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A DICHOTOMY FOR MINIMAL SETS OF FIBRE-PRESERVING MAPS IN GRAPH BUNDLES

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Abstract. Minimal sets are studied for a compact dynamical system given by a fibre-preserving continuous map F in a graph bundle E (i.e., F is a skew product map). Given a closed set M in E we define the set of so-called star-like interior points of M. A point in M is called an end-point of M if it is not star-like interior. It is assumed that the base system is minimal (this assumption does not restrict generality when we are interested in the topological structure of minimal sets).

The main theorem says that if M is a minimal set of F then the following dichotomy holds: The set of end-points of M is either dense in M (and then M is nowhere dense in E), or it is empty (and then M has nonempty interior in E). Further, it is proved that in the latter case $F|_M$ is monotone on some sub-arcs of the fibres. Since minimal sets of fibre-preserving maps in tree bundles necessarily have end-points, this theorem implies that these sets are nowhere dense, which in particular solves a problem of J. Smítal on minimal sets of triangular maps in the square.

1. Introduction

1.1. **Problem of the classification of minimal sets.** A pair (X, f) where X is a (usually Hausdorff or even metrizable) topological space and f is a *continuous* map $X \to X$ can be viewed as a *dynamical system* in which the *orbit* of a point $x \in X$ is defined to be the set $\{x, f(x), f^2(x), \dots\}$. So, by an orbit we mean a forward orbit rather than a full orbit, even if f is a homeomorphism. A system (X, f) is called *minimal* if there is no proper subset $M \subseteq X$ which is nonempty, closed and f-invariant (i.e., $f(M) \subseteq M$). In such a case we also say that the map f itself is minimal. Clearly, a system (X, f) is minimal if and only if the orbit of every point $x \in X$ is dense in X. Another equivalent definition is that for every point $x \in X$, its *ω-limit set* $\omega_f(x) := \bigcap_{m\geq 0} \overline{\bigcup_{n\geq m}\{f^n(x)\}}$ equals the whole space X.

The classification, i.e. the full topological characterization, of (at least compact metrizable) spaces admitting minimal maps is a well-known open problem in topological dynamics (see, e.g., [9], the entry 'Minimal set' written by D. V. Anosov and 'Expert Comments' to it). For the state of the art of the problem see [1], [3], [4] and [19].

The basic and well known fact due to G. D. Birkhoff is that any compact dynamical system (X, f) has minimal (closed) subsystems $(M, f|_M)$. Such closed sets M are called *minimal sets* of f or, more precisely, of (X, f). Thus, though a (noncompact) dynamical system need not have any minimal sets, these in fact appear very often since if an orbit has compact closure then this closure contains at least one minimal set of f. The minimal sets, as 'irreducible' parts of a system, attract much attention and the question of their topological structure is central in topological dynamics.

It seems that Y. N. Dowker [7] and M. L. Cartwright [5] were the first who studied the *topological structure of minimal sets* (of homeomorphisms). Since then it is a topic of constant interest. It is folklore that if X is a compact *zero-dimensional* space, $f : X \to X$ is continuous and $M \subseteq X$ is a minimal set of f then M is either a finite set (a periodic orbit of f) or a Cantor set and this is in

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fact a characterization because also conversely, whenever $M \subseteq X$ is a finite or a Cantor set then there is a continuous map $f : X \to X$ such that M is a minimal set of f. Among *one-dimensional* spaces, the characterization of minimal sets is known for *graphs* — minimal sets on graphs are characterized as finite sets, Cantor sets and unions of finitely many pairwise disjoint simple closed curves, see [2] or [23]. On *dendrites* and on *local dendrites* the problem is very difficult and the full characterization of minimal sets has been found just recently, see [1].

In *higher dimensions* the topological structure of minimal sets is much more complicated and, besides some important examples, only few results are known. However, for some classes of maps which are special from the dynamical or topological point of view the structure of minimal sets can be partially described regardless of the phase space of the system. One result of this kind is that if a dynamical system (X, f) is *topologically transitive* then every minimal set of f is either nowhere dense or it is the whole space X. Another simple result concerns homeomorphisms. If (X, h) is a dynamical system and h is a *homeomorphism* then the boundary of a minimal set M is h-invariant (and closed), hence is equal to the set M or is empty. Thus, a minimal set of a homeomorphism either has empty interior (i.e., it is nowhere dense in X) or it is a clopen subset of X. Consequently, if X is connected, then the homeomorphism h has only nowhere dense minimal sets, with one possible exception when the whole space X is minimal for h.

Concerning minimal sets of (not necessarily invertible) continuous maps on *manifolds* we know, due to [21], that if $\mathcal{M}^2$ is a compact connected 2-dimensional manifold, with or without boundary, $f : \mathcal{M}^2 \to \mathcal{M}^2$ is a continuous map and $M \subseteq \mathcal{M}^2$ is a minimal set of the dynamical system $(\mathcal{M}^2, f)$ then either $M = \mathcal{M}^2$ or M is a nowhere dense subset of $\mathcal{M}^2$. Moreover, by [3], the former case is possible only if $\mathcal{M}^2$ is a torus or a Klein bottle. To find a full topological characterization of minimal sets on compact, connected 2-manifolds is a very difficult task. Of course, some examples of 'strange' minimal sets of continuous maps on 2-manifolds are scattered in the literature (e.g., the *Sierpiński curve* on the 2-torus, see [4], or the *pseudocircle*, see [16]). One can also think of embedding known one-dimensional minimal systems into a 2-manifold. But all this is far from giving a characterization of minimal sets. In dimensions higher than 2 the tori and we know from [11] that also the odd-dimensional spheres admit minimal diffeomorphisms. However, it is an open problem whether, for $n > 2$, on compact connected n-dimensional manifolds *proper* minimal sets with nonempty interior exist.

1.2. **Motivation for the study of minimal sets of fibre-preserving maps.** Quite often, an important example of a dynamical system is obtained as an extension of another one. Recall that a dynamical system (E, F) is called an *extension* of a *base* dynamical system (B, f) if there is a continuous surjective map $\pi : E \to B$ such that $\pi \circ F = f \circ \pi$. We also say that the base (B, f) is a *factor* of (E, F). A special case is when E is a cartesian product, $E = B \times Y$, and $F(x, y) = (f(x), g(x, y))$; then the map F is called a *skew product map* or sometimes a *triangular map*.[1] A factor of a minimal system is minimal, so one needs to start with a minimal base system (B, f) if an extension (E, F) of it should have a chance to be also minimal.

So called *Floyd-Auslander minimal systems* (see [15]) are homeomorphisms which are extensions of Cantor minimal homeomorphisms and their phase spaces are subsets of the unit square which are nonhomogeneous — some fibres are compact intervals while the others are singletons. Modifying the construction, one can obtain also a noninvertible nonhomogeneous system of this kind, see [27]. Note that, by the extension lemma from [18], all these systems can be embedded into systems given by triangular selfmaps of the square. So, not only general continuous maps in the square but even triangular selfmaps of the square admit many nonhomogeneous minimal sets. This is perhaps one of the possible reasons why the problem of characterizing minimal sets of higher dimensional

[1]The latter name is usually used if E is a square, see e.g. [17], or at least if Y is a real interval. The former name is sometimes used even in a more general setting when F is a fibre-preserving map in a fibre space, see below for a definition; in fact a skew product is in topology an outdated name for a fibre space.

continuous maps, even skew product maps, is still open. The aim of the present paper is to shed some light on this problem by studying minimal sets of continuous fibre-preserving maps in graph bundles. (It does not seem easy to generalize the results to more general bundles.)

1.3. **Terminology – star-like interior points and end-points in graph bundles.** First recall some terminology. A *fibre space* is an object (E, B, p) where E and B are topological spaces and $p : E \to B$ is a continuous surjection. Here E, B and p are called the total space, the base (space) and the projection (map) of the fibre space, respectively, and $p^{-1}(b)$ is called the fibre over the point $b \in B$. If Γ is another topological space, the fibre space (E, B, p) is called a *fibre bundle* (or a locally trivial fibre space) with fibre Γ, and denoted by (E, B, p, Γ), if the projection map $p : E \to B$ satisfies the following condition of local triviality: For every point $b \in B$ there is an open neighbourhood U of b (which will be called a *trivializing neighbourhood*) and a homeomorphism $h : p^{-1}(U) \to U \times \Gamma$ such that on $p^{-1}(U)$ it holds $\mathrm{pr}_1 \circ h = p$. Here $\mathrm{pr}_1 : U \times \Gamma \to U$ is the canonical projection onto the first factor. Unlike the general fibre spaces, in the fibre bundle (E, B, p, Γ) it holds that for any $b \in B$ the fibre over b, $p^{-1}(b)$, is homeomorphic to the same space Γ. We will always assume that **both E and B are compact metric spaces** (not necessarily connected) and so we will speak on compact fibre spaces (bundles).

Of course, a special case of the described fibre bundle is the *cartesian product* $E = B \times \Gamma$. Note that besides "product" graph bundles there are "twisted" graph bundles such as the Möbius band (interval bundle over a circle), the Klein bottle (circle bundle over a circle) or the 3-dimensional sphere $\mathbb{S}^3$ (an $\mathbb{S}^1$-bundle over $\mathbb{S}^2$; so-called Hopf fibration).

Given a fibre space (E, B, p), consider two dynamical systems (E, F) and (B, f) such that $p \circ F = f \circ p$. Thus, (E, F) is an extension of (B, f) and (B, f) is a factor of (E, F), the projection map p being the factor map. For every $b \in B$ we have $F(p^{-1}(b)) \subseteq p^{-1}(f(b))$, i.e., F sends the fibre over b into the fibre over $f(b)$. Therefore F is said to be *fibre-preserving*.

From now on, a *graph* is a (nonempty) compact metric space which can be written as the union of finitely many arcs any two of which are either disjoint or intersect only in one or both of their end-points. Note that a graph need not be connected and that a singleton is not a graph. By a *circle* we will mean a *simple closed curve*, i.e., a space homeomorphic to the unit circle $\mathbb{S}^1$. A graph contains only finitely many circles. A *tree* is a graph containing no circle. The number of arcs which emanate from a point x of a graph G is called the *order* of the point and is denoted by $\mathrm{ord}(x, G)$. Points of order 1 are called *end-points* of G and points of order at least 3 are called *ramification points* of G.

For $n \geq 1$ we will consider the notion of the *n-star* S_n, which can be described as the set of all complex numbers z such that z^n is in the real unit interval $[0, 1]$, i.e., a central point (the origin) with n copies of the interval $[0, 1]$ attached to it. We will view the n-star as a tree with $n+1$ vertices, one of them (the central point) having order n and the other n vertices (the end-points of S_n) having order 1. Since we will often be in a situation when we can identify homeomorphic graphs, any set homeomorphic to S_n will also be called an n-star and also denoted by S_n. So, S_1 and S_2 are homeomorphic to a closed interval, S_3 and S_4 are homeomorphic to the letter Y and to the letter X, respectively. By the/an *open n-star* Σ_n we will mean S_n without its n end-points. In particular, Σ_2 is homeomorphic to an open interval (while Σ_1 to a half-closed interval).

Let Γ be a graph and $Z \subseteq \Gamma$ be closed. A point $x \in Z$ is said to be a *star-like interior point of Z* if there exists a Z-open neighbourhood of x (i.e., the intersection of Z and a Γ-open neighbourhood of x) which is homeomorphic to Σ_k for $k \geq 2$. Figure 1 shows that a star-like interior point of Z need not be an interior point of Z in the topology of Γ and an interior point of Z need not be a star-like interior point of Z.

If $x \in Z$ is not a star-like interior point of Z we say that it is an *end-point of Z*. Let $\mathrm{Sint}(Z)$ and $\mathrm{End}(Z)$ denote the set of all star-like interior points of Z and the set of all end points of Z, respectively. The set $\mathrm{Sint}(Z)$ is open in Z (but not necessarily in Γ) and so the set $\mathrm{End}(Z)$ is closed

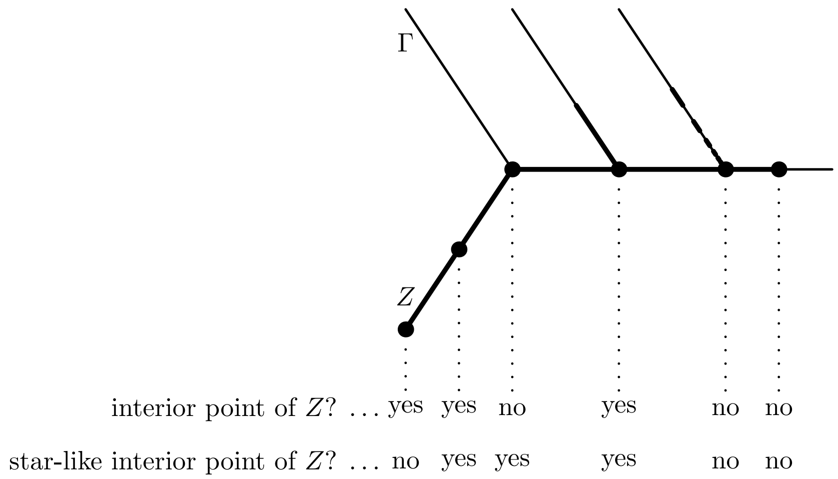

FIGURE 1. There is no connection between interior and star-like interior points.

in Z (hence closed in Γ). If Z is a subgraph of Γ, the set $\mathrm{End}(Z)$ coincides with the usual set of end-points of the graph Z and so no confusion with the graph terminology should arise.

A *graph bundle* is a fibre bundle whose fibre Γ is a graph. Given a graph bundle (E, B, p, Γ), for $M \subseteq E$ and $b \in B$ we denote $M_b = M \cap p^{-1}(b)$; this set is said to be the *fibre of* M *over* b. If $M \subseteq E$ and $U \subseteq B$, we denote $M_U = M \cap p^{-1}(U)$.

Given a closed set M in a compact graph bundle (E, B, p, Γ) we define the set of *star-like interior points of* M and the set of *end-points of* M by

$$\mathrm{Sint}(M) = \bigcup_{b \in B} \mathrm{Sint}(M_b) \quad \text{and} \quad \mathrm{End}(M) = \bigcup_{b \in B} \mathrm{End}(M_b) \ ,$$

respectively. Of course, $\mathrm{End}(M) = M \setminus \mathrm{Sint}(M)$.

This terminology is sufficient for the statement of our main result (though not yet for our proof of it).

1.4. **Main results.** Our main result is the following theorem.

Theorem A. *Let (E, B, p, Γ) be a compact graph bundle, (E, F) and (B, f) dynamical systems with $p \circ F = f \circ p$. Suppose that the base system (B, f) is minimal. Let $M \subseteq E$ be a minimal set of the system (E, F). Then $p(M) = B$ and one of the following holds:*

(i) *either $\overline{\mathrm{End}(M)} = M$ (and then M is nowhere dense in E), or*
(ii) *$\mathrm{End}(M) = \emptyset$ (and then M has nonempty interior in E).*

In particular, the fibre preserving maps in tree bundles have only nowhere dense minimal sets.

The last claim is obvious — if Γ is a tree then, since $p(M) = B$, each of the sets $\mathrm{End}(M_b)$ is nonempty and so we are in the case (i).

The assumption that the base system (B, f) is minimal is not restrictive. In fact, if M is a minimal set of (E, F) then its projection $p(M)$ is obviously a minimal set of (B, f) and so one can pass to the sub-bundle over $p(M)$ and to consider, instead of (E, F), the system $(E^*, F|_{E^*})$ where $E^* = p^{-1}(p(M))$. As a simple application of this fact we get that though a minimal set of a triangular, i.e. skew product, map in the square can contain a vertical interval (so that in general $\mathrm{End}(M) \neq M$ in the case (i), see an example in Subsection 1.2), the following corollary holds (I denotes a real compact interval and pr_1 is the projection onto the first coordinate).

Corollary B. *Let $F(x, y) = (f(x), g(x, y))$ be a continuous triangular map in the square I^2 and let M be a minimal set of F. Then M is nowhere dense in the space $\mathrm{pr}_1(M) \times I$.*

It follows from the characterization of minimal sets on the interval that here $\mathrm{pr}_1(M)$ is either a finite set or a Cantor set. In the latter case the result in the corollary is nontrivial, it strengthens Theorem 1 from [13] (where the same result is obtained for a very particular and small subclass of the class of triangular selfmaps of the square) and answers in negative the question posed by J. Smítal whether a minimal set M of a triangular map in the square can have nonempty interior in the space $\mathrm{pr}_1(M) \times I$.

The problem of characterization of minimal sets of fibre-preserving maps in graph bundles (even the particular problem of characterization of minimal sets of triangular maps in the square) is difficult. Our Theorem A is a key result which enables to address it.

The paper is organized as follows. In Section 2 we continue the discussion of Theorem A. Then, in Section 3 we introduce the key notion of our paper, namely that of a strongly star-like interior point of a subset of a graph bundle, and we study the structure of open neighbourhoods of those compact subsets of a fibre which entirely consist of strongly star-like interior points of a given subset of the bundle. The description of such neighbourhoods plays a key role in our proof of Theorem A. The proof itself is given in Section 4, together with Proposition 16 which provides an additional information on the behaviour of a fibre-preserving map in a graph bundle on a minimal set.

2. Remarks, examples and open problems related to Theorem A

2.1. **Remarks, examples and open problems to the case (i) in Theorem A.** The case (i) occurs for instance if Γ is a tree. An example of a minimal set M satisfying (i) can for instance be obtained if we continuously extend a Floyd-Auslander minimal system (M, H) onto the product of the Cantor set (the projection of M) and a compact interval. (Though in this example H is a homeomorphism on M, it is not true in general that if f is a homeomorphism then $F|_M$ is monotone — to see it, replace (M, H) in this construction by a noninvertible modification of it from [27].) Other examples can be obtained in a similar way, by replacing a Floyd-Auslander minimal system by some other cantoroids (for the definition of a cantoroid see [1]). However, even the problem of finding a full topological characterization of minimal sets of triangular maps in the square is still open.

2.2. **Remarks, examples and open problems to the case (ii) in Theorem A.** Note that in the case (ii) the graph Γ contains a circle. To give an example when (ii) in Theorem A holds true, it is sufficient to consider an irrational rotation of the torus (M is the whole torus).

One can easily construct also systems in which $M = B \times K$ where K is a union of any number of pairwise disjoint circles (not greater than the maximal number of such circles in Γ). This is trivial if the base B is a singleton, i.e. the system is just a graph map, see [2]. If B is a circle and $E = B \times \Gamma$, still it is trivial to construct such an example. In fact, let $f : B \to B$ be minimal. Then f is conjugate to an irrational rotation by angle α. Let $K \subseteq \Gamma$ be a disjoint union of m circles. Then one can define a minimal map $g : K \to K$ which cyclically permutes the m circles in K and the restriction of g^m to each of these circles is conjugate to an irrational rotation by angle β such that α/β is irrational. Put $G(x, y) = (f(x), g(y))$. Then G is minimal on $B \times K$ and can be continuously extended to a direct product map on E (since g can be continuously extended to Γ).

To produce examples falling within the case (ii) with B being a general compact metric space admitting a minimal map (not just a point or a circle), one can use Proposition 1 and Corollary 3 below. To prove Proposition 1, let us start by recalling a theorem due to H. Weyl (see e.g. [22, Theorem 4.1]) saying that if $(a_n)_{n=1}^{\infty}$ is a sequence of mutually distinct real numbers then for almost all (with respect to the Lebesgue measure) real numbers x the sequence $(a_n x)_{n=1}^{\infty}$ is uniformly distributed modulo 1. As an obvious consequence of this theorem we get that for any sequence of positive integers $n_1 < n_2 < \dots$ there is an angle α such that the rotation g of $\mathbb{S}^1$ by the angle

α is *minimal with respect to the sequence* $(n_k)_{k=1}^{\infty}$. This means that for every $s \in \mathbb{S}^1$ the set $\{g^{n_k}(s) : k = 1, 2, \dots\}$ is dense in $\mathbb{S}^1$. Of course, any such rotation g is necessarily irrational.

The following simple proposition dealing with direct product maps (rather than with skew product minimal systems as for instance in [14] or [6]) is, though not most general possible, pretty sufficient for our purposes – besides giving examples of minimal systems falling within the case (ii) in the form of a direct product, it will be also used in the construction of a minimal set M in Theorem A, case (ii), which is not a sub-bundle of E, see Example 4 below. We present here a short proof, based on the Weyl's theorem mentioned above.

Proposition 1. *Let (B, f) be a minimal dynamical system, B being a metric space. Then there exists an irrational rotation g of the circle $\mathbb{S}^1$ such that the direct product system $(B \times \mathbb{S}^1, f \times g)$ is minimal.*

By the way, Proposition 1 can be reformulated by saying that for any minimal system there exists an irrational rotation of the circle disjoint with it. The notion of disjointness was introduced in [12].

Proof. Fix $x_0 \in B$ and positive integers $n_1 < n_2 < \dots$ such that $f^{n_k}(x_0) \to x_0$ when $k \to \infty$. By the Weyl's theorem, there is an irrational rotation g of $\mathbb{S}^1$ such that for every $s \in \mathbb{S}^1$ the set $\{g^{n_k}(s) : k = 1, 2, \dots\}$ is dense in $\mathbb{S}^1$. We claim that $F = f \times g$ is minimal. It is sufficient to prove that the ω-limit set $\omega_F(x, s) = B \times \mathbb{S}^1$ for every $(x, s) \in B \times \mathbb{S}^1$.

From the choice of x_0 and g it follows that for every $y \in \mathbb{S}^1$, $\omega_F(x_0, y) \supseteq \{x_0\} \times \mathbb{S}^1$. Since the f-orbit of x_0 is dense in B and $F(\omega_F(x_0, y)) \subseteq \omega_F(x_0, y)$ and g is surjective, the closed set $\omega_F(x_0, y)$ contains the union of a dense family of fibres. We have thus proved that $\omega_F(x_0, y) = B \times \mathbb{S}^1$ for every $y \in \mathbb{S}^1$.

Now fix any point $(x, s) \in B \times \mathbb{S}^1$. Since $\omega_f(x) = B$ and $\mathbb{S}^1$ is compact, the set $\omega_F(x, s)$ contains at least one point $(x_0, y) \in \{x_0\} \times \mathbb{S}^1$. Then $\omega_F(x, s) \supseteq \omega_F(x_0, y) = B \times \mathbb{S}^1$. $\square$

Corollary 2. *Let $E = B \times \Gamma$ be a graph bundle such that B is a compact metric space admitting a minimal map and Γ be a graph containing a circle C. Then there exists a fibre-preserving map $F : E \to E$ such that $B \times C$ is a minimal set of F. (Note that $B \times C$ has nonempty interior in E, so we are in case (ii).)*

Proof. Using Proposition 1 extend a minimal map $f : B \to B$ to a minimal map $f \times g : B \times C \to B \times C$. Then use the fact that there is a retraction $r : \Gamma \to C$ and put $F = f \times (g \circ r)$. $\square$

However, for a general graph bundle (E, B, p, Γ), where B is a compact metric space admitting a minimal map and Γ contains a circle, the existence of examples falling within the case (ii) in Theorem A is not clear at all. For instance, the construction of a fibre-preserving minimal map is not easy already on the Klein bottle (see the construction of such a minimal homeomorphism on the Klein bottle in [8] or in [26]). We do not know whether in any graph bundle which is not a tree bundle and whose base admits a minimal map there exists a fibre-preserving map having a minimal set with nonempty interior.

Corollary 3. *Let (B, f) be a totally minimal dynamical system, B being a metric space. Let Γ be a graph which (possibly properly) contains m disjoint circles. Denote the union of these circles by S. Then there exists a continuous map $h : \Gamma \to \Gamma$ such that $B \times S$ is a minimal set in the direct product system $(B \times \Gamma, f \times h)$.*

Proof. Let g be the irrational rotation by angle α, which can be assigned to the minimal system (B, f^m) by Proposition 1. Fix a circle C in S. Let $\widetilde{g}$ be the map $S \to S$ whose restriction to C is (conjugate to) g and which is identity on $S \setminus C$. Then compose $\widetilde{g}$ with a homeomorphism on S, which cyclically permutes the m circles in S. Finally, extend the continuous selfmap of S obtained in such a way to a continuous selfmap h of Γ (this is always possible, see e.g. [2]). By Proposition 1, the set $B \times C$ is minimal for $(f \times h)^m = f^m \times h^m$ because $h^m|_C$ is (conjugate to) g. It follows that $B \times S$ is minimal for $f \times h$. □

Finally, we are going to construct an example when M in Theorem A, case (ii), is not a subbundle of E. In it, given a set $A \subseteq \mathbb{R}^k$ and a vector $v \in \mathbb{R}^k$, by $A+v$ we mean the set $\{a+v : a \in A\}$. Similarly we define $A - v$.

Example 4. Let (C, f), with C being a subset of the real line, be a Cantor minimal system such that one point has two pre-images and all the other points have only one pre-image each. Besides examples in symbolic dynamics, such systems appear for instance in interval dynamics when a suitable unimodal map is restricted to the ω-limit set of the critical point (see [24], or [20, p. 142] for other references and related examples).

Another way how to see that such a system exists, is as follows. Start with the dyadic adding machine on the Cantor ternary set. Recall that it is often viewed as a restriction of an interval map to the invariant Cantor set, usually a restriction of the map shown for instance in [27, Fig. 1]; notice that then the adding machine is increasing at each point except of the rightmost one where it is decreasing. Choose a point a in this Cantor set which does not belong to the countable set consisting of the endpoints of the contiguous intervals (including the leftmost and the rightmost points of the Cantor set). Hence the points $a_{-j} := f^{-j}(a)$, $j = 1, 2, \dots$ do not belong to this countable set, too. Now blow up the backward orbit of a, i.e., for $j = 1, 2, \dots$, replace the point a_{-j} by a compact interval with length L_{-j} with convergent sum $\sum_{j=1}^{\infty} L_{-j}$ and remove the interior of this interval. This means that the points a_{-j}, $j = 1, 2, \dots$ are "doubled", i.e. replaced by pairs of points $a^-_{-j} < a^+_{-j}$. If we wish, we can do this in such a way that the point a does not change its position on the real line. What we get is again a Cantor set. Consider the dynamics on it which is inherited from the adding machine, except for the "new" points a^-_{-j}, a^+_{-j}, $j = 1, 2, \dots$ where we still need to define the dynamics. To this end, send both a^-_{-1} and a^+_{-1} to a and, since the adding machine is increasing at each a_{-j} and we want a continuous dynamics, for $j = 2, 3, \dots$ send a^-_{-j} to a^-_{-j+1} and a^+_{-j} to a^+_{-j+1}. The map defined in such a way is continuous and the system is minimal (every orbit is dense).

Recall that, up to a homeomorphism, there is only one Cantor set and it is homogeneous. Therefore, no matter which of the Cantor minimal systems (C, f) (such that one point has two pre-images and all the other points have only one pre-image) we choose, we may think of C as a Cantor set on the real line, with the point having two pre-images being for instance the rightmost point of C (though this is not important for us). For the same reason we can also assume that the two-preimages, denote them $c_1 < c_2$, are the endpoints of a contiguous interval (this is important for geometry of our construction below).

Applying Proposition 1 we extend (C, f) to a minimal system $(C \times S_1, f \times g)$ where g is an irrational rotation of the circle $S_1 = \{(y, z) \in \mathbb{R}^2 : y^2 + z^2 = 1\}$. Denote by a_1 and b_1 the g-images of the points $(0, 1)$ and $(0, -1)$, respectively. Let J_1 be one of the half-circles determined by a_1, b_1.

The set C is the union of $C_L = \{x \in C : x \le c_1\}$ and $C_R = \{x \in C : x \ge c_2\}$. Put $C_1 = C_L$ and $C_2 = C_R - (c_2 - c_1)$. Then $C_1 \cup C_2$ is a Cantor set with $C_1 \cap C_2 = \{c_1\}$. Further put $S_2 = S_1 + (0, 3)$, $a_2 = a_1 + (0, 3)$, $b_2 = b_1 + (0, 3)$ and $J_2 = J_1 + (0, 3)$. Finally, denote $M = (C_1 \times S_1) \cup (C_2 \times S_2)$. The dynamical system $(C \times S_1, f \times g)$ induces in a natural way a (minimal) dynamical system (M, F) which is topologically conjugate to $(C \times S_1, f \times g)$ and is obtained from $(C \times S_1, f \times g)$ by just replacing $(C_R \times S_1)$ by its translate $(C_2 \times S_2)$, 'without changing dynamics'. In the new system (M, F) the map F preserves 'vertical' fibres; the fibre over c_1 consists of two circles, each of the other fibres is just a circle. Denote by φ the base map of F. It is clear that (M, F) can be considered as a minimal extension of the dynamical system $(C_1 \cup C_2, \varphi)$ obtained from (C, f) by identifying points c_1 and c_2. Let $\Gamma = S_1 \cup I \cup S_2$ where $I \subseteq \mathbb{R}^2$ is the 'vertical' interval with end-points $(0,1)$ and $(0,2)$. Put $E = (C_1 \cup C_2) \times \Gamma$. Then Γ is a connected graph and E is a graph bundle with fibre Γ.

We claim that the map F can be extended to a continuous fibre-preserving map $G : E \to E$. We are going to define G. Of course, $G|_M = F$. Further, for $x \in C_1 \setminus \{c_1\}$ and $(y,z) \in S_2$ put $G(x,y,z) = F(x,y,z-3)$ and for $x \in C_2 \setminus \{c_1\}$ and $(y,z) \in S_1$ put $G(x,y,z) = F(x,y,z+3)$. So, G is already defined on $C \times (S_1 \cup S_2)$. It remains to define G on $C \times (I \setminus \{(0,1),(0,2)\})$. So, fix $x \in C$. Then $G(\{x\} \times (S_1 \cup S_2)) = \{\varphi(x)\} \times S_i$ for some $i \in \{1,2\}$. Further, $G(x,0,1) = \{\varphi(x)\} \times \{a_i\}$ and $G(x,0,2) = \{\varphi(x)\} \times \{b_i\}$. For $1 < z < 2$ let $G(x,0,z)$ be the point of $\{\varphi(x)\} \times J_i$ such that the length of the sub-arc of $\{\varphi(x)\} \times J_i$ with end-points $\{\varphi(x)\} \times \{a_i\}$ and $G(x,0,z)$ equals $\pi \cdot (z-1)$.

The map G maps E continuously onto M and M is a minimal set for G. Here M_{c_1} is the union of two circles and M_b for $b \neq c_1$ is a circle. So, M is not a sub-bundle of E.

By a slight modification of Example 4 one can obtain for instance an example where M_{c_1} is the 'figure eight' (i.e. the union of two circles which intersect just in one point) or the figure Θ (i.e. the union of two circles whose intersection is an arc) or the figure consisting of two circles having two points in common, and M_b for $b \neq c_1$ is a circle. We will not go into details; in a forthcoming paper we will try to characterize or at least partially describe the topological structure of minimal sets in case (ii).

In Example 4 the base map is noninvertible and $F|_M$ is not monotone. The case (ii) of Theorem A cannot be strengthened by adding that if the base map f is a homeomorphism then $F|_M$ is also a homeomorphism (consider the noninvertible skew product minimal map on the 2-torus from [20], an extension of an irrational rotation of the circle).

2.3. **Theorem A does not give a characterization of minimal sets.** Our Theorem A gives only a necessary condition for a set to be minimal for a fibre-preserving map in a graph bundle. Even if the base B is a singleton (and so E is just Γ), Theorem A does not give a characterization of minimal sets in the graph Γ. We only get that either $\overline{\mathrm{End}(M)} = M$ or $\mathrm{End}(M) = \emptyset$. However, neither of these two conditions is sufficient for a subset M of a graph Γ to be a minimal set of a continuous selfmap of Γ (the characterization of minimal sets on graphs was mentioned in Introduction).

2.4. **On possible generalizations of Theorem A.** The core part of Theorem A is the dichotomy $\overline{\mathrm{End}(M)} = M$ or $\mathrm{End}(M) = \emptyset$. Since the notion of $\mathrm{End}(M)$ can be carried over from the graph bundles to some more general fibre spaces, it is worth of trying to check whether such a dichotomy for minimal sets of fibre-preserving maps can be carried over to such spaces. For instance, one can consider bundles with the fibre Γ being a local dendrite, or at least a local dendrite with a nowhere dense set of ramification points. Or, one can consider fibre spaces whose fibres are graphs but not necessarily homeomorphic ones.

When trying to generalize Theorem A, singleton fibres cause problems and so it is reasonable to consider only fibre spaces with non-degenerate fibres. This is indicated by the fact that, in the 'non-core' part of Theorem A, the implication "$\overline{\mathrm{End}(M)} = M \Rightarrow M$ is nowhere dense in E" does not work for Floyd-Auslander minimal systems mentioned above. Such a system is a fibre space (but not a bundle) with the base being a Cantor set and with all the fibres being arcs or singletons, and the whole space $E = M$ is minimal. So, M is not nowhere dense in E, though $\overline{\mathrm{End}(M)} = M$.

The fibre-preserving maps in tree bundles have only nowhere dense minimal sets. The assumption that the map is fibre-preserving plays a crucial role in the proof. In spite of this fact, one can ask whether this assumption cannot be removed. The answer is negative. In fact, if $\mathbb{S}^1$ is a circle and H is the Hilbert cube then the space $P = \mathbb{S}^1 \times H$ admits a continuous minimal map (in the form of a skew product map with an irrational rotation in the base $\mathbb{S}^1$ and homeomorphisms $H \to H$ as fibre maps, see [14] and [6]). However, P can be written in the form $P = (\mathbb{S}^1 \times H) \times I$ where $I = [0,1]$. Thus we have an interval bundle admitting a minimal (of course not fibre-preserving) map. Still one can ask whether it is true that all minimal (not necessarily fibre-preserving) maps in interval bundles $X \times I$ have only nowhere dense minimal sets if we additionally assume that X has finite dimension. (Recall that, by the result from [21] mentioned in Introduction, this is true if X is a one-dimensional manifold and so $X \times I$ is a 2-manifold with boundary.)

3. Strongly star-like interior points

We introduce the notion of a *strongly* star-like interior point which is more restrictive than that of a star-like interior point of M and, though not appearing in the statement of Theorem A, will play a key role in the proof of it.

First of all recall that, when speaking on a graph bundle, we always assume that it is a (compact) *metric space*, as it was already said in Introduction.

To avoid cumbersome formulations, we will often make no distinction between homeomorphic spaces. If (E, B, p, Γ) is a graph bundle and $Q \subseteq E$ and $Z \subseteq \Gamma$, then we say that Q is *canonically homeomorphic* to $U \times Z$, if $p(Q) = U$ and there is a homeomorphism $h : Q \to U \times Z$ such that on Q we have $\mathrm{pr}_1 \circ h = p$ (here h is said to be a *canonical homeomorphism*). Notice that, in this terminology, in the above definition of the fibre bundle it is required that $p^{-1}(U)$ be canonically homeomorphic to $U \times \Gamma$.

Recall that if (E, B, p, Γ) is a graph bundle and $M \subseteq E$ and $b \in B$, then the the fibre of M over b is $M_b = M \cap p^{-1}(b)$. Further, by Γ_b we will denote the set $p^{-1}(b)$, the fibre over b (now we slightly abuse the already adopted notation M_b, since Γ is not a subset of E). Note that $\Gamma_b = E_b \subseteq E$ is a graph homeomorphic to Γ and if $E = B \times \Gamma$ then $\Gamma_b = \{b\} \times \Gamma$. Also subsets of Γ_b will be sometimes denoted by, say, I_b, T_b, etc. We believe that this will not cause any misunderstanding because always when using notation like X_b it will be clear what kind of a set it is. Recall also that if $M \subseteq E$ and $U \subseteq B$, we denote $M_U = M \cap p^{-1}(U)$.

By an arc we mean a homeomorphic image of a compact real interval. Sometimes we call it a closed arc, since in an obvious way we also use the notions of an open or a half-closed arc. We often make no distinction between a point x and the singleton $\{x\}$. For $N \geq n \geq 2$ let $\Sigma_n \subseteq \Sigma_N$ be two open stars with the same central point. Suppose that Σ_n is the union of some of the half-closed branches of Σ_N (i.e., Σ_n is obtained from Σ_N by removing $N - n \geq 0$ open branches of Σ_N). Then we will say that Σ_n is a *full sub-star of* Σ_N. Here 'full' does not mean that $n = N$; it just refers to the fact that Σ_n consists of 'whole' branches of Σ_N (rather than of just subsets of them) and so it can be $n < N$. Note also that we consider only the case when $N \geq n \geq 2$ (though, formally, such a definition would have a good sense for $N \geq n \geq 1$).

We are now ready to introduce the notion of a strongly star-like interior point of M. For simplicity, first suppose that M is a closed subset of a *product* graph bundle $E = B \times \Gamma$. We are going to define $\mathrm{Sint}_s(M)$, the set of strongly star-like interior points of M. A point $x = (x_1, x_2) \in M$ is said to be a *strongly star-like interior point of* M, if

- x has order $N \geq 2$ in the graph $\Gamma_{x_1} = \{x_1\} \times \Gamma$ (so, $\mathrm{ord}(x_2, \Gamma) = N \geq 2$), and
- there exists an E-open neighbourhood $O \times \Sigma_N$ of x such that x_2 is the central point of Σ_N and the corresponding M-open neighbourhood $\mathcal{G} = M \cap (O \times \Sigma_N)$ of x has the following structure:

  $\mathcal{G}_{x_1} = \{x_1\} \times \Sigma_k$ where $k \geq 2$ and Σ_k is a full sub-star of Σ_N, and for every $z \in p(\mathcal{G}) \subseteq O$ we have $\mathcal{G}_z = \{z\} \times \Sigma_{k(z),z} \subseteq \{z\} \times \Sigma_k$, where $k(z) \in \{2, \dots, k\}$ and $\Sigma_{k(z),z}$ is a full sub-star of Σ_k. (Notice that $\Sigma_{k(x_1),x_1} = \Sigma_k$.) We will say that $\mathcal{G}$ is a *canonical* $\mathrm{Sint}_s(M)$*-neighbourhood of* x (note that, among others, $\mathcal{G} \subseteq \mathrm{Sint}_s(M)$).

The following example illustrates the notion.

Example 5. Let $E = B \times \Gamma$ where $B = [0,1]$ and $\Gamma = ([-1,1] \times \{0\}) \cup (\{0\} \times [0,1])$. Put $A = [0,1] \times [-1,1] \times \{0\}$ and

$$\begin{aligned} M^1 &= A \cup \{(x,0,x) : x \in [0,1]\}, \\ M^2 &= M^1 \cup \{(0,0,z) : z \in [0,1]\}, \\ M^3 &= A \cup \{(0,0,z) : z \in [0,1]\}, \\ M^4 &= A \cup \{(x,0,1-x) : x \in [0,1]\} \ . \end{aligned}$$

Then $(0,0,0) \notin \mathrm{Sint}_s(M^i)$ for $i = 1,2$ and $(0,0,0) \in \mathrm{Sint}_s(M^i)$ for $i = 3,4$.

In the definition we write $\Sigma_{k(z),z}$ rather than $\Sigma_{k(z)}$ because it may happen that $\Sigma_{k(z_1),z_1}$ and $\Sigma_{k(z_2),z_2}$, considered as subgraphs of Γ, are different even when $k(z_1) = k(z_2)$. The following instructive example illustrates this fact.

Example 6. Let $E = B \times S_4$ where $B = [0,1]$. Let $(C_n)_{n=1}^{\infty}$ be a sequence of pairwise disjoint Cantor sets in $(0,1]$ converging, in the Hausdorff metric, to the singleton $\{0\}$. Denote three of the four closed branches of S_4 by J_1, J_2, J_3 and the central point of S_4 by c. Let M be the set with

$$M_x = \begin{cases} \{x\} \times (J_1 \cup J_2 \cup J_3) & \text{if } x = 0 \\ \{x\} \times (J_1 \cup J_2) & \text{if } x \in C_n \text{ for } n \equiv 1 \mod 3, \\ \{x\} \times (J_2 \cup J_3) & \text{if } x \in C_n \text{ for } n \equiv 2 \mod 3, \\ \{x\} \times (J_3 \cup J_1) & \text{if } x \in C_n \text{ for } n \equiv 0 \mod 3, \\ \emptyset & \text{otherwise ,} \end{cases}$$

see Fig.2. Then M is compact and $\{0\} \times \{c\} \in \mathrm{Sint}_s(M)$. In fact all the points of M except of the end-points of the stars M_x, $x \in p(M)$, belong to $\mathrm{Sint}_s(M)$.

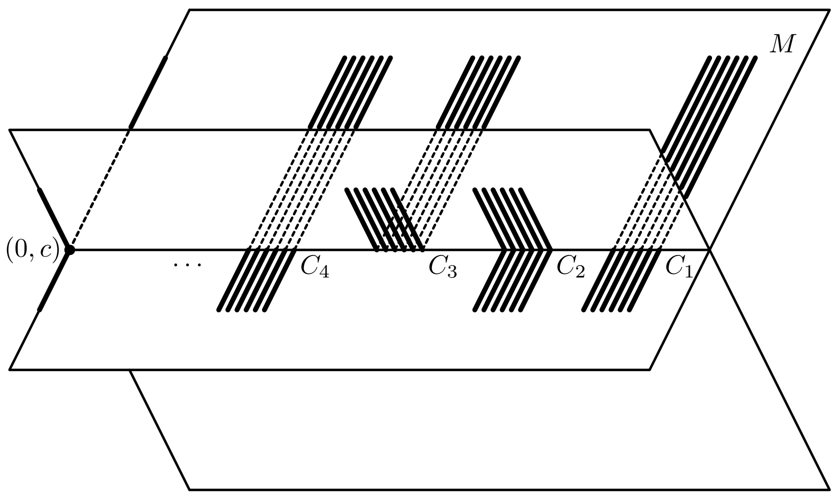

FIGURE 2. $(0;c)$ is a strongly star-like interior point of M.

Above, $\mathrm{Sint}_s(M)$ was defined for a closed subset M of $E = B \times \Gamma$. However, since each graph bundle is locally trivial and the above definition has a local character, the concept of $\mathrm{Sint}_s(M)$ has an obvious extension to the case when the graph bundle E is not a direct product space. For a closed set M in an arbitrary graph bundle we set $\mathrm{End}_s(M) = M \setminus \mathrm{Sint}_s(M)$. Again, it is easy to check that $\mathrm{Sint}_s(M)$ is open in M (but not necessarily in E) and $\mathrm{End}_s(M)$ is closed in M (hence closed in E). Observe that

$$\mathrm{Sint}_s(M) \subseteq \mathrm{Sint}(M) = \bigcup_{b\in B} \mathrm{Sint}(M_b) \qquad \text{and} \qquad \mathrm{End}_s(M) \supseteq \mathrm{End}(M) = \bigcup_{b\in B} \mathrm{End}(M_b). \tag{3.1}$$

In general neither of these two inclusions is an equality. For $M \subseteq E$ and $b \in B$ we will further use the notation

$$M_b^{S_s} = M_b \cap \mathrm{Sint}_s(M) = (\mathrm{Sint}_s(M))_b \ .$$

Example 7. Let $E = B \times \Gamma$ with $B = [-1, 1]$ and $\Gamma = [0, 3]$. Let C be a Cantor set with $\min C = 0$, $\max C = 1$ and let $M = ([-1, 0] \times [0, 1]) \cup (C \times [1, 2]) \cup (\{0\} \times [2, 3])$. Then $\mathrm{Sint}_s(M) = ([-1, 0] \times (0, 1)) \cup (C \times (1, 2)) \cup (\{0\} \times (2, 3))$. So, $M_0^{S_s} = \{0\} \times ((0, 1) \cup (1, 2) \cup (2, 3))$ while $\mathrm{Sint}(M_0) = \{0\} \times (0, 3)$.

Lemma 8. *Let (E, B, p, Γ) be a compact graph bundle and $M \subseteq E$ a compact set. Then*

$$\mathrm{End}_s M = \overline{\mathrm{End}\, M}.$$

Proof. Without loss of generality we may assume that $E = B \times \Gamma$. One inclusion is trivial by (3.1). To prove the other one, suppose that there is a point $x \in \mathrm{End}_s(M) \setminus \overline{\mathrm{End}\, M}$. Then, if the second coordinate of x has order m in Γ, we have $m \geq 2$ (otherwise x would be in $\mathrm{End}(M)$) and some E-open neighbourhood $O \times \Sigma_m$ of x is disjoint with $\mathrm{End}(M)$. Hence, if $z \in O$ then the set $(\{z\} \times \Sigma_m) \cap M$ is empty or is of the form $\{z\} \times \Sigma_{k(z),z}$ where $k(z) \in \{2, \dots, m\}$ and $\Sigma_{k(z),z}$ is a full sub-star of Σ_m (otherwise it would necessarily contain a point from $\mathrm{End}(M_z)$). It follows that $x \in \mathrm{Sint}_s(M)$, a contradiction. □

Lemma 9. *Let (E, B, p, Γ) be a compact graph bundle and $M \subseteq E$ a compact set. If $\mathrm{End}_s(M) = M$ then M is nowhere dense in E.*

Proof. If M is somewhere dense in E then, being closed, has nonempty interior in E. It is clear that this interior contains a point which belongs to $\mathrm{Sint}_s(M)$. □

Lemma 10. *Let (E, B, p, Γ) be a compact graph bundle and $M \subseteq E$ a compact set with $p(M) = B$. If $\mathrm{End}(M) = \emptyset$ then M has nonempty interior in E.*

Proof. Without loss of generality we may assume that $E = B \times \Gamma$ (in fact, in what follows it would be sufficient to replace B by the closure of one trivializing neighbourhood). Let $K_1, K_2, \dots, K_k$ be the (finite) list of circles in Γ. For $i = 1, 2, \dots, k$, let $B^{(i)}$ be the set of points $b \in B$ such that M_b contains $\{b\} \times K_i$. The set M is closed and so all the sets $B^{(i)}$ are closed. Since $p(M) = B$ and $\mathrm{End}(M) = \emptyset$ we have $B = \bigcup_{i=1}^k B^{(i)}$ and since the metric space B is compact (hence second category), there is $j \in \{1, 2, \dots, k\}$ such that the (closed) set $B^{(j)}$ has nonempty interior. Since Γ is a graph, it follows that M has nonempty interior in E. □

Trivial examples show that the converse statements to the previous two lemmas are not true.

Lemma 11. *Let $E = B \times \Gamma$ be a compact graph bundle, $M \subseteq E$ a compact set and $a \in B$. Suppose that $\Delta = \{a\} \times \Delta^\Gamma$ is a compact subset of $M_a^{S_s}$. If W is a sufficiently small open neighbourhood of a and U is a sufficiently small open neighbourhood of Δ^Γ then the E-open neighbourhood $W \times U$ of Δ has the following properties:*

- *The corresponding M-open neighbourhood $\mathcal{D} = M \cap (W \times U)$ of Δ is a subset of $\mathrm{Sint}_s(M)$.*
- *If we write $\mathcal{D}_z = \{z\} \times \mathcal{D}_z^\Gamma$, then $\mathcal{D}_z^\Gamma \subseteq \mathcal{D}_a^\Gamma$ and $\overline{\mathcal{D}_z^\Gamma} \setminus \mathcal{D}_z^\Gamma \subseteq \overline{\mathcal{D}_a^\Gamma} \setminus \mathcal{D}_a^\Gamma$ whenever $z \in p(\mathcal{D})$.*
- *The set $p(\mathcal{D})$ is closed in W (not necessarily closed in B), hence it is a Baire space.*

Proof. Since Δ is compact, it can be covered by a finite family of M-open sets $\mathcal{G}^j = M \cap (O_j \times \Sigma_{N(j)})$, $j = 1, \dots, r$, where $\mathcal{G}^j$ are some canonical $\mathrm{Sint}_s(M)$-neighbourhoods of points in Δ. Put $W = \bigcap_{j=1}^r O_j$ and $U = \bigcup_{j=1}^r \Sigma_{N(j)}$. We prove that $\mathcal{D} = M \cap (W \times U)$ satisfies all the requirements. First, it is obvious that $\mathcal{D}$ is an M-open neighbourhood of Δ and $\mathcal{D} \subseteq \mathrm{Sint}_s(M)$. Further notice that if we denote, for $j = 1, \dots, r$,

$$M^j = M \cap (W \times \Sigma_{N(j)})$$

then $a \in p(M^j)$, $(M^j)_a = \{a\} \times \Sigma^j$ where Σ^j is a full sub-star of $\Sigma_{N(j)}$ and, for $z \in p(M^j)$, $(M^j)_z = \{z\} \times \Sigma_{j(z),z}$ where $\Sigma_{j(z),z}$ is a full sub-star of $\Sigma^j = \Sigma_{j(a),a}$. Thus

$$\mathcal{D} = M \cap (W \times U) = \bigcup_{j=1}^r M^j = \bigcup_{j=1}^r \bigcup_{z \in p(M^j)} (\{z\} \times \Sigma_{j(z),z}) .$$

Fix $z \in p(\mathcal{D}) = \bigcup_{j=1}^r p(M^j)$. Since

$$\mathcal{D}_z^\Gamma = \bigcup_{j=1}^r \Sigma_{j(z),z}, \quad \text{in particular} \quad \mathcal{D}_a^\Gamma = \bigcup_{j=1}^r \Sigma^j,$$

we get $\mathcal{D}_z^\Gamma \subseteq \mathcal{D}_a^\Gamma$. Hence $\overline{\mathcal{D}_z^\Gamma} \subseteq \overline{\mathcal{D}_a^\Gamma}$ and so, to prove that $\overline{\mathcal{D}_z^\Gamma} \setminus \mathcal{D}_z^\Gamma \subseteq \overline{\mathcal{D}_a^\Gamma} \setminus \mathcal{D}_a^\Gamma$, it is sufficient to show that the assumption that some point $q \in \overline{\mathcal{D}_z^\Gamma} \setminus \mathcal{D}_z^\Gamma$ belongs to $\mathcal{D}_a^\Gamma$, leads to a contradiction. To this end consider such a point q. Since $q \in \mathcal{D}_a^\Gamma$, there is $j \in \{1, \dots, r\}$ such that $q \in \Sigma^j$ and so $q \in U$. On the other hand, $q \in \overline{\mathcal{D}_z^\Gamma}$ and so $(z, q) \in \overline{\mathcal{D}_z} \subseteq \overline{M} = M$. Also, $(z, q) \in W \times U$ because $z \in p(\mathcal{D}) \subseteq W$ and $q \in U$. Thus, $(z, q) \in M \cap (W \times U) = \mathcal{D}$ which implies that $q \in \mathcal{D}_z^\Gamma$, a contradiction.

Now we prove that $p(\mathcal{D})$ is closed in W. It can immediately be seen from the definition that if $\mathcal{G}$ is a canonical $\mathrm{Sint}_s(M)$-neighbourhood of a point $x \in M \subseteq B \times \Gamma$ and for each $z \in p(\mathcal{G})$ we put $\mathcal{G}_z = \{z\} \times \mathcal{G}_z^\Gamma$, then the family $\{\mathcal{G}_z^\Gamma : z \in p(\mathcal{G})\}$ is finite. Since $\mathcal{D}$ was defined using only finitely many such canonical $\mathrm{Sint}_s(M)$-neighbourhoods, we get that also the family $\{\mathcal{D}_z^\Gamma : z \in p(\mathcal{D})\}$ is finite. Therefore, if $p(\mathcal{D}) \ni z_n \to z \in W$, we may (passing to a subsequence if necessary) assume that all sets $\mathcal{D}_{z_n}^\Gamma$ are the same. But then, since M is closed, obviously $\mathcal{D}_z$ is nonempty and so $z \in p(\mathcal{D})$.

So, the set $p(\mathcal{D})$ is closed (hence is of type G_δ) in the metric space W. Since W is open in B, this implies that $p(\mathcal{D})$ is G_δ in the complete (even compact) space B. Thus $p(\mathcal{D})$ is a topologically complete (i.e. completely metrizable) space, hence a Baire space (see, e.g., [25, Theorems 12.1 and 9.1]). □

In the situation from Lemma 11, let $\Delta \subseteq M_a^{S_s}$ be *connected*. Then it is a (closed) graph and obviously there exist $m, n \geq 0$ such that every sufficiently small connected open Γ_a-neighbourhood V of Δ has the following properties:

- V is connected and (see Lemma 11) $M \cap V \subseteq M_a^{S_s}$,
- $V \setminus \Delta$ consists of pairwise disjoint open arcs $\{a\} \times I_1^\Gamma, \dots, \{a\} \times I_m^\Gamma, \{a\} \times J_1^\Gamma, \dots, \{a\} \times J_n^\Gamma$ where the arcs $\{a\} \times I_i^\Gamma$ are subsets of $M_a^{S_s}$ and the arcs $\{a\} \times J_i^\Gamma$ are disjoint from M_a. Each of these arcs is attached to Δ at an end-point of Δ or at a ramification point of Γ_a (an end-point of Δ can simultaneously be a ramification point of Γ_a).

We extend the notion of a ramification point as follows. If G is a (not necessarily closed and not necessarily connected) subset of a graph Γ and $g \in G$, we say that g is a *ramification point of* G if there is a G-open neighbourhood of g which has the form of an open r-star with $r \geq 3$ and with central point g.

By an *open graph* we mean a graph without its end-points if it has any. So, since a graph is a union of finitely many connected graphs, an open graph is a union of finitely many connected open graphs, whose closures are pairwise disjoint. Notice that, by this definition, a graph having no end-points (in particular, a circle) is also an open graph and that a circle with one point removed is not an open graph. If an open graph G is a subset of a graph Γ then G need not be an open set in Γ. Each ramification point of G is a ramification point of Γ but the converse is not true in general. If Γ is a graph and $G \subseteq \Gamma$ is an open graph, by the end-points of G we mean the end-points of the (closed) graph $\overline{G}$. It follows from the definition of strongly star-like interior points that the set $M_a^{S_s}$ is open in the topology of M_a (though not necessarily open in the topology of Γ_a). Its connected components are not necessarily open graphs. For instance, $M_a^{S_s}$ can be a circle with one point removed. In any case, $M_a^{S_s}$ is a subset of Γ_a and so the notion of a ramification point can be applied to it.

In the following lemma we keep the notation from Lemma 11.

Lemma 12. *Let $E = B \times \Gamma$ be a compact graph bundle, $M \subseteq E$ a compact set and $a \in B$. Suppose that $\Delta = \{a\} \times \Delta^\Gamma$ is a compact* connected *subset of $M_a^{S_s}$. Then for any sufficiently small open neighbourhood W of a and any sufficiently small connected open neighbourhood U of Δ^Γ in Lemma 11, the following holds.*

(a) *If Δ is an arc or a circle and does not contain any ramification point of $M_a^{S_s}$, then $\mathcal{D} = W^* \times U^*$, i.e. $\mathcal{D}$ has the structure of a direct product. Here $a \in W^* \subseteq W$ is some not necessarily B-open set. If Δ is a circle then $\{a\} \times U^*$ coincides with Δ and if Δ is an arc then $\{a\} \times U^*$ is an open arc containing Δ (and still containing no ramification point of $M_a^{S_s}$).*

(b) *If Δ is a graph possibly degenerate to a singleton (and possibly containing ramification points of $M_a^{S_s}$, which may or may not be ramification points of Δ), then:*
- *$\big(\{a\} \times (U \setminus \Delta^\Gamma)\big) \cap M \subseteq M_a^{S_s}$ is empty or consists of pairwise disjoint open arcs $\{a\} \times I_1^\Gamma, \dots, \{a\} \times I_m^\Gamma$ ($m \geq 0$ being finite and independent on U, since U is small enough; $m = 0$ means that the described set is empty).*
- *For each $i = 1, \dots, m$ the open arc $\{a\} \times I_i^\Gamma$ is attached to Δ at a point $p_i = (a, p_i^\Gamma)$ which is an end-point of Δ or a ramification point of $M_a^{S_s}$ (an end-point of Δ can simultaneously be a ramification point of $M_a^{S_s}$ and it can be $p_i = p_j$ even if $i \neq j$), and at each of the end-points of Δ there is at least one such open arc attached to it. Here for every i, the closure of $\{a\} \times I_i^\Gamma$ is an arc and any two of the sets $\Delta, \overline{\{a\} \times I_i^\Gamma}$, $i = 1, \dots, m$ are either disjoint or intersect only at one of the 'attaching' points p_i.*
- *$\mathcal{D}_a = \Delta$ or $\mathcal{D}_a = \Delta \cup \bigcup_{i=1}^m (\{a\} \times I_i^\Gamma)$, depending on whether $m = 0$ or $m \geq 1$. So, $\mathcal{D}_a$ is an open graph.*
- *The structure of the corresponding M-open neighbourhood $\mathcal{D} = M \cap (W \times U) \subseteq \mathrm{Sint}_s(M)$ of Δ is such that for any $z \in p(\mathcal{D})$, $\mathcal{D}_z^\Gamma$ is a union of finitely many open graphs whose closures are pairwise disjoint, $\mathcal{D}_z^\Gamma \subseteq \mathcal{D}_a^\Gamma$ and $\mathrm{End}(\overline{\mathcal{D}_z^\Gamma}) \subseteq \mathrm{End}(\overline{\mathcal{D}_a^\Gamma})$.*
- *For any $z \in p(\mathcal{D})$, each of the connected components of $\mathcal{D}_z$ is the union of a (nonempty, closed) possibly degenerate subgraph of $\{z\} \times \Delta^\Gamma$ and some (possibly zero) of the open arcs $\{z\} \times I_i^\Gamma$ with the 'attaching' points (z, p_i^Γ) belonging to $\mathcal{D}_z$. If this subgraph is nondegenerate and does have one or more end-points, then at each of these end-points there is at least* one *of these open arcs attached to it. If the subgraph is a singleton (which may happen even if Δ is nondegenerate) then at least* two *of these open arcs are attached to it.*

In particular, if Δ is a tree, possibly degenerate to a singleton, then:
- *For each $z \in p(\mathcal{D})$, the set $\mathcal{D}_z$ contains (a nonempty closed subgraph of $\{z\} \times \Delta^\Gamma$, possibly disconnected, possibly degenerate to a finite set, and) at least* two *of the open arcs $\{z\} \times I_i^\Gamma$, with the 'attaching' points (z, p_i^Γ) belonging to $\mathcal{D}_z$.*

Of course, if Δ is a singleton, then the last statement of the lemma does not say anything more than the definition of a strongly star-like interior point of M.

Proof. (a) In this case, every point from Δ has an M_a-neighbourhood in the form of an open arc and so, since Δ is a subset of $\mathrm{Sint}_s(M)$, Δ can be covered by a finite family of canonical $\mathrm{Sint}_s(M)$-neighbourhoods of points from Δ which have the form (see the proof of Lemma 11)

$$\mathcal{G}^j = M \cap (O_j \times \Sigma_{N(j)}) = V^j \times \Sigma_2^j \ .$$

Here V^j is a (not necessarily B-open) set containing a and Σ_2^j is an open arc in Γ such that $\{a\} \times \Sigma_2^j \subseteq \mathrm{Sint}_s(M)$ contains no ramification point of $M_a^{S_s}$.

If two open arcs Σ_2^j and Σ_2^i intersect and $z \in \bigcap O_j$ then $z \in V^j$ if and only if $z \in V^i$. This together with the fact that Δ is connected gives that if $z \in \bigcap O_j$ then z belongs to all of the sets V^j whenever it belongs to one of them. Now let U be any sufficiently small connected open neighbourhood of Δ^Γ so that $(\{a\} \times U) \cap M_a \subseteq \{a\} \times \bigcup \Sigma_2^j$. Further, let $W \subseteq \bigcap O_j$ be any open neighbourhood of a. Then the claim holds with $U^* = U \cap \bigcup \Sigma_2^j$ and

$$W^* = \{z \in W : z \in V^j \text{ for some (hence for all) } j\} \ .$$

(b) The first three parts are just consequences of our definitions of $M_a^{S_s}$, ramification points, endpoints and open graphs. The rest follows from Lemma 11 and the remarks above Lemma 12, and also the already proved part (a) is helpful. Note that a key role is played by the fact that $\mathcal{D} \subseteq \mathrm{Sint}_s(M)$. For instance when describing the structure of a connected component of $\mathcal{D}_z$, if the intersection of $\mathcal{D}_z$ with $\{z\} \times \Delta^\Gamma$ is a singleton, then at least two open arcs have to be attached to this singleton, otherwise $\mathcal{D}_z$ could not be a subset of $\mathrm{Sint}_s(M)$. □

Example 13. Consider the same situation as in Example 6. Denoting by Δ an arc in M_0 containing the ramification point c, we see that without assuming that Δ contains no ramification point of $M_a^{S_s}$, in Lemma 12(a) one cannot ensure the existence of $\mathcal{D}$ in the form of a direct product. Further, if Δ does not contain the ramification point c and is a sub-arc of, say, J_1 we can see that one cannot claim that W^* exists in the class of B-open sets.

Example 14. Consider the same situation as in Example 7 and put $\Delta = \{0\} \times \{1/2, 3/2, 5/2\}$. Then $\Delta \subseteq M_0^{S_s}$ and it does not contain any ramification point of $M_0^{S_s}$ (even any ramification point of Γ_0). However, Δ is disconnected and there is no M-open neighbourhood of Δ of the product form $W^* \times U^*$.

4. Proof of Theorem A

A set $G \subseteq X$ is said to be a *redundant open set for a map* $f : X \to X$ if G is nonempty, open and $f(G) \subseteq f(X \setminus G)$ (i.e., its removal from the domain of f does not change the image of f). For a minimal map f there is no such set. We state this simple fact as a lemma, because we will use it repeatedly.

Lemma 15 ([20])**.** *Let X be a compact Hausdorff space and $f : X \to X$ continuous. Suppose that there is a redundant open set for f. Then the system (X, f) is not minimal.*

Recall also that (X, f) is minimal if and only if no proper, closed nonempty subset A of X is such that $f(A) \supseteq A$ (see for instance [3, Lemma 3.10]).

We will use the notation $F_z = F|_{\Gamma_z}$. So, F_z is a map from Γ_z into $\Gamma_{f(z)}$.

The following result partially describes F on its minimal sets (in case (ii) in our Theorem A, since in case (i) no fibre contains an arc lying in $\mathrm{Sint}_s(M)$). Another reason why we prove it, is that its use slightly simplifies arguments in the proof of Theorem A.

Proposition 16. *Let the assumptions of Theorem A be satisfied. Let I_a be a closed arc and T_b be a tree such that $I_a \subseteq M_a^{S_s}$, $T_b \subseteq M_b$ and $F(I_a) \subseteq T_b$. If the interior of I_a does not contain any ramification point of $M_a^{S_s}$ then $F|_{I_a}$ is monotone (hence $F(I_a)$ is an arc or a point).*

The statement in the parentheses is obvious since a monotone image of an arc cannot be a nondegenerate tree. Both cases (i.e., $F(I_a)$ is an arc or a point) occur in the example of a noninvertible fibre preserving minimal map on the torus in [20] (the base is a 'horizontal' circle, the fibres are 'vertical' circles). Since in this example there is a vertical arc mapped by F into a point while the vertical circle containing this arc is mapped onto a circle, the example also shows that the proposition would not be true if T_b were allowed to contain a circle.

Proof. It is sufficient to prove a weaker version of the proposition which is obtained by adding the assumption that neither the end-points of I_a are ramification points of $M_a^{S_s}$. For if one or both end points of I_a are ramification points of $M_a^{S_s}$ then, by applying such a weaker proposition to all sub-arcs J_a of I_a which do not contain end-points of I_a, we get the monotonicity of F on the whole interior of I_a. Since the F-image of this interior is a point or a (not necessarily closed) arc and T_b does not contain a circle, F is obviously monotone on I_a.

So, let I_a contain no ramification point of $M_a^{S_s}$ and suppose, on the contrary, that $F|_{I_a}$ is not monotone. Then there exists $q \in T_b$ such that $(F|_{I_a})^{-1}(q) \subseteq I_a$ is not connected. Take two points u, v in two different connected components of $(F|_{I_a})^{-1}(q)$ and consider the (unique) arc $J_a \subseteq I_a$ with the end-points u, v. From the choice of u, v it follows that there is a point $w \in J_a$ with $F(w) \neq q$. This point w partitions J_a into two nondegenerate closed sub-arcs J_a^1 and J_a^2. The set $F(J_a) = F_a(J_a) \subseteq T_b$ is a nontrivial continuum (hence a tree) and each of the sets $F(J_a^1)$ and $F(J_a^2)$ contains the (unique) arc in T_b having the end-points $F(w)$ and q. It follows that the arc J_a contains two disjoint closed nondegenerate sub-arcs T_a^1, T_a^2 such that $F(T_a^1)$ and $F(T_a^2)$ are closed arcs with $F(T_a^1) \subseteq \operatorname{Int} F(T_a^2)$ (where by $\operatorname{Int} F(T_a^2)$ we mean the arc $F(T_a^2)$ without its end-points).

Now, since we will work only with some neighbourhood of a, without loss of generality we may assume that E has the structure of a product space, i.e. $E = B \times \Gamma$. So I_a has the form $\{a\} \times I$ and similarly $T_a^1 = \{a\} \times T^1$ and $T_a^2 = \{a\} \times T^2$. By Lemma 12 (a), there is an M-open neighbourhood $\mathcal{D}$ of I_a which has the product form $\mathcal{D} = W^* \times U^*$ for some (not necessarily B-open) set $W^* \ni a$ and some open arc U^* containing I.

Since $F_a(\{a\} \times T^1) \subseteq \operatorname{Int} F_a(\{a\} \times T^2)$ and since (by replacing T^1 by a smaller arc if necessary) we may assume that the arc $F_a(\{a\} \times T^1)$ does not contain any ramification point of Γ_b, we have $F_x(\{x\} \times T^1) \subseteq \operatorname{Int} F_x(\{x\} \times T^2)$ also for all x sufficiently close to a. By replacing W^* by its intersection with a small open neighbourhood of a if necessary, we may assume that this is the case for all $x \in W^*$. Then

$$F|_M(W^* \times \operatorname{Int} T^1) \subseteq F|_M(W^* \times T^2) \subseteq F|_M(M \setminus (W^* \times \operatorname{Int} T^1)) .$$

Hence the nonempty M-open set $W^* \times \operatorname{Int} T^1$ is redundant for $F|_M$ which contradicts the minimality of $F|_M$. $\square$

When $M \subseteq E$ and $\beta \in \operatorname{End}(M)$, i.e. $\beta \in \operatorname{End}(M_b)$ where $b = p(\beta)$, then still it can happen that there is an open arc $J \subseteq M_b$ such that $\beta \in J$ (e.g., let Γ_b be a 3-star S_3 with central point β, M_b be the union of a 2-star S_2 with the same central point β and a sequence of points lying in $S_3 \setminus S_2$ and converging to β). However, it holds the following lemma.

Lemma 17. *Let the assumptions of Theorem A be satisfied. Suppose that there exists a point in $\operatorname{End}(M) \setminus F(\operatorname{End}_s(M))$. Then in the same fibre there exists also a point $\beta \in \operatorname{End}(M) \setminus F(\operatorname{End}_s(M))$ such that no open arc containing β exists in M_b, $b = p(\beta)$.*

Proof. Choose any $\beta' \in \operatorname{End}(M) \setminus F(\operatorname{End}_s(M))$ and denote $p(\beta') = b$. Suppose that β' is contained in an open arc $J \subseteq M_b$. Then, since $\beta' \notin \operatorname{Sint}(M_b)$, the point β' is necessarily a ramification point of Γ_b and in one of the small open branches emanating from β' there are both a sequence of points in M_b converging to β' and a sequence of points in $\Gamma_b \setminus M_b$ converging to β'. Then this branch obviously contains also a sequence of points $\beta_n \to \beta'$ such that, for every n, $\beta_n \in \mathrm{End}(M_b)$ and no open arc in M_b contains β_n. Now it is sufficient to put $\beta = \beta_n$ for a sufficiently large n, because $F(\mathrm{End}_s(M))$ is a closed set which does not contain β'. $\square$

We are finally ready to prove our Theorem A.

Theorem A. *Let (E, B, p, Γ) be a compact graph bundle, (E, F) and (B, f) dynamical systems with $p \circ F = f \circ p$. Suppose that the base system (B, f) is minimal. Let $M \subseteq E$ be a minimal set of the system (E, F). Then $p(M) = B$ and one of the following holds:*

(i) *either* $\overline{\mathrm{End}(M)} = M$ *(and then M is nowhere dense in E), or*
(ii) $\mathrm{End}(M) = \emptyset$ *(and then M has nonempty interior in E).*

In particular, the fibre preserving maps in tree bundles have only nowhere dense minimal sets.

Proof. It is clear that $p(M) = B$. Also the last claim is obvious, since if Γ is a tree then $\mathrm{End}(M) \neq \emptyset$ and we are therefore in the case (i). Thus, taking into account Lemmas 8, 9 and 10, it remains to prove the dichotomy: either $\overline{\mathrm{End}(M)} = M$ or $\mathrm{End}(M) = \emptyset$. To this end suppose that $\mathrm{End}\, M \neq \emptyset$. To prove that then $\overline{\mathrm{End}(M)} = M$, it is sufficient to show that every point from $\mathrm{End}(M)$ has an F-pre-image in $\mathrm{End}_s(M)$. In fact, suppose for a moment that we have proved the inclusion $F(\mathrm{End}_s(M)) \supseteq \mathrm{End}(M)$. Then $F(\mathrm{End}_s(M)) \supseteq \overline{\mathrm{End}(M)} = \mathrm{End}_s(M)$ (see Lemma 8). It follows that the nonempty and closed set $\mathrm{End}_s(M)$ is not a proper subset of M (otherwise $(M, F|_M)$ would not be minimal, see the remark after Lemma 15). So, $\mathrm{End}_s(M) = M$ whence by Lemma 8 we get $\overline{\mathrm{End}(M)} = M$.

Thus, to finish the proof, we suppose that there exists a point $\beta \in \mathrm{End}(M) \setminus F(\mathrm{End}_s(M))$ and we want to get a contradiction. If we denote $p(\beta) = b$, by Lemma 17 we can assume that

(4.1) there is no open arc in M_b containing β.

Since $F(M) = M$ and $\beta \notin F(\mathrm{End}_s(M))$, there is a point $\alpha \in \mathrm{Sint}_s(M)$ with $F(\alpha) = \beta$. Denote $p(\alpha) = a$. From now on we will work only with neighbourhoods of Γ_a and Γ_b and so, due to the local triviality of the graph bundle, we may assume that $E = B \times \Gamma$. Let $\mathrm{ord}(\beta, \Gamma_b) = r \geq 1$, i.e. $\beta = (b, \beta^\Gamma)$ where β^Γ is the central point of an open r-star in Γ. Since the set $F(\mathrm{End}_s(M))$ is closed in E and does not contain β, for some B-open neighbourhood O of b and some open r-star Σ_r with the central point β^Γ the open E-neighbourhood $\mathcal{O}^* = O \times \Sigma_r$ and hence also the M-open neighbourhood $\mathcal{O} = \mathcal{O}^* \cap M$ of β are disjoint from $F(\mathrm{End}_s(M))$. In view of (4.1),

(4.2) the connected component of $M_b \cap \mathcal{O}$ containing β is either the singleton β or a (half-closed or closed) arc whose one end-point is β.

For $z \in B$ put $F_z = F|_{\Gamma_z}$. Consider the map $F_a : \Gamma_a \to \Gamma_b$ and choose that connected component Δ of the set $F_a^{-1}(\beta) \cap M$ which contains the point α. Since $\beta \notin F(\mathrm{End}_s(M))$, we have $\Delta \subseteq \mathrm{Sint}_s(M)$. The set Δ is closed, so it is the singleton α or a (nondegenerate closed) connected subgraph of Γ_a containing α. Let Δ^Γ be the counterpart of Δ in Γ, i.e., $\Delta = \{a\} \times \Delta^\Gamma$.

Let W be a B-open neighbourhood of a and U be a connected Γ-open neighbourhood of Δ^Γ, both as small as Lemma 12(b) requires. In what follows, $\mathcal{D} = M \cap (W \times U) \subseteq \mathrm{Sint}_s(M)$, I_i^Γ and $p_i = (a, p_i^\Gamma)$ will have the meaning from this lemma. We will also consider the half-closed arcs $A_i^\Gamma = \{p_i^\Gamma\} \cup I_i^\Gamma$, $i = 1, \ldots, m$. Since $F(\Delta)$ is just the singleton β, we may also assume that W and U are small enough to give

(4.3) $F(\mathcal{D}) \subseteq \mathcal{O}$, hence none of the sets $F(\mathcal{D}_z)$, $z \in W$, contains a circle.

Claim. There is $d \in p(\mathcal{D})$ such that $\mathcal{D}_d^\Gamma$ contains no circle (and each component of $\mathcal{D}_d^\Gamma$ is nondegenerate since $\mathcal{D} \subseteq \mathrm{Sint}_s(M)$ and $\mathcal{D}$ is M-open). Moreover, $m \geq 2$ and $\mathcal{D}_d$ contains at least two different half-closed arcs from the list $\{d\} \times A_i^\Gamma$, $i = 1, \dots, m$.

Prof of Claim. Let $C_1^\Gamma, \dots, C_r^\Gamma$, $r \geq 0$, be the list of all (not necessarily pairwise disjoint) circles in Δ^Γ. If $z \in p(\mathcal{D})$ then, by Lemma 12(b), $\mathcal{D}_z^\Gamma \subseteq \mathcal{D}_a^\Gamma = \Delta^\Gamma \cup \bigcup_{i=1}^m I_i^\Gamma$ and $\mathcal{D}_a^\Gamma$ contains only those circles which are contained in Δ^Γ. So, if D_z^Γ contains a circle, it is necessarily a circle from the list $C_1^\Gamma, \dots, C_r^\Gamma$. Denote

$$K_i = \{z \in p(\mathcal{D}) : \mathcal{D}_z^\Gamma \supseteq C_i^\Gamma\}, \quad i = 1, \dots, r \ .$$

To prove the claim suppose, on the contrary, that for every $z \in p(\mathcal{D})$, $\mathcal{D}_z^\Gamma$ contains a circle. Then $r \geq 1$ and

$$p(\mathcal{D}) = \bigcup_{i=1}^r K_i \ .$$

Each of the sets K_i, $i = 1, \dots, r$, is obviously closed in the set $p(\mathcal{D})$ which is, by Lemma 11, a Baire space. Hence there is $s \in \{1, \dots, r\}$ with

$$\mathrm{Int}_{\,p(\mathcal{D})} K_s \neq \emptyset. \tag{4.4}$$

Now fix an arbitrary $j \in \{1, \dots, r\}$ and an open arc L_j^Γ in C_j^Γ such that the closure of L_j^Γ contains only points of order 2 in Γ (in particular, L_j^Γ has positive distance from the set $\{p_i^\Gamma : i = 1, \dots, m\}$). Observe that then for every $z \in K_j$ the map F_z is, by Proposition 16 (see also (4.3)), monotone on $\{z\} \times L_j^\Gamma$ and so $F_z(\{z\} \times L_j^\Gamma)$ is an open, closed or half-closed arc, possibly degenerate to a point. Since $F_z(\mathcal{D}_z)$ is by (4.3) a tree (which is a uniquely arcwise connected space), we have that $F_z(\{z\} \times (C_j^\Gamma \setminus L_j^\Gamma)) \supseteq F_z(\{z\} \times L_j^\Gamma)$. Hence

$$F(S \times L_j^\Gamma) \subseteq F(M \setminus (S \times L_j^\Gamma)) \quad \text{for any set } S \subseteq K_j \ , \ j \in \{1, \dots, r\} \ . \tag{4.5}$$

Note also that here $S \times L_j^\Gamma \subseteq M$.

Then by (4.5), for $j = s$ and $S = \mathrm{Int}_{p(\mathcal{D})} K_s$ we obtain $F(\mathrm{Int}_{p(\mathcal{D})} K_s \times L_s^\Gamma) \subseteq F(M \setminus (\mathrm{Int}_{p(\mathcal{D})} K_s \times L_s^\Gamma))$. Therefore, since the set $\emptyset \neq \mathrm{Int}_{p(\mathcal{D})} K_s \times L_s^\Gamma \subseteq M$ is obviously open in the topology of M, the set $\mathrm{Int}_{p(\mathcal{D})} K_s \times L_s^\Gamma$ is a redundant open set for $F|_M$, which contradicts the minimality of $F|_M$. We have thus proved that there exists $d \in p(\mathcal{D})$ such that $\mathcal{D}_d^\Gamma$ contains no circle.

Applying now the last assertion of Lemma 12, we find that $\mathcal{D}_d$ contains at least two different half-closed arcs from the list $\{d\} \times A_i^\Gamma$, $i = 1, \dots, m$. Thus $m \geq 2$ which finishes the proof of Claim. ✓

Next, we will replace W by a smaller open neighbourhood of a and U by a smaller connected open neighbourhood of Δ^Γ so that $\mathcal{D}$ have an additional nice property. We are going to show how to do that. (Note also that Claim will still work.)

Recall that, by Claim, $m \geq 2$. The attaching points $p_i = (a, p_i^\Gamma)$, $i = 1, 2 \dots, m$ belong to Δ and so are mapped to the point β. On the other hand, Δ is disjoint with the open arcs $\{a\} \times I_i^\Gamma$. Therefore each of the sets $F(\{a\} \times A_i^\Gamma)$ is a *nondegenerate* connected set in M_b containing β. Taking into account (4.2), we see that each of these sets is in fact a closed or half-closed arc containing β as one of its end-points [2] and $F(\{a\} \times A_i^\Gamma) \subseteq F(\{a\} \times A_j^\Gamma)$ or $F(\{a\} \times A_j^\Gamma) \subseteq F(\{a\} \times A_i^\Gamma)$ whenever $i, j \in \{1, \dots, m\}$. By replacing the half-closed arcs A_i^Γ by shorter ones (i.e., by replacing U by a smaller connected open neighbourhood of Δ^Γ) if necessary, we may assume that each of the half-closed arcs $\{a\} \times A_i^\Gamma$ is monotonically (see (4.3) and Proposition 16) mapped by F onto the *same* half closed arc H with the end-point $\beta \in F(\{a\} \times A_i^\Gamma)$ and another end-point $\beta^* \notin F(\{a\} \times A_i^\Gamma)$.

[2] thus, since $m \geq 2$, the connected component of $M_b \cap \mathcal{O}$ containing β cannot be just the singleton β; see (4.2).

Now fix $k \in \{1, \dots, m\}$ and choose a small open arc $J_k = \{a\} \times J_k^\Gamma$ such that the closure of J_k lies in the interior of $\{a\} \times A_k^\Gamma$ and the closure of $F(J_k)$ lies in the interior of H. Then

$$\text{the closure of } F(\{a\} \times J_k^\Gamma) \text{ lies in the interior of } F(\{a\} \times A_i^\Gamma) \text{ for every } i = 1, 2 \dots, m. \tag{4.6}$$

By continuity, and replacing W by a smaller neighbourhood of a if necessary, we may assume that

$$F(\{z\} \times J_k^\Gamma) \subseteq F(\{z\} \times A_i^\Gamma) \text{ for every } z \in W \text{ and } i = 1, 2 \dots, m.^3 \tag{4.7}$$

Note that this holds (i.e., such a J_k^Γ exists) for any $k \in \{1, \dots, m\}$.

Now we are in the position to finish the proof. By Claim, there exists $d \in p(\mathcal{D})$ such that $\mathcal{D}_d$ does not contain any circle and contains at least two different half-closed arcs, say $\{d\} \times A_1^\Gamma$ and $\{d\} \times A_2^\Gamma$. Both these properties are shared by all the points $z \in p(\mathcal{D})$ sufficiently close to the point d. In fact, M is closed and Γ contains only finitely many circles and so, if $z \in p(\mathcal{D})$ is close to d, neither the set $\mathcal{D}_z$ can contain a circle. But then, using the same argument as for the point d (see the very end of the proof of Claim), the set $\mathcal{D}_z$ also contains at least *two* of the half-closed arcs $\{z\} \times A_i^\Gamma$. It follows that for any $z \in p(\mathcal{D})$ close to d there is at least one $i \neq 1$ such that $\{z\} \times A_i^\Gamma \subseteq M$ and so, regardless of whether $\{z\} \times J_1^\Gamma \subseteq \{z\} \times A_1^\Gamma$ is a subset of M or is disjoint from M, the condition (4.7) applied to $k = 1$ gives $F(M_z \setminus (\{z\} \times J_1^\Gamma)) \supseteq F(M_z \cap (\{z\} \times J_1^\Gamma))$. Hence, for sufficiently small neighbourhood $W_1 \subset W$ of d we have $F(M \setminus (W_1 \times J_1^\Gamma)) \supseteq F(M \cap (W_1 \times J_1^\Gamma))$ and so the nonempty M-open set $M \cap (W_1 \times J_1^\Gamma)$ is redundant for $F|_M$, a contradiction with minimality of $F|_M$. □

References

[1] F. Balibrea, T. Downarowicz, R. Hric, Ľ. Snoha, V. Špitalský, *Almost totally disconnected minimal systems*, Ergodic Theory Dynam.Systems, **29** (2009), no. 3, 737–766.

[2] F. Balibrea, R. Hric, Ľ. Snoha, *Minimal sets on graphs and dendrites.* Dynamical systems and functional equations (Murcia, 2000), Internat. J. Bifur. Chaos Appl. Sci. Engrg., **13** (2003), no. 7, 1721–1725.

[3] A. Blokh, L. Oversteegen, E. D. Tymchatyn, *On minimal maps of 2-manifolds*, Ergodic Theory Dynam. Systems **25** (2005), no. 1, 41–57.

[4] H. Bruin, S. Kolyada, Ľ. Snoha, *Minimal nonhomogeneous continua*, Colloq. Math. **95** (2003), no. 1, 123–132.

[5] M. L. Cartwright, *Equicontinuous mappings of plane minimal sets*, Proc. London Math. Soc. **(3)14a** (1965), 51–54.

[6] M. Dirbák, P. Maličký, *On the construction of non-invertible minimal skew products*, J. Math. Anal. Appl. **375** (2011), 436–442.

[7] Y. N. Dowker, *On minimal sets in dynamical systems*, Quart. J. Math. Oxford Ser. **(2)7** (1956), 5–16.

[8] R. Ellis, *The construction of minimal discrete flows*, Amer. J. Math. **87** (1965), 564–574.

[9] *Encyclopaedia of mathematics. Vol. 6.* Lobachevskii criterion (for convergence)–Optional sigma-algebra. Translated from the Russian. Translation edited by M. Hazewinkel. Kluwer Academic Publishers, Dordrecht, 1990.

[10] R. Engelking, *General topology*, Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.

[11] A. Fathi and M. R. Herman, *Existence de difféomorphismes minimaux*, Astérisque **49** (1977), 37–59.

[12] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Systems Theory 1 (1967), 1–49.

[13] G. L. Forti, L. Paganoni, J. Smítal, *Dynamics of homeomorphisms on minimal sets generated by triangular mappings*, Bull. Austral. Math. Soc. **59** (1999), no. 1, 1–20.

[14] S. Glasner, B. Weiss, *On the construction of minimal skew products*, Israel J. Math. **34** (1979), no. 4, 321–336.

[15] K. N. Haddad, A. S. A. Johnson, *Auslander systems*, Proc. Amer. Math. Soc. **125** (1997), no. 7, 2161–2170

[16] M. Handel, *A pathological area preserving C^∞ diffeomorphism of the plane*, Proc. Amer. Math. Soc. **86** (1982), no. 1, 163–168.

[17] S. F. Kolyada, *On dynamics of triangular maps of the square*, Ergodic Theory Dynam. Systems **12** (1992), no. 4, 749–768.

[18] S. F. Kolyada, Ľ. Snoha, *On ω-limit sets of triangular maps*, Real Anal. Exchange **18** (1992/93), no. 1, 115–130.

[19] S. Kolyada, Ľ. Snoha, *Minimal dynamical systems*, Scholarpedia **4(11):5803** (2009), `http://www.scholarpedia.org/article/Minimal_dynamical_systems`

[20] S. Kolyada, Ľ. Snoha, S. Trofimchuk, *Noninvertible minimal maps*, Fund. Math. **168** (2001), no. 2, 141–163.

[21] S. Kolyada, Ľ. Snoha, S. Trofimchuk, *Proper minimal sets on compact connected* 2*-manifolds are nowhere dense*, Ergodic Theory Dynam. Systems **28** (2008), no. 3, 863–876.

[3] In other words, (4.6) holds for all $z \in W$ and not only for $z = a$. However, there is a difference here. While we know that $\{a\} \times A_i^\Gamma$, $i = 1, \dots, m$, are subsets of M, even subsets of $\mathrm{Sint}_s(M)$, we do not know whether also for $z \neq a$ the sets $\{z\} \times A_i^\Gamma$, $i = 1, \dots, m$ are subsets of M or not.

[22] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.
[23] J.- H. Mai, *Pointwise-recurrent graph maps*, Ergodic Theory Dynam. Systems **25** (2005), no. 2, 629–637.
[24] Z. Nitecki, *Topological dynamics on the interval.* Ergodic theory and dynamical systems, II (College Park, Md., 1979/1980), pp. 173, Progr. Math., 21, Birkhuser, Boston, Mass., 1982.
[25] J. C. Oxtoby, *Measure and category. A survey of the analogies between topological and measure spaces.* Second edition. Graduate Texts in Mathematics, 2. Springer-Verlag, New York-Berlin, 1980.
[26] W. Parry, *A note on cocycles in ergodic theory*, Contemp. Math. **28** (1974), 343–350.
[27] Ľ. Snoha, V. Špitalský, *Recurrence equals uniform recurrence does not imply zero entropy for triangular maps of the square*, Discrete Contin. Dyn. Syst. **14** (2006), no. 4, 821–835.

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