generality. Our point of view is to consider the field K which is obtained by adjoining to k new indeterminates $\{u_{ij}\}$, $i = 1,\ldots,s$ and $j = 0,\ldots,n$. Then we prove that the points $P_1,\ldots,P_s$ with homogeneous coordinates $P_i := (u_{io},\ldots,u_{in})$ are in generic position in $\mathbb{P}^n_k$ and verify the Cohen-Macaulay type conjecture.

Since this is equivalent to the fact that a certain matrix, whose entries are monomials in the u_{ij}'s , is of maximal tank, our result proves, by specialisation, that almost every set of s points in $\mathbb{P}^n_k$ which are in generic position verify the Cohen-Macaulay type conjecture.

§ 2. Main result

Let k be an algebraically closed field and let $\{u_{ij}\}$ $i = 1,\dots,s$ $j = 0,\dots,n$, be a set of indeterminates over k . Let K be the field obtained by adjoining these indeterminates to k .

Let $X = \{P_1,\dots,P_s\}$ be the set of the K-rational points in $\mathbb{P}^n_K$ whose coordinates are given by $P_i := (u_{io},\dots,u_{in})$. If we denote by R the polynomial ring $K[x_0,\dots,x_n]$ and by I the defining ideal of X in R , then $A := R/I$ is the homogeneous coordinate ring of X . The ring A is a K-graded algebra whose Hilbert function $H_A(t)$ is defined as $H_A(t) := \dim_K A_t = \dim_K(R_t/I_t)$.

In the following we consider a total order $<$ on the set of monomials of R which is sensitive to the degree. This induces in a canonical way an order on the monomials in $k[u_{io},\dots,u_{in}]$ for every $i = 1,\dots,s$: $u_{io}^{b_0} \dots u_{in}^{b_n}$ corresponds to $X_0^{b_0} \dots X_n^{b_n}$. Further, if α is a monomial in $k[u_{ij}]$, it is clear that $\alpha = V_1 \dots V_s$ where V_i is a monomial in $u_{io},\dots,u_{in}$. Hence we get an induced order on the monomials of $k[u_{ij}]$ by letting $\alpha = V_1 \dots V_s > \beta = W_1 \dots W_s$ if for some $i = 1,\dots,s-1$ we have $V_1 = W_1,\dots,V_i = W_i$ and $V_{i+1} > W_{i+1}$.

If F is an homogeneous polynomial of degree t in R , we can write $F = \sum_i \alpha_i M_i$ where $\alpha_i \in K$ and $M_1 > M_2 > \dots > M_r$ are the monomials of degree t in R . Hence $F \in I$ if and only if $\sum_i \alpha_i M_i(P_j) = 0$ for all $j = 1,\dots,s$. Thus, if we let $a_{ij} := M_j(P_i)$ and $\rho := \operatorname{rank}(a_{ij})$, it is clear that

$$\dim_K I_t = \binom{n+t}{n} - \rho \quad \text{for all} \quad t \geq 0 .$$

The size of the matrix (a_{ij}) is $s \times \binom{n+t}{n}$ and we claim that it has maximal rank, namely

$$\rho = \min\left\{s, \binom{n+t}{n}\right\}$$

The claim can be proved in the following way. If for example we assume $s \leq \binom{n+t}{n}$ and consider the $s \times s$ minor of (a_{ij}) involving the first s columns, then its determinant D is not zero since $D = \sum_{\sigma} (-1)^{\sigma} a_{1,\sigma(1)} \cdots a_{s,\sigma(s)}$ and it is clear that, accordingly to the order we have defined, $a_{11}a_{22} \cdots a_{ss} > a_{1,\sigma(1)} \cdots a_{s,\sigma(s)}$ for every $\sigma \neq id$. The same if $s > \binom{n+t}{n}$.

In this way we get a complete description of the Hilbert function of A , namely

$$H_A(t) = \min\left\{s, \binom{n+t}{n}\right\} \quad \text{for all} \quad t \geq 0 .$$

It turns out that the points $P_1, \ldots, P_s$ are in generic position, and this gives a very easy proof that almost every set of s points in $\mathbb{P}^n_k$ (k infinite) is in generic position (see [G-O$_1$]).

Let now d be the least integer such that $s \leq \binom{d+n}{n}$, then d is also the least integer such that $H_A(d) = s$.

Hence, if $B = \overset{\infty}{\underset{i=0}{\oplus}} B_i$ denotes the graded artinian ring $B := A/X_0A = R/(I,X_0)$, the socle $s(B) = (I/X_0) : R_1/(I,X_0)$ is concentrated in degree d and $d-1$. Also we get $\dim_K B_d = \dim_K s(B)_d = s - \binom{n+d-1}{n}$ and $\dim_K B_{d-1} = \binom{n+d-2}{n-1}$. We let $q := \binom{n+d-2}{n-1}$, $p := \binom{n+d-1}{n}$ and $S := K[x_1,\ldots,x_n]$.

<u>Theorem</u>. Under the above assumptions and notations we have:

$$\dim_K s(B)_{d-1} = \max\{0, q+np-ns\} .$$

<u>Proof</u>. We consider in $R = K[X_0,\ldots,X_n]$ the order defined on monomials of the same degree by

$$X_0^{a_0} \ldots X_n^{a_n} > X_0^{b_0} \ldots X_n^{b_n}$$

if the first non zero entry of the vector $(a_0-b_0,\ldots,a_n-b_n)$ is negative. For example we have the following chain of monomials in degree say m :

$$X_n^m > X_n^{m-1}X_{n-1} > X_n^{m-2}X_{n-1}^2 > \ldots > X_{n-1}^m > X_n^{m-1}X_{n-2} > X_n^{m-2}X_{n-1}X_{n-2} > \ldots$$

$$\ldots > X_nX_0^{m-1} > \ldots > X_1X_0^{m-1} > X_0^m .$$

This order induces a total order also on the monomials of S ; hence we let $M_1 > M_2 > \ldots > M_q$ be the monomials of degree $d-1$ in S .

Now it is clear that $s(B)_{d-1} = \left[(I,X_0) : R_1/(X_0)\right]_{d-1}$, hence $s(B)_{d-1} \cong \left[(I,X_0) \underset{S}{:} S_1\right]_{d-1}$ as K-vector spaces. Let $N_1 > N_2 > \dots > N_p$ be the monomials of degree d in R containing X_0 .

If $F \in S_{d-1}$ and $FS_1 \in (I,X_0)$, we can write

$$F = \sum_{i=1}^{q} \alpha_i M_i \ , \ \alpha_i \in K$$

and we can find elements $\beta_{ri} \in K$, $i = 1,\dots,p$, $r = 1,\dots,n$ such that

$$X_r F + \sum_{i=1}^{p} \beta_{ri} N_i \in I \ , \text{ for all } r = 1,\dots,n \ .$$

If we let $a_{ij} := M_i(P_j)$ and $b_{ij} := N_i(P_j)$, we get a system of homogeneous equations

$$X_r F(P_j) : u_{jr} \sum_{i=1}^{q} \alpha_i a_{ij} + \sum_{i=1}^{p} \beta_{ri} b_{ij} = 0$$

$r = 1,\dots,n$ and $j = 1,\dots,s$.

The matrix associated to the above system is an $ns \times (q + up)$ matrix which we write in the following way.

	$\alpha_1 \cdots\cdots \alpha_q$	$\beta_{n1}\cdots\beta_{np}$			$\beta_{11}\cdots\beta_{1p}$
$X_1F(P_1)$	$a_{11}u_{1n}\cdots a_{q1}u_{1n}$	$b_{11}\cdots b_{p1}$	○	....	
⋮	⋮ ⋮	○	$b_{11}\cdots b_{p1}$		○
$X_1F(P_1)$	$a_{11}u_{11}\cdots a_{q1}u_{11}$		○	....	$b_{11}\cdots b_{p1}$
$X_nF(P_2)$	$a_{12}u_{2n}\cdots a_{q2}u_{2n}$	$b_{12}\cdots b_{p2}$	○	....	
⋮	⋮ ⋮	○	$b_{12}\cdots b_{p2}$		○
$X_1F(P_2)$	$a_{12}u_{21}\cdots a_{q2}u_{21}$		○	....	$b_{12}\cdots b_{p2}$
⋮	⋮ ⋮	⋮ ⋮	⋮ ⋮	⋮ ⋮	⋮ ⋮
$X_mF(P_s)$	$a_{1s}u_{sn}\cdots a_{qs}u_{sn}$	$b_{1s}\cdots b_{ps}$	○		
⋮	⋮ ⋮	○	$b_{1s}\cdots b_{ps}$		○
$X_1F(P_s)$	$a_{1s}u_{s1}\cdots a_{qs}u_{s1}$		○		$b_{1s}\cdots b_{ps}$

<u>Claim</u>. This matrix has maximal rank given by the minor involving all the last np columns (note that $ns > np$; in fact, $s > p$ by the minimality of d).

First we prove that the claim gives the conclusion. In fact if we consider our matrix as the matrix associated to a morphism of K-vector spaces

$$\varphi : K^{q+np} \longrightarrow K^{ns}$$

the claim implies $\dim \operatorname{Ker} \varphi = \min(0,q+np-ns)$ and also that the canonical projection $\pi : K^{q+np} \longrightarrow K^{q}$ is injective on $\operatorname{Ker} \varphi$. Hence we get $\dim_K {}^{s}f(B)_{d-1} = \dim_K \pi(\operatorname{Ker} \varphi) = \dim_K \operatorname{Ker} \varphi =$ $= \min(0,q+np-ns)$, as wanted.

<u>Proof of the claim.</u>

<u>Case $ns \leq q+np$</u> .

If we let $t = s-p$, we get $t > 0$ and $q \geq nt$. We prove that the following $ns \times ns$ matrix M , which is obtained by deleting the columns corresponding to $\alpha_{nt+1},\dots,\alpha_q$, has nonzero determinant.

	$\alpha_1 \cdots\cdots \alpha_{nt}$	$\beta_{n1}\cdots\beta_{np}$			$\beta_{11}\cdots\beta_{1p}$
$X_nF(P_1)$	$a_{11}u_{1n}\cdots a_{nt1}u_{1n}$	$b_{11}\cdots b_{p1}$	O	....	
⋮	⋮ ⋮	O	$b_{11}\cdots b_{p1}$		O
$X_1F(P_1)$	$a_{11}u_{11}\cdots a_{nt1}u_{11}$		O	....	$b_{11}\cdots b_{p1}$
$X_nF(P_2)$	$a_{12}u_{2n}\cdots a_{nt2}u_{2n}$	$b_{12}\cdots b_{p2}$	O	....	
⋮	⋮ ⋮	O	$b_{12}\cdots b_{p2}$		O
$X_1F(P_2)$	$a_{12}u_{21}\cdots a_{nt2}u_{21}$		O	....	$b_{12}\cdots b_{p2}$
⋮	⋮ ⋮	⋮ ⋮	⋮ ⋮		⋮ ⋮
$X_nF(P_s)$	$a_{1s}u_{sn}\cdots a_{nts}u_{sn}$	$b_{1s}\cdots b_{ps}$	O		
⋮	⋮ ⋮	O	$b_{1s}\cdots b_{ps}$		O
$X_1F(P_s)$	$a_{\ s}u_{s1}\cdots a_{nts}u_{s1}$		O		$b_{1s}\cdots b_{ps}$

We recall that if $M = (m_{ij})$ is a square matrix of size say v, an M-product is an element $(-1)^{\sigma} m_{1\sigma(1)} \cdots m_{v\sigma(v)}$ where σ is a permutation of $\{1,2,\ldots,v\}$. Thus $\det M$ is the sum of the M-products.

Now let $D = \det M$; since every entry of the row corresponding to $X_i F(P_j)$ is a monomial of degree d in $u_{jo},\ldots,u_{jn}$, every M-product is a monomial $j = V_1 \ldots V_s$ where, for every $\gamma = 1,\ldots,s$, V_j is a monomial for degree nd in $u_{j0},\ldots,u_{jn}$.

We prove that $D \neq 0$ by checking that there exists a M-product which is greater, in the given order on the monomials of $k[u_{ij}]$, than any other M-product which does not cancel out in the presentation of D.

We denote by M_α the submatrix of M corresponding to the first nt columns and by M_β that corresponding to the last np.

We have two important remarks.

1. Every M-product which, inside M_α, involves two rows corresponding to the same point, can be deleted.

This is clear since for every $1 \leq j \leq s$ and every $1 \leq k < i \leq n$,

$$\begin{bmatrix} a_{1j}u_{ji} & \cdots & a_{ntj}u_{ji} \\ \vdots & & \vdots \\ a_{1j}u_{jk} & \cdots & a_{ntj}u_{jk} \end{bmatrix}$$

is a matrix of rank one. Hence such an M-product cancels out in the presentation of D .

2. If γ is a M-product which, inside M_α and for some $1 \leq i \leq n$, involves less than t rows corresponding to the variable X_i , then $\gamma = 0$.

For example, if γ , inside M_α , involves v rows corresponding to X_1 , with $v < t$, then, inside $M_{\beta,\alpha}$ involves $s-v$ rows corresponding to X_1 . Since $s-v > s-t = p$, this implies $\gamma = 0$.

Thus if γ is a M-product which is not 0 , then γ , inside M_α , involves exactly the rows corresponding to each variable $X_n,\dots,X_1$.

According to the above rules, we consider the M-product γ which is obtained using the following correspondence

$$\alpha_i \longleftrightarrow X_{n-v+1}F(P_i)$$

$$\beta_{ij} \longleftrightarrow \begin{cases} X_i F(P_j) & \text{if } j \leq (n-i)t \\ X_i F(P_{j+t}) & \text{if } j > (n-i)t \end{cases}$$

where v is the least integer $\geq (i/t)$.

In other words, we associate to the columns $\alpha_1,\dots,\alpha_t$ the rows $X_nF(P_1),\dots,X_nF(P_t)$, to the columns $\alpha_{t+1},\dots,\alpha_{2t}$ the rows $X_{n-1}F(P_{t+1}),\dots,X_{n-1}F(P_{2t})$, and so on up to the columns $\alpha_{(n-1)t+1},\dots,\alpha_{nt}$ to which we associate the rows $X_1F(P_{(n-1)t+1})$,

$X_1F(P_{(n-1)t+1}),\dots,X_1F(P_{nt})$. As for the remaining $n-1$ rows corresponding to each point $P_1,\dots,P_{nt}$, we choose in M_β , starting from the top, the first admissible nonzero entry on the left. At this point we are left with the rows corresponding to the last $s-nt$ point and with the last $p-(n-1)t = s-nt$ columns in each vertical section of M_β . Hence we can choose, for $X_nF(P_{nt+1}),\dots,X_1F(P_{nt+1})$, the columns $\beta_{n,(n-1)t+1},\dots,\beta_{1,(n-1)t+1}$, for $X_nF(P_{nt}+2),\dots,X_1F(P_{nt+2})$, the columns $\beta_{n,(n-1)t+2},\dots,\beta_{1,(n-1)t+2}$ and so on up to $X_nF(P_{nt+s-nt}),\dots,X_1F(P_{nt+1-nt})$ for which we choose the columns $\beta_{n,(n-1)t+s-nt} = \beta_{np},\dots,\beta_{1,(n-1)t+s.nt} = \beta_{1p}$.

It is not difficult to see that we have $\gamma = V_1\dots V_s$ where

$$\begin{cases} V_i = a_{ii}u_{i,n-v+1}\, b_{i-t}^{v-1}\, b_{ii}^{n-v} & \text{if } 1 \le i \le nt \\ \\ V_i = b_{i-t,i}^{n} & \text{if } nt+1 \le i \le s \ . \end{cases}$$

Since in each horizontal section of M we have

$$\begin{array}{l|lllllllll} X_nF(P_i) & a_{1i}u_{in} > \dots\dots > a_{nti}u_{in} > b_{1i} > b_{2i} > \dots\dots > b_{pi} \ \dots\dots \\ \vdots & \vee \qquad\quad \vee \quad \vee \qquad \vee \\ \vdots & \vdots \qquad\quad \vdots \quad \vdots \qquad \vdots \\ \vdots & \vee \qquad\quad \vee \quad \vee \qquad \vee \qquad\qquad\qquad 0 \\ X_1F(P_i) & a_{1i}u_{i1} > \dots\dots > a_{nti}u_{i1} \qquad\qquad\qquad\qquad\qquad \dots\dots \end{array}$$

it is clear that the monomial γ is greater than any other M-product which does not cancel out.

Case $ns > q+np$.

First we remark that it is sufficient to prove the result for the smallest s such that $\binom{n+d-1}{n} < s \leq \binom{n+d}{n}$ and $ns \geq q+np$. For such s we let as before $t = s-p$ and $m = nt-q \geq 0$; then $m < n$ (otherwise, $nt-g \geq n$ implies $ns-np-q \geq n$, hence $n(s-1) \geq q+np$, a contradiction).

We prove that if we delete the rows corresponding to $X_1F(P_1),\dots,X_mF(P_1)$, we get a square matrix N whose determinant is nonzero.

As before we denote by N_α and N_β respectively the "α-part" and "β-part" of N .

It is clear that the first rule does not change, while the second one should be red in the following way.

Each N-product which is not zero, involves, inside N_α , exactly t rows corresponding to each variable $X_n,\dots,X_{m+1}$ and $t-1$ rows corresponding to each variable $X_m,\dots,X_1$.

Note that $t(n-m) + m(t-1) = q$.

According to this remark, we consider the N-product γ which is determined in the following way.

We associate to the columns $\alpha_1,\dots,\alpha_t$ the rows $X_nF(P_1),\dots,X_nF(P_t)$, to the columns $\alpha_{t+1},\dots,\alpha_{2t}$ the rows $X_{n-1}F(P_{t+1}),\dots,X_{n-1}F(P_{2t})$ and so as up to the columns $\alpha_{(n-m-1)t+1},\dots,\alpha_{(n-m)t}$ to which we associate the rows $X_{m+1}F(P_{(n-m-1)t+1}),\dots,X_{m+1}F(P_{(n-m)t})$. Then we are left in N_α with $q-(n-m)t = m(t-1)$ columns. Hence we associate to the columns $\alpha_{(n-m)t+1},\dots,\alpha_{(n-m)t+t-1}$ the rows

$X_{m-1}F(P_{(n-m)t+t}),\dots,X_{m-1}F(P_{(n-m)t+2t-2})$ and so on up to the columns $\alpha_{t(n-m)+(m-1)t-(m-1)+1},\dots,\alpha_{t(n-m)+mt-m} = \alpha_q$ to which we associate the rows $X_1F(P_{t(n+m)+(m-1)t-(m-1)+1}),\dots,X_1F(P_q)$.

As for the remaining $n-m-1$ rows corresponding to P_1 and the remaining $n-1$ rows corresponding to each point $P_2,\dots,P_q$, we choose in N_β , starting from the top, the first admissible nonzero entry on the left.

In this way we involved each variable $X_n,\dots,X_{m+1}$ t times in N_α , hence $q-t$ times in N_β ; as for $X_m,\dots,X_1$ they have been involved $t-1$ times in N_α , hence $q-(t-1)-1 = q-t$ times in N_β (note that the rows corresponding to the point P_1 do not involve $X_m,\dots,X_1$).

Thus we are left with the rows corresponding to the last $s-q$ points and with the last $p-(q-t) = s-q$ columns in each vertical section of N_β . Hence we can choose for $X_nF(P_{q+1}),\dots,X_1F(P_{q+1})$, the columns $\beta_{n,q-t+1},\dots,\beta_{1,q-t+1}$, for $X_nF(P_{q+2}),\dots,X_1F(P_{q+2})$, the columns $\beta_{n,q-t+2},\dots,\beta_{1,q-t+2}$ and so on up to $X_nF(P_{q+s-q}),\dots,X_1F(P_{q+s-q})$ for which we choose the columns $\beta_{n,q-t+s-q} = \beta_{n,p},\dots,\beta_{1,q-t+s-q} = \beta_{1,p}$.

As before it is clear that the monomial γ is bigger than any other N-product which does not cancel out. This includes the proof of the theorem.

<u>Remark</u> In order to further clarify the argument of the proof, we give in the Appendix a picture of the matrices M and N corresponding to the case $n = 3$, $s = 23$ and $n = 3$, $s = 24$ (see Figures (1) and (2) of the Appendix). Of course we consider

only the top part of the matrices, where our choice of the factors of γ is much more subtle.

Note that in the Figures the dots correspond to the nonzero entries of the matrices, while the entries labelled with an X are the factors of γ .

Fig. 1

	123456789	123456 20	123456 20	123456 20
31	X........	....		
21	....		X...................	
11	....			X...................
32	.X.......	....		
22	....		.X..................	
12	....			.X..................
33	..X......	....		
23	....		..X.................	
13	....			..X.................
34	....	X...................		
24	...X.....		....	
14	....			...X................
35	....	.X..................		
25	....X....		....	
15	....			....X...............
36	....	..X.................		
26	.....X...		....	
16	....			.....X..............
37	....	...X................		
27	....		...X................	
17	......X..			....
38	....	....X...............		
28	....		....X...............	
18	.......X.			....
39	....	.....X..............		
29	....		.....X..............	
19	X			....

$n = 3,\ s = 23,\ d = 4,\ q = 10,\ p = 20,\ t = 3$.

Fig. 2

	12345678910	123456 … 20	123456 … 20	123456 … 20
31	X.........	....	....	....
32	.X........	....		
22	....		X...................	
12	....			X...................
33	..X.......	....		
23	....		.X..................	
13	....			.X..................
34	...X......	....		
24	....		..X.................	
14	....			..X.................
35	....	X...................		
25	....X.....		....	
15	....			...X................
36	....	.X..................		
26	.....X....		....	
16	....			....X...............
37	....	..X.................		
27	......X...		....	
17	....			.....X..............
38	....	...X................		
28	....		...X................	
18	.......X..			....
39	....	....X...............		
29	....		....X...............	
19	........X.			....
310	....	.....X..............		
210	....		.....X..............	
110	X			....

n = 3, s = 24, d = 4, q = 10, p = 20, t = 4, m = 2 .

§ 3. The case $d = 2$

It is clear that the proof of the theorem does not identify in some concrete geometric or algebraic way the non-empty Zariski open set $U_{n,s} \subseteq (\mathbb{P}_k^n)^s$ where the Cohen-Macaulay type takes its minimal value. Hence it could be of some interest the following result.

We recall that a set of points $\{P_1,\ldots,P_s\}$ in $\mathbb{P}_k^n$ is said to be in "general position" if no subset of $n+1$ points lies on an hyperplane. For example, it is well known that, if V is a reduced irreducible nondegenerate variety in $\mathbb{P}_k^m$ of dimension d and degree s and L is a generic linear subspace of dimension $n = m-d$ in $\mathbb{P}_k^m$, then the section $V \cap L$ consists of s points in general position in $L \cong \mathbb{P}_k^n$.

<u>Proposition</u> Let $\{P_1,\ldots,P_s\}$ in $\mathbb{P}_k^n$ be a set of points in generic and general position. If $n+1 < s \leq \binom{n+2}{2}$, then $\tau(A) = s-n-1$.

<u>Proof</u> We have $H_A(0) = 1$, $H_A(1) = n+1$, $H_A(2) = s$, hence if L is a linear form which is a non zero divisior modula I, we have with $B = A/LA = R/(I,L)$, $H_B(0) = 1$, $H_B(1) = n$, $H_B(2) = s-n-1$. Thus, we must show that the socle of B is concentrated in degree 2.

We may choose homogeneous coordinates in $\mathbb{P}^n$ so that

$$P_0 := (1,0,\ldots,0)\ P_1 := (0,1,0,\ldots,0),\ldots,\ P_n := (0,0,\ldots,1)\ .$$

Let $F \in R_1$ and $FR_1 \subseteq (I,L)$, then we can write for $i = 0,\dots,n$,

$$X_i F = G_i + H_i L ,$$

where $G_i \in I$, $H_i \in R_1$. This gives us a system of homogeneous equations

$$\delta_{ij} F(P_j) = L(P_j) H_i(P_j), \ i = 0,\dots,n \quad j = 0,\dots,n .$$

Now it is clear that $H_i = \sum_{j=0}^{n} H_i(P_j) X_j = \alpha_i X_i$ if we let $\alpha_i := F(P_i)/L(P_i)$ for all i.

It follows that

$$X_i(F-\alpha_i L) \in I \quad \text{for} \quad i = 0,\dots,n .$$

Since the points are in general position and $s > n+1$, we can find a new point, say Q, such that $Q := (\beta_0,\dots,\beta_n)$ with $\beta_i \neq 0$ for all i. Hence $F(Q) = \alpha_i L(Q)$ for all $i = 0,\dots,n$. This implies $\alpha_i L(Q) = \alpha_j L(Q)$, hence $\alpha_i = \alpha_j$ for all i and j. Thus $F - \alpha_0 L \in I : R_1 = I$, and $F \in (I,L)$ as wanted.

We remark that in the above proposition we cannot delete any of the assumptions.

For example, if $P_0 = (1,0,0)$, $P_1 = (0,1,0)$, $P_2 = (0,0,1)$ and $P_3 = (1,1,0)$, then the Hilbert function of A is

$H_A(0) = 1$, $H_A(1) = 3$, $H_A(2) = 4$ hence the points are in generic position in $\mathbb{P}^2$. But if $\partial(A) = 1$ then this set of points should be the complete intersection of two conics, a contradiction to the fact that P_0, P_1 and P_3 are on the same line.

On the other hand, if we take 6 points on an irreducible conic in $\mathbb{P}^2$, then this set of points is in general position. Since they are the complete intersection of the given conic with a cubic, we have $\partial(A) = 1$ while $s-n-1 = 3$.

Finally, in [G-M] an example is given of 12 points in $\mathbb{P}^2$ which are in generic and uniform position but which do not verify the expected value for the type (see [G-M], ex. 4.1).

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