RESOLUTION OF SINGULARITIES OF GERMS IN CHARACTERISTIC POSITIVE ASSOCIATED WITH VALUATION RINGS OF ITERATED DIVISOR TYPE

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Abstract. In this paper we show that any hypersurface singularities of germs of varieties in positive characteristic can be resolved by iterated monoidal transformations with centers in smooth subvarieties, if we have a valuation ring of iterated divisor type associated with the germ. Besides, we introduce fundamental concepts for the study of resolution of singularities of germs such as space germs, iterated analytic monoidal transforms with a normal crossing, Weierstrass representations, reduction sequences, and so forth.

1. Introduction

In this paper we consider the problem of resolution of singularities in the case of positive characteristic. Since Hironaka's success in resolution of singularities in characteristic zero (Hironaka [5].), the solution of this problem has long been expected. However, it is still open.

This is perhaps because the problem is hard by nature.

Therefore, we consider the problem in the category of germs of varieties. Besides, we assume that the singularity is a hypersurface.

Recently very clear exposition was given in Appendix of Abhyankar [2] for resolution of hypersurface singularities in characteristic zero. I have noticed that most of Abhyankar's ideas are effective even in characteristic positive. I supply the missing part to his ideas, and develop the theory of positive characteristic in this paper.

The main result is the following: The algebraically closed ground field of characteristic positive is denoted by k. By B and B' we denote rings isomorphic to the formal power series ring $k[[x_0, x_1, \dots, x_r]]$ over k with $r+1$ variables $x_0, x_1, \dots, x_r$ for some non-negative integer r.

Theorem 1.1. *Let (B, W) be a space germ over k. Assume that the valuation ring W is of iterated divisor type. Let $f_1, f_2, \dots, f_\ell$ be a finite number of arbitrary non-zero elements in the ring B. Then, there exists an iterated analytic monoidal transform (B', W') of (B, W) such that the product $f'_1 f'_2 \cdots f'_\ell \in B'$ has a normal crossing, and the ideal in B' generated by any non-empty subset of $\{f'_1, f'_2, \dots, f'_\ell\}$ is a principal ideal, where for every $1 \le i \le \ell$, $f'_i \in B'$ denotes the image of $f_i \in B$ by the associated homomorphism $B \to B'$.*

Terminologies will be defined in this paper. I can give only brief explanation for them here. By W we denote a valuation ring of the quotient field of B. We assume

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that W dominates B and $\dim_k W = 0$. The pair (B, W) is called a space germ over k. The valuation ring W is said to be of iterated divisor type, if $\dim B = \operatorname{rank} W$ holds.

Let $\sigma : \Sigma \to \mathrm{Spec}(B)$ be a blowing-up of the scheme $\mathrm{Spec}(B)$ with center in some non-singular closed subscheme. It is known that the morphism $\mathrm{Spec}(W) \to \mathrm{Spec}(B)$ induced by the inclusion $B \subset W$ of rings has a unique lift $\mathrm{Spec}(W) \to \Sigma$ such that the composition $\mathrm{Spec}(W) \to \Sigma \to \mathrm{Spec}(B)$ coincides with $\mathrm{Spec}(W) \to \mathrm{Spec}(B)$. Let $s \in \Sigma$ be the image of the unique closed point of $\mathrm{Spec}(W)$ by $\mathrm{Spec}(W) \to \Sigma$, B_1 be the completion of the local ring $\mathcal{O}_{\Sigma,s}$, and W_1 be a valuation ring of the quotient field of B_1 such that $\dim_k W_1 = 0$ and W_1 dominates W. The pair (B_1, W_1) is called an elementary monoidal transform of (B, W). Iterating this procedure, we define an iterated analytic monoidal transform.

Now, applying a blowing-up to a germ, the obtained exceptional variety is a global object. To treat resolution of singularities in the category of germs, we have to choose a closed point lying over the origin on the exceptional variety at random. The valuation ring W in the second term of a space germ plays the role of being who chooses a closed point on the exceptional variety at random.

Our formulation of space germs is similar to Zariski's uniformization. (Abhyankar [1], Zariski [9], [10].) However, it differs from uniformization in that the ring B is complete. There are two reasons. First, treating complete rings, we can make it clear that we treat the problem locally in the category of germs. Second, we apply the Weierstrass preparation theorem in the essential part of our theory. If the ring is not complete, Weierstrass' theorem does not hold.

The above theorem solves only a small part of the problem. However, it is the first result on resolution of singularities in positive characteristic treating arbitrary dimensions, using monoidal transformations with smooth centers and giving concrete descriptions for the choice of centers of monoidal transformations. I hope that this paper together with several related results in a few years (Kuhlmann [6], Spivakovsky [8].) will play a key role to open the door toward the final solution.

Throughout this paper we fix the ground field k. We assume that k is an algebraically closed field and the characteristic number p of k is positive. The prime number p is used to denote the characteristic number of k.

Let Γ be a totally ordered abelian group. Often in this paper we associate a special element ∞ called the *infinity* with Γ. We assume that the element ∞ satisfies the following two conditions:

1. $\infty \geq \infty$, $\infty + \infty = \infty$, and $n\infty = \infty n = \infty$ for every positive integer n.
2. For every element $\gamma \in \Gamma$ $\gamma \neq \infty$, $\gamma < \infty$ and $\gamma + \infty = \infty + \gamma = \infty$.

Any positive integer m can be written uniquely in the form $m = lq$ where l is an integer prime to p and $q = p^\delta$ is a power of p. The exponent δ is a non-negative integer. The pair (l, q) is called the *p-decomposition* of m. For simplicity we say that $m = lq$ is the p-decomposition of m, or that $m = lp^\delta$ is the p-decomposition of m.

By $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ we denote the ring of integers, the rational number field and the real number field respectively. By $\mathbb{Z}_0$ we denote the set of non-negative integers.

The number of elements in a finite set X is denoted by $\sharp X$.

Section 1 is the introduction. Sections 2, 3, 4 are preliminary parts. We collect results on commutative rings, totally ordered abelian groups and valuations in respective sections. In Section 5 we give definitions for fundamental objects in our formulation such as a space germ, a framed space germ, a basic space germ, an elementary monoidal transform, an iterated elementary monoidal transform (an IEMT, in short), an iterated analytic monoidal transform (an IAMT, in short), and so forth. The concept of Weierstrass representations is introduced in Section 6. This concept plays the central role in our theory. The Abhyankar condition, the strong Abhyankar condition and the condition "reductive" are defined for Weierstrass representations. The condition "not reductive" is essential in characteristic positive. Let $G_1, G_2, \ldots, G_m \in Z = k[[x_1, x_2, \ldots, x_n]]$ and $g = t^m + \sum_{j=1}^{m} G_j t^{m-j}$. We define $\mathrm{w}(g,\, t) = \min\{\mathrm{ord}(G_j)/j \mid 1 \le j \le m\} \in \mathbb{Q} \cup \{\infty\}$. Note that for any $H \in Z$ there are elements $G'_1, G'_2, \ldots, G'_m \in Z$ with $g = (t+H)^m + \sum_{j=1}^{m} G'_j (t+H)^{m-j}$. Thus $\mathrm{w}(g,\, t+H) = \min\{\mathrm{ord}(G'_j)/j \mid 1 \le j \le m\}$ is defined. The pair (g, t) is said to be *reductive*, if there is an element $H \in Z$ with $\mathrm{w}(g,\, t) < \mathrm{w}(g,\, t+H)$. In Section 7 we show that if a Weierstrass representation satisfies both the Abhyankar condition and the condition "not reductive", then the singularity can be resolved even in characteristic positive. By this result our goal becomes to modify a given Weierstrass representation to satisfy both conditions. In Section 8 properties of the collection $\mathcal{C}_0$ of all framed space germs of iterated divisor type are studied. An effective tool called a reduction sequence is introduced in Section 9 to treat both the Abhyankar condition and the condition "not reductive" simultaneously. Central idea of a reduction sequence is like the following: Let $Z_0 = Z$. We consider a pair $(g, t_0) = (g, t)$ as above. Assume that we have already obtained rings $Z_0, Z_1, \ldots, Z_{\nu-1}$ and pairs $(g, t_0), (g, t_1), \ldots, (g, t_{\nu-1})$. If $\mathrm{w}(g, t_{\nu-1}) = \infty$, then $g = t_{\nu-1}^m$ is a function with a normal crossing. We can treminate our procedure. We consider the case $\mathrm{w}(g, t_{\nu-1}) < \infty$ below. By the induction hypothesis on dimensions there exists an IAMT Z_ν of $Z_{\nu-1}$ such that the pair $(g, t_{\nu-1})$ satisfies the Abhyankar condition, if we regard the coefficient function sequence $G_{\nu-1,1}, G_{\nu-1,2}, \ldots, G_{\nu-1,m} \in Z_{\nu-1}$ of g as elements in Z_ν by the associated homomorphism $Z_{\nu-1} \to Z_\nu$. If $(g, t_{\nu-1})$ is not reductive when we regard $G_{\nu-1,j}$'s as elements in Z_ν, then we terminate our procedure. Otherwise, we choose an element $H_\nu \in Z_\nu$ with $\mathrm{w}(g, t_{\nu-1}) < \mathrm{w}(g, t_{\nu-1} + H_\nu)$ and put $(g, t_\nu) = (g, t_{\nu-1} + H_\nu)$. The sequence obtained by repeating this procedure is called a *reduction sequence*. We will see that any member $G_{\nu,j}$ of the coefficient function sequence of (g, t_ν) has a special form. (By this form it seems intuitively that $\lim_{\nu\to\infty} G_{\nu,j} = 0$ holds for every $1 \le j \le m$. However, it is difficult to give a meaning to this limit. We apply the valuation theory for this purpose.) In Section 10 we show essential results in the ring theory for the final goal. These results are necessary to avoid purely inseparable extensions of the function field. In the final section Section 11 we show that it is possible to modify a given Weierstrass representation to satisfy both the Abhyankar condition and the condition "not reductive", if we assume that the corresponding valuation ring is of iterated divisor type. In other words, we show that there exists a reduction sequence with finite length under the assumption

After I sent the first draft of this paper to the AMS preprint server, Joseph Lipman kindly informed me that Section 4 of Bierstone and Milman [3] (and Section 3 of Bierstone and Milman [4]) contains the same results as ones in Appendix of Abhyankar [2], and I received envelope containing a lot of results by Bierstone and Milman from Bierstone. By Lemma 4.7 in Bierstone and Milman [3] (or Lemma 3.12 in Bierstone and Milman [4]) I could improve some descriptions in Section 5 and Section 6 of my manuscript. Because of difference on how to choose centers of monoidal transformations among in [2], in [3] and in [4], I take Abhyankar [2] as the basis of our theory in this paper. (The methods in [2] and [3] are simpler, but they choose centers even outside the zero-locus of the function under consideration. Two methods are slightly different. In the third method in [4] they choose centers inside the singular sets of the zero-locus of the function. The third can be regarded as the improvement of the second one in [3].)

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2. Ring theory

In this section we review several important points in the ring theory and explain non-standard concepts.

By a ring we mean a commutative ring with the identity. We use standard terminologies in the commutative ring theory. (Matsumura [7]. Zariski-Samuel [11].) We shall denote the identity by the symbol 1, while the element zero will be denoted by 0. A ring homomorphism $\varphi : A \to B$ from a ring A to a ring B is always assumed to satisfy $\varphi(1_A) = 1_B$, where 1_A and 1_B denote the identities of A and B respectively. When a ring homomorphism $\varphi : A \to B$ is given, we say that B is an *A-algebra*.

Let A be a ring. By A^* we denote the set of invertible elements of A, i.e.,

$$A^* = \{a \in A \mid \text{ There exists an element } b \in A \text{ with } ab = 1\}.$$

Note that A^* is an abelian group with respect to the multiplication. Note also that $E^* = E - \{0\}$ for a field E.

A ring with a unique maximal ideal is called a *local ring*, while a ring with a finite number of maximal ideals is called a *semi-local ring*. For a local ring A by $M(A)$ we denote the maximal ideal of A. It is easy to see that a local ring A is the disjoint union of A^* and $M(A)$.

Let A and B be local rings and $\varphi : A \to B$ be a ring homomorphism. We can check that $\varphi(M(A)) \subset M(B)$ if and only if $\varphi^{-1}(M(B)) = M(A)$. We say that φ is a *local homomorphism*, or it is a *homomorphism of local rings*, if these equivalent conditions are satisfied.

Assume that local rings A and B are subrings of a field E. We say that B *dominates* A, if $A \subset B$ and if $M(A) = M(B) \cap A$. The ring B dominates A, if and only if the inclusion homomorphism $A \hookrightarrow B$ is defined and the inclusion homomorphism is local.

A subset S of a ring A is said to be a *multiplicative subset*, if $1 \in S$ and $0 \notin S$ and if the product of any two elements in S is again in S.

Any A-submodule $\mathfrak{i}$ of a ring A with $\mathfrak{i} \neq A$ is called an *ideal* of A. An ideal $\mathfrak{p}$ of a ring A is a *prime* ideal, if the complement $A - \mathfrak{p}$ is a multiplicative subset. The set of prime ideals of a ring A is denoted by $\mathrm{Spec}(A)$. Any homomorphism $\varphi : A \to B$ of rings induces a map $\varphi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ in the opposite direction. The image of $\mathfrak{q} \in \mathrm{Spec}(B)$ is defined to be $\varphi^{-1}(\mathfrak{q}) \in \mathrm{Spec}(A)$. Note that $\varphi^{-1}(\mathfrak{q}) = \mathfrak{q} \cap A$ if φ is injective. Obviously $\mathrm{id}_A^* = \mathrm{id}_{\mathrm{Spec}(A)}$ and $(\psi\varphi)^* = \varphi^*\psi^*$, where id_X denotes the identity map of a set X, and $\varphi : A \to B$, $\psi : B \to C$ are homomorphisms of rings.

For any multiplicative subset S of a ring A and for any A-module M, we can define the *localization* $S^{-1}M$ of M with respect to S. (Matsumura [7] p.6–p.7 (1.F).) If $S = A - \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, we write $M_{\mathfrak{p}}$ for $S^{-1}M$.

A finite sequence $\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n$ of $n+1$ prime ideals of a ring A is called a *prime chain* of *length* n, if $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ and if any two of $\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n$ are different. The prime ideal $\mathfrak{p}_0$ is called the *bottom member* of the chain, while $\mathfrak{p}_n$ is called the *top member* of the chain.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$ be a prime ideal. The supremum of the length of prime chains with the top member $\mathfrak{p}$ is called the *height* of a prime ideal $\mathfrak{p}$. Let $\mathfrak{i}$ be a ideal of A. The *height* of the ideal $\mathfrak{i}$ is defined to be the minimum of the heights of prime ideals containing $\mathfrak{i}$, and it is denoted by $\mathrm{ht}(\mathfrak{i})$. In case where $\mathfrak{i}$ is prime, two definitions of height of $\mathfrak{i}$ coincide.

The *dimension* of a ring A is defined to be the supremum of heights of the prime ideals in A, and is denoted by $\dim A$. If A is a noetherian semi-local ring, $\dim A$ is finite. (Matsumura [7], p.74, (12.E), LEMMA 1.) Since $A_{\mathfrak{p}}$ is a local ring for any $\mathfrak{p} \in \mathrm{Spec}(A)$, and since $\mathrm{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}}$ by definition, $\mathrm{ht}(\mathfrak{i})$ is a finite integer for any ideal $\mathfrak{i}$ of a noetherian ring A.

A prime chain $\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n$ is said to be *maximal*, if there is no prime chain $\mathfrak{p}'_0, \mathfrak{p}'_1, \dots, \mathfrak{p}'_m$ such that $\mathfrak{p}_0 = \mathfrak{p}'_0$, $\mathfrak{p}_n = \mathfrak{p}'_m$, $n < m$ and $\{\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subset \{\mathfrak{p}'_0, \mathfrak{p}'_1, \dots, \mathfrak{p}'_m\}$. For any pair of ideals $\mathfrak{i}$, $\mathfrak{j}$ with $\mathfrak{i} \supset \mathfrak{j}$, the height of the ideal $\mathfrak{i}/\mathfrak{j}$ in the residue ring $A/\mathfrak{j}$ is denoted by $\mathrm{ht}(\mathfrak{i}/\mathfrak{j})$.

A ring A is said to be *catenary*, if for each pair of prime ideals $\mathfrak{p}$, $\mathfrak{q}$ with $\mathfrak{p} \supset \mathfrak{q}$, the length of any maximal chain of prime ideals with top member $\mathfrak{p}$ and with the bottom member $\mathfrak{q}$ is finite, and if it does not depend on the choice of the maximal chain satisfying the conditions.

A ring without a zero divisor is called an *integral domain* or a *domain* in short.

A noetherian domain A is catenary, if and only if, for any pair of prime ideals $\mathfrak{p}$, $\mathfrak{q}$ with $\mathfrak{p} \supset \mathfrak{q}$, we have $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}) + \mathrm{ht}(\mathfrak{p}/\mathfrak{q})$.

A ring A is said to be *universally catenary*, if A is noetherian and if every finitely generated A-algebra is catenary. (Matsumura [7], p.84, (14.B))

Let B be a domain, and A be a subdomain. The quotient field L of B contains the quotient field K of A. The transcendence degree $\mathrm{tr.deg}_K L$ of L over K is defined. By putting $\mathrm{tr.deg}_A B = \mathrm{tr.deg}_K L$ we define the transcendence degree $\mathrm{tr.deg}_A B$ of B over A.

The following theorem plays an important role in this paper.

Theorem 2.1. *Let B be a domain and A be a subdomain. We assume that A is noetherian and B is a finitely generated A-algebra. Let $\mathfrak{q} \in \mathrm{Spec}(B)$ be a prime ideal in B. Then we have*

$$\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{q} \cap A) + \mathrm{tr.deg}_A B - \mathrm{tr.deg}_{(A/A\cap\mathfrak{q})}(B/\mathfrak{q}).$$

And the equality holds, if A is universally catenary, or if B is a polynomial ring $A[x_1, x_2, \dots, x_n]$ over A with a finite number of algebraically independent elements $x_1, x_2, \dots, x_n$.

Proof. See Matsumura [7], p.84–p.86, (14.C), THEOREM 23, and Zariski-Samuel [11], Vol. II, p.321–p.329, APPENDIX 1. $\square$

We have a class of rings called *excellent rings*. The exact characterization of excellent rings is complicated. (Matsumura [7], p.259, (34.A).) We will apply only the properties of excellent rings in the theorem below.

Theorem 2.2. 1. *Any ring isomorphic to an excellent ring is excellent.*

2. *An excellent ring is noetherian.*
3. *A noetherian complete semi-local ring is excellent. In particular, a field is excellent.*
4. *If a ring A is excellent, then for any finite number of algebraically independent elements $x_1, x_2, \dots, x_n$ over A and for any ideal $\mathfrak{i}$ in the polynomial ring $A[x_1, x_2, \dots, x_n]$ over A, the residue ring $A[x_1, x_2, \dots, x_n]/\mathfrak{i}$ is excellent.*
5. *If a ring A is excellent, then for any multiplicative subset S of A the localization $S^{-1}A$ of A with respect to S is excellent.*
6. *An excellent ring is universally catenary.*
7. *Let A be an excellent domain with the quotient field K. Let L be a field containing K. If L is a finite algebraic extension of K, then the integral closure of A in L is a finitely generated A-module.*
8. *The completion of an excellent local domain is reduced.*
9. *The completion of an excellent normal local domain is a normal local domain.*
10. *An excellent ring A is a G-ring. In other words, A has the following property: for any two prime ideals $\mathfrak{p}$, $\mathfrak{q}$ of A with $\mathfrak{p} \supset \mathfrak{q}$, for any field L such that L contains the quotient field of $A/\mathfrak{q}$ and L is finitely generated over the quotient field of $A/\mathfrak{q}$ and for any prime ideal $\mathfrak{Q}$ of the ring $\widehat{A_\mathfrak{p}} \otimes_A L$ the localization $(\widehat{A_\mathfrak{p}} \otimes_A L)_\mathfrak{Q}$ is a regular local ring, where $\widehat{A_\mathfrak{p}}$ denotes the completion of the local ring $A_\mathfrak{p}$.*

Proof. See Matsumura [7], p.259–p.260, (34.A) and (34.B). □

By $\mathbb{Z}_0$ we denote the set of non-negative integers.

Let A be a ring. The binomial coefficient $\binom{r}{j}$ is the mapping $(r, j) \in \mathbb{Z}_0^2 \mapsto \binom{r}{j} \in A$ satisfying the following two conditions:

1. If $0 \leq r < j$, then $\binom{r}{j} = 0$.
2. For every non-negative integer r the equality
$$(t+1)^r = \sum_{j=0}^{r} \binom{r}{j} t^j$$
holds in the ring $A[t]$, where t is an algebraically independent element over A.

Lemma 2.3. 1.
$$\binom{r}{0} = \binom{r}{r} = 1.$$

2.
$$\binom{r+1}{j+1} = \binom{r}{j} + \binom{r}{j+1}.$$

3. *If $0 \leq j \leq r$, then*
$$\binom{r}{j} = \binom{r}{r-j}.$$

4.
$$\binom{r}{1} = r,$$
where the image of $r \in \mathbb{Z}$ by the canonical map $\mathbb{Z} \to A$ is denoted by the same symbol $r \in A$ in the right-hand side.

5.
$$\binom{r+s}{j} = \sum_{i=0}^{j} \binom{r}{j-i}\binom{s}{j}.$$

6. *Let B be a ring containing A. For every $x, y \in B$*
$$(x+y)^r = \sum_{j=0}^{r} \binom{r}{j} x^{r-j} y^j.$$

7. *If $0 \leq a \leq b$ and $a \leq r$, then*
$$\binom{b}{a}\binom{r}{b} = \binom{r}{a}\binom{r-a}{b-a}.$$

Proof. Let t, u be algebraically independent elements over A. Computing $(tu+u+1)^r$ in two different ways we obtain claim 7. ☐

Lemma 2.4. *Assume that a ring A contains a field of characteristic p.*

1. *For $1 \leq j \leq p-1$*
$$\binom{r}{j} = \frac{r(r-1)\cdots(r-j+1)}{j\,!}.$$

2. *Let δ be a non-negative integer. If $1 \leq j \leq p^\delta - 1$, then*
$$\binom{p^\delta}{j} = 0.$$
Let t be an algebraically independent element over A.
$$(t+1)^{p^\delta} = t^{p^\delta} + 1.$$

3. *Let δ and l be non-negative integers. If a non-negative integer j is not a multiple of p^δ, then*
$$\binom{lp^\delta}{j} = 0.$$
If there is a non-negative integer i with $j = ip^\delta$, then
$$\binom{lp^\delta}{j} = \binom{lp^\delta}{ip^\delta} = \binom{l}{i}.$$

4. *Let δ, a and b be non-negative integers, and c and d be integers with $0 \leq c \leq p^\delta - 1$, $0 \leq d \leq p^\delta - 1$.*
$$\binom{ap^\delta + c}{bp^\delta + d} = \binom{a}{b}\binom{c}{d}.$$

Proof. Not difficult. ☐

Definition 2.5. Let A be a k-algebra and δ be a non-negative integer. The map

$$F_\delta : A \to A$$

defined by $F_\delta(a) = a^{p^\delta}$ for $a \in A$ is called the *Frobenius homomorphism* of degree δ over A. The Frobenius homomorphism of degree 1 is called the *Frobenius homomorphism* and is denoted by F.

Lemma 2.6.
1. *The Frobenius homomorphism $F_\delta : A \to A$ of degree δ is a ring homomorphism, but is not a homomorphism of k-algebras. $F_\delta = F^\delta$.*
2. *For every homomorphism $\varphi : A \to B$ of k-algebras we have $F_\delta \varphi = \varphi F_\delta$.*
3. *If A is a domain, then F_δ is injective.*
4. *If A is a topological ring with $\mathfrak{i}$-adic topology defined by an ideal $\mathfrak{i}$, then F_δ is continuous.*

Let n be a non-negative integer, and $x_0, x_1, \dots, x_n$ be $n+1$ of variables. We consider the formal power series ring $k[[x]] = k[[x_0, x_1, \dots, x_n]]$ over k. Any element $f \in k[[x]]$ can be written uniquely in the form

$$f = \sum_{\alpha \in \mathbb{Z}_0^{n+1}} f_\alpha x^\alpha,$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{Z}_0^{n+1}$, $\alpha_0, \alpha_1, \dots, \alpha_n$ are non-negative integers, $f_\alpha \in k$ for any $\alpha \in \mathbb{Z}_0^{n+1}$, and $x^\alpha = x_0^{\alpha_0} x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Lemma 2.7.
1. *The ring $k[[x]]$ is a noetherian complete regular local ring of dimension $n+1$, and is a unique factorization domain (UFD in short.) with the unique maximal ideal $M(k[[x]]) = \{f \in k[[x]] \mid f_{(0,0,\dots,0)} = 0\}$. The set $\{x_0, x_1, \dots, x_n\}$ is a parameter system of $k[[x]]$.*
2. *Let $y_0, y_1, \dots, y_n$ be $n+1$ elements in $k[[x]]$. The following three conditions are equivalent:*
   (a) *The set $\{y_0, y_1, \dots, y_n\}$ is a parameter system of $k[[x]]$.*
   (b) *There exists a unique continuous isomorphism $\varphi : k[[x]] \to k[[x]]$ of k-algebras satisfying $\varphi(x_i) = y_i$ for every $0 \le i \le n$.*
   (c) *$y_0, y_1, \dots, y_n \in M(k[[x]])$ and $\det(\partial y_i / \partial x_j) \notin M(k[[x]])$.*

Definition 2.8.
1. Let $Q = \{x_0, x_1, \dots, x_n\}$ be a parameter system of the ring $k[[x]]$ and $f \in k[[x]]$. We say that f has a *normal crossing with respect to Q*, if there are non-negative integers $\beta_0, \beta_1, \dots, \beta_n$ with
$$\frac{f}{x_0^{\beta_0} x_1^{\beta_1} \cdots x_n^{\beta_n}} \in k[[x]]^*.$$
2. An element $f \in k[[x]]$ is said to have a *normal crossing*, if there is a parameter system Q of $k[[x]]$ such that f has a normal crossing with respect to Q.

Lemma 2.9. *Let Q be a parameter system of $k[[x]]$, and $f \in k[[x]]$, $g \in k[[x]]$.*
1. *the following two conditions are equivalent:*
   (a) *The product fg has a normal crossing with respect to Q.*
   (b) *Both of f and g have a normal crossing with respect to Q.*
2. *If the product fg has a normal crossing, then both of f and g have a normal crossing.*

Proof. It follows from the uniqueness of the irreducible decomposition. ∎

Let $w = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_0^{n+1}$, and $Q = \{x_0, x_1, \dots, x_n\}$ be a parameter system of $k[[x]]$. We will define a map

$$\mathrm{ord}_w : k[[x]] - \{0\} \to \mathbb{Z}_0$$

associated with w and Q. The value $\mathrm{ord}_w(f) \in \mathbb{Z}_0$ for $f \in k[[x]]$ with $f \neq 0$ is called the *order* of f with *weight* w. Let $f = \sum_{\alpha \in \mathbb{Z}_0^{n+1}} f_\alpha x^\alpha \in k[[x]]$ be an element with $f \neq 0$. Let

$$ex(f) = \{\alpha \in \mathbb{Z}_0^{n+1} \mid f_\alpha \neq 0\}.$$

Since $f \neq 0$, the set $ex(f)$ is not empty. We define

$$\mathrm{ord}_w(f) = \min\{\sum_{i=0}^{n} w_i \alpha_i \mid (\alpha_0, \alpha_1, \dots, \alpha_n) \in ex(f)\}.$$

Lemma 2.10. 1. *If $f, g \in k[[x]] - \{0\}$, then $\mathrm{ord}_w(fg) = \mathrm{ord}_w(f) + \mathrm{ord}_w(g)$.*

2. *If $f, g \in k[[x]] - \{0\}$ and if $f + g \in k[[x]] - \{0\}$, then $\mathrm{ord}_w(f+g) \geq \min\{\mathrm{ord}_w(f), \mathrm{ord}_w(g)\}$.*
3. *Assume that $w_i = 0$ for $0 \leq i \leq m$ and $w_i > 0$ for $m+1 \leq i \leq n$. We have $\mathrm{ord}_w(f) = 0$ if and only if $f_{(\alpha_0, \alpha_1, \dots, \alpha_m, 0, 0, \dots, 0)} \neq 0$ for some $m+1$ non-negative integers $\alpha_0, \alpha_1, \dots, \alpha_m$.*
4. *Assume that $w_i >$ for every $0 \leq i \leq n$. We have $\mathrm{ord}_w(f) = 0$ if and only if $f \in k[[x]]^*$.*

Let E denote the quotient field of $k[[x]]$. We can define an extended map

$$\mathrm{ord}_w : E^* \to \mathbb{Z}$$

by putting $\mathrm{ord}_w(h) = \mathrm{ord}_w(f) - \mathrm{ord}_w(g)$, where $h \in E^*$ and $f, g \in k[[x]] - \{0\}$ are elements with $h = f/g$. By Lemma 2.10.1 $\mathrm{ord}_w(h)$ does not depend on the choice of elements $f, g \in k[[x]] - \{0\}$ satisfying $h = f/g$.

Besides, by putting $\mathrm{ord}_w(0) = \infty$ we have the extension

$$\mathrm{ord}_w : E \to \mathbb{Z} \cup \{\infty\}.$$

If $w_i = 1$ for every $0 \leq i \leq n$, we write ord instead of ord_w. The value $\mathrm{ord}(f)$ is called the *order* of f.

Definition 2.11. Let E be a field and Γ be a totally ordered abelian group. A map $v : E \to \Gamma \cup \{\infty\}$ is called *valuation* of E, if it satisfies the following conditions:

1. $v(h) = \infty$ if and only if $h = 0$.
2. For every $f, g \in E$ $v(fg) = v(f) + v(g)$.
3. For every $f, g \in E$ $v(f + g) \geq \min\{v(f), v(g)\}$.

The group Γ is called the *value group* of the valuation v.

Lemma 2.12. 1. *Let E be the quotient field of $k[[x]]$. The map $\mathrm{ord}_w : E \to \mathbb{Z} \cup \{\infty\}$ associated with an element $w \in \mathbb{Z}_0^{n+1}$ and a parameter system Q is a valuation of E with the value group $\mathbb{Z}$.*

2. *$\mathrm{ord}_w(f) \geq 0$ for every $f \in k[[x]]$.*
3. *Assume that $w_i > 0$ for every $0 \leq i \leq n$. If $f \in k[[x]]$ and $\mathrm{ord}_w(f) = 0$, then $f \in k[[x]]^*$.*
4. *Assume $w_i = 1$ for every $0 \leq i \leq n$. Then, $\mathrm{ord}_w = \mathrm{ord}$ does not depend on the choice of the parameter system Q.*

Proof. We consider claim 4. Under the assumption we have $\mathrm{ord}_w(f) = \max\{\nu \mid f \in M(k[[x]])^\nu\}$, which shows the claim. $\square$

Definition 2.13. Let $f \in k[[x]]$ be an element with $f \neq 0$. Let $f = ug_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ be the irreducible decomposition of f, where $g_1, g_2, \dots, g_r \in k[[x]]$ are irreducible elements such that any two of them are coprime, $u \in k[[x]]^*$ and $\beta_1, \beta_2, \dots, \beta_r$ are positive integers. We assume that $\mathrm{ord}(g_i) > 1$ for $1 \leq i \leq s$ and $\mathrm{ord}(g_i) = 1$ for $s+1 \leq i \leq r$.

1. A triplet (v, h_s, h_n) satisfying the following conditions are called the *singular non-singular decomposition* of f.
   (a) $f = vh_sh_n$. $v \in k[[x]]^*$. $h_s \in k[[x]]$. $h_n \in k[[x]]$.
   (b)
   $$\frac{h_s}{g_1^{\beta_1} g_2^{\beta_2} \cdots g_s^{\beta_s}} \in k[[x]]^*.$$
   (c)
   $$\frac{h_n}{g_{s+1}^{\beta_{s+1}} g_{s+2}^{\beta_{s+2}} \cdots g_r^{\beta_r}} \in k[[x]]^*.$$
2. Let (v, h_s, h_n) be the singular non-singular decomposition of f. We call $\mathrm{ord}(h_s)$ the *singular order* of f, and denote it by $\mathrm{sord}(f)$. We call $\mathrm{ord}(h_n)$ the *non-singular order* of f, and denote it by $\mathrm{nord}(f)$.

Lemma 2.14. 1. *The singular order* $\mathrm{sord}(f)$ *and the non-singular order* $\mathrm{nord}(f)$ *of an element* $f \in k[[x]] - \{0\}$ *are well-defined. They do not depend on the choice of the singular non-singular decomposition* (v, h_s, h_n) *of* f *in the definition.*
2. $\mathrm{sord}(f)$ *and* $\mathrm{nord}(f)$ *are non-negative integers.* $\mathrm{ord}(f) = \mathrm{sord}(f) + \mathrm{nord}(f)$. $\mathrm{sord}(f) \neq 1$. $\mathrm{sord}(fg) = \mathrm{sord}(f) + \mathrm{sord}(g)$. $\mathrm{nord}(fg) = \mathrm{nord}(f) + \mathrm{nord}(g)$.

Let $w = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_0^{n+1}$, and $Q = \{x_0, x_1, \dots, x_n\}$ be a parameter system of $k[[x]]$.

Let $f = \sum_{\alpha \in \mathbb{Z}_0^{n+1}} f_\alpha x^\alpha \in k[[x]]$ be an element with $f \neq 0$. Let $iex(f) = \{\alpha \in ex(f) \mid \sum_{i=0}^n w_i \alpha_i = \mathrm{ord}_w(f)\}$. Putting

$$\mathrm{in}_w(f) = \sum_{\alpha \in iex(f)} f_\alpha x^\alpha,$$

we call $\mathrm{in}_w(f)$ the *w-initial polynomial* of f. For $f = 0 \in k[[x]]$ we define $\mathrm{in}_w(0) = 0$.

We can check that $\mathrm{in}_w(fg) = \mathrm{in}_w(f)\mathrm{in}_w(g)$ for any $f, g \in k[[x]]$.

Assume that $w_i = 0$ for $0 \leq i \leq m$ and $w_i > 0$ for $m+1 \leq i \leq n$. One sees that $\mathrm{in}_w(f) \in k[[x_0, x_1, \dots, x_m]][x_{m+1}, x_{m+2}, \dots, x_n]$.

We say that f is a *w-homogeneous polynomial*, if $f = \mathrm{in}_w(f)$. If $w_i > 0$ for every $0 \leq i \leq n$, then any w-homogeneous polynomial contains only a finite number of terms, and it is a polynomial with coefficients in k.

Consider the case where $w_i = 1$ for every $0 \leq i \leq n$ in particular. In this case $\mathrm{in}_w(f)$ does not depend on the choice of the parameter system Q. We write $\mathrm{in}(f)$ instead of $\mathrm{in}_w(f)$, and call $\mathrm{in}(f)$ the *initial polynomial* of f. It has only a finite number of terms. Also in this case a w-homogeneous polynomial is called a *homogeneous polynomial.*

In the theorem below one of $n+1$ variables plays a special role. Putting $t = x_0$, and $y_i = x_i$ for $1 \leq i \leq n$, we write

$$\begin{aligned} k[[t, y]] &= k[[t, y_1, y_2, \dots, y_n]] = k[[x]], \\ k[[y]] &= k[[y_1, y_2, \dots, y_n]]. \end{aligned}$$

Theorem 2.15 (Weierstrass division theorem). *Let* $w = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_0^{n+1}$. *We assume that* $w_i > 0$ *for every* $0 \le i \le n$. *Let* $f \in k[[t, y]]$. *Assume that* $\mathrm{in}_w(f)(1, 0, 0, \dots, 0) \neq 0$, *where* $\mathrm{in}_w(f)(1, 0, 0, \dots, 0)$ *denotes the result of substitution* $t = 1$, $y_1 = y_2 = \cdots = y_n = 0$ *in the* w*-initial polynomial of* f.

1. *The quotient* $m = \mathrm{ord}_w(f)/\mathrm{ord}_w(t)$ *is a non-negative integer.*
2. *For every* $h \in k[[t, y]]$ *there exists a unique set of an element* $e \in k[[t, y]]$ *and* m *numbered elements* $H_1, H_2, \dots, H_m \in k[[y]]$ *satisfying*
$$h = ef + \sum_{j=1}^{m} H_j t^{m-j}.$$

Proof. Not difficult. □

Corollary 2.16 (Weierstrass preparation theorem). *Let* $w = (w_0, w_1, \dots, w_n) \in \mathbb{Z}_0^{n+1}$. *We assume that* $w_i > 0$ *for every* $0 \le i \le n$. *Let* $f \in k[[t, y]]$. *Assume that* $\mathrm{in}_w(f)(1, 0, 0, \dots, 0) \neq 0$. *Let* $m = \mathrm{ord}_w(f)/\mathrm{ord}_w(t)$, *which is a non-negative integer.*

1. *There exists a unique set of an element* $e \in k[[t, y]]$ *and* m *numbered elements* $G_1, G_2, \dots, G_m \in k[[y]]$ *satisfying*
$$ef = t^m + \sum_{j=1}^{m} G_j t^{m-j}.$$
2. $e \in k[[t, y]]^*$, *and* $\mathrm{ord}_w(G_j) \ge j\,\mathrm{ord}_w(t)$ *for every* $1 \le j \le m$.
3. *The element* $g = t^m + \sum_{j=1}^{m} G_j t^{m-j} \in k[[t, y]]$ *belongs to the subring* $k[[y]][t]$, *or the ring of polynomials with variable* t *with coefficients in* $k[[y]]$. $m\,\mathrm{ord}_w(t) = \mathrm{ord}_w(g)$.

Proof. It follows from the special case of Theorem 2.15 where $h = t^m$. □

Lemma 2.17. *Let* $f \in k[[x]] = k[[x_0, x_1, \dots, x_n]]$ *be an element with* $f \neq 0$.

1. *There exist* n *elements* $a_1, a_2, \dots, a_n \in k$ *with* $\mathrm{in}(f)(1, a_1, a_2, \cdots, a_n) \neq 0$, *where* $\mathrm{in}(f)(1, a_1, a_2, \cdots, a_n)$ *denotes the result of substitution* $x_0 = 1, x_1 = a_1, x_2 = a_2, \dots, x_n = a_n$ *in* $\mathrm{in}(f)$.
2. *Assume that* $\mathrm{in}(f)(1, a_1, a_2, \cdots, a_n) \neq 0$ *for* $a_1, a_2, \dots, a_n \in k$. *Let* $t = x_0$ *and* $y_i = x_i - a_i x_0$ *for* $1 \le i \le n$. *Then* $t, y_1, y_2, \dots, y_n$ *is a parameter system of* $k[[x]]$, *and the result of substitution* $t = 1, y_1 = y_2 = \cdots = y_n = 0$ *in* $\mathrm{in}(f)$ *is not zero.*

Proposition 2.18. *Let* $f_1, f_2, \dots, f_\ell \in k[[x]] = k[[x_0, x_1, \dots, x_n]]$ *be a finite number of elements with* $f = f_1 f_2 \cdots f_\ell \neq 0$. *Let* $m_\nu = \mathrm{ord}(f_\nu)$ *for* $1 \le \nu \le \ell$.

Let $a_1, a_2, \dots, a_n \in k$ *be elements with* $\mathrm{in}(f)(1, a_1, a_2, \dots, a_n) \neq 0$.

Let $t, y_1, y_2, \dots, y_n$ *be the parameter system of* $k[[x]]$ *satisfying* $t = x_0$ *and* $y_i = x_i - a_i x_0$ *for* $1 \le i \le n$. *We denote* $k[[t, y]] = k[[t, y_1, y_2, \dots, y_n]] = k[[x]]$ *and* $k[[y]] = k[[y_1, y_2, \dots, y_n]]$. *For every* $1 \le \nu \le \ell$ *the following 1–3 hold:*

1. *The result of substitution* $t = 1$, $y_1 = y_2 = \cdots = y_n = 0$ *in the initial polynomial of* f_ν *is not equal to* 0.
2. *There exists a unique set of an element* $e_\nu \in k[[t, y]]$ *and* m_ν *numbered elements* $G_{\nu,1}, G_{\nu,2}, \dots, G_{\nu,m_\nu} \in k[[y]]$ *satisfying* $e_\nu f_\nu = t^{m_\nu} + \sum_{j=1}^{m_\nu} G_{\nu,j} t^{m_\nu - j}$.
3. $e_\nu \in k[[t, y]]^*$, *and* $\mathrm{ord}(G_{\nu,j}) \ge j$ *for every* $1 \le j \le m_\nu$.

Proof. It follows from Lemma 2.17 and Corollary 2.16. □

Remark . Let $f = ug_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ be the irreducible decomposition of above f. For every $1 \leq \nu \leq r$ we define a k-linear map

$$\lambda_\nu : k^{n+1} \to k[[x]]/g_\nu k[[x]]$$

by $\lambda_\nu(a_0, a_1, a_2, \ldots, a_n) = \sum_{i=0}^{n} a_i(\partial g_\nu/\partial x_i) \mod g_\nu k[[x]]$. We can show $\lambda_\nu \neq 0$ for every $1 \leq \nu \leq r$. For each $1 \leq \nu \leq r$ we choose a homogeneous polynomial h_ν of degree 1 whose zero-locus contains the kernel of λ_ν. Let $\Psi = \prod_{\nu=1}^{r} \mathrm{in}(g_\nu) h_\nu$.

We can show that if elements $a_1, a_2, \ldots, a_n \in k$ satisfy a stronger condition $\Psi(1, a_1, a_2, \ldots, a_n) \neq 0$, then for every $1 \leq \nu \leq r$ any irreducible component of the polynomial $t^{m_\nu} + \sum_{j=1}^{m_\nu} G_{\nu,j} t^{m_\nu - j}$ in the ring $D[t]$ is a separable polynomial, where D denotes the quotient field of $k[[y]]$.

Definition 2.19.

1. Assume that a parameter system Q of the ring $k[[x]]$ is given. We call a pair $(g,\, t)$ of elements $g \in k[[x]]$ and $t \in Q$ a *semi-Weierstrass pair* of $k[[x]]$ *under* Q, if there exist a non-negative integer m and m elements $G_1, G_2, \ldots, G_m \in k[[y]] = k[[y_1, y_2, \ldots, y_n]]$ satisfying $g = t^m + \sum_{j=1}^{m} G_j t^{m-j}$, where $y_1, y_2, \ldots,\, y_n$ are elements in Q except t.
2. A non-empty subset of a parameter system of a regular local ring is called a *partial parameter system.*
3. Assume that a partial parameter system P of the ring $k[[x]]$ with $\sharp\, P = \dim k[[x]] - 1$ is given. We call a pair $(g,\, t)$ of elements $g \in k[[x]]$ and $t \in k[[x]]$ a *semi-Weierstrass pair* of $k[[x]]$ *over* P, if $t \notin P$ and the union $P \cup \{t\}$ is a parameter system of $k[[x]]$, and if there exist a non-negative integer m and m elements $G_1, G_2, \ldots, G_m \in k[[y]] = k[[y_1, y_2, \ldots, y_n]]$ satisfying $g = t^m + \sum_{j=1}^{m} G_j t^{m-j}$, where $P = \{y_1, y_2, \ldots, y_n\}$.
4. The integer m is called the *degree* of $(g,\, t)$, and is denoted by $\deg(g,\, t)$.
5. The m-tuple $(G_1, G_2, \ldots, G_m)$ of elements in $k[[y]]$ is called the *coefficient function sequence* of $(g,\, t)$.
6. If $m = \deg(g,\, t) = 0$, we put $\mathrm{w}(g,\, t) = \infty$. If $m = \deg(g,\, t) > 0$, we put $\mathrm{w}(g,\, t) = \min\{\mathrm{ord}(G_j)/j \mid 1 \leq j \leq m\}$. We call $\mathrm{w}(g,\, t) \in \mathbb{Q} \cup \{\infty\}$ the *weight* of $(g,\, t)$.
7. For $1 \leq j \leq m = \deg(g,\, t)$, let

$$\overline{G_j} = \begin{cases} 0 & \text{if } \mathrm{ord}(G_j) > j\,\mathrm{w}(g,\, t) \\ \mathrm{in}(G_j) & \text{if } \mathrm{ord}(G_j) = j\,\mathrm{w}(g,\, t). \end{cases}$$

   The polynomial $t^m + \sum_{j=1}^{m} \overline{G_j} t^{m-j}$ is called the *initial polynomial* of $(g,\, t)$, and is denoted by $\mathrm{in}(g,\, t)$.
8. If the coefficient function sequence $(G_1, G_2, \ldots, G_m)$ of a semi-Weierstrass pair $(g,\, t)$ satisfies $\mathrm{ord}(G_j) \geq j$ for every $1 \leq j \leq m$, then we say that the pair $(g,\, t)$ is a *Weierstrass pair.*

Lemma 2.20. *Assume either that a parameter system Q of the ring $k[[x]]$ is given, or that a partial parameter system P of the ring $k[[x]]$ with $\sharp\, P = \dim k[[x]] - 1$ is given. Let $(g,\, t)$ be a semi-Weierstrass pair either under Q or over P.*

1. *The degree* $\deg(g,\, t)$*, the coefficient function sequence* $(G_1,\, G_2, \ldots, G_m)$*, the weight* $\mathrm{w}(g,\, t)$ *and the initial polynomial* $\mathrm{in}(g,\, t)$ *of* $(g,\, t)$ *are uniquely defined depending on the pair* $(g,\, t)$*.*
2. $\deg(g,\, t) \geq \mathrm{ord}(g)$.
3. *Let* $m = \deg(g,\, t)$*. The following three conditions are equivalent:*
   (a) $\mathrm{w}(g,\, t) = \infty$.

   (b) $G_j = 0$ *for every* $1 \leq j \leq m$.
   (c) $\mathrm{in}(g, t) = t^m$.
4. *The following three conditions are equivalent:*
   (a) *The pair* (g, t) *is a Weierstrass pair.*
   (b) $\deg(g, t) = \mathrm{ord}(g)$.
   (c) $\mathrm{w}(g, t) \geq 1$.

Lemma 2.21. *Let* Q *be a parameter system of the ring* $k[[x]]$, *and* (g, t) *and* (g', t) *be semi-Weierstrass pairs under* Q *with the common second term* $t \in Q$. *Let* D *denote the quotient field of* $k[[y]] = k[[y_1, y_2, \dots, y_n]]$ *where* $y_1, y_2, \dots, y_n$ *are elements in* Q *except* t. *Let* $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ *be the irreducible decomposition in the ring* $D[t]$. *We assume that the coefficient of the term with the highest degree in* $g_\nu \in D[t]$ *is equal to* 1 *for every* $1 \leq \nu \leq r$.

1. *The pair* (gg', t) *is a semi-Weierstrass pair.* $\deg(gg', t) = \deg(g, t) + \deg(g', t)$. *The pair* (gg', t) *is a Weierstrass pair, if and only if, both of* (g, t) *and* (g', t) *are Weierstrass pairs.*
2. *For every* $1 \leq \nu \leq r$, $g_\nu \in k[[y]][t] \subset k[[x]]$.
3. *Assume that* (g, t) *is a Weierstrass pair. For every* $1 \leq \nu \leq r$, (g_ν, t) *is a Weierstrass pair, and the decomposition* $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ *gives the irreducible decomposition in the ring* $k[[x]]$

Proof. The claim 2 follows from Lemma of Gauss. (Zariski-Samuel[11], Vol.I, p.32.) □

3. Totally ordered abelian groups

In this section we consider totally ordered abelian groups, or abelian groups with total order compatible with the group structure. We prepare for treating the valuation theory in the next section.

Let Γ be an abelian group. For a subset $\Lambda \subset \Gamma$ we write $-\Lambda = \{-\gamma \mid \gamma \in \Lambda\}$. We assume that total order $\geq$ is defined on Γ. Either $\alpha \geq \beta$ or $\beta \geq \alpha$ holds for any two elements α, β. We write also $\beta \leq \alpha$ instead of $\alpha \geq \beta$. We denote $\alpha > \beta$ or $\beta < \alpha$, if $\alpha \geq \beta$ and $\alpha \neq \beta$.

If the following condition **toag** is satisfied, then we call Γ a *totally ordered abelian group.*

toag: If $\alpha, \beta, \gamma \in \Gamma$ and $\alpha \geq \beta$, then $\alpha + \gamma \geq \beta + \gamma$.

By Γ we denote a totally ordered abelian group below.

Let Γ' be another totally ordered abelian group, and $\varphi : \Gamma \to \Gamma'$ be a map. If $\varphi(\alpha) \geq \varphi(\beta)$ for every element $\alpha, \beta \in \Gamma$ with $\alpha \geq \beta$, then we say that φ *preserves order.* A homomorphism of abelian groups preserving order is called a *homomorphism of totally ordered abelian groups.*

Lemma 3.1. *Let* α, β *be elements in* Γ.

1. $\alpha \geq \beta$ *if and only if* $\alpha - \beta \geq 0$.
2. $\alpha > 0$ *if and only if* $-\alpha < 0$.
3. *If* $\alpha \geq 0$ *and* $\beta \geq 0$, *then* $\alpha + \beta \geq 0$.
4. *A totally ordered abelian group* Γ *contains no elements of finite order except* 0.
5. *Let* $\varphi : \Gamma \to \Gamma'$ *be a homomorphism of abelian groups. The homomorphism* φ *preserves order, if and only if, we have* $\varphi(\gamma) \geq 0$ *for every element* $\gamma \in \Gamma$ *with* $\gamma \geq 0$.

Lemma 3.2. *Let G be an abelian group, and G_+ be a subset such that $G_+ \cup (-G_+) = G$, $G_+ \cap (-G_+) = \{0\}$, and $\alpha + \beta \in G_+$ for every $\alpha \in G_+$ and $\beta \in G_+$.*

Then, we can define total order $\geq$ on G such that $\alpha \geq \beta$ for $\alpha, \beta \in G$ if $\alpha - \beta \in G_+$. With this order $\geq$ the group G is a totally ordered abelian group, and G_+ coincides with the set $\{\gamma \in G \mid \gamma \geq 0\}$ of non-negative elements.

Definition 3.3.
1. A subgroup Δ of Γ is said to be an *isolated subgroup* of Γ, if for any elements $\alpha \in \Delta$, $\beta \in \Gamma$ with $\alpha \geq \beta \geq 0$ it follows that $\beta \in \Delta$.
2. The cardinal number of the set of all isolated subgroups Δ of Γ with $\Delta \neq \Gamma$ is called the *rank* of Γ, and is denoted by $\operatorname{rank}\Gamma$.
3. The dimension of the $\mathbb{Q}$-vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is called the *rational rank* of Γ, and is denoted by $\operatorname{r.rank}\Gamma$.
4. We say that Γ is *archimedean*, if for every element $\alpha, \beta \in \Gamma$ with $\alpha \geq 0$ and $\beta > 0$ we have a positive integer m with $m\beta \geq \alpha$.

Lemma 3.4.
1. *Let Δ and Δ' be isolated subgroups of Γ. Either $\Delta \subset \Delta'$ or $\Delta \supset \Delta'$ holds.*
2. *Let $\alpha \in \Gamma$ be an element with $\alpha > 0$. The set of elements $\gamma \in \Gamma$ such that there is a positive integer m with $m\alpha \geq \gamma \geq -m\alpha$ is a non-zero isolated subgroup of Γ.*
3. *If* $\operatorname{r.rank}\Gamma$ *is finite, then*
$$\operatorname{rank}\Gamma \leq \operatorname{r.rank}\Gamma.$$
4. *Let Δ be an isolated subgroup of Γ. We have unique total order on the residue group Γ/Δ such that Γ/Δ is a totally ordered abelian group and the canonical surjective homomorphism $\pi : \Gamma \to \Gamma/\Delta$ preserves order.*
$$\operatorname{rank}\Gamma = \operatorname{rank}\Delta + \operatorname{rank}\Gamma/\Delta, \quad \operatorname{r.rank}\Gamma = \operatorname{r.rank}\Delta + \operatorname{r.rank}\Gamma/\Delta.$$
5. *Let $\varphi : \Gamma \to \Gamma'$ be a homomorphism of totally ordered abelian groups. The kernel $\Delta = \varphi^{-1}(0)$ is an isolated subgroup of Γ. The induced map $\Gamma/\Delta \to \Gamma'$ is an injective homomorphism of totally ordered abelian groups.*
6. *Let $\varphi : \Gamma \to \Gamma'$ be an injective homomorphism of totally ordered abelian groups. The inverse image $\varphi^{-1}(\Delta') = \Delta' \cap \Gamma$ is an isolated subgroup of Γ for any isolated subgroup Δ' of Γ'. For every isolated subgroup Δ of Γ there exists an isolated subgroup Δ' of Γ' with $\Delta = \Delta' \cap \Gamma$.*
$$\operatorname{rank}\Gamma \leq \operatorname{rank}\Gamma', \quad \operatorname{r.rank}\Gamma \leq \operatorname{r.rank}\Gamma'.$$

Proof. 3. See Zariski-Samuel [11], Vol. II, p.50, NOTE.
4. Apply Lemma 3.2.
6. The existence of an isolated subgroup Δ' of Γ' with $\Delta = \Delta' \cap \Gamma$ follows from Zorn's lemma and 2 above. $\square$

Let Γ' and Γ'' be totally ordered abelian groups. We can define *lexicographic order* $\geq$ on the direct sum $\Gamma' + \Gamma''$. Let α', $\beta' \in \Gamma'$ and α'', $\beta'' \in \Gamma''$. We define $(\alpha', \alpha'') \geq (\beta', \beta'')$ with respect to the lexicographic order, if either $\alpha' > \beta'$, or $\alpha' = \beta'$ and $\alpha'' \geq \beta''$.

Lemma 3.5. *Let Δ be an isolated subgroup of Γ. Assume that there exists a homomorphism $\sigma : \Gamma/\Delta \to \Gamma$ of abelian groups such that the composition $\pi\sigma$ is the identity map of Γ/Δ, where $\pi : \Gamma \to \Gamma/\Delta$ denotes the canonical surjective homomorphism. Then, Γ is isomorphic to the direct sum $(\Gamma/\Delta)+\Delta$ with the lexicographic order as totally ordered abelian groups.*

Lemma 3.6. *The following three conditions are equivalent:*

1. Γ *is archimedean.*
2. $\operatorname{rank}\Gamma \leq 1$.
3. *There exists an injective homomorphism* $\Gamma \to \mathbb{R}$ *of totally ordered abelian groups.*

Proof. The implication 1⇒3 follows from the theory of Dedekind cut. (Zariski-Samuel [11], Vol. II, p.45.) ☐

Let n be a positive integer. We can define the *lexicographic order* $\geq$ on the abelian group $\mathbb{Z}^n$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}^n$, where $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n$ are integers. If there is a number i with $1 \leq i \leq n$ such that $\alpha_j = \beta_j$ for $1 \leq j < i$ and $\alpha_i > \beta_i$, then we write $\alpha > \beta$. We define $\alpha \geq \beta$, if $\alpha > \beta$ or $\alpha = \beta$. We can check that with the lexicographic order the group $\mathbb{Z}^n$ is a totally ordered abelian group with $\operatorname{rank}\mathbb{Z}^n = n$. Let $\Delta_i = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n \mid \alpha_j = 0 \text{ for } 1 \leq j \leq i.\}$ for $0 \leq i \leq n$. We have

$$\mathbb{Z}^n = \Delta_0 \supset \Delta_1 \supset \cdots \supset \Delta_{n-1} \supset \Delta_n = \{0\},$$

and Δ_i's are isolated subgroups of $\mathbb{Z}^n$.

Unless it is mentioned otherwise, we assume below that $\mathbb{Z}^n$ has the lexicographic order.

Remark . We can define total order on $\mathbb{Z}^n$ essentially different from the lexicographic order, and with it $\mathbb{Z}^n$ is a totally ordered abelian group.

Let $\theta_1 = 1, \theta_2, \theta_3, \ldots, \theta_n \in \mathbb{R}$ be real numbers linearly independent over $\mathbb{Q}$. Giving the restriction of the order on $\mathbb{R}$, the subgroup $\Gamma_0 = \sum_{i=1}^n \mathbb{Z}\theta_i$ of $\mathbb{R}$ is a totally ordered abelian group. The group Γ_0 is isomorphic to $\mathbb{Z}^n$ as abelian groups.

However, by Lemma 3.6 we have $\operatorname{rank}\Gamma_0 = 1$.

Thus, even if a totally ordered abelian group Γ is isomorphic to $\mathbb{Z}^n$ as abelian groups, Γ is not necessarily isomorphic to $\mathbb{Z}^n$ as totally ordered abelian groups.

4. Valuation theory

We treat the valuation theory in this section. (Zariski-Samuel [11], Vol.II, Chapter VI and Appendix 2.) By E we denote a fixed field.

We have defined the concept of valuations in Definition 2.11.

Lemma 4.1. *Let* $v : E \to \Gamma \cup \{\infty\}$ *be a valuation.*

1. $v(f) = v(-f)$.
2. *Let* $f_1, f_2, \ldots, f_r \in E$. *We have*
$$v(\sum_{i=1}^r f_i) \geq \min\{v(f_i) \mid 1 \leq i \leq r\}.$$
If, moreover, $v(f_1) < v(f_i)$ *for* $2 \leq i \leq r$, *then we have*
$$v(\sum_{i=1}^r f_i) = v(f_1).$$
3. *Let* $R_v = \{f \in E \mid v(f) \geq 0\}$. *The set* R_v *is a normal local subdomain of* E *with the maximal ideal* $M(R_v) = \{f \in E \mid v(f) > 0\}$. *If* $f \in E$ *and* $f \notin R_v$, *then* $1/f \in R_v$.

4. *Let D be a subfield of E. The restricted map $v \mid D : D \to \Gamma \cup \{\infty\}$ is a valuation of D.*
5. *The image $v(E^*)$ is a subgroup of Γ.*

Proof. We consider the normality in claim 3. Assume that an element $f \in E$ is integral over R_v. We have a positive integer m and m elements $a_1, a_2, \dots, a_m \in R_v$ with $f^m + \sum_{i=1}^m a_i f^{m-i} = 0$. By 1 and 2 above, one obtains $mv(f) \geq \min\{(m-i)v(f) + v(a_i) \mid 1 \leq i \leq m\}$. Thus, for some integer i with $1 \leq i \leq m$ we have $mv(f) \geq (m-i)v(f) + v(a_i) \geq (m-i)v(f)$, which implies $v(f) \geq 0$ and $f \in R_v$. □

Definition 4.2. Let $v : E \to \Gamma \cup \{\infty\}$ be a valuation.

1. The local domain $R_v = \{f \in E \mid v(f) \geq 0\}$ is called the *valuation ring* of v.
2. Putting $\Gamma_1 = v(E^*)$ we have a surjective induced map by v
$$v_1 : E \to \Gamma_1 \cup \{\infty\}.$$
Obviously also v_1 is a valuation of E. We call v_1 the *surjective reduction* of v.
Below by $v_1 : E \to \Gamma_1 \cup \{\infty\}$ we denote the surjective reduction of v.
3. If $\Gamma_1 = \{0\}$, then we say that the valuation v is *trivial*.
4. Let D be a subfield of E. If the restricted map $v \mid D : D \to \Gamma \cup \{\infty\}$ is a trivial valuation, then we say that v is a valuation *over* D.
5. The rank $\operatorname{rank} \Gamma_1$ of the totally ordered abelian group Γ_1 is called the *rank* of v, and is denoted by $\operatorname{rank} v$.
6. The rational rank $\operatorname{r.rank} \Gamma_1$ of the totally ordered abelian group Γ_1 is called the *rational rank* of v, and is denoted by $\operatorname{r.rank} v$.
7. Let $v' : E \to \Gamma' \cup \{\infty\}$ be another valuation of the field E. Let $v'_1 : E \to \Gamma'_1 \cup \{\infty\}$ be the surjective reduction of v'. We say that v and v' are *equivalent*, if there exists an isomorphism $\varphi : \Gamma_1 \to \Gamma'_1$ of totally ordered abelian groups such that $\varphi v_1 = v'_1$, where we denote the extension $\Gamma_1 \cup \{\infty\} \to \Gamma'_1 \cup \{\infty\}$ of φ satisfying $\varphi(\infty) = \infty$ by the same symbol φ.

Lemma 4.3. *Equivalent valuations have the same valuation ring, the same rank, and the same rational rank.*

Definition 4.4. Let $v : E \to \Gamma \cup \{\infty\}$ and $v' : D \to \Gamma' \cup \{\infty\}$ be valuations. We say that the valuation v is an *extension* of v', or v' is a *restriction* of v, if D is a subfield of a field E and if there is an injective homomorphism $\varphi : \Gamma' \to \Gamma$ of totally ordered abelian groups such that $\varphi v' = v \mid D$, where we denote the extension $\Gamma' \cup \{\infty\} \to \Gamma \cup \{\infty\}$ of φ satisfying $\varphi(\infty) = \infty$ by the same symbol φ.

Lemma 4.5. *Assume that a valuation v' of a field D is a restriction of a valuation v. We have $R_{v'} = R_v \cap D$, where $R_{v'}$ and R_v denote the valuation rings of v' and v respectively.*

Lemma 4.6.
1. *A valuation v of a field E is trivial, if and only if, the valuation ring R_v coincides with E.*
2. *A valuation v is one over a subfield D, if and only if, the valuation ring R_v contains D.*

Definition 4.7. A subdomain V of E is said to be a *valuation ring* of E, if for every element $f \in V$ with $f \notin V$ it follows that $1/f \in V$.

Remark . By the above condition E coincides with the quotient field of V.

Below in this section until Theorem 4.24, by V we denote a valuation ring of the field E.

Lemma 4.8. 1. *Let $\Gamma = E^*/V^*$ and $\Gamma_+ = (V - \{0\})/V^*$. We can define total order on Γ such that Γ_+ coincides with the set of non-negative elements. With this total order Γ is a totally ordered abelian group.*
2. *Let $v : E^* \to E^*/V^* = \Gamma$ be the canonical surjective homomorphism as abelian groups. By putting $v(0) = \infty$ we define the extension $v : E \to \Gamma \cup \{\infty\}$, which is denoted by the same symbol. Then v is a valuation of E.*

Proof. Claim 1 follows from Lemma 3.2. □

The above map $v : E \to (E^*/V^*) \cup \{\infty\}$ is called the *valuation defined by a valuation ring V.*

Lemma 4.9. 1. *Let $v : E \to \Gamma \cup \{\infty\}$ be a valuation of E, R_v be a valuation ring of v, and $v_1 : E \to (E^*/R_v^*) \cup \{\infty\}$ be the valuation defined by the valuation ring R_v. Then, v and v_1 are equivalent.*
2. *Let V be a valuation ring of E, and $v : E \to (E^*/V^*) \cup \{\infty\}$ be the valuation defined by V. Then, we have $V = R_v$.*

Corollary 4.10. 1. *The set of equivalence classes of valuations of E and the set of valuation rings of E have one-to-one correspondence described above.*
2. *Any valuation ring of E is a normal local domain.*

Definition 4.11. Let $\Gamma = E^*/V^*$.
1. The rank $\operatorname{rank}\Gamma$ of the totally ordered abelian group Γ is called *the rank of V*, and is denoted by $\operatorname{rank} V$.
2. The rational rank $\operatorname{r.rank}\Gamma$ of the totally ordered abelian group Γ is called *the rational rank of V*, and is denoted by $\operatorname{r.rank} V$.
3. Assume that V contains a subfield D. We identify D and the image of D by the canonical surjective homomorphism $V \to V/M(V)$. The transcendence degree $\operatorname{tr.deg}_D V/M(V)$ of the residue field $V/M(V)$ over D is called the *dimension* of V over D, and it is denoted by $\dim_D V$.

Lemma 4.12. *Let $v : E \to \Gamma \cup \{\infty\}$ be the valuation defined by V.*
1. $\operatorname{rank} V = \operatorname{rank} v = \operatorname{rank}\Gamma$, $\operatorname{r.rank} V = \operatorname{r.rank} v = \operatorname{r.rank}\Gamma$.
2. *If* $\operatorname{r.rank} V$ *is finite, then* $\operatorname{rank} V \leq \operatorname{r.rank} V$.

Proof. Claim 2 follows from Lemma 3.4.3. □

Lemma 4.13. *Let D be a subfield of E.*
1. *The intersection $V' = V \cap D$ is a valuation ring of D. $V'^* = V^* \cap D$. $M(V') = M(V) \cap D$.*
2. *The valuation $v : E \to (E^*/V^*) \cup \{\infty\}$ defined by V is an extension of the valuation $v' : D \to (D^*/V'^*) \cup \{\infty\}$ defined by V'.*
3. $\operatorname{rank} V' \leq \operatorname{rank} V$, $\operatorname{r.rank} V' \leq \operatorname{r.rank} V$.

Proof. Claim 3 follows from Lemma 3.4.6. □

Lemma 4.14. *Let $v : E \to \Gamma \cup \{\infty\}$ be the valuation defined by V.*
1. *Let Δ be an isolated subgroup of Γ. Then, $\mathfrak{p} = \{f \in V \mid v(f) \notin \Delta\}$ is a prime ideal in V. Besides, we have $V - \mathfrak{p} = \{f \in V \mid v(f) \in \Delta\}$.*

2. *Let* $\mathfrak{p}$ *be a prime ideal of* V. *Then,* $\Delta = v(V_{\mathfrak{p}}^{*})$ *is an isolated subgroup of* Γ. *Besides, we have* $V_{\mathfrak{p}}^{*} = v^{-1}(\Delta)$.
3. *By the correspondence in* 1 *and* 2 *above, isolated subgroups of* Γ *and prime ideals of* V *have one-to-one correspondence, under which inclusion relations are reversed.*

Corollary 4.15. 1. *The rank* $\operatorname{rank} V$ *of a valuation ring* V *is equal to the cardinal number of the set of all non-zero prime ideals of* V.
2. *Let* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *be prime ideals of a valuation ring* V. *Either* $\mathfrak{p} \supset \mathfrak{p}'$ *or* $\mathfrak{p} \subset \mathfrak{p}'$ *holds.*
3. *Let* D *be a subfield of* E, *and* $V' = V \cap D$. *The map* $\operatorname{Spec}(V) \to \operatorname{Spec}(V')$ *induced by the inclusion homomorphism* $V' \hookrightarrow V$ *is surjective.*

Proof. Claim 2 follows from Lemma 3.4.1. Claim 3 follows from Lemma 3.4.6. □

Remark . By the above corollary $\operatorname{rank} V$ coincides with the dimension of V for a valuation ring V. Therefore, we should write $\dim V$ instead of $\operatorname{rank} V$ for consistency. However, for valuation rings the word "dimension" is used for the different meaning as in Definition 4.11.3. This is because of a historical reason.

Lemma 4.16. *Let* $v : E \to \Gamma \cup \{\infty\}$ *be the valuation defined by* V, *and* $\mathfrak{p}$ *be a prime ideal of* V.

1. *The residue ring* $\overline{V} = V/\mathfrak{p}$ *of* V *by* $\mathfrak{p}$ *is a valuation ring of the field* $V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}}$. *Let* $\overline{v} : V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}} \to \overline{\Gamma} \cup \{\infty\}$ *denote the valuation defined by* $\overline{V}$. *Let* $\Delta = v(V_{\mathfrak{p}}^{*})$. *There exists an isomorphism* $\Delta \xrightarrow{\sim} \overline{\Gamma}$ *of totally ordered abelian groups which makes the following diagram commutative:*

$$\begin{array}{ccc} V_{\mathfrak{p}}^{*} & \longrightarrow & (V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}})^{*} \\ \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle \overline{v}} \\ \Delta & \xrightarrow{\sim} & \overline{\Gamma} \end{array}$$

The map $V_{\mathfrak{p}}^{*} \to (V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}})^{*}$ *is the one induced by the canonical surjective homomorphism* $V_{\mathfrak{p}} \to V_{\mathfrak{p}}/\mathfrak{p}V_{\mathfrak{p}}$. *Note that it is surjective.*

If V *contains a subfield* D, *then we have* $\dim_D V = \dim_D \overline{V}$.

2. *The localization* $V_1 = V_{\mathfrak{p}}$ *of* V *by* $\mathfrak{p}$ *is a valuation ring of* E. *Let* $v_1 : E \to \Gamma_1 \cup \{\infty\}$ *denote the valuation defined by* V_1. *Let* $\Delta = v(V_{\mathfrak{p}}^{*})$. *There exists an isomorphism* $\Gamma/\Delta \xrightarrow{\sim} \Gamma_1$ *of totally ordered abelian groups which makes the following diagram commutative:*

$$\begin{array}{ccc} E^{*} & = & E^{*} \\ \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle v_1} \\ \Gamma & \longrightarrow & \Gamma_1 \end{array}$$

The map $\Gamma \to \Gamma_1$ *is the composition of the canonical surjective homomorphism* $\Gamma \to \Gamma/\Delta$ *and the isomorphism* $\Gamma/\Delta \xrightarrow{\sim} \Gamma_1$.

Corollary 4.17. *Let* $\mathfrak{p}$ *be a prime ideal of* V. *We have*

$$\operatorname{rank} V = \operatorname{rank}(V/\mathfrak{p}) + \operatorname{rank} V_{\mathfrak{p}}, \quad \operatorname{r.rank} V = \operatorname{r.rank}(V/\mathfrak{p}) + \operatorname{r.rank} V_{\mathfrak{p}}.$$

Proof. It follows from Lemma 3.4.4. □

Theorem 4.18 (Zariski-Samuel [11], Vol.II. p.8, THEOREM 3.). *Let V' be a subring of E with $V \subset V' \subset E$. Then, we have $V' = V_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ of V.*

Corollary 4.19. *Let V' be a local subring of E dominating V. Then, we have $V' = V$.*

Remark . Recall that V' dominates V, if $V \subset V'$ and $M(V') \cap V = M(V)$. (Refer to the beginning part of Section 2.)

Theorem 4.20 (Zariski-Samuel [11], Vol.II, p.12, THEOREM 5.). *Let A be a local subring of E. Then, there exists a valuation ring of E dominating A.*

Theorem 4.21.

1. (Zariski-Samuel [11], Vol.II, p.331, PROPOSITION 2.) *Let A be a noetherian local domain with the quotient field E, and assume that V dominates A. We have*
$$\mathrm{r.rank}\, V + \mathrm{tr.deg}_{(A/M(A))}(V/M(V)) \leq \dim A.$$
2. (Zariski-Samuel [11], Vol.II, p.335 PROPOSITION 3.) *If the equality holds in the above, then the value group of the valuation defined by V is isomorphic to $\mathbb{Z}^r$ as abelian groups (not necessarily as totally ordered abelian groups) where $r = \mathrm{r.rank}\, V$, and the residue field $V/M(V)$ is finitely generated over $A/M(A)$.*

Corollary 4.22.

1. *Under the same assumptions as in the theorem we have*
$$\mathrm{rank}\, V + \mathrm{tr.deg}_{(A/M(A))}(V/M(V)) \leq \dim A.$$
2. *If the equality holds in the above, then the value group of the valuation defined by V is isomorphic to $\mathbb{Z}^r$ as totally ordered abelian groups where $r = \mathrm{rank}\, V$, and the residue field $V/M(V)$ is finitely generated over $A/M(A)$.*

Proof. Claim 1 follows from Theorem 4.21.1 and Lemma 4.12.2. Claim 2 follows from Theorem 4.21.2 and Lemma 3.5. □

Lemma 4.23. *Let D be a subfield of E. Let $U = V \cap D$.*

1. (Zariski-Samuel [11], Vol.II, p.25, COROLLARY 1.)
$$\mathrm{tr.deg}_{(U/M(U))}(V/M(V)) \leq \mathrm{tr.deg}_D E.$$
2. (Zariski-Samuel [11], Vol.II, p.333, LEMMA 1.) *Assume that* $\mathrm{tr.deg}_D E$ *is finite. We have*
$$\mathrm{r.rank}\, V - \mathrm{r.rank}\, U \leq \mathrm{tr.deg}_D E - \mathrm{tr.deg}_{(U/M(U))}(V/M(V)).$$

Theorem 4.24 (Zariski-Samuel [11], Vol.II, p.41, THEOREM 16.). *A valuation ring V is noetherian, if and only if, either $V = E$ or the value group of the valuation defined by V is isomorphic to $\mathbb{Z}$.*

Proposition 4.25. *Let V be a ring. The following three conditions are equivalent:*

1. *V is a noetherian normal local ring of dimension one.*
2. *V is a regular local ring of dimension one.*
3. *V is a domain, it is a valuation ring of the quotient field, and the value group of the valuation defined by V is isomorphic to $\mathbb{Z}$.*

Proof. 1⇔2. See Matsumura [7], p.124, (17.H), THEOREM 37.
2⇔3. Easy exercise in the ring theory. □

Lemma 4.26. *Let V be a regular local ring of dimension one with the quotient field E. Let $v : E \to \Gamma \cup \{\infty\}$ be the valuation defined by V. Let $\pi \in M(V)$ be a generator of the maximal ideal $M(V)$. Then, $\Gamma = \mathbb{Z}v(\pi)$, and $v(f) = \nu v(\pi)$ for $f \in E^*$ and $\nu \in \mathbb{Z}$, if and only if, there exists an element $u \in V^*$ with $f = u\pi^\nu$.*

Proof. Since V is a UFD, (Matsumura [7], p. 142, (19.B), THEOREM 48.), every element $f \in E^*$ can be written uniquely in the form $f = u\pi^\nu$ with $u \in V^*$ and $\nu \in \mathbb{Z}$. Thus the correspondence $f \in E^* \mapsto \nu \in \mathbb{Z}$ induces an isomorphism $\Gamma = E^*/V^* \xrightarrow{\sim} \mathbb{Z}$ of abelian groups. □

Definition 4.27. Let A be a noetherian local domain with the quotient field E, and V be a valuation ring of E dominating A. We say that the pair (A, V) is *of iterated divisor type*, if the equality $\operatorname{rank} V = \dim A$ holds.

If a given ring A is obvious, we say that a valuation ring V is of iterated divisor type, if the pair (A, V) is so.

Remark . Assume that a pair (A, V) is of iterated divisor type. Let $n = \operatorname{rank} V = \dim A$. V has $n + 1$ prime ideals $\mathfrak{p}_i$, $1 \leq i \leq n$ with

$$\{0\} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n = M(V).$$

By Corollary 4.22, and Theorem 4.24 and Proposition 4.25 for every $1 \leq i \leq n$ the ring $V_{\mathfrak{p}_i}/\mathfrak{p}_{i-1}V_{\mathfrak{p}_i}$ is a regular local ring of dimension one, and the residue field $V/M(V)$ is a finite algebraic extension of $A/M(A)$.

We can show that if A is excellent, then the set of all valuation rings V of iterated divisor type is dense in the Zariski-Riemann space of A. (Zariski-Samuel [11], Vol.II. p.110–p.123, §17.) We call what Zariski called "the Riemann surface" the Zariski-Riemann space, since the word "surface" is strange. In Zariski's notation $E(A)$ is our Zariski-Riemann space of A.

The converse of Lemma 4.16 holds.

Lemma 4.28. *Let V_1 be a valuation ring of E and $\overline{V}$ be a valuation ring of the residue field $V_1/M(V_1)$. Let V be the inverse image of $\overline{V}$ by the canonical surjective homomorphism $V_1 \to V_1/M(V_1)$. Then, V is a valuation ring of E with $M(V_1) \subset V \subset V_1 \subset E$. Besides, $\mathfrak{p} = M(V_1) \subset V$ is a prime ideal of V, $V_1 = V_\mathfrak{p}$, $\overline{V} \cong V/\mathfrak{p}$, and $V/M(V) \cong \overline{V}/M(\overline{V})$.*

Remark . If $V_1 \neq E$ and $\overline{V} \neq V_1/M(V_1)$, then the valuation ring V is called the *composite* with V_1 and $\overline{V}$.

Corollary 4.29. *Let V be a valuation ring, and $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of V with $\mathfrak{p} \subset \mathfrak{q}$. Then, $\mathfrak{p} = \mathfrak{p}V_\mathfrak{q}$ as sets.*

Proof. Obviously $\mathfrak{p} \subset \mathfrak{p}V_\mathfrak{q} \subset \mathfrak{p}V_\mathfrak{p}$. Let $V_1 = V_\mathfrak{p}$. By Lemma 4.28 we have $M(V_1) = \mathfrak{p}V_\mathfrak{p} = \mathfrak{p}V_\mathfrak{p} \cap V = \mathfrak{p}$. □

Definition 4.30. Let $v : E \to \Gamma \cup \{\infty\}$ be a valuation of E with the valuation ring $V = R_v$, and D be a subfield of E. We assume that E is a finite algebraic extension of D.

1. The index of the subgroup $v(D^*)$ in $v(E^*)$ is called the *ramification index* of v (or of V) over D.
2. The relative degree $[V/M(V) : (V \cap D)/M(V \cap D)]$ of residue fields is called the *relative degree* of v (or of V) over D.

Theorem 4.31. *Let D be a subfield of E. Assume that E is a finite algebraic extension of D. Let V' be a valuation ring of D. Let $V_1, V_2, \dots, V_g$ be valuation rings of E with $V_i \cap D = V'$ for $1 \le i \le g$. We assume $V_i \ne V_j$ if $i \ne j$. Let e_i and f_i denote the ramification index and the relative degree of V_i over D respectively for $1 \le i \le g$.*

1. (Zariski-Samuel [11], Vol. II, p.55, THEOREM 19.) *We have*
$$\sum_{i=1}^{g} e_i f_i \le [E : D].$$
2. (Zariski-Samuel [11], Vol. II, p.17, THEOREM 8 and p.60-p.61, THEOREM 20.) *Assume that any valuation ring V of E with $V \cap D = V'$ coincides with one of V_i's. Then, $\overline{V} = \cap_{i=1}^{g} V_i$ coincides with the integral closure of V' in E. If $\overline{V}$ is a finitely generated V'-module, then the equality holds in the above.*
3. (Zariski-Samuel [11], Vol. I, p.285, THEOREM 21.) *Assume moreover that V' is a regular local ring of dimension one. If the equality above holds, then the integral closure of V' in E is a finitely generated V'-module.*

Lemma 4.32. *Let A be a noetherian local domain with quotient field E, and V be a valuation ring of E dominating A. Assume that the completion $\hat{A}$ of A is a domain. Then, there exists a valuation ring $\hat{V}$ of the quotient field $\hat{E}$ of $\hat{A}$ such that $\hat{V}$ dominates $\hat{A}$ and $\hat{V} \cap E = V$.*

Proof. This is Lemma 13 of Abhyankar [1] p.513. □

Lemma 4.33. *In above Lemma 4.32, if A contains the ground field k and if $\dim_k V = 0$, then there exists a valuation ring $\hat{V}$ of the quotient field $\hat{E}$ of $\hat{A}$ such that $\hat{V}$ dominates $\hat{A}$, $\dim_k \hat{V} = 0$ and $\hat{V} \cap E = V$.*

Proof. First, note that $A/M(A) \cong \hat{A}/M(\hat{A}) \cong V/M(V) \cong k$.

Since $\text{tr.deg}_k(V/M(V)) = \dim_k V = 0$ and since k is algebraically closed, we have $V/M(V) \cong k$. Since V dominates A, we have $k \subset A/M(A) \subset V/M(V) \cong k$. Since $A/M(A) \cong \hat{A}/M(\hat{A})$ by definition, we obtain the desired claim.

By Lemma 4.32 there exists a valuation ring $\hat{V}_0$ of $\hat{E}$ dominating $\hat{A}$ with $V = \hat{V}_0 \cap E$. Let $L = \hat{V}_0/M(\hat{V}_0)$. By Theorem 4.21 we have $r = \text{tr.deg}_k L \le \dim \hat{A}$. Let $x_1, x_2, \dots, x_r \in L$ be algebraically independent elements over k. The field $k(x_1, x_2, \dots, x_r)$ is isomorphic to the rational function field over k with r variables, and there is a valuation ring $\overline{V}_1$ of $k(x_1, x_2, \dots, x_r)$ such that $k \subset \overline{V}_1$ and $\dim_k \overline{V}_1 = 0$. By Theorem 4.20 there is a valuation ring $\overline{V}$ of L dominating $\overline{V}_1$. By Corollary 4.19 one knows $\overline{V}_1 = \overline{V} \cap k(x_1, x_2, \dots, x_r)$. By Lemma 4.23.1 one has $0 \le \text{tr.deg}_{(\overline{V}_1/M(\overline{V}_1))}(\overline{V}/M(\overline{V})) \le \text{tr.deg}_{k(x_1,x_2,\dots,x_r)} L = 0$ and $\dim_k \overline{V} = 0$.

Let $\pi : \hat{V}_0 \to L = \hat{V}_0/M(\hat{V}_0)$ denote the canonical surjective homomorphism. Let $\hat{V} = \pi^{-1}(\overline{V})$ be the inverse image of $\overline{V} \subset L$ by π. By Lemma 4.28 $\hat{V}$ is a valuation ring of E with $\dim_k \hat{V} = \dim_k \overline{V} = 0$.

$\pi(\hat{A}) \cong \hat{A}/(M(\hat{V}_0) \cap \hat{A}) \cong \hat{A}/M(\hat{A}) \cong k$. Thus $\hat{V} = \pi^{-1}(\overline{V}) \supset \pi^{-1}(k) \supset \hat{A}$. Since $M(\hat{V}) \supset M(\hat{V}_0) \supset M(\hat{A})$ by Corollary 4.29, one knows that $\hat{V}$ dominates $\hat{A}$.

Since $\pi(V) \cong V/(M(\hat{V}_0) \cap V) = V/M(V) \cong k$, one has $\hat{V} = \pi^{-1}(\overline{V}) \supset \pi^{-1}(k) \supset V$. Putting $V' = \hat{V} \cap E$, one has $V \subset V'$. Since $M(V) \subset M(\hat{V}_0) \subset M(\hat{V})$, we have $M(V) \subset M(\hat{V}) \cap E = M(V')$, and V' dominates V. By Corollary 4.19 one concludes $\hat{V} \cap E = V' = V$. □

5. Space germ

In this section we give the definition of our main objects, a space germ, a framed space germ, a basic space germ, an elementary monoidal transform, an iterated elementary monoidal transform (an IEMT, in short), an iterated analytic monoidal transform (an IAMT, in short) and so on.

Let $k[[x]] = k[[x_0, x_1, \dots, x_r]]$ denote the formal power series ring over k with $r+1$ variables $x_0, x_1, \dots, x_r$. We give the $M(k[[x]])$-adic topology to the ring $k[[x]]$. Recall that a non-empty subset of a parameter system of a regular local ring is called a *partial parameter system.* (Definition 2.19.2.)

Definition 5.1. 1. A pair (B, W) is called a *space germ* over k, if it satisfies the following two conditions:
   (a) The first term B is a topological local ring. Ring B is isomorphic to $k[[x]] = k[[x_0, x_1, \dots, x_r]]$ for some non-negative integer r.
   (b) The second term W is a valuation ring of the quotient field of B. We assume that W dominates B, and $\dim_k W = 0$.
2. A partial parameter system Q of B is called a *frame* of (B, W).
3. For any partial parameter system $Q = \{y_0, y_1, \dots, y_n\}$ of B by $k[[Q]]$ we denote the minimal closed k-subalgebra $k[[y_0, y_1, \dots, y_n]]$ of B generated by $y_0, y_1, \dots, y_n$.
4. A triplet
$$(B, W, Q),$$
where the pair (B, W) is a space germ over k, and Q is a frame of (B, W), is called a *framed space germ* over k.
5. The number $\dim B = r+1$ is called the *dimension* of a space germ (B, W) or a framed space germ (B, W, Q).
6. The number $\sharp Q$ of elements in Q is called the *basic dimension* of a framed space germ (B, W, Q).

Lemma 5.2. *Let (B, W) be a space germ.*
1. *$W/M(W) \cong k$.*
2. *Let Q be a frame of (B, W), $A = k[[Q]]$, and V be the intersection of W and the quotient field of A. The triplet (A, V, Q) is a framed space germ over k such that Q is a parameter system of A.*

Proof. 1. Since $\mathrm{tr.deg}_k(W/M(W)) = \dim_k W = 0$, and since k is an algebraically closed field, we have $W/M(W) \cong k$.
2. Obviously the pair (A, V) satisfies the first condition of a space germ, and Q is a parameter system of A. By Lemma 4.13, V is a valuation ring of the quotient field E of A. Since $M(A) \subset M(B) \subset M(W)$, we have $M(A) \subset M(W) \cap E = M(V)$. Thus, V dominates A. Since $k \subset V/M(V) \subset W/M(W)$, we have $0 \le \dim_k V \le \dim_k W = 0$. □

Definition 5.3. 1. Let (B, W, Q) be a framed space germ. The space germ (B, W) without a frame is called the *underlying space germ* of (B, W, Q). We write $(B, W) = (B, W, Q)_\bullet$ to represent that (B, W) is the underlying space germ of (B, W, Q).
2. We call the framed space germ (A, V, Q) described in Lemma 5.2.2 the *basic framed space germ* of a framed space germ (B, W, Q), or the basic framed space germ of the space germ (B, W) with respect to the frame Q. We write

$(A, V, Q) = \underline{(B, W, Q)}$ to represent that (A, V, Q) is the basic framed space germ of (B, W, Q).

3. Let (B, W) and (B', W') be space germs. A morphism $\varphi : F \to F'$ of k-algebras from the quotient field F of B to the quotient field F' of B' satisfying the following conditions is called a *morphism* of space germs from (B, W) to (B', W')
   (a) $\varphi(B) \subset B'$ and $\varphi(W) \subset W'$.
   (b) The induced homomorphisms $\varphi : B \to B'$ and $\varphi : W \to W'$ are local.
   (c) The homomorphism $\varphi : B \to B'$ is continuous.
   We denote $\varphi : (B, W) \to (B', W')$.
4. Let (B, W, Q) and (B', W', Q') be framed space germs. A morphism $\varphi : F \to F'$ of k-algebras from the quotient field F of B to the quotient field F' of B' satisfying the following conditions is called a *morphism* of framed space germs from (B, W, Q) to (B', W', Q')
   (a) $\varphi(B) \subset B'$, $\varphi(W) \subset W'$, and $\varphi(k[[Q]]) \subset k[[Q']]$.
   (b) The induced homomorphisms $\varphi : B \to B'$, $\varphi : W \to W'$ and $\varphi : k[[Q]] \to k[[Q']]$ are local.
   (c) The homomorphisms $\varphi : B \to B'$ $\varphi : k[[Q]] \to k[[Q']]$ are continuous.
   (d) There exists a parameter system R of B such that $Q \subset R$, $\varphi(R-Q) \cap Q' = \emptyset$, and $\varphi(R - Q) \cup Q'$ is a parameter system of B'.
   We denote $\varphi : (B, W, Q) \to (B', W', Q')$.

The above notation $(A, V, Q) = \underline{(B, W, Q)}$ in Definition 5.3.2 will be used frequently.

Lemma 5.4. *Let (B, W, Q) be a framed space germ, and $(A, V, Q) = \underline{(B, W, Q)}$.*

1. *For the framed space germ (A, V, Q), Q is a parameter system of A.*
2. *The extension of the inclusion homomorphism $A \to B$ to the quotient fields defines a morphism $(A, V) \to (B, W)$ of space germs.*
3. *Let $R = \{x_0, x_1, \dots, x_r\}$ be a parameter system of B containing the parameter system $Q = \{x_0, x_1, \dots, x_n\}$ of A. Let $R' = \{y_0, y_1, \dots, y_s\}$ be a finite subset of B containing a parameter system $Q' = \{y_0, y_1, \dots, y_n\}$ of A. Then, R' is a parameter system of B, if and only if, $s = r$, $R' \subset M(B)$ and $\det(\partial y_j / \partial x_i)_{n<i, j\le r} \notin M(B)$.*

We call the morphism $(A, V) \to (B, W)$ in Lemma 5.4.2 the *inclusion morphism*.

Lemma 5.5. *Let $\varphi : (B, W, Q) \to (B', W', Q')$ be a morphism of framed space germs. Let $(A, V, Q) = \underline{(B, W, Q)}$, and $(A', V', Q') = \underline{(B', W', Q')}$.*

1. *The morphism φ induces an injective homomorphism from the quotient field F of B to the quotient field F' of B' such that the image of the quotient field E of A is contained in the quotient field E' of A'. We can regard F as a subfield of F', and E as a subfield of E'. We have $W = W' \cap F$ and $V = V' \cap E$.*
2. *For any parameter system R_1 of B containing a parameter system Q_1 of A and for any parameter system Q_1' of A', the union $\varphi(R_1 - Q_1) \cup Q_1'$ is a parameter system of B'.*
3. *The morphism φ induces a morphism $\varphi_\bullet : (B, W) \to (B', W')$ of underlying space germs.*
4. *The morphism φ induces a morphism $\underline{\varphi} : (A, V, Q) \to (A', V', Q')$ of basic framed space germs.*

5. *As the composition of above two correspondences, the morphism φ induces a morphism $\underline{\varphi}_\bullet : (A, V) \to (A', V')$.*
6. *The following diagram is commutative:*

$$\begin{array}{ccc} (A,V) & \longrightarrow & (B,W) \\ \big\downarrow{\scriptstyle \underline{\varphi}_\bullet} & & \big\downarrow{\scriptstyle \varphi_\bullet} \\ (A',V') & \longrightarrow & (B',W') \end{array}$$

Horizontal arrows are inclusion morphisms.

Proof. 1. The equalities $W = W' \cap F$ and $V = V' \cap E$ follow from Corollary 4.19.
2. It follows from Lemma 5.4.3. □

We call $\varphi_\bullet$ the *underlying morphism* of φ, while we call $\underline{\varphi}$ the *basic morphism* of φ.

Lemma 5.6. *Let $\varphi : (B, W, Q) \to (B', W', Q')$ be a morphism of framed space germs. Let Q_1 be a partial parameter system of B with $Q \subset Q_1$.*

Let $Q'_1 = Q' \cup \varphi(Q_1 - Q)$. The set Q'_1 is a partial parameter system of B'. It satisfies $Q' \subset Q'_1$ and $\sharp(Q_1 - Q) = \sharp(Q'_1 - Q')$. There exists a unique morphism $\varphi_1 : (B, W, Q_1) \to (B', W', Q'_1)$ of framed space germs such that $\varphi_\bullet = (\varphi_1)_\bullet$ and the following diagram is commutative: We denote the basic framed space germs of (B, W, Q), (B, W, Q_1), (B', W', Q'), and (B', W', Q'_1) by (A, V, Q), (A_1, V_1, Q_1), (A', V', Q'), and (A'_1, V'_1, Q'_1) respectively.

$$\begin{array}{ccccc} (A,V) & \longrightarrow & (A_1,V_1) & \longrightarrow & (B,W) \\ \big\downarrow{\scriptstyle \underline{\varphi}_\bullet} & & \big\downarrow{\scriptstyle (\underline{\varphi}_1)_\bullet} & & \big\downarrow{\scriptstyle \varphi_\bullet = (\varphi_1)_\bullet} \\ (A',V') & \longrightarrow & (A'_1,V'_1) & \longrightarrow & (B',W') \end{array}$$

Proof. The claim on Q'_1 follows from Lemma 5.5.2. The remaining parts are easy. □

Definition 5.7. Let (B, W, Q) be a framed space germ, and Q' be an arbitrary parameter system of $k[[Q]]$. The triplet (B, W, Q') is a framed space germ, and we call it a *coordinate transform* of (B, W, Q).

We associate a morphism $(B, W, Q) \to (B, W, Q')$ of space germs induced by the identity homomorphisms of the quotient field of B.

Note that the associated morphism $(B, W, Q) \to (B, W, Q')$ can be defined above, since for any parameter system R of B containing Q, the union $(R - Q) \cup Q'$ is a parameter system of B containing Q'. (Lemma 5.4.3.)

Also note $(B, W, Q)_\bullet = (B, W, Q')_\bullet$ and $\underline{(B, W, Q)}_\bullet = \underline{(B, W, Q')}_\bullet$, if (B, W, Q') is a coordinate transform of (B, W, Q).

We will define an elementary monoidal transform of a framed space germ.

Let (B, W, Q) be a framed space germ, and $C \subset Q$ be a subset containing at least two elements. Let $C = \{x_0, x_1, \dots, x_t\}$ with $t \geq 1$. By F we denote the quotient field of B. Let $v : F \to \Gamma \cup \{\infty\}$ be the valuation defined by W. Since $Q \subset M(B) \subset M(W)$, we have $v(x) > 0$ for every $x \in Q$. We have at least one number j with $0 \leq j \leq t$ satisfying $v(x_j) = \min\{v(x_i) \mid 0 \leq i \leq t\}$. By exchanging the numbers of x_i's, we can assume $v(x_0) = \min\{v(x_i) \mid 0 \leq i \leq t\}$. Consider a subring

$$B_1 = B[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \cdots, \frac{x_t}{x_0}]$$

of F generated by B and t elements $x_1/x_0, x_2/x_0, \cdots, x_t/x_0$. We have $v(x_i/x_0) \geq 0$ for $1 \leq i \leq t$, and $B_1 \subset W$. Let $\mathfrak{m} = M(W) \cap B_1$. Since $k \subset B_1/\mathfrak{m} \subset W/M(W) \cong k$, $B_1/\mathfrak{m} \cong k$ and $\mathfrak{m}$ is a maximal ideal of B_1.

Besides, since $W/M(W) \cong k$, for every number i with $1 \leq i \leq t$, we have a unique element $\lambda_i \in k$ with $v((x_i/x_0) - \lambda_i) > 0$. Let $y_0 = x_0$ and $y_i = (x_i/x_0) - \lambda_i$ for $1 \leq i \leq t$. Let $B_2 = (B_1)_{\mathfrak{m}}$, and $Q' = (Q - C) \cup \{y_0, y_1, \ldots, y_t\}$. One sees by Theorem 2.2.4 and 5 that B_2 is a excellent regular local ring, Q' is a partial parameter system of B_2, and W dominates B_2.

Let B' be the completion of B_2, and F' be the quotient field of B'. Ring B' is a complete regular local ring and Q' is a partial parameter system of B'. Ring B' coincides with the completion of B_1 with respect to the ideal $\mathfrak{m}$. The inclusion homomorphism $B \subset B_2$ and the canonical homomorphism $B_2 \to B'$ to the completion define an injective homomorphism $B \to B'$ of rings. Since B and B_2 have common quotient field F, an injective homomorphism $F \to F'$ of fields is induced. We can regard F as a subfield of F'. By Lemma 4.33 there exists a valuation ring W' of F' such that W' dominates B', $\dim_k W' = 0$ and $W' \cap F = W$.

The triplet (B', W', Q') is a framed space germ over k.

Let $A = k[[Q]]$, $A' = k[[Q']]$ $A_1 = A[x_1/x_0,\ x_2/x_0, \ldots, x_t/x_0]$, and $\mathfrak{l} = A_1 \cap M(W)$. One sees easily that the completion of A_1 with respect to $\mathfrak{l}$ coincides with A'. Thus composing the inclusion homomorphism $A \to A_1$ and the canonical homomorphism $A_1 \to A'$ to the completion, we obtain an injective ring homomorphism $A \to A'$. Obviously $A \to A'$ coincides with the restriction of the homomorphism $F \to F'$ to A.

The homomorphism $F \to F'$ induces a morphism $(B, W, Q) \to (B', W', Q')$ of framed space germs.

Definition 5.8. Let (B, W, Q) be a framed space germ over k, F be a quotient field of B, and $v : F \to \Gamma \cup \{\infty\}$ be the valuation defined by W.

1. A triplet (C, x_0, W') satisfying the following three conditions is called a *datum of a monoidal transformation* over (B, W, Q).
   (a) $C \subset Q$. C contains at least two elements.
   (b) $x_0 \in C$. $v(x_0) = \min\{v(x) \mid x \in C\}$.
   (c) Let B_1 be a subring of F generated by B and the set $\{x/x_0 \mid x \in C\}$, B' denote the completion of B_1 with respect to the ideal $B_1 \cap M(W)$, and F' denote the quotient field of B'. W' is a valuation ring of F' such that W' dominates B', $\dim_k W' = 0$ and $W' \cap F = W$.
2. Let (C, x_0, W') be a datum of a monoidal transformation over (B, W, Q). Let B' denote the ring in the above 1.(c). For every $x \in C$ we choose an element $\lambda_x \in k$ with $v((x/x_0) - \lambda_x) > 0$, and put $y_x = (x/x_0) - \lambda_x$. Let $Q' = (Q - C) \cup \{x_0\} \cup \{y_x \mid x \in C, x \neq x_0\}$. The framed space germ (B', W', Q') is called the *elementary monoidal transform* of (B, W, Q) associated with a datum (C, x_0, W').
3. The induced morphism $(B, W, Q) \to (B', W', Q')$ described above is called the *associated morphism* with the elementary monoidal transform (B', W', Q').

4. A space germ (B', W') without a frame is called an elementary monoidal transform of (B, W), if there exist a frame Q of B and a datum (C, x_0, W') of a monoidal transformation over (B, W, Q) such that the underlying space germ of the elementary monoidal transform (B', W', Q') of (B, W, Q) associated with (C, x_0, W') coincides with (B', W'). The underlying morphism $(B, W) \to (B', W')$ of the associated morphism $(B, W, Q) \to (B', W', Q')$ is called the associated morphism with the elementary monoidal transform (B', W') of (B, W).

Definition 5.9. Let (B, W, Q) and (B', W', Q') be framed space germs over k. Assume that there exists a finite sequence $\{(B_\nu, W_\nu, Q_\nu) \mid 0 \le \nu \le \omega\}$ of framed space germs satisfying the following two conditions:

1. $(B, W, Q) = (B_0, W_0, Q_0)$, $(B', W', Q') = (B_\omega, W_\omega, Q_\omega)$.
2. For every number ν with $1 \le \nu \le \omega$, (B_ν, W_ν, Q_ν) is an elementary monoidal transform of $(B_{\nu-1}, W_{\nu-1}, Q_{\nu-1})$ associated with some datum. (resp. an elementary monoidal transform of $(B_{\nu-1}, W_{\nu-1}, Q_{\nu-1})$ associated with some datum, or a coordinate transform of $(B_{\nu-1}, W_{\nu-1}, Q_{\nu-1})$.)

Then, we say that (B', W', Q') is an *iterated elementary monoidal transform*, or an *IEMT* in short (resp. an *iterated analytic monoidal transform*, or an *IAMT* in short) of (B, W, Q). The composition $(B, W, Q) = (B_0, W_0, Q_0) \to (B_\omega, W_\omega, Q_\omega) = (B', W', Q')$ of ω associated morphisms $(B_{\nu-1}, W_{\nu-1}, Q_{\nu-1}) \to (B_\nu, W_\nu, Q_\nu)$, $1 \le \nu \le \omega$ is called the associated morphism with the IEMT (resp. IAMT) (B', W', Q') of (B, W, Q).

A space germ (B', W') without a frame is called an IAMT of (B, W), if there exist a frame Q of B and an IAMT (B', W', Q') of (B, W, Q) such that the underlying space germ of (B', W', Q') coincides with (B', W'). The underlying morphism $(B, W) \to (B', W')$ of the associated morphism $(B, W, Q) \to (B', W', Q')$ is called the associated morphism with the IAMT (B', W') of (B, W).

Definition 5.10. 1. Let $\varphi : (B, W, Q) \to (B', W', Q')$ be a morphism of framed space germs. If, for every $x \in Q$, the image $\varphi(x) \in k[[Q']]$ has a normal crossing with respect to Q', then, we say that the morphism φ *preserves normal crossings*, or φ is a *morphism preserving normal crossings.*

2. Let (B', W', Q') be an IAMT of a framed space germ (B, W, Q). We say that (B', W', Q') is an IAMT *with a normal crossing* of (B, W, Q), if the associated morphism $(B, W, Q) \to (B', W', Q')$ preserves normal crossings.

Lemma 5.11. 1. *If a morphism $\varphi : (B, W, Q) \to (B', W', Q')$ preserves normal crossings, then, for any parameter system R of B containing Q and for every $x \in R$, $\varphi(x) \in B'$ has a normal crossing with respect to the parameter system $\varphi(R - Q) \cup Q'$.*

2. *If two morphisms $\varphi : (B, W, Q) \to (B', W', Q')$ and $\psi : (B', W', Q') \to (B'', W'', Q'')$ of framed space germs preserve normal crossings, then the composition $\psi\varphi : (B, W, Q) \to (B'', W'', Q'')$ also preserves normal crossings.*
3. *A morphism associated with an IEMT preserves normal crossings.*

Proof. 1, 2. Easy.

3. By 2 we can consider only the case of an elementary monoidal transform.

Assume that $y_0, y_1, \ldots, y_t$ is a subset of a parameter system R' of some complete regular local ring. Assume moreover that elements $x_0, x_1, \ldots, x_t$ satisfy relations $y_0 = x_0$ and $y_i = (x_i/x_0) - \lambda_i$ for $1 \le i \le t$, where λ_i's are elements in k.

Then, we have $x_0 = y_0$ and $x_i = y_0(\lambda_i + y_i)$ for $1 \le i \le t$. One sees that every x_i has a normal crossing with respect to R'.

The case of an elementary monoidal transform follows from this observation. $\square$

We can explain our goal of this paper here. Let (B, W, Q) be a framed space germ over k. We consider the following statement $RS(B, W, Q)$ on (B, W, Q):

$\mathbf{RS(B, W, Q)}$: Assume that a finite number of arbitrary non-zero elements $f_1, f_2, \ldots, f_\ell$ in $k[[Q]]$ are given. Then, there exists an IAMT (B', W', Q') of (B, W, Q) such that for every $1 \le i \le \ell$ the image $f_i' \in k[[Q']]$ of $f_i \in k[[Q]]$ by the associated homomorphism $k[[Q]] \to k[[Q']]$ has a normal crossing with respect to Q', and the ideal in $k[[Q']]$ generated by any non-empty subset of $\{f_1', f_2', \ldots, f_\ell'\}$ is a principal ideal.

Also we consider another statement $RW(B, W, Q)$ on (B, W, Q):

$\mathbf{RW(B, W, Q)}$: Let $f \in k[[Q]]$ be a non-zero element. Then, there exists an IAMT (B', W', Q') of (B, W, Q) such that the image $f' \in k[[Q']]$ of f by the induced homomorphism $k[[Q]] \to k[[Q']]$ has a normal crossing with respect to Q'.

Obviously $RS(B, W, Q)$ implies $RW(B, W, Q)$. Also the converse holds. The following lemma can be shown easily:

Lemma 5.12 (Bierstone and Milman [3], p.25, Lemma 4.7)**.** *Let (B', W', Q') be a framed space germ over k, and $Q' = \{y_0, y_1, \ldots, y_n\}$. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$, $\beta_0, \beta_1, \ldots, \beta_n$, $\gamma_0, \gamma_1, \ldots, \gamma_n$ be non-negative integers, and $a, b, c \in k[[Q']]^*$. If*

$$a y_0^{\alpha_0} y_1^{\alpha_1} \cdots y_n^{\alpha_n} - b y_0^{\beta_0} y_1^{\beta_1} \cdots y_n^{\beta_n} = c y_0^{\gamma_0} y_1^{\gamma_1} \cdots y_n^{\gamma_n},$$

then either $\alpha_i \le \beta_i$ for every $0 \le i \le n$, or $\alpha_i \ge \beta_i$ for every $0 \le i \le n$.

Let $f_1, f_2, \ldots, f_\ell \in k[[Q]]$ be finite elements in $RS(B, W, Q)$. We can assume that any two of them are different. Let $f = \left(\prod_{i=1}^{\ell} f_i\right) \left(\prod_{1 \le i < j \le \ell} (f_i - f_j)\right) \in k[[Q]]$. By Lemma 2.9 and Lemma 5.12 one sees that $RW(B, W, Q)$ implies $RS(B, W, Q)$.

Corollary 5.13. *Let (B, W, Q) be a framed space germ over k. Two statements on (B, W, Q), $RS(B, W, Q)$ and $RW(B, W, Q)$ are equivalent.*

We hope to show that for any framed space germ (B, W, Q) the statement $RW(B, W, Q)$ is true.

Note that if $\sharp\, Q = \dim k[[Q]] = 1$, then every non-zero element in $k[[Q]]$ has a normal crossing with respect to Q.

Lemma 5.14. *If the basic dimension $\sharp\, Q$ of a framed space germ (B, W, Q) is equal to one, then $RW(B, W, Q)$ is true.*

6. Weierstrass representation

We hope to apply induction on the basic dimension $\sharp\, Q$ to show the statement $RW(B, W, Q)$. The Weierstrass preparation theorem (Corollary 2.16) provides us a method for induction.

Definition 6.1. Let (B, W, Q) be a framed space germ with $\sharp\, Q \ge 2$, and $A = k[[Q]]$. A triplet (h, g, t) satisfying the following four conditions is called a *Weierstrass representation under* (B, W, Q).

1. $h \in A$, $g \in A$ and $t \in Q$.
   We write $Q = \{t, x_1, x_2, \dots, x_n\}$, and $Z = k[[Q - \{t\}]]$ below. Note that $A = k[[t, x_1, x_2, \dots, x_n]]$ and $Z = k[[x_1, x_2, \dots, x_n]]$.
2. There exist non-negative integers $s_1, s_2, \dots, s_n$ with
$$\frac{h}{x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}} \in A^*.$$
3. The pair (g, t) is a semi-Weierstrass pair of the ring A under the parameter system Q. In other words, there are a non-negative integer m and m elements $G_1, G_2, \dots, G_m \in Z$ with
$$g = t^m + \sum_{j=1}^{m} G_j t^{m-j}.$$
4. For every integer j with $1 \le j \le m$
$$\mathrm{ord}\,(G_j) \ge j.$$

If (h, g, t) satisfies only conditions 1–3, then we call (h, g, t) a *semi-Weierstrass representation*. The frame Q of the space germ (B, W) is called the *upper frame* of (h, g, t), while the frame $P = Q - \{t\}$ of the space germ (B, W) is called the *lower frame* of (h, g, t). The framed space germ (B, W, Q) is called the *upper framed space germ* of (h, g, t), and $(B, W, Q - \{t\})$ is called the *lower framed space germ* of (h, g, t).

Let $P = Q - \{t\}$. We say also that (h, g, t) is a semi-Weierstrass representation *over* (B, W, P), referring to the lower framed space germ (B, W, P), not to the upper framed space germ (B, W, Q). If (h, g, t) is one *over* (B, W, P), then (B, W, P) is the lower framed space germ of (h, g, t), $t \notin P$, and the union $P \cup \{t\}$ is a partial parameter system of B. If it is one *under* (B, W, Q), then (B, W, Q) is the upper framed space germ of (h, g, t) and $t \in Q$.

Let $f \in A$ be a non-zero element. If a semi-Weierstrass representation (h, g, t) satisfies $f = hg$, then we say that (h, g, t) is a semi-Weierstrass representation *of* f.

If a semi-Weierstrass pair (g, t) of A under Q has a certain property X, then, for simplicity, we say that the semi-Weierstrass representation (h, g, t) under (B, W, Q) has a property X.

Lemma 6.2. *Let (B, W, Q) be a framed space germ with $\sharp Q \ge 2$, and $f \in k[[Q]]$ be a non-zero element.*

1. *There exists a coordinate transform (B, W, Q') of (B, W, Q) such that there exists a Weierstrass representation (h, g, t) of f under (B, W, Q') satisfying $h \in k[[Q]]^*$.*
2. *Any semi-Weierstrass representation (h, g, t) of f satisfies* $\mathrm{ord}(f) = \mathrm{ord}(g) + \mathrm{ord}(h)$, $\mathrm{sord}(h) = 0$, $\mathrm{sord}(f) = \mathrm{sord}(g)$, *and* $\mathrm{nord}(f) = \mathrm{nord}(g) + \mathrm{nord}(h)$.
3. *Any Weierstrass representation (h, g, t) of f with $h \in k[[Q]]^*$ satisfies* $\mathrm{ord}(f) = \mathrm{ord}(g) = \deg(g,\, t)$, $\mathrm{sord}(f) = \mathrm{sord}(g)$, *and* $\mathrm{nord}(f) = \mathrm{nord}(g)$.

Proof. 1. Note that $\underline{(B, W, Q')}_\bullet = \underline{(B, W, Q)}_\bullet$ by definition of a coordinate transform. It follows from Proposition 2.18.
2. Refer to Definition 2.13 and Lemma 2.14 for definition of sord and nord.
3. It follows from 2 and Lemma 2.20.4. $\square$

Lemma 6.3. *Let (h, g, t) be a semi-Weierstrass representation over a framed space germ (B, W, P). Let $\varphi : (B, W, P) \to (B', W', P')$ be a morphism preserving normal crossings from the lower framed space germ.*

1. *Triplet $(\varphi(h), \varphi(g), \varphi(t))$ is a semi-Weierstrass representation over (B', W', P') and under (B', W', Q'), where $Q' = P' \cup \{\varphi(t)\}$.*
2. $\mathrm{w}(g,\, t) \leq \mathrm{w}(\varphi(g),\, \varphi(t))$, $\mathrm{ord}(g) \leq \mathrm{ord}(\varphi(g))$, $\deg(g,\, t) = \deg(\varphi(g),\, \varphi(t))$.
3. *Assume moreover that (h, g, t) is a Weierstrass representation. Also $(\varphi(h), \varphi(g), \varphi(t))$ is a Weierstrass representation.* $\mathrm{ord}(g) = \mathrm{ord}(\varphi(g))$, $\mathrm{nord}(g) \leq \mathrm{nord}(\varphi(g))$, $\mathrm{sord}(g) \geq \mathrm{sord}(\varphi(g))$.

Proof. 1. It is easy to check the first and the third condition in Definition 6.1. Since φ preserves normal crossings, also the second condition is satisfied.
2. For every $f \in k[[P \cup \{t\}]]$ we have $\mathrm{ord}(f) \leq \mathrm{ord}(\varphi(f))$.
3. By Lemma 2.20.4 we have $\deg(g,\, t) = \mathrm{ord}(g)$. By 2 and Lemma 2.20.2 we have $\mathrm{ord}(g) \leq \mathrm{ord}(\varphi(g)) \leq \deg(\varphi(g), \varphi(t)) = \deg(g,\, t) = \mathrm{ord}(g)$. We have $\deg(\varphi(g), \varphi(t)) = \mathrm{ord}(\varphi(g))$, and again by Lemma 2.20.4 one concludes that $(\varphi(h), \varphi(g), \varphi(t))$ is a Weierstrass representation. The equality $\mathrm{ord}(g) = \mathrm{ord}(\varphi(g))$ follows from the above. Let D denote the quotient field of $k[[P]]$, and D' denote the quotient field of $k[[P']]$. Let $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ be the irreducible decomposition of g in $D[t]$. We assume that the coefficient of the term with the highest degree in $g_\nu \in D[t]$ is equal to 1 for every $1 \leq \nu \leq r$, and that for $1 \leq \nu \leq s$ $\deg(g_\nu, t) = 1$ and for $s < \nu \leq r$ $\deg(g_\nu, t) > 1$. Since $\deg(g,\, t) = \mathrm{ord}(g)$, by Lemma 2.21.3 we have $\deg(g_\nu, t) = \mathrm{ord}(g_\nu)$ for every $1 \leq \nu \leq r$, and $g = g_1^{\beta_1} g_2^{\beta_2} \cdots g_r^{\beta_r}$ gives the irreducible decomposition in $k[[P \cup \{t\}]]$. Thus $\mathrm{nord}(g) = \sum_{\nu=1}^{s} \beta_\nu$. On the other hand, we have a decomposition $\varphi(g) = \varphi(g_1)^{\beta_1} \varphi(g_2)^{\beta_2} \cdots \varphi(g_r)^{\beta_r}$ in $D'[\varphi(t)]$. Since $\deg(\varphi(g), \varphi(t)) = \mathrm{ord}(\varphi(g))$, by Lemma 2.21.1 we have $\deg(\varphi(g_\nu), \varphi(t)) = \mathrm{ord}(\varphi(g_\nu))$ for every $1 \leq \nu \leq r$. Thus $\mathrm{ord}(\varphi(g_\nu)) = \deg(\varphi(g_\nu), \varphi(t)) = \deg(g_\nu, t) = 1$ for $1 \leq \nu \leq s$. One concludes $\mathrm{nord}(\varphi(g)) \geq \sum_{\nu=1}^{s} \beta_\nu = \mathrm{nord}(g)$.

The last equality follows from Lemma 2.14.2. □

The semi-Weierstrass representation $(\varphi(h), \varphi(g), \varphi(t))$ is called the *pull-back* of (h, g, t) by φ. If (h, g, t) is a semi-Weierstrass (resp. Weierstrass) representation of $f \in k[[P \cup \{t\}]]$, then the pull-back $(\varphi(h), \varphi(g), \varphi(t))$ is a semi-Weierstrass (resp. Weierstrass) representation of $\varphi(f) \in k[[P' \cup \{\varphi(t)\}]]$.

We denote the integral part of a real number r by $[r]$ (The Gauss symbol), and the part under the decimal point by $\langle r \rangle$.

$$
\begin{aligned}
[r] &= \max\{i \in \mathbb{Z} \mid i \leq r\}, \\
\langle r \rangle &= r - [r].
\end{aligned}
$$

Definition 6.4. Let (h, g, t) be a semi-Weierstrass representation over a space germ (B, W, P). We write $P = \{x_1, x_2, \ldots, x_n\}$, $Z = k[[P]]$. Let $m = \deg(g, t)$ and $G_1, G_2, \ldots, G_m \in Z$ be the coefficient function sequence of (g, t).

$$
g = t^m + \sum_{j=1}^{m} G_j t^{m-j}.
$$

1. We say that a semi-Weierstrass pair (g, t) satisfies the *Abhyankar condition*, if the following two conditions are simultaneously satisfied:
   (a) $m = \deg(g,\, t) \geq 1$.
   (b) There exist n non-negative integers $b_1, b_2, \ldots, b_n$ such that

(i)
$$\frac{G_j}{(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j} \in Z$$
for every $1 \leq j \leq m$, and

(ii)
$$\min\{\frac{1}{j}\operatorname{ord}(\frac{G_j}{(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j}) \mid 1 \leq j \leq m\} < 1.$$

The non-negative integers $b_1, b_2, \ldots, b_n$ in (b) are called the *characteristic numbers* of (g, t).

2. We say that (g, t) satisfies the *strong Abhyankar condition*, if the following three conditions are simultaneously satisfied:
   (c) Let $J = \{j \in \mathbb{Z} \mid 1 \leq j \leq m, G_j \neq 0\}$. For every $j \in J$ there exist n integers $b_{j1}, b_{j2}, \ldots, b_{jn}$ with
   $$\frac{G_j}{x_1^{b_{j1}} x_2^{b_{j2}} \cdots x_n^{b_{jn}}} \in Z^*.$$
   (d) The subset
   $$E = \{e \in J \mid \frac{G_j^{m!/j}}{G_e^{m!/e}} \in Z \text{ for every } j \in J\}$$
   of J is not empty.
   (e) There exists an element $e \in E$ with
   $$\sum_{i=1}^{n} \langle \frac{b_{ei}}{e} \rangle < 1.$$

Remark . The strong Abhyankar condition is due to Abhyankar. (Abhyankar [2], p.285–p.299, Appendix.)

Lemma 6.5. *We consider conditions in Definition 6.4.*

1. *If (g, t) satisfies the Abhyankar condition, then characteristic numbers $b_1, b_2, \ldots, b_n$ are uniquely defined depending on (g, t). In other words, n numbered non-negative integers $b_1, b_2, \ldots, b_n$ satisfying* (b) *are unique, if they exist.*
2. *If (g, t) satisfies the Abhyankar condition, and if (g, t) is a Weierstrass pair, then at least one of characteristic numbers $b_1, b_2, \ldots, b_n$ is not zero.*
3. *If (g, t) satisfies the Abhyankar condition, then* $\mathrm{w}(g, t) < \infty$.
4. *Non-negative integers b_{ji}, $j \in J$, $1 \leq i \leq n$ in* (c) *are uniquely determined depending on (g, t), if they exist. Therefore, if* (c) *is satisfied, then* (e) *has definite meaning.*
5. *If* (c) *and* (d) *are satisfied, then for a fixed number i with $1 \leq i \leq n$, the rational number*
   $$\frac{b_{ei}}{e}$$
   does not depend on the choice of an element $e \in E$. Therefore, the inequality in (e) *holds for every element $e \in E$.*

Proposition 6.6. *Let (g, t) be a semi-Weierstrass pair of a ring $k[[x]]$ over a partial parameter space P. If (g, t) satisfies the strong Abhyankar condition, then (g, t) satisfies the Abhyankar condition. In other words, the strong Abhyankar condition implies the Abhyankar condition.*

Theorem 6.7 (Abhyankar [2], p.285–p.299, Appendix.). *Let (h, g, t) be a Weierstrass representation over a framed space germ (B, W, P) satisfying* $\mathrm{w}(g,\, t) < \infty$.

1. *Assume that $RW(B, W, P)$ is true. Then, there exist an IAMT (B', W', P') with a normal crossing of (B, W, P) such that the pull-back of (h, g, t) by $(B, W, P) \to (B', W', P')$ is a Weierstrass representation over (B', W', P') satisfying the condition* (c) *and* (d) *in Definition 6.4.*
2. *Assume that (h, g, t) satisfies* (c) *and* (d) *in Definition 6.4. Then, there exist an IEMT (B', W', P') of (B, W, P) such that the pull-back of (h, g, t) by the associated morphism $(B, W, P) \to (B', W', P')$ is a Weierstrass representation over (B', W', P') satisfying the strong Abhyankar condition.*

Proof. Let $m = \deg(g,\, t)$, $G_1, G_2, \ldots, G_m \in k[[P]]$ be the coefficient function sequence of $(g,\, t)$, $P = \{x_1, x_2, \ldots, x_n\}$, and $J = \{j \in \mathbb{Z} \mid 1 \le j \le m, G_j \ne 0\}$. Since $\mathrm{w}(g,\, t) < \infty$, we have $J \ne \emptyset$.
1. Since $RW(B, W, P)$ is true, also $RS(B, W, P)$ is true by Corollary 5.13. Consider finite elements $G_j^{m!/j}$, $j \in J$ and $x_1, x_2, \ldots, x_n$. Claim 1 follows from $RS(B, W, P)$.
2. It follows from Lemma 6.8 below. □

Lemma 6.8 (Abhyankar [2], p.290, Reduction Lemma). *Let (B, W, P) be a framed space germ over k, $G_1, G_2, \ldots, G_m \in k[[P]]$ be a finite number of non-zero elements, and e be a positive integer. Assume that the product $G = G_1 G_2 \cdots G_m$ has a normal crossing with respect to P.*

Then, there exists an IEMT (B', W', P') of (B, W, P) such that the non-negative integers $b_{11}, b_{12}, \ldots, b_{1n}$ defined by the condition

$$\frac{G_1'}{y_1^{b_{11}} y_2^{b_{12}} \cdots y_n^{b_{1n}}} \in k[[P']]^*$$

satisfy

$$\sum_{i=1}^{n} \langle \frac{b_{1i}}{e} \rangle < 1,$$

where we write $P' = \{y_1, y_2, \ldots, y_n\}$, and for $1 \le i \le m$ by $G_i' \in k[[P']]$ we denote the image of $G_i \in k[[P]]$ by $k[[P]] \to k[[P']]$.

Remark . Since (B', W', P') is an IEMT of (B, W, P), the associated homomorphism $k[[P]] \to k[[P']]$ preserves normal crossings (Lemma 5.11.3) and the image $G' = G_1' G_2' \cdots G_m' \in k[[P']]$ of G by $k[[P]] \to k[[P']]$ has a normal crossing with respect to P'.

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be positive integers. Assume, moreover, that the ideal in $k[[P]]$ generated by $G_1^{\alpha_1}, G_2^{\alpha_2}, \ldots, G_m^{\alpha_m}$ is a principal ideal generated by $G_1^{\alpha_1}$. Then, obviously, the ideal in $k[[P']]$ generated by $(G_1')^{\alpha_1}, (G_2')^{\alpha_2}, \ldots, (G_m')^{\alpha_m}$ is a principal ideal generated by $(G_1')^{\alpha_1}$, since there is a ring homomorphism $k[[P]] \to k[[P']]$.

We can check that Abhyankar's proof is valid under our formulation with framed space germs.

Corollary 6.9. *Consider the same situation as in Theorem 6.7. We assume that $RW(B, W, P)$ is true and $\mathrm{w}(g,\, t) < \infty$. Then, there exist an IAMT (B', W', P') with a normal crossing of (B, W, P) such that the pull-back of (h, g, t) by $(B, W, P) \to (B', W', P')$ is a Weierstrass representation over (B', W', P') satisfying the strong Abhyankar condition.*

We introduce another condition "reductive" for semi-Weierstrass pairs, which is essential in the theory of positive characteristic.

Lemma 6.10. *Let (h, g, t) be a semi-Weierstrass representation over a framed space germ (B, W, P), $m = \deg(g, t)$, and $G_1, G_2, \dots, G_m \in k[[P]]$ be the coefficient function sequence of the semi-Weierstrass pair (g, t).*

1. *For every element $H \in k[[P]]$ the triplet $(h, g, t + H)$ (resp. the pair $(g, t + H)$) is also a semi-Weierstrass representation over the framed space germ (B, W, P) (resp. a semi-Weierstrass pair of $k[[P \cup \{t\}]]$ over P). $\deg(g, t) = \deg(g, t + H)$.*
2. *Let $H \in k[[P]]$, and $G'_1, G'_2, \dots, G'_m \in k[[P]]$ be the coefficient function sequence of the semi-Weierstrass pair $(g, t + H)$. Let $G_0 = G'_0 = 1$. We have*

$$\begin{aligned} G'_j &= \sum_{i=0}^{j} \binom{m-i}{m-j} (-H)^{j-i} G_i, \\ G_j &= \sum_{i=0}^{j} \binom{m-i}{m-j} H^{j-i} G'_i, \end{aligned}$$

   for every $1 \le j \le m$.

Definition 6.11. Let (B, W, P) be a framed space germ over k.

1. Let $t \in B$ be an element such that $t \notin P$ and $P \cup \{t\}$ is a partial parameter system of B. The coordinate transformation $(B, W, P \cup \{t\}) \mapsto (B, W, P \cup \{t + H\})$ defined for any element $H \in k[[P]]$ is called the *parallel coordinate transformation* of $(B, W, P \cup \{t\})$ along H.
2. Let (h, g, t) be a semi-Weierstrass representation over (B, W, P), and $H \in k[[P]]$. The semi-Weierstrass representation $(h, g, t + H)$ (resp. the semi-Weierstrass pair $(g, t + H)$) is called the *parallel coordinate transform* of (h, g, t) (resp. (g, t)) along H.
3. An element $H \in k[[P]]$ with $\mathrm{w}(g, t) < \mathrm{w}(g, t + H)$ is called a *reducing* element of (h, g, t) or of (g, t).
4. If (h, g, t) or (g, t) has a reducing element $H \in k[[P]]$, then we say that (h, g, t) or (g, t) is *reductive*.

Lemma 6.12. *Let n be a positive integer and A be a ring isomorphic to the formal power series ring over k with $n + 1$ variables. Let $P = \{x_1, x_2, \dots, x_n\}$ be a partial parameter system of A with $\sharp P = n$, and (g, t) be a semi-Weierstrass pair of A over P. Let $m = \deg(g, t)$ and $G_1, G_2, \dots, G_m \in k[[P]]$ be the coefficient function sequence of (g, t).*

1. *For any reducing element $H \in k[[P]]$ of (g, t), we have $\mathrm{ord}(H) = \mathrm{w}(g, t) < \infty$.*
2. *For an element $H \in k[[P]]$ the following two conditions are equivalent:*
   (a) *H is a reducing element of (g, t).*
   (b) *$m \ge 1$, $H \ne 0$ and $\mathrm{in}(g, t) = (t + \mathrm{in}(H))^m$.*
3. *The following four conditions are equivalent:*
   (a) *Pair (g, t) is reductive.*
   (b) *Pair (g, t) is reductive, $\mathrm{w}(g, t) < \infty$ and $m \ge 1$.*
   (c) *$m \ge 1$, there exists an element $H_0 \in k[[P]]$ such that $H_0 \ne 0$ and $\mathrm{in}(g, t) = (t + H_0)^m$.*
   (d) *There exists a unique element $H_0 \in k[[P]]$ satisfying $\mathrm{in}(g, t) = (t + H_0)^m$, and H_0 is a non-zero homogeneous polynomial.*

4. *One and only one of following two claims holds:*
   (a) *Pair* $(g,\, t)$ *is not reductive.*
   (b) *Pair* $(g,\, t)$ *is reductive, and there exists an element* $H \in k[[P]]$ *such that* $(g,\, t+H)$ *is not reductive.*

Below we assume that $(g,\, t)$ *is reductive. Let* $m = lq$ *be the* p*-decomposition of* m*. (Refer to the latter half of* INTRODUCTION *for the definition.)*

5. *Weight* $\mathrm{w}(g,\, t)$ *is an integer.*
6. *For an element* $H \in k[[P]]$ *the following two conditions are equivalent:*
   (a) H *is a reducing element of* $(g,\, t)$*.*
   (b) $H \neq 0$ *and* $\mathrm{in}(G_q) = l\, \mathrm{in}(H)^q$*.*

Proof. Note Lemma 6.10.2 and Lemma 2.4.3. For every integer j with $1 \leq j \leq m$ and $j \not\equiv 0 \mod q$ we have $\binom{m}{j} = 0$. For an integer j with $1 \leq j \leq m$ and $j \equiv 0 \mod q$ we have $\binom{m}{j} = \binom{lq}{iq} = \binom{l}{i}$, where $i = j/q$. In particular, $\binom{m}{q} = \binom{l}{1} = l \neq 0$.

By Lemma 2.3.7, we obtain implication 2.(b)⇒(a). □

7. THE FIRST REDUCTION

In this section following Abhyankar's ideas in [2], p.285–p.299, Appendix, we add our concept "reductive" to his ideas. We will show that the Abhyankar condition plus the condition "not reductive" implies that the singularity can be resolved even in characteristic positive.

Theorem 7.1 (Abhyankar [2], p.292–p.293, Good Point Lemma.). *Let* (B, W, Q) *be a framed space germ,* $f \in k[[Q]]$ *be a non-zero element, and* (h, g, t) *be a Weierstrass representation of* f *under* (B, W, Q)*.*

We write $Q - \{t\} = P = \{x_1, x_2, \ldots, x_n\}$*, and* $(Z, U, P) = \underline{(B, W, P)}$*. Let* F *denote the quotient field of* B*,* $v : F \to \Gamma \cup \{\infty\}$ *be the valuation defined by* W*,* $m = \deg(g,\, t)$*, and* $G_1, G_2, \ldots, G_m \in Z$ *be the coefficient function sequence of* $(g,\, t)$*.*

Assume the following two conditions:

$\langle A \rangle$ *For every integer* j *with* $1 \leq j \leq m$ $G_j/x_1^j \in Z$*.*
$\langle B \rangle$ $v(t) \geq v(x_1)$*.*

Let (B', W', Q') *be the elementary monoidal transform of* (B, W, Q) *associated with the datum* $(\{t, x_1\}, x_1, W')$*. We identify elements in* F *and their images in the quotient field of* B' *by the associated homomorphism.*

1. *There exists a unique element* $\lambda \in k$ *satisfying* $Q' = \{(t/x_1) - \lambda\} \cup P$*.* $\lambda = 0$ *if and only if* $v(t) > v(x_1)$*.*
2. $(Z, U, P) = \underline{(B', W', P)}$*.*
3. *Let*
$$t' = \frac{t}{x_1} - \lambda,\ g' = \frac{g}{x_1^m},\ h' = hx_1^m.$$
The triplet (h', g', t') *is a semi-Weierstrass representation of* f *over* (B', W', P')*.* $\deg(g',\, t') = \deg(g,\, t)$*.*
4. $\mathrm{ord}(g') \leq \mathrm{ord}(g)$*. Besides,* $\mathrm{ord}(g') = \mathrm{ord}(g)$ *if and only if* (h', g', t') *is a Weierstrass representation of* f*.*

5. *If* (h, g, t) *is not reductive and if* $\operatorname{ord}(g') = \operatorname{ord}(g) > 0$*, then* $\lambda = 0$ *and* (h', g', t') *is not reductive, either.*
6. *If* $\lambda = 0$ *and if* (h, g, t) *satisfies the Abhyankar condition, then also* (h', g', t') *satisfies the Abhyankar condition.*
7. *If* $\lambda = 0$ *and if* $\mathrm{w}(g,\, t) < \infty$*, then* $\mathrm{w}(g',\, t') = \mathrm{w}(g,\, t) - 1$*. If* $\lambda = 0$ *and if* $\mathrm{w}(g,\, t) = \infty$*, then* $\mathrm{w}(g',\, t') = \infty$*.*
8. $\operatorname{sord}(g') \leq \operatorname{sord}(g)$*.*

Proof. We have $k[[Q']] = k[[t', x_1, x_2, \dots, x_n]] \supset k[[Q]] = k[[t, x_1, x_2, \dots, x_n]] \supset Z = k[[x_1, x_2, \dots, x_n]]$.
3. Obviously $f = h'g'$. Since $g = t^m + \sum_{j=1}^m G_j t^{m-j}$ and $t = (t' + \lambda)x_1$, we have

$$g' = \frac{g}{x_1^m} = (t' + \lambda)^m + \sum_{j=1}^m \frac{G_j}{x_1^j}(t' + \lambda)^{m-j}.$$

By assumption $\langle A \rangle$ one knows $g' \in k[[Q']]$.

Since (h, g, t) is a Weierstrass representation, we have non-negative integers $s_1, s_2, \dots, s_n$ with

$$\frac{h'}{x_1^{s_1+m} x_2^{s_2} \cdots x_n^{s_n}} = \frac{h}{x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}} \in k[[Q]]^* \subset k[[Q']]^*.$$

Let

$$G'_j = \binom{m}{m-j}\lambda^m + \sum_{i=1}^j \binom{m-i}{m-j}\lambda^{j-i}\frac{G_j}{x_1^i}$$

for $1 \leq j \leq m$. By $\langle A \rangle$ one has $G'_j \in Z$ for $1 \leq j \leq m$. One can check $g' = (t')^m + \sum_{i=1}^m G'_j (t')^{m-j}$.
4. By 3 and by Lemma 2.20.2 $\operatorname{ord}(g') \leq \deg(g',\, t') = m = \operatorname{ord}(g)$.
5. Since $m = \operatorname{ord}(g') = \operatorname{ord}(g) > 0$, for every $1 \leq j \leq m$ we have $\operatorname{ord}(G'_j) \geq j$, and $G'_j \in M(Z)$.

On the other hand, we have

$$\frac{G_j}{x_1^j} = \binom{m}{m-j}(-\lambda)^j + \sum_{i=1}^j \binom{m-i}{m-j}(-\lambda)^{j-i} G'_i.$$

Thus, for every $1 \leq j \leq m$ there is an element $R_j \in M(Z)$ with

$$G_j = \binom{m}{m-j}(-\lambda)^j x_1^j + x_1^j R_j.$$

Assume $\lambda \neq 0$. We have $\operatorname{in}(g,\, t) = (t - \lambda x_1)^m$. By Lemma 6.12.3 $(g,\, t)$ is reductive, which contradicts the assumption.

Assume $\lambda = 0$. For every $1 \leq j \leq m$ we have

$$(*) \qquad G'_j = \frac{G_j}{x_1^j}.$$

Let $\varphi(t) = \operatorname{in}(g,\, t)$. By (*) one has $\varphi(x_1 t') = x_1^m \operatorname{in}(g',\, t')$.

Assume that (h', g', t') is reductive. By Lemma 6.12.3 there is an element $H \in Z$ with $H \neq 0$ and $\operatorname{in}(g',\, t') = (t' + H)^m$. We have $\operatorname{in}(g,\, t) = (t + x_1 H)^m$. Since $0 \neq x_1 H \in Z$, we conclude (h, g, t) is reductive, which contradicts the assumption.
8. We can consider the order $\operatorname{ord}_w : k[[Q]] - \{0\} \to \mathbb{Z}_0$ associated with the weight $w = (1, 1, 0, 0, \dots, 0)$ and the parameter system $Q = \{x_0 = t, x_1, x_2, \dots, x_n\}$ of $k[[Q]]$. Since $G_j / x_1^j \in Z$ for $1 \leq j \leq m$, we have $\operatorname{ord}_w(g) = \deg(g,\, t) = m$.

Let $\delta = \mathrm{nord}(g)$. By Lemma 2.21.3 we can write

$$g = \left(\prod_{\nu=1}^{\delta}(t - H_\nu)\right) g_s, \quad g_s = t^{m-\delta} + \sum_{j=1}^{m-\delta} L_j t^{m-\delta-j}$$

with some elements $H_1, H_2, \ldots, H_\delta, L_1, L_2, \ldots, L_{m-\delta} \in Z$. We have $\sum_{\nu=1}^{\delta} \mathrm{ord}_w(t - H_\nu) + \mathrm{ord}_w(g_s) = \mathrm{ord}_w(g) = \deg(g, t) = \sum_{\nu=1}^{\delta} \deg(t - H_\nu, t) + \deg(g_s, t)$. Since $\mathrm{ord}_w(t - H_\nu) \leq \deg(t - H_\nu, t)$ and $\mathrm{ord}_w(g_s) \leq \deg(g_s, t)$ by definition, we know $\mathrm{ord}_w(t - H_\nu) = \deg(t - H_\nu, t) = 1$ for $1 \leq \nu \leq \delta$ and $\mathrm{ord}_w(g_s) = \deg(g_s, t) = m - \delta$. The last claim is equivalent to the following two conditions hold:

1. $H'_\nu = H_\nu / x_1 \in Z$ for every $1 \leq \nu \leq \delta$.
2. $L'_j = L_j / x_1^j \in Z$ for every $1 \leq j \leq m - \delta$.

Thus we can write

$$g = \left(\prod_{\nu=1}^{\delta}(t + x_1 H'_\nu)\right) g_s, \quad g_s = t^{m-\delta} + \sum_{j=1}^{m-\delta} x_1^j L'_j t^{m-\delta-j}.$$

Substituting $t = (t' + \lambda)x_1$, we obtain

$$g' = \left(\prod_{\nu=1}^{\delta}(t' + \lambda + H'_\nu)\right) g'_s, \quad g'_s = (t' + \lambda)^{m-\delta} + \sum_{j=1}^{m-\delta} L'_j (t' + \lambda)^{m-\delta-j}.$$

Since $\mathrm{ord}(t' + \lambda + H'_\nu) = 0$ or 1 for every $1 \leq \nu \leq \delta$, we have $\mathrm{sord}(g') = \mathrm{sord}(g'_s) \leq \mathrm{ord}(g'_s) \leq m - \delta = \mathrm{ord}(g) - \mathrm{nord}(g) = \mathrm{sord}(g)$. $\square$

Theorem 7.2. *Let (B, W, Q) be a framed space germ, $f \in k[[Q]]$ be a non-zero element, and (h, g, t) be a Weierstrass representation of f under (B, W, Q).*

Assume that (h, g, t) satisfies the Abhyankar condition and is not reductive.

Then, for some elementary monoidal transform (B', W', Q') of (B, W, Q) and for some elements $h' \in k[[Q']]$, $g' \in k[[Q']]$ and $t' \in Q'$ one of the following four conditions holds: By $f' \in k[[Q']]$ we denote the image of $f \in k[[Q]]$ by the associated homomorphism $k[[Q]] \to k[[Q']]$.

- $\langle\alpha\rangle$ *The element f' has a normal crossing.*
- $\langle\beta\rangle$ $\mathrm{sord}(f') < \mathrm{sord}(f)$.
- $\langle\gamma\rangle$ $\mathrm{sord}(f') = \mathrm{sord}(f)$, $\mathrm{nord}(g') < \mathrm{nord}(g)$ *and (h', g', t') is a Weierstrass representation of f' under (B', W', Q').*
- $\langle\delta\rangle$ $\mathrm{sord}(f') = \mathrm{sord}(f)$, $\mathrm{nord}(g') = \mathrm{nord}(g)$ *and (h', g', t') is a Weierstrass representation of f' under (B', W', Q'). Besides, (h', g', t') satisfies the Abhyankar condition, (h', g', t') is not reductive and $1 \leq \mathrm{w}(g', t') = \mathrm{w}(g, t) - 1$.*

Remark . By assumption and by Lemma 6.2.2 $\mathrm{sord}(f) = \mathrm{sord}(g)$. If (h', g', t') is a semi-Weierstrass representation of f', then $\mathrm{sord}(f') = \mathrm{sord}(g')$ by Lemma 6.2.2. Therefore, in condition $\langle\gamma\rangle$ and $\langle\delta\rangle$, the condition $\mathrm{sord}(f') = \mathrm{sord}(f)$ is equivalent to $\mathrm{sord}(g') = \mathrm{sord}(g)$.

Proof. We write $Q = \{t, x_1, x_2, \ldots, x_n\}$, $Z = k[[x_1, x_2, \ldots, x_n]]$, $m = \deg(g, t) = \mathrm{ord}(g)$. Let $G_1, G_2, \ldots, G_m \in Z$ be the coefficient function sequence of (g, t), F be the quotient field of B, and $v : F \to \Gamma \cup \{\infty\}$ be the valuation defined by W.

First, note that $m \geq 1$ and $\mathrm{w}(g, t) < \infty$ by the Abhyankar condition.

By Lemma 6.5.2 there is an integer i with $1 \leq i \leq n$ such that $G_j / x_i^j \in Z$ for every $1 \leq j \leq m$.

Exchanging the numbering of x_i's, we can assume that $G_j/x_1^j \in Z$ for every $1 \leq j \leq m$. The assumption $\langle A \rangle$ in Theorem 7.1 is satisfied.

Case $\mathbf{v(t) < v(x_1)}$. Let (B', W', Q') be the elementary monoidal transform of (B, W, Q) associated with the datum $(\{t, x_1\}, t, W')$. We identify elements in F and their images in the quotient field of B' by the associated morphism. Let $t' = t$, $x'_1 = x_1/t$ and $x'_i = x_i$ for $2 \leq i \leq n$. We have $Q' = \{t', x'_1, x'_2, \dots, x'_n\}$, and $k[[Q']] = k[[t', x'_1, x'_2, \dots, x'_n]] \supset k[[Q]] = k[[t, x_1, x_2, \dots, x_n]] \supset Z$.

$$g = t^m + \sum_{j=1}^{m} \frac{G_j}{x_1^j} \cdot x_1^j t^{m-j} = (t')^m \left\{ 1 + \sum_{j=1}^{m} \frac{G_j}{x_1^j} (x'_1)^j \right\}$$

Thus $g/(t')^m \in k[[Q']]^*$. On the other hand,

$$\frac{h}{x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}} = \frac{h}{(t')^{s_1} (x'_1)^{s_1} (x'_2)^{s_2} \cdots (x'_n)^{s_n}} \in k[[Q]]^* \subset k[[Q']]^*,$$

where $s_1, s_2, \dots, s_n$ are non-negative integers in Definition 6.1.2. Thus $f' = f = hg$ has a normal crossing in $k[[Q']]$. The condition $\langle \alpha \rangle$ holds in this case.

Case $\mathbf{v(t) \geq v(x_1)}$. The assumption $\langle B \rangle$ in Theorem 7.1 is satisfied in this case, and we can apply Theorem 7.1. Let (B', W', Q') be the elementary monoidal transform of (B, W, Q) associated with the datum $(\{t, x_1\}, x_1, W')$. Let t', g', and h' be the elements in Theorem 7.1.3. By Theorem 7.1.3 (h', g', t') is a semi-Weierstrass representation of f'. We have $\mathrm{sord}(f') = \mathrm{sord}(g')$.

First, we show that in the case $\mathrm{ord}(g') = \mathrm{ord}(g)$ one of $\langle \alpha \rangle$, $\langle \beta \rangle$, $\langle \delta \rangle$ holds.

By Theorem 7.1.4 (h', g', t') is a Weierstrass representation of f'. By Theorem 7.1.5 and by the assumption we have $\lambda = 0$. By Theorem 7.1.5, 6, 7, (h', g', t') satisfies the Abhyankar condition, (h', g', t') is not reductive, and $\mathrm{w}(g', t') = \mathrm{w}(g, t) - 1$. By Lemma 2.20.4 we have $\mathrm{w}(g', t') \geq 1$.

Assume that $\langle \alpha \rangle$ and $\langle \beta \rangle$ do not hold. Since $\langle \beta \rangle$ does not hold, we have $\mathrm{sord}(f') \geq \mathrm{sord}(f)$, which is equivalent to $\mathrm{sord}(g') \geq \mathrm{sord}(g)$. On the other hand, by Theorem 7.1.8, we have $\mathrm{sord}(g') \leq \mathrm{sord}(g)$. Thus $\mathrm{sord}(g') = \mathrm{sord}(g)$, which is equivalent to $\mathrm{sord}(f') = \mathrm{sord}(f)$. By assumption and Lemma 2.14.2 we obtain $\mathrm{nord}(g') = \mathrm{nord}(g)$.

Second, we show that if $\mathrm{ord}(g') \neq \mathrm{ord}(g)$, then one of $\langle \alpha \rangle$, $\langle \beta \rangle$, $\langle \gamma \rangle$ holds. By Theorem 7.1.4 we have $\mathrm{ord}(g') < \mathrm{ord}(g)$

Assume that $\langle \alpha \rangle$ and $\langle \beta \rangle$ do not hold. By the same argument as above, we obtain $\mathrm{sord}(f') = \mathrm{sord}(g') = \mathrm{sord}(f) = \mathrm{sord}(g)$ by Theorem 7.1.8. Since $\mathrm{ord}(g') < \mathrm{ord}(g)$, we have $\mathrm{nord}(g') = \mathrm{ord}(g') - \mathrm{sord}(g') < \mathrm{ord}(g) - \mathrm{sord}(g) = \mathrm{nord}(g)$.

Below, we will show that there are elements $h'_1, g'_1 \in k[[Q']]$ such that the triplet (h'_1, g'_1, t') is a Weierstrass representation of f' under (B', W', P') and $\mathrm{nord}(g'_1) \leq \mathrm{nord}(g')$. The triplet (h'_1, g'_1, t') satisfies $\langle \gamma \rangle$.

Let $\delta = \mathrm{nord}(g)$. By the proof of Theorem 7.1.8 we know that there are elements $H''_1, H''_2, \dots, H''_\delta, L''_1, L''_2, \dots, L''_{m-\delta} \in Z$ satisfying

$$g' = \left(\prod_{\nu=1}^{\delta} (t' + H''_\nu) \right) g'_s, \quad g'_s = (t')^{m-\delta} + \sum_{j=1}^{m-\delta} L''_j (t')^{m-\delta-j}.$$

We have $\mathrm{sord}(g') \le \mathrm{sord}(g'_s) \le \mathrm{ord}(g'_s) \le m - \delta = \mathrm{sord}(g)$. Since $\mathrm{sord}(g') = \mathrm{sord}(g)$ under our assumption, we have $\mathrm{sord}(g'_s) = \mathrm{ord}(g'_s) = m - \delta = \deg(g'_s,\, t')$. In particular, $(g'_s,\, t')$ is a Weierstrass pair.

Let

$$\begin{aligned}\Delta &= \{\nu \in \mathbb{Z} \mid 1 \le \nu \le \delta, H''_\nu \notin M(Z)\}\\ \Gamma &= \{\nu \in \mathbb{Z} \mid 1 \le \nu \le \delta, H''_\nu \in M(Z)\}\end{aligned}$$

$$h'_1 = h' \left(\prod_{\nu\in\Delta} (t' + H''_\nu) \right), \quad g'_1 = \left(\prod_{\nu\in\Gamma} (t' + H''_\nu) \right) g'_s.$$

One can check that (h'_1, g'_1, t') is a Weierstrass representation of f' under (B', W', P'). We have $\mathrm{nord}(g') = \mathrm{nord}(g'_1) + \sum_{\nu\in\Delta} \mathrm{nord}(t' + H''_\nu) = \mathrm{nord}(g'_1)$. $\square$

Definition 7.3. A collection $\mathcal{C}$ of framed space germs (B, W, Q) over k is called an *admissible class* of framed space germs over k, if $\mathcal{C}$ satisfies the following two conditions:

1. If $(B, W, Q) \in \mathcal{C}$, then for every non-zero element $f \in k[[Q]]$ there exist a coordinate transform (B, W, Q') of (B, W, Q) and a Weierstrass representation (h, g, t) of f under (B, W, Q') such that $(B, W, Q' - \{t\}) \in \mathcal{C}$.
2. If $(B, W, Q) \in \mathcal{C}$, and if (B', W', Q') is an IAMT of (B, W, Q), then $(B', W', Q') \in \mathcal{C}$.

The collection of all framed space germs over k is, of course, an admissible class.

Let $\mathcal{C}_0$ be the collection of framed space germs (B, W, Q) over k such that the underlying space germ $(A, V) = \underline{(B, W, Q)}_\bullet$ of the basic framed space germ of it is of iterated divisor type. Recall Definition 4.27 here. A space germ (A, V) is said to be of iterated divisor type, if it satisfies $\dim A = \mathrm{rank}\, V$. In the next section we will show that the collection $\mathcal{C}_0$ is an admissible class. (Corollary 8.3.)

Let (B, W, P) be a framed space germ over k. We consider the following statement $AN(B, W, P)$ on (B, W, P).

$\mathbf{AN(B, W, P)}$: For every Weierstrass representation (h, g, t) over (B, W, P), there exist an IAMT (B', W', P') with a normal crossing of (B, W, P) and an element $H' \in k[[P']]$ satisfying the following condition: Let (h', g', t') denote the pull-back of (h, g, t) by $(B, W, P) \to (B', W', P')$. The triplet $(h', g', t' + H')$ is a Weierstrass representation over (B', W', P'). Besides, if $\mathrm{w}(g',\, t' + H') < \infty$, then $(h', g', t' + H')$ satisfies the Abyankar condition and moreover $(h', g', t' + H')$ is not reductive.

Recall the statement $RW(B, W, Q)$ we gave before Lemma 5.12.

Theorem 7.4. *Let $\mathcal{C}$ be an admissible class of framed space germs over k, and r be a positive integer. Assume that $AN(B, W, P)$ is true for every framed space germ (B, W, P) such that $(B, W, P) \in \mathcal{C}$ and $\sharp\, P \le r$. Then, $RW(B, W, Q)$ is true for every framed space germ (B, W, Q) such that $(B, W, Q) \in \mathcal{C}$ and $\sharp\, Q \le r + 1$.*

Proof. Assume that there is a framed space germ (B, W, Q) such that $(B, W, Q) \in \mathcal{C}$, $\sharp\, Q \le r + 1$, and $RW(B, W, Q)$ is false. There is a non-zero element $f \in k[[Q]]$ such that for any IAMT (B', W', Q') of (B, W, Q) the image $f' \in k[[Q']]$ by the associated homomorphism $k[[Q]] \to k[[Q']]$ does not have a normal crossing.

Note that the following three claims hold under these assumptions:

1. For any IAMT (B', W', Q') of (B, W, Q) and for any Weierstrass representation (h', g', t') of f' under (B', W', Q'), we have $\mathrm{w}(g',\, t') < \infty$.
2. For any IAMT (B', W', Q') of (B, W, Q), there exist a coordinate transform (B', W', Q'_1) of (B', W', Q') and a Weierstrass representation (h', g', t') of f' under (B', W', Q'_1) such that $(B', W', Q'_1 - \{t'\}) \in \mathcal{C}$.
3. For any IAMT (B', W', Q') of (B, W, Q) and for any Weierstrass representation (h', g', t') of f' under (B', W', Q') such that $(B', W', Q' - \{t'\}) \in \mathcal{C}$, there exist an IAMT (B'', W'', Q'') of (B', W', Q') and a Weierstrass representation (h'', g'', t'') of f'' under (B'', W'', Q'') such that $(B'', W'', Q'' - \{t''\}) \in \mathcal{C}$, (h'', g'', t'') satisfies the Abhyankar condition, (h'', g'', t'') is not reductive, $\mathrm{ord}(g'') = \mathrm{ord}(g')$, and $\mathrm{sord}(g'') \leq \mathrm{sord}(g')$, where by $f'' \in k[[Q'']]$ we denote the image of $f' \in k[[Q']]$ by the associated homomorphism $k[[Q']] \to k[[Q'']]$.

We will show claim 1. Assume that for some IAMT (B', W', Q') of (B, W, Q) and for some Weierstrass representation (h', g', t') of f' under (B', W', Q'), we have $\mathrm{w}(g',\, t') = \infty$. Then, $g' = (t')^m$ for some non-negative integer m. Let $Q' = \{t', x'_1, x'_2, \dots, x'_n\}$. Since (h', g', t') is a Weierstrass representation, there are non-negative integers $s_1, s_2, \dots, s_n$ with

$$\frac{h'g'}{(t')^m (x'_1)^{s_1} (x'_2)^{s_2} \cdots (x'_n)^{s_n}} \in k[[Q']]^*,$$

Thus $f' = h'g' \in k[[Q']]$ has a normal crossing, which contradicts the assumption.

Claim 2 follows from Definition 7.3.

We proceed to claim 3. Let $P' = Q' - \{t'\}$. Since $(B', W', P') \in \mathcal{C}$ and since $\sharp P' = \sharp Q - 1 \leq r$, $AN(B', W, P')$ is true. By $AN(B', W, P')$ and by 1 above, there are an IAMT (B'', W'', P'') with a normal crossing of (B', W', P') and an element $H'' \in k[[P'']]$ such that the triplet $(h'', g'', t''_1 + H'')$ is a Weierstrass representation over (B'', W'', P''), $(h'', g'', t''_1 + H'')$ satisfies the Abhyankar condition, and $(h'', g'', t''_1 + H'')$ is not reductive, where (h'', g'', t''_1) denotes the pull-back of (h', g', t') by the associated morphism $(B', W', P') \to (B'', W'', P'')$. Since $\mathcal{C}$ is an admissible class, we have $(B'', W'', P'') \in \mathcal{C}$.

Let $t'' = t''_1 + H''$ and $Q'' = P'' \cup \{t''\}$. The triplet (B'', W'', Q'') is an IAMT of (B', W', Q'), and $(B'', W'', Q'' - \{t''\}) = (B'', W'', P'') \in \mathcal{C}$. One sees that the triplet (h'', g'', t'') satisfies the desired conditions by Lemma 6.3.3.

Let $\mathcal{X}$ be the set of pairs $((B', W', Q'), (h', g', t'))$ satisfying the following conditions:

(a) The first term (B', W', Q') is an IAMT of (B, W, Q).
(b) The second term (h', g', t') is a Weierstrass representation of f' under (B', W', Q'), where $f' \in k[[Q']]$ denotes the image of $f \in k[[Q]]$ by the associated morphism $k[[Q]] \to k[[Q']]$. Moreover, we assume that $(B', W', Q' - \{t'\}) \in \mathcal{C}$, (h', g', t') satisfies the Abhyankar condition, and (h', g', t') is not reductive.

By 2 and 3 above one knows $\mathcal{X} \neq \emptyset$.

We consider the totally ordered abelian group $\mathbb{Z}^3$ with the lexicographic order. By 1 with every $X = ((B', W', Q'), (h', g', t')) \in \mathcal{X}$ we can associate the element

$$c(X) = (\mathrm{sord}(g'), \mathrm{nord}(g'), [\mathrm{w}(g',\, t')]) \in \mathbb{Z}^3,$$

where $[\ ]$ denotes the Gauss symbol. By definition of $\mathcal{X}$ we have $\mathrm{sord}(g') \geq 0$, $\mathrm{nord}(g') \geq 0$ and $[\mathrm{w}(g',\, t')] \geq 1$. Since $\mathbb{Z}$ is discrete, there exists an element $X_0 = ((B_0, W_0, Q_0), (h_0, g_0, t_0)) \in \mathcal{X}$ such that $c(X) \geq c(X_0)$ for every $X \in \mathcal{X}$. Let $f_0 = h_0 g_0 \in k[[Q_0]]$ denote the image of $f \in k[[Q]]$.

By Theorem 7.2, there is an elementary monoidal transform (B_1, W_1, Q_1) of (B_0, W_0, Q_0) and an element $h_1 \in k[[Q_1]]$, $g_1 \in k[[Q_1]]$ and $t_1 \in Q_1$ and one of the four cases $\langle\alpha\rangle$, $\langle\beta\rangle$, $\langle\gamma\rangle$, $\langle\delta\rangle$ holds. Let $f_1 \in k[[Q_1]]$ denote the image of $f_0 \in k[[Q_0]]$.

Assume that case $\langle\alpha\rangle$ holds. Then f_1 has a normal crossing, which contradicts our assumption.

Assume that $\langle\beta\rangle$ holds. We have $\mathrm{sord}(f_1) < \mathrm{sord}(f_0) = \mathrm{sord}(g_0)$. By 2 and 3 there are an element $g_1' \in k[[Q_1]]$ with $\mathrm{sord}(g_1') = \mathrm{sord}(f_1)$ and an element $X_2 = ((B_2, W_2, Q_2), (h_2, g_2, t_2)) \in \mathcal{X}$ such that (B_2, W_2, Q_2) is an IAMT of (B_1, W_1, Q_1), $f_2 = h_2 g_2 \in k[[Q_2]]$ coincides the image of $f_1 \in k[[Q_1]]$ by $k[[Q_1]] \to k[[Q_2]]$, and $\mathrm{sord}(g_2) \leq \mathrm{sord}(g_1')$. We have $\mathrm{sord}(g_2) < \mathrm{sord}(g_0)$, and $c(X_2) < c(X_0)$, which contradicts the property of X_0.

Assume that $\langle\gamma\rangle$ holds. The triplet (h_1, g_1, t_1) is a Weierstrass representation of f_1, $\mathrm{sord}(g_1) = \mathrm{sord}(g_0)$, and $\mathrm{nord}(g_1) < \mathrm{nord}(g_0)$. By 3 there is an element $X_2 = ((B_2, W_2, Q_2), (h_2, g_2, t_2)) \in \mathcal{X}$ such that (B_2, W_2, Q_2) is an IAMT of (B_1, W_1, Q_1), $f_2 = h_2 g_2 \in k[[Q_2]]$ coincides the image of $f_1 \in k[[Q_1]]$ by $k[[Q_1]] \to k[[Q_2]]$, $\mathrm{ord}(g_2) = \mathrm{ord}(g_1)$. and $\mathrm{sord}(g_2) \leq \mathrm{sord}(g_1)$. If $\mathrm{sord}(g_2) < \mathrm{sord}(g_1)$, then we have $\mathrm{sord}(g_2) < \mathrm{nord}(g_0)$, and $c(X_2) < c(X_0)$. If $\mathrm{sord}(g_2) = \mathrm{sord}(g_1)$, then we have $\mathrm{nord}(g_2) = \mathrm{ord}(g_2) - \mathrm{sord}(g_2) = \mathrm{ord}(g_1) - \mathrm{sord}(g_1) = \mathrm{nord}(g_1) < \mathrm{nord}(g_0)$. Thus we have $c(X_2) < c(X_0)$ also in this case. Anyway we obtain $c(X_2) < c(X_0)$, which contradicts the property of X_0.

Assume that $\langle\delta\rangle$ holds. We have $X_1 = ((B_1, W_1, Q_1), (h_1, g_1, t_1)) \in \mathcal{X}$ and $\mathrm{sord}(g_1) = \mathrm{sord}(g_0)$, $\mathrm{nord}(g_1) = \mathrm{nord}(g_0)$, and $[\mathrm{w}(g_1, t_1)] = [\mathrm{w}(g_0, t_0)] - 1$. Thus we have $c(X_1) < c(X_0)$, which contradicts the property of X_0.

In every case we obtained a contradiction. Thus our theorem is true. □

Let $\mathcal{C}$ be an admissible class of framed space germs over k. By Theorem 7.4 it turns out that the problem below depending on $\mathcal{C}$ is essential.

Problem($\mathcal{C}$): Let r be a positive integer. Show that $AN(B, W, P)$ is true for every framed space germ (B, W, P) such that $(B, W, P) \in \mathcal{C}$ and $\sharp P \leq r$, assuming that $RW(B, W, P)$ is true for every framed space germ (B, W, P) such that $(B, W, P) \in \mathcal{C}$ and $\sharp P \leq r$.

If we could solve Problem($\mathcal{C}$), then $RW(B, W, Q)$ would be true for *every* $(B, W, Q) \in \mathcal{C}$. Note that if we drop the last condition "$(h', g', t' + H')$ is not reductive" from $AN(B, W, P)$, then Corollary 6.9 solves Problem($\mathcal{C}$) for the admissible class of all framed space germs.

In the remaining part of this paper, we solve the above problem for the admissible class $\mathcal{C}_0$.

Remark . In the essential parts until here, we have never used the assumption that the characteristic number p of the ground field k is positive.

If the characteristic number of the ground field k is zero, then we can easily solve Problem($\mathcal{C}$) for the admissible class of all framed space germs.

Let (g, t) be a Weierstrass pair with $\mathrm{w}(g, t) < \infty$, and $G_1, G_2, \ldots, G_m$ be the coefficient function sequence of (g, t). If the characteristic number is zero, then we can apply the parallel coordinate transformation along $H = G_1/m$ to (g, t). As the result, we can assume that (g, t) is a Weierstrass pair with $G_1 = 0$ from the beginning. It is easy to see that if $G_1 = 0$ and if the characteristic number is zero, then any pull-back (g',t') of (g,t) is not reductive. Thus, by Corollary 6.9 Problem$(\mathcal{C})$ can be solved.

Any hypersurface singularity of a germ of a variety in characteristic zero can be resolved by iterated monoidal transformations.

In Abhyankar [2] the parallel coordinate transformation along $H = G_1/m$ is called the Shreedharacharya transformation.

8. The admissible class $\mathcal{C}_0$

Recall that by $\mathcal{C}_0$ we denoted the collection of all framed space germs (B,W,Q) over k such that the equality $\dim A = \operatorname{rank} V$ holds for $(A,V) = \underline{(B,W,Q)}_{\bullet}$. (a paragraph after Definition 7.3.)

Lemma 8.1. *Let (B,W,Q) be a framed space germ with $(B,W,Q) \in \mathcal{C}_0$, and $f \in k[[Q]]$ be a non-zero element. There are a coordinate transform (B,W,Q') and a Weierstrass representation (h,g,t) of f under (B,W,Q') such that $h \in k[[Q']]^*$ and $(B,W,Q'-\{t\}) \in \mathcal{C}_0$.*

Proof. Let $(A,V) = \underline{(B,W,Q)}_{\bullet}$ and $n+1 = \dim A = \operatorname{rank} V$. $A = k[[Q]]$. By 1 and 2 of Corollary 4.15 the valuation ring V has $n+2$ prime ideals $\mathfrak{p}_i$, and we can assume that they satisfy

$$\{0\} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{i-1} \subset \mathfrak{p}_i \subset \cdots \subset \mathfrak{p}_{n+1} = M(V).$$

For every integer i with $1 \leq i \leq n+1$ we choose an element $a_i \in \mathfrak{p}_i - \mathfrak{p}_{i-1}$. Let $b \in A$ be a non-zero element with $ba_i \in A$ for every $1 \leq i \leq n+1$. Since A and V have the common quotient field, there exists b satisfying this condition. Let $c_i = ba_i \in A$ for every $1 \leq i \leq n+1$. We have $a_i \neq 0$ and $c_i \neq 0$, since $a_i \notin \mathfrak{p}_0 = \{0\}$.

By Proposition 2.18 there exist a parameter system $Q' = \{t, x_1, x_2, \ldots, x_n\}$ and elements $h, g, \epsilon, \tilde{b}, \delta_1, \tilde{c}_1, \delta_2, \tilde{c}_2, \ldots, \delta_{n+1}, \tilde{c}_{n+1}$ satisfying the following three conditions:

1. $h, \epsilon, \delta_1, \delta_2, \ldots, \delta_{n+1} \in A^*$.
2. $f = hg$, $b = \epsilon\tilde{b}$, and $c_i = \delta_i \tilde{c}_i$ for every $1 \leq i \leq n+1$.
3. (g,t), $(\tilde{b},t)$ and $(\tilde{c}_i,t)$ for $1 \leq i \leq n+1$ are Weierstrass pairs of A under Q'.

The triplet (h,g,t) is a Weierstrass representation of f under (B,W,Q') such that $h \in A^* = k[[Q']]^*$.

Let $P = Q' - \{t\}$. Let $(Z,U,P) = \underline{(B,W,P)}$, and D denote the quotient field of Z. Below we will show $(B,W,P) \in \mathcal{C}_0$, or in other words, $\dim Z = \operatorname{rank} U$.

Let $\tilde{A} = Z[t]$, $\tilde{E} = D(t)$ be the quotient field of $\tilde{A}$, and $\tilde{V} = V \cap \tilde{E}$. $\tilde{A}$ is a finitely generated Z-subalgebra of A, $\tilde{E}$ is a subfield of the quotient field of A, $D \subset \tilde{E}$, and $\operatorname{tr.deg}_D \tilde{E} = 1$. By Lemma 4.13.1 $\tilde{V}$ is a valuation ring of the field $\tilde{E}$. $g, \tilde{b}, \tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{n+1} \in \tilde{A} \subset \tilde{V}$.

We will show $\operatorname{rank} \tilde{V} = n+1$.

Let $\tilde{\mathfrak{p}}_i = \mathfrak{p}_i \cap \tilde{V}$ for every integer i with $0 \leq i \leq n+1$. For every i $\tilde{\mathfrak{p}}_i$ is a prime ideal of $\tilde{V}$, and they satisfy

$$\{0\} = \tilde{\mathfrak{p}}_0 \subset \tilde{\mathfrak{p}}_1 \subset \cdots \subset \tilde{\mathfrak{p}}_{i-1} \subset \tilde{\mathfrak{p}}_i \subset \cdots \subset \tilde{\mathfrak{p}}_{n+1}.$$

Since $k \subset \tilde{V}/\tilde{\mathfrak{p}}_{n+1} \subset V/\mathfrak{p}_{n+1} = V/M(V) \cong k$, one knows $\tilde{\mathfrak{p}}_{n+1} = M(\tilde{V})$, and $\tilde{V}/\tilde{\mathfrak{p}}_{n+1} \cong k$. $\tilde{b} \neq 0$, since $\epsilon\tilde{b} = b \neq 0$. Let $\tilde{a}_i = \tilde{c}_i/\tilde{b} \in \tilde{E}$ for every $1 \leq i \leq n+1$. We have $\epsilon/\delta_i \in A^* \subset V^*$, since $\epsilon \in A^*$ and $\delta_i \in A^*$. Thus $\tilde{a}_i = (\epsilon/\delta_i)a_i \in (\mathfrak{p}_i - \mathfrak{p}_{i-1}) \cap \tilde{E} = \tilde{\mathfrak{p}}_i - \tilde{\mathfrak{p}}_{i-1}$ for every $1 \leq i \leq n+1$. One knows $\tilde{\mathfrak{p}}_i \neq \tilde{\mathfrak{p}}_{i-1}$ for every $1 \leq i \leq n+1$. Since $\tilde{V}$ contains mutually different $n+2$ prime ideals $\tilde{\mathfrak{p}}_i$, one concludes $\operatorname{rank} \tilde{V} \geq n+1$.

Let $\mathfrak{q} = M(\tilde{V}) \cap \tilde{A}$. We apply Theorem 2.1 to $\tilde{A} \supset Z$ and a prime ideal $\mathfrak{q} \in \operatorname{Spec}(\tilde{A})$. Note that Z is universally catenary by Theorem 2.2. Since $k \subset Z/\mathfrak{q} \cap Z \subset \tilde{A}/\mathfrak{q} \subset \tilde{V}/M(\tilde{V}) \cong k$, one has $Z/\mathfrak{q} \cap Z \cong \tilde{A}/\mathfrak{q} \cong k$. $\operatorname{tr.deg}_Z \tilde{A} = \operatorname{tr.deg}_D \tilde{E} = 1$. Since $Z/\mathfrak{q} \cap Z \cong k$, we have $\mathfrak{q} \cap Z = M(Z)$ and $\operatorname{ht}(\mathfrak{q} \cap Z) = \operatorname{ht}(M(Z)) = \dim Z = n$.

By Theorem 2.1 one concludes $\operatorname{ht}(\mathfrak{q}) = \operatorname{ht}(\mathfrak{q} \cap Z) + \operatorname{tr.deg}_Z \tilde{A} - \operatorname{tr.deg}_{(Z/\mathfrak{q}\cap Z)}(\tilde{A}/\mathfrak{q}) = n+1$ and $\dim \tilde{A}_{\mathfrak{q}} = n+1$. Since the valuation ring $\tilde{V}$ dominates the noetherian local domain $\tilde{A}_{\mathfrak{q}}$, we have $\operatorname{rank} \tilde{V} \leq n+1$ by Corollary 4.22. Thus $\operatorname{rank} \tilde{V} = n+1$.

By Lemma 4.12.2 and Theorem 4.21 $n+1 = \operatorname{rank} \tilde{V} \leq \operatorname{r.rank} \tilde{V} \leq \dim \tilde{A}_{\mathfrak{q}} = n+1$, and $\operatorname{rank} U \leq \operatorname{r.rank} U \leq \dim Z = n$.

On the other hand, since $k \subset U/M(U) \subset \tilde{V}/M(\tilde{V}) \cong k$, we have $U/M(U) \cong \tilde{V}/M(\tilde{V}) \cong k$ and $\operatorname{tr.deg}_{(U/M(U))}(\tilde{V}/M(\tilde{V})) = 0$. Since $\operatorname{tr.deg}_D \tilde{E} = 1$, we obtain $n+1 - \operatorname{r.rank} U = \operatorname{r.rank} \tilde{V} - \operatorname{r.rank} U \leq \operatorname{tr.deg}_D \tilde{E} - \operatorname{tr.deg}_{(U/M(U))}(\tilde{V}/M(\tilde{V})) = 1$ by Lemma 4.23.2. We obtain $\operatorname{rank} U \leq \operatorname{r.rank} U = n$.

Assume that $r = \operatorname{rank} U < n$.

Let $\tilde{v} : \tilde{E} \to \Gamma \cup \{\infty\}$ and $u : D \to \bar{\Gamma} \cup \{\infty\}$ denote the valuations defined by valuation rings $\tilde{V}$ and U respectively. By Lemma 4.13.2 we can regard $\bar{\Gamma}$ as a subgroup of Γ and $\tilde{v} \mid D = u$.

By Theorem 4.21.2 there exists an isomorphism $\bar{\Gamma} \cong \mathbb{Z}^n$ as abelian groups. By Corollary 4.22.2 there exists an isomorphism $\Gamma \cong \mathbb{Z}^{n+1}$ as totally ordered abelian groups. Below by $\operatorname{rank}_{\mathbb{Z}} L$ we denote the rank over $\mathbb{Z}$ of a free $\mathbb{Z}$-module L.

Let

$$\bar{\Gamma} = \bar{\Delta}_0 \supset \bar{\Delta}_1 \supset \cdots \supset \bar{\Delta}_{i-1} \supset \bar{\Delta}_i \supset \cdots \supset \bar{\Delta}_r = \{0\}$$

be the sequence of isolated subgroups of $\bar{\Gamma}$. Since $\sum_{i=1}^{r} \operatorname{rank}_{\mathbb{Z}}(\bar{\Delta}_{i-1}/\bar{\Delta}_i) = \operatorname{rank}_{\mathbb{Z}} \bar{\Gamma} = n$, there is an integer i_0 such that $1 \leq i_0 \leq r$ and $\operatorname{rank}_{\mathbb{Z}}(\bar{\Delta}_{i_0-1}/\bar{\Delta}_{i_0}) \geq 2$.

By Lemma 4.14.2 for every integer j with $0 \leq j \leq n+1$, $\Delta_j = \tilde{v}(\tilde{V}^*_{\tilde{\mathfrak{p}}_j})$ is an isolated subgroup of Γ.

$$\Gamma = \Delta_0 \supset \Delta_1 \supset \cdots \supset \Delta_{i-1} \supset \Delta_i \supset \cdots \supset \Delta_{n+1} = \{0\}.$$

For every integer j with $1 \leq j \leq n+1$ we have $\operatorname{rank}_{\mathbb{Z}}(\Delta_{j-1}/\Delta_j) = 1$.

For every $0 \leq i \leq r$ the set $\{j \mid 0 \leq j \leq n+1, \Delta_j \cap \bar{\Gamma} = \bar{\Delta}_i\}$ is not empty by Corollary 3.4.6. Let

$$\begin{aligned} j_1 &= \min\{j \mid 0 \leq j \leq n+1, \Delta_j \cap \bar{\Gamma} = \bar{\Delta}_{i_0}\}, \text{ and} \\ j_2 &= \max\{j \mid 0 \leq j \leq n+1, \Delta_j \cap \bar{\Gamma} = \bar{\Delta}_{i_0-1}\}. \end{aligned}$$

We can easily show $j_2 = j_1 - 1$.

Since $\bar{\Delta}_{i_0} = \Delta_{j_1} \cap \bar{\Gamma}$, there exists an injective homomorphism

$$\bar{\Delta}_{i_0-1}/\bar{\Delta}_{i_0} \to \Delta_{j_2}/\Delta_{j_1} = \Delta_{j_1-1}/\Delta_{j_1} \cong \mathbb{Z}$$

of abelian groups. We have $2 \leq \operatorname{rank}_{\mathbb{Z}}(\bar{\Delta}_{i_0-1}/\bar{\Delta}_{i_0}) \leq 1$, which is a contradiction.

We conclude $\operatorname{rank} U = n = \dim Z$ and $(B, W, P) \in \mathcal{C}_0$. □

Lemma 8.2. *Let (B, W, P) be a space germ over k. Let $(Z, U, P) = \underline{(B, W, P)}$, $P = \{x_1, x_2, \ldots, x_n\}$, $r = \operatorname{rank} U$ and*

$$\{0\} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{i-1} \subset \mathfrak{q}_i \subset \cdots \subset \mathfrak{q}_r = M(U)$$

be the sequence of prime ideals of U. Consider the following three conditions on (B, W, P).

$\langle\alpha\rangle$ $\dim Z = \operatorname{rank} U$.

$\langle\beta\rangle$ *The morphism* $\operatorname{Spec}(U) \to \operatorname{Spec}(Z)$ *induced by the inclusion homomorphism* $Z \subset U$ *is injective.*

$\langle\gamma\rangle$ *Under an appropriate numbering of* x_i*'s, for every integer* i *with* $1 \le i \le r$ *the prime ideal* $\mathfrak{q}_i \cap Z$ *of* Z *is generated by* i *elements* $x_1, x_2, \ldots, x_i$.

1. *Assume that* (B, W, P) *satisfies* $\langle\alpha\rangle$. *Let* (B', W', P') *be an IAMT of* (B, W, P). *Also* (B', W', P') *satisfies* $\langle\alpha\rangle$. *Let* $(Z', U', P') = \underline{(B', W', P')}$, *and*
$$\{0\} = \mathfrak{q}'_0 \subset \mathfrak{q}'_1 \subset \cdots \subset \mathfrak{q}'_{i-1} \subset \mathfrak{q}'_i \subset \cdots \subset \mathfrak{q}'_n = M(U')$$
denote the sequence of prime ideals of U'. *For every* $0 \le i \le n$ *we have* $\mathfrak{q}_i = \mathfrak{q}'_i \cap U$ *and* $(\mathfrak{q}'_i \cap Z') \cap Z = \mathfrak{q}_i \cap Z$.
2. *Assume that* (B, W, P) *satisfies* $\langle\alpha\rangle$ *and* $\langle\beta\rangle$. *Let* (B', W', P') *be an IAMT of* (B, W, P). *Also* (B', W', P') *satisfies* $\langle\alpha\rangle$ *and* $\langle\beta\rangle$. *For every* $0 \le i \le n$ *we have* $\operatorname{ht}(\mathfrak{q}'_i \cap Z') = i$, *where* $\mathfrak{q}'_i$ *and* Z' *denote the same as above.*
3. *If* (B, W, P) *satisfies* $\langle\alpha\rangle$, $\langle\beta\rangle$ *and* $\langle\gamma\rangle$, *then any IAMT* (B', W', P') *with a normal crossing of* (B, W, P) *satisfies* $\langle\alpha\rangle$, $\langle\beta\rangle$ *and* $\langle\gamma\rangle$.

Proof. 1. By Lemma 5.5.1 and Lemma 4.13.3 we have $\operatorname{rank} U' \ge \operatorname{rank} U$. By Corollary 4.22.1 $\operatorname{rank} U' \le \dim Z' = \dim Z = \operatorname{rank} U$. Thus $\operatorname{rank} U' = \operatorname{rank} U = \dim Z = \dim Z' = n$.

By Lemma 5.5.1 and Corollary 4.15.3 the morphism $\operatorname{Spec}(U') \to \operatorname{Spec}(U)$ is bijective. Since the morphism $\operatorname{Spec}(U') \to \operatorname{Spec}(U)$ preserves inclusion relations, we have $\mathfrak{q}_i = \mathfrak{q}'_i \cap U$ for every $0 \le i \le n$. Besides, $(\mathfrak{q}'_i \cap Z') \cap Z = (\mathfrak{q}'_i \cap U) \cap Z = \mathfrak{q}_i \cap Z$.
2. By 1 (B', W', P') satisfies $\langle\alpha\rangle$. The following diagram is commutative:

$$\begin{array}{ccc} \operatorname{Spec}(U') & \longrightarrow & \operatorname{Spec}(U) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(Z') & \longrightarrow & \operatorname{Spec}(Z) \end{array}$$

By 1 the morphism $\operatorname{Spec}(U') \to \operatorname{Spec}(U)$ is bijective. By $\langle\beta\rangle$ for (B, W, P) the morphism $\operatorname{Spec}(U) \to \operatorname{Spec}(Z)$ is injective. Thus $\operatorname{Spec}(U') \to \operatorname{Spec}(Z')$ is injective.

Obviously we have inclusion relations
$$\{0\} = \mathfrak{q}'_0 \cap Z' \subset \mathfrak{q}'_1 \cap Z' \subset \cdots \subset \mathfrak{q}'_{i-1} \cap Z' \subset \mathfrak{q}'_i \cap Z' \subset \cdots \subset \mathfrak{q}'_n \cap Z'.$$
By $\langle\beta\rangle$ for (B', W', P'), which we have shown above, any two of $n+1$ prime ideals $\mathfrak{q}'_i \cap Z'$ never coincide. Since $\dim Z' = n$, one knows that $\operatorname{ht}(\mathfrak{q}'_i \cap Z') = i$ for every $0 \le i \le n$.
3. By 1 and 2 (B', W', P') satisfies $\langle\alpha\rangle$ and $\langle\beta\rangle$.

We will show $\langle\gamma\rangle$ for (B', W', P'). Let Z', U' and $\mathfrak{q}'_i$ be the same as in 1. We write $P' = \{y_1, y_2, \cdots, y_n\}$. By $\langle\gamma\rangle$ for (B, W, P) we can assume that for every $1 \le i \le n$ the prime ideal $\mathfrak{q}_i \cap Z$ is generated by $x_1, x_2, \ldots, x_i$. We assume this condition.

We apply induction. Let j be an integer with $1 \le j \le n$. We assume that under an appropriate renumbering of y_i's, for every integer i with $0 \le i < j$ the prime ideal $\mathfrak{q}'_i \cap Z'$ is generated by $y_1, y_2, \ldots, y_i$. In case $i = 0$, the ideal generated by the empty set is the ideal $\{0\}$ containing only one element 0.

Since (B', W', P') is an IAMT with a normal crossing of (B, W, P), there are an element $u \in (Z')^*$ and non-negative integers $s_1, s_2, \ldots, s_n$ with $x_j = u y_1^{s_1} y_2^{s_2} \cdots y_n^{s_n}$.

We infer $s_1 = s_2 = \cdots = s_{j-1} = 0$. Assume the contrary. Since $\mathfrak{q}'_{j-1} \cap Z'$ is generated by $y_1, y_2, \ldots, y_{j-1}$, we have $x_j \in (\mathfrak{q}'_{j-1} \cap Z') \cap Z = \mathfrak{q}_{j-1} \cap Z$. We used 1 here. The ideal $\mathfrak{q}_{j-1} \cap Z$ is generated by $x_1, x_2, \dots, x_{j-1}$. In the ring $Z = k[[x_1, x_2, \dots, x_n]]$ the element x_j belongs to the ideal generated $x_1, x_2, \dots, x_{j-1}$, which is a contradiction.

By the above we can write $x_j = u y_j^{s_j} y_{j+1}^{s_{j+1}} \cdots y_n^{s_n}$. Since $x_j \in \mathfrak{q}_j \cap Z = (\mathfrak{q}'_j \cap Z') \cap Z \subset \mathfrak{q}'_j \cap Z'$, and since $\mathfrak{q}'_j \cap Z'$ is a prime ideal of Z', one of $y_j, y_{j+1}, \dots, y_n$ belongs to $\mathfrak{q}'_j \cap Z'$. Fixing the numbers of $y_1, y_2, \dots, y_{j-1}$ and exchanging the numbers $y_j, y_{j+1}, \dots, y_n$, we can assume that $y_j \in \mathfrak{q}'_j \cap Z'$.

We consider the residue ring $\bar{Z} = Z'/(\mathfrak{q}'_{j-1} \cap Z')$ and the prime ideal $\bar{\mathfrak{q}} = (\mathfrak{q}'_j \cap Z')/(\mathfrak{q}'_{j-1} \cap Z')$ of $\bar{Z}$. By $\bar{e} \in \bar{Z}$ we denote the image of an element $e \in Z'$ by the canonical surjective homomorphism $Z' \to \bar{Z}$.

$\bar{y}_j \in \bar{\mathfrak{q}}$. Since $\mathfrak{q}'_{j-1} \cap Z'$ is the ideal in $Z' = k[[y_1, y_2, \dots, y_n]]$ generated by $y_1, y_2, \dots, y_{j-1}$, we have $y_j \notin \mathfrak{q}'_{j-1} \cap Z'$ and $\bar{y}_j \neq 0$. Thus $\{0\} \neq \bar{y}_j \bar{Z} \subset \bar{\mathfrak{q}}$.

Since $\bar{Z}/\bar{y}_j \bar{Z} \cong k[[y_{j+1}, y_{j+2}, \dots, y_n]]$, $\bar{y}_j \bar{Z}$ is a prime ideal of $\bar{Z}$.

By 3 and 6 of Theorem 2.2 Z' is a catenary noetherian domain. Thus by 2 we have $\mathrm{ht}(\bar{\mathfrak{q}}) = \mathrm{ht}(\mathfrak{q}'_j \cap Z') - \mathrm{ht}(\mathfrak{q}'_{j-1} \cap Z') = j - (j-1) = 1$, and we can conclude $\bar{y}_j \bar{Z} = \bar{\mathfrak{q}}$. Thus $\mathfrak{q}'_j \cap Z'$ is generated by $y_1, y_2, \dots, y_j$.

In the above renumbering procedure, we fixed the numbers of $y_1, y_2, \dots, y_{j-1}$. Thus one can conclude that under an appropriate renumbering of y_i's, for every integer i with $0 \leq i < j+1$ the prime ideal $\mathfrak{q}'_i \cap Z'$ is generated by $y_1, y_2, \dots, y_i$.

For $j = 1$ the claim is true, since $\mathfrak{q}'_0 \cap Z' = \{0\}$. Thus by induction one concludes that $\langle\gamma\rangle$ holds for (B', W', P'). □

Corollary 8.3. *The collection $\mathcal{C}_0$ of all framed space germs (B, W, Q) over k such that $(A, V) = \underline{(B, W, Q)}_{\bullet}$ satisfies* $\dim A = \mathrm{rank}\, V$ *is an admissible class.*

Proof. It follows Lemma 8.1 and Lemma 8.2.1. □

Recall again the statement $RW(B, W, P)$ we gave before Lemma 5.12.

Lemma 8.4. *Let r be a positive integer. Assume that $RW(B, W, P)$ is true for every space germ (B, W, P) such that $(B, W, P) \in \mathcal{C}_0$ and $\sharp P \leq r$.*

Then, for every $(B, W, P) \in \mathcal{C}_0$ satisfying $\sharp P \leq r$, there exists an IAMT (B', W', P') with a normal crossing of (B, W, P) such that (B', W', P') satisfies the three conditions $\langle\alpha\rangle$, $\langle\beta\rangle$ and $\langle\gamma\rangle$ in Lemma 8.2.

Proof. Let (B, W, P) be a framed space germ over k such that $(B, W, P) \in \mathcal{C}_0$ and $\sharp P \leq r$. By assumption $RW(B, W, P)$ is true. Let $(Z, U, P) = \underline{(B, W, P)}$ and $n = \dim Z = \mathrm{rank}\, U$. Let

$$\{0\} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{i-1} \subset \mathfrak{q}_i \subset \cdots \subset \mathfrak{q}_n = M(U)$$

be the sequence of prime ideals of U.

For every $1 \leq i \leq n$ we choose an element $a_i \in \mathfrak{q}_i - \mathfrak{q}_{i-1}$. We have $a_i \neq 0$, since $a_i \notin \mathfrak{q}_0 = \{0\}$.

Let (B', W', P') be an arbitrary IAMT of (B, W, P). Let $(Z', U', P') = \underline{(B', W', P')}$. We identify $a_i \in U$ and the image of a_i by the associated homomorphism $U \to U'$. Define

$$\beta(B', W', P') = \sharp\{a_i \mid 1 \leq i \leq n, a_i \notin Z'\}.$$

By definition $\beta(B', W', P')$ is a non-negative integer.

First, by induction on $\beta(B', W', P')$ we will show that there exists an IAMT (B', W', P') with a normal crossing of (B, W, P) such that $\beta(B', W', P') = 0$.

If $\beta(B,W,P)=0$, then $(B',W',P')=(B,W,P)$ satisfies the condition.

Assume $\beta(B,W,P)>0$. There exists an integer i such that $1\leq i\leq n$ and $a_i\notin Z$. Since Z and U have the common quotient field, there are elements $b,c\in Z$ with $a_i=c/b$, $c\neq 0$ and $b\neq 0$. By Corollary 5.13 $RS(B,W,P)$ is true. Thus there exists an IAMT (B_1,W_1,P_1) with a normal crossing of (B,W,P) such that one of the following (a), (b) holds: Let $(Z_1,U_1,P_1)=\underline{(B_1,W_1,P_1)}$.

(a) There exists an element $e\in Z_1$ with $c=eb$.
(b) There exists an element $e\in M(Z_1)$ with $b=ec$.

Assume that (b) holds. $e\in M(Z_1)\subset M(U_1)$. $1/e=c/b=a_i\in U\subset U_1$. Thus we have $1=e\cdot(1/e)\in M(U_1)$, which is a contradiction. Thus (a) holds, and $a_i=c/b=e\in Z_1$. Since $Z\subset Z_1$, we have $\beta(B,W,P)>\beta(B_1,W_1,P_1)$. By Lemma 8.2.1 $(B_1,W_1,P_1)\in\mathcal{C}_0$ and $\sharp P_1=\sharp P\leq r$. By assumption $RW(B_1,W_1,P_1)$ is true. By induction hypothesis there exists an IAMT (B',W',P') with a normal crossing of (B_1,W_1,P_1) with $\beta(B',W',P')=0$.

Obviously (B',W',P') is an IAMT with a normal crossing of (B,W,P). By Lemma 8.2.1 (B',W',P') satisfies $\langle\alpha\rangle$. We will show that (B',W',P') satisfies $\langle\beta\rangle$, too.

Let $(Z',U',P')=\underline{(B',W',P')}$, and

$$\{0\}=\mathfrak{q}'_0\subset\mathfrak{q}'_1\subset\cdots\subset\mathfrak{q}'_{i-1}\subset\mathfrak{q}'_i\subset\cdots\subset\mathfrak{q}'_n=M(U')$$

be the sequence of prime ideals of U'. By Lemma 8.2.1 we have $\mathfrak{q}_i=\mathfrak{q}'_i\cap U$ for every $0\leq i\leq n$. Obviously $\mathfrak{q}'_{i-1}\cap Z'\subset\mathfrak{q}'_i\cap Z'$ for every $1\leq i\leq n$. Since $\beta(B',W',P')=0$, we have $a_i\in Z'$ for every $1\leq i\leq n$. Thus $a_i\in\mathfrak{q}_i\cap Z'=(\mathfrak{q}'_i\cap U)\cap Z'\subset\mathfrak{q}'_i\cap Z'$. Assume $a_i\in\mathfrak{q}'_{i-1}\cap Z'$. We have $a_i\in\mathfrak{q}'_{i-1}\cap U=\mathfrak{q}_{i-1}$, which contradicts $a_i\notin\mathfrak{q}_{i-1}$. Thus, for every $1\leq i\leq n$ we have $a_i\notin\mathfrak{q}'_{i-1}\cap Z'$ and $\mathfrak{q}'_{i-1}\cap Z'\neq\mathfrak{q}'_i\cap Z'$. We conclude that the morphism $\operatorname{Spec}(U')\to\operatorname{Spec}(Z')$ is injective. (B',W',P') satisfies $\langle\beta\rangle$.

An IAMT (B'',W'',P'') with a normal crossing of (B',W',P') is an IAMT with a normal crossing of (B,W,P). Besides, $(B',W',P')\in\mathcal{C}_0$ and $\sharp P'=\sharp P\leq r$. Thus, exchanging (B,W,P) and (B',W',P'), we can assume that (B,W,P) satisfies $\langle\alpha\rangle$ and $\langle\beta\rangle$. By induction of the basic dimension $\sharp P$, we will show that there exists an IAMT (B',W',P') with a normal crossing of (B,W,P) satisfying $\langle\alpha\rangle$, $\langle\beta\rangle$, and $\langle\gamma\rangle$ under this stronger assumption.

If $\sharp P=1$, then $(B',W',P')=(B,W,P)$ satisfies $\langle\alpha\rangle$, $\langle\beta\rangle$, and $\langle\gamma\rangle$.

Assume $r\geq\sharp P\geq 2$. By Lemma 8.2.2 we have $\operatorname{ht}(\mathfrak{q}_1\cap Z)=1$ and there is an element f with $0\neq f\in\mathfrak{q}_1\cap Z$. By assumption $RW(B,W,P)$ is true, and there exists an IAMT (B_1,W_1,P_1) with a normal crossing of (B,W,P) such that the image of f in $Z_1=k[[P_1]]$ has a normal crossing with respect to P_1. Let $(Z_1,U_1,P_1)=\underline{(B_1,W_1,P_1)}$, and $P_1=\{y_1,y_2,\dots,y_n\}$. We have an element $u\in Z_1^*$ and non-negative integers $s_1,s_2,\dots,s_n$ with $f=uy_1^{s_1}y_2^{s_2}\cdots y_n^{s_n}$. Let

$$\{0\}=\mathfrak{q}_{1,0}\subset\mathfrak{q}_{1,1}\subset\cdots\subset\mathfrak{q}_{1,i-1}\subset\mathfrak{q}_{1,i}\subset\cdots\subset\mathfrak{q}_{1,n}=M(U_1)$$

be the sequence of prime ideals of U_1. By Lemma 8.2.1 we have $f=uy_1^{s_1}y_2^{s_2}\cdots y_n^{s_n}\in\mathfrak{q}_1\cap Z=(\mathfrak{q}_{1,1}\cap Z_1)\cap Z\subset\mathfrak{q}_{1,1}\cap Z_1$. Since $\mathfrak{q}_{1,1}\cap Z_1$ is a prime ideal of Z_1, there is an integer i such that $1\leq i\leq n$ and $y_i\in\mathfrak{q}_{1,1}\cap Z_1$. By exchanging numbering of y_i's, we can assume that $y_1\in\mathfrak{q}_{1,1}\cap Z_1$. y_1Z_1 is a non-zero prime ideal of Z_1, and $y_1Z_1\subset\mathfrak{q}_{1,1}\cap Z_1$. By Lemma 8.2.2 we know $\operatorname{ht}(\mathfrak{q}_{1,1}\cap Z_1)=1$. Thus we conclude $y_1Z_1=\mathfrak{q}_{1,1}\cap Z_1$.

Let $O_1 = P_1 - \{y_1\} = \{y_2, y_3, \dots, y_n\}$, and we consider the framed space germ (B_1, W_1, O_1).

Let (B', W', O') be an arbitrary IAMT of (B_1, W_1, O_1), $(Z', U', O' \cup \{y_1\}) = \underline{(B', W', O' \cup \{y_1\})}$, and $(Y', T', O') = \underline{(B', W', O')}$. We identify $y_1 \in B_1$ and the image in B' of y_1 here. Let

$$\{0\} = \mathfrak{q}'_0 \subset \mathfrak{q}'_1 \subset \cdots \subset \mathfrak{q}'_{i-1} \subset \mathfrak{q}'_i \subset \cdots \subset \mathfrak{q}'_n = M(U')$$

be the sequence of prime ideals of U'.

The following five claims hold:

1. $\mathfrak{q}'_1 \cap Z' = y_1 Z'$.
2. The prime ideal $\mathfrak{q}'_i \cap Z'$ of Z' is generated by y_1 and $\mathfrak{q}'_i \cap Y'$ for every $1 \leq i \leq n$.
3. $\mathfrak{q}_{i-1} \cap Y' \neq \mathfrak{q}_i \cap Y'$ for every $2 \leq i \leq n$.
4. $\operatorname{rank} T' = \dim Y' = n - 1$. $\operatorname{Spec}(T') = \{\mathfrak{q}'_i \cap T' \mid 1 \leq i \leq n\}$.
5. The morphism $\operatorname{Spec}(T') \to \operatorname{Spec}(Y')$ is injective.

We will show claims 1–5.

By Lemma 8.2.1 $y_1 \in \mathfrak{q}_{1,1} \cap Z_1 = (\mathfrak{q}'_1 \cap Z') \cap Z_1 \subset \mathfrak{q}'_1 \cap Z'$. Thus $y_1 Z' \subset \mathfrak{q}'_1 \cap Z'$. Since y_1 is a member of a parameter system of a regular local ring Z', $y_1 Z'$ is a non-zero prime ideal of Z'. By Lemma 8.2.2 $\operatorname{ht}(\mathfrak{q}'_1 \cap Z') = 1$. We have $y_1 Z = \mathfrak{q}'_1 \cap Z'$. We obtain claim 1.

Let i be an integer with $1 \leq i \leq n$. $y_1 \in \mathfrak{q}'_1 \cap Z' \subset \mathfrak{q}'_i \cap Z'$. $\mathfrak{q}'_i \cap Y' \subset \mathfrak{q}'_i \cap Z'$. Thus the prime ideal $\mathfrak{q}'_i \cap Z'$ contains the ideal generated by y_1 and $\mathfrak{q}'_i \cap Y'$.

Let $g \in \mathfrak{q}'_i \cap Z'$ be an arbitrary element. Since $Y' = k[[O']]$ and $Z' = k[[O' \cup \{y_1\}]]$, we have elements g', g'' such that

$$g = g' y_1 + g'', \quad g' \in Z', \quad g'' \in Y'.$$

We have $g'' = g - g' y_1 \in (\mathfrak{q}'_i \cap Z') \cap Y' = \mathfrak{q}'_i \cap Y'$. Thus $\mathfrak{q}'_i \cap Z'$ is contained in the ideal generated by y_1 and $\mathfrak{q}'_i \cap Y'$. We obtain claim 2.

By Lemma 8.2.1 Any two of n prime ideals $\mathfrak{q}'_i \cap Z'$, $1 \leq i \leq n$ never coincide. We obtain claim 3 by claim 2.

Since $\mathfrak{q}'_i \cap Y' = (\mathfrak{q}'_i \cap T') \cap Y'$, any two of n prime ideals $\mathfrak{q}'_i \cap T'$, $1 \leq i \leq n$ of T' never coincide. Thus $\operatorname{rank} T' \geq n - 1 = \dim Y'$. On the other hand, by Corollary 4.22.1 we have $\operatorname{rank} T' \leq \dim Y'$. We obtain claim 4.

Claim 5 follows from claim 3 and claim 4.

We go back to the consideration of (B_1, W_1, P_1) and $O_1 = P_1 - \{y_1\}$. Applying above claim 4 and claim 5 for $(B', W', O') = (B_1, W_1, O_1)$, one knows that (B_1, W_1, O_1) satisfies $\langle\alpha\rangle$ and $\langle\beta\rangle$. Since $\sharp\, O_1 = \sharp\, P - 1$, we can apply the induction hypothesis.

Let (B', W', O') be an appropriate IAMT with a normal crossing of (B_1, W_1, O_1). Let $(Y', T', O') = \underline{(B', W', O')}$, $O' = \{z_2, z_3, \dots, z_n\}$, and

$$\{0\} = \bar{\mathfrak{q}}'_1 \subset \bar{\mathfrak{q}}'_2 \subset \cdots \subset \bar{\mathfrak{q}}'_{i-1} \subset \bar{\mathfrak{q}}'_i \subset \cdots \subset \bar{\mathfrak{q}}'_n = M(T')$$

be the sequence of prime ideals of T'. By the induction hypothesis, the prime ideal $\bar{\mathfrak{q}}'_i \cap Y'$ of Y' is generated by $z_2, z_3, \dots, z_i$ for every $2 \leq i \leq n$. Let $z_1 \in B'$ be the image of $y_1 \in B_1$ by the associated homomorphism $B_1 \to B'$. By Lemma 5.6 $O' \cup \{z_1\}$ is a frame of B'. Let $(Z', U', O' \cup \{z_1\}) = \underline{(B', W', O' \cup \{z_1\})}$. $(B', W', O' \cup \{z_1\})$ is an IAMT with a normal crossing of $(B_1, W_1, O_1 \cup \{y_1\})$. Denoting the sequence of prime ideals of U' by $\mathfrak{q}'_i$, $0 \leq i \leq n$, by claim 4 above one knows that $\bar{\mathfrak{q}}'_i = \mathfrak{q}'_i \cap T'$ for every $1 \leq i \leq n$.

By claim 2, the prime ideal $\mathfrak{q}'_i \cap Z'$ of Z' is generated by z_1 and $\mathfrak{q}'_i \cap Y'$ for every $1 \leq i \leq n$. Besides, $\mathfrak{q}'_i \cap Y' = (\mathfrak{q}'_i \cap T') \cap Y' = \bar{\mathfrak{q}}'_i \cap Y'$.

Thus, for every $2 \leq i \leq n$, $\mathfrak{q}'_i \cap Z'$ is generated by z_1 and $\bar{\mathfrak{q}}'_i \cap Y'$. Since $\bar{\mathfrak{q}}'_i \cap Y'$ is generated by $z_2, z_3, \ldots, z_i$, $\mathfrak{q}'_i \cap Z'$ is generated by $z_1, z_2, \ldots, z_i$.

By claim 1 $\mathfrak{q}'_1 \cap Z'$ is generated by $z_1 = y_1$.

We conclude that $(B', W', O' \cup \{z_1\})$ satisfies $\langle\gamma\rangle$. By Lemma 8.2.2 $(B', W', O' \cup \{z_1\})$ satisfies also $\langle\alpha\rangle$ and $\langle\beta\rangle$. $(B', W', O' \cup \{z_1\})$ is an IAMT with a normal crossing of (B, W, P). ☐

Definition 8.5. Let (B, W, P) be a framed space germ over k. Let $(Z, U, P) = \underline{(B, W, P)}$, $P = \{x_1, x_2, \ldots, x_n\}$ and

$$\{0\} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{i-1} \subset \mathfrak{q}_i \subset \cdots \subset \mathfrak{q}_n = M(U)$$

be the sequence of prime ideals of U. We call U the *basic valuation ring* of (B, W, P). If (B, W, P) satisfies the following $\langle A\rangle$ and $\langle B\rangle$, then we say that (B, W, P) *has the basic valuation ring of normal crossing type*, or U is *of normal crossing type.*

- $\langle A\rangle$ $\operatorname{rank} U = \dim Z$.
- $\langle B\rangle$ Under an appropriate numbering of x_i's, the prime ideal $\mathfrak{q}_i \cap Z$ of Z is generated by $x_1, x_2, \ldots, x_i$ for every $1 \leq i \leq n$.

Obviously (B, W, P) has the basic valuation ring of normal crossing type, if and only if, (B, W, P) satisfies conditions $\langle\alpha\rangle$, $\langle\beta\rangle$ and $\langle\gamma\rangle$ in Lemma 8.2.

Corollary 8.6. *Let r be a positive integer. Assume that $RW(B, W, P)$ is true for every space germ (B, W, P) such that $(B, W, P) \in \mathcal{C}_0$ and $\sharp P \leq r$.*

Then, for every $(B, W, P) \in \mathcal{C}_0$ satisfying $\sharp P \leq r$, there exists an IAMT (B', W', P') with a normal crossing of (B, W, P) such that (B', W', P') has the basic valuation ring of normal crossing type.

Lemma 8.7. *Let (B, W, P) be a framed space germ over k. Assume that (B, W, P) has the basic valuation ring of normal crossing type. Let $(Z, U, P) = \underline{(B, W, P)}$, $P = \{x_1, x_2, \ldots, x_n\}$ and*

$$\{0\} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{i-1} \subset \mathfrak{q}_i \subset \cdots \subset \mathfrak{q}_n = M(U)$$

be the sequence of prime ideals of the basic valuation ring U.

1. *For every $0 \leq i \leq n$, the residue ring $Z/\mathfrak{q}_i \cap Z$ is a normal domain, and the quotient field of $Z/\mathfrak{q}_i \cap Z$ and the quotient field of $U/\mathfrak{q}_i$ coincide.*
2. *For every $1 \leq i \leq n$,*
$$Z_{(\mathfrak{q}_i \cap Z)}/(\mathfrak{q}_{i-1} \cap Z)Z_{(\mathfrak{q}_i \cap Z)} \cong U_{\mathfrak{q}_i}/\mathfrak{q}_{i-1}U_{\mathfrak{q}_i}.$$
3. *Let (B', W', P') be an IAMT with a normal crossing of (B, W, P). Then, also (B', W', P') has the basic valuation ring of normal crossing type.*

Proof. 1. $Z = Z/\mathfrak{q}_0 \cap Z = k[[x_1, x_2, \ldots, x_n]]$. For every integer i with $1 \leq i \leq n$ we have $Z/\mathfrak{q}_i \cap Z \cong k[[x_{i+1}, x_{i+2}, \ldots, x_n]]$ by definition of normal crossing type. Thus, for every $0 \leq i \leq n$, $Z/\mathfrak{q}_i \cap Z$ is a regular local ring, and in particular it is a normal domain.

By induction on i we will show that $Z/\mathfrak{q}_i \cap Z$ and $U/\mathfrak{q}_i$ have a common quotient field.

In case $i = 0$, by definition of space germs, the quotient field of $Z = Z/\mathfrak{q}_0 \cap Z$ and the quotient field of $U = U/\mathfrak{q}_0$ coincide.

Assume that $1 \leq i \leq n$ and the quotient field of $Z/\mathfrak{q}_{i-1} \cap Z$ and the quotient field of $U/\mathfrak{q}_{i-1}$ coincide. The valuation ring $U_{\mathfrak{q}_i}/\mathfrak{q}_{i-1}U_{\mathfrak{q}_i}$ dominates the local ring $\bar{Z}_i = Z_{(\mathfrak{q}_i \cap Z)}/(\mathfrak{q}_{i-1} \cap Z)Z_{(\mathfrak{q}_i \cap Z)}$ and these two rings have the same quotient field. On the other hand, by the assumption of normal crossing type, we have $\bar{Z}_i \cong k[[x_i, x_{i+1}, \ldots, x_n]]_{(x_i)}$, where (x_i) denotes the prime ideal in $k[[x_i, x_{i+1}, \ldots, x_n]]$ generated by one element x_i. Thus, $\bar{Z}_i$ is a regular local ring of dimension one. By Proposition 4.25 $\bar{Z}_i$ is a valuation ring of the quotient field. By Corollary 4.19 we have $\bar{Z}_i = U_{\mathfrak{q}_i}/\mathfrak{q}_{i-1}U_{\mathfrak{q}_i}$. In particular, the residue field of the local ring $\bar{Z}_i$ coincides with the residue field of the local ring $U_{\mathfrak{q}_i}/\mathfrak{q}_{i-1}U_{\mathfrak{q}_i}$. The quotient field of the ring $Z/\mathfrak{q}_i \cap Z$ coincides with the residue field of $\bar{Z}_i$, The residue field of $U_{\mathfrak{q}_i}/\mathfrak{q}_{i-1}U_{\mathfrak{q}_i}$ coincides with the quotient field of $U/\mathfrak{q}_i$. Thus the quotient fields of $Z/\mathfrak{q}_i \cap Z$ and $U/\mathfrak{q}_i$ coincide.
2. It has already been shown in the above proof of the latter half of 1.
3. It is equivalent to Lemma 8.2.3. □

9. Reduction sequence

We introduce an effective tool to treat the Abhyankar condition and the condition "not reductive" simultaneously for a Weierstrass representation.

Definition 9.1. Let (B, W, P) be a framed space germ over k, (h, g, t) be a Weierstrass representation over (B, W, P), and ω be either a positive integer or symbol ∞. We consider a sequence

$$\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \omega + 1\}$$

of quadruplets $(B_\nu, W_\nu, P_\nu, H_\nu)$ numbered by integers ν with $1 \leq \nu < \omega + 1$. Let $(B_0, W_0, P_0) = (B, W, P)$ and $(h_0, g_0, t_0) = (h, g, t)$. If $\mathcal{S}$ satisfies the following conditions 0–4 for every integer ν with $1 \leq \nu < \omega + 1$, then we call $\mathcal{S}$ a *reduction sequence* of (h, g, t). We call ω the *length* of $\mathcal{S}$.

0. (B_ν, W_ν, P_ν) is a framed space germ over k. $H_\nu \in k[[P_\nu]]$.
1. (B_ν, W_ν, P_ν) is an IAMT with a normal crossing of $(B_{\nu-1}, W_{\nu-1}, P_{\nu-1})$.

To state conditions 2–4 we define two Weierstrass representations $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ and (h_ν, g_ν, t_ν) over (B_ν, W_ν, P_ν) by the following conditions $\langle A\rangle$ and $\langle B\rangle$:

$\langle A\rangle$ $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ is the pull-back of the Weierstrass representation $(h_{\nu-1}, g_{\nu-1}, t_{\nu-1})$ over $(B_{\nu-1}, W_{\nu-1}, P_{\nu-1})$ by the associated morphism $(B_{\nu-1}, W_{\nu-1}, P_{\nu-1}) \to (B_\nu, W_\nu, P_\nu)$.

$\langle B\rangle$ $(h_\nu, g_\nu, t_\nu) = (h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1} + H_\nu)$.

2. $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ satisfies the Abhyankar condition.
3. If $\nu \neq \omega$, then $\mathrm{w}(g'_{\nu-1},\, t'_{\nu-1}) < \mathrm{w}(g_\nu,\, t_\nu)$.
4. If $\nu = \omega$ and $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ is not reductive, then $H_\nu = 0$. If $\nu = \omega$ and $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ is reductive, then $\mathrm{w}(g_\nu,\, t_\nu) = \infty$.

Besides, If ω is a positive integer and if $\mathcal{S}$ satisfies the conditions 0, 1, 2, 3 plus the following 4* for every integer ν with $1 \leq \nu < \omega + 1$, then we say that $\mathcal{S}$ is a *semi-reduction sequence* of (h, g, t) of *length* ω.

4*. If $\nu = \omega$, then $\mathrm{w}(g'_{\nu-1},\, t'_{\nu-1}) < \mathrm{w}(g_\nu,\, t_\nu) < \infty$.

We call the empty set $\mathcal{S} = \emptyset = \{\}$ *semi-reduction sequence of length* 0.

We call the above sequence $\{(h_\nu, g_\nu, t_\nu) \mid 0 \leq \nu < \omega + 1\}$ of Weierstrass representations the *transform Weierstrass representation sequence* of (h, g, t) along $\mathcal{S}$. We call the sequence $\{(B_\nu, W_\nu, P_\nu) \mid 0 \leq \nu < \omega + 1\}$ the *lower framed space germ* *sequence* along $\mathcal{S}$, while we call $\{(B_\nu, W_\nu, P_\nu \cup \{t_\nu\}) \mid 0 \le \nu < \omega + 1\}$ the *upper framed space germ sequence* along $\mathcal{S}$.

If $\omega = \infty$, then we say that $\mathcal{S}$ has *infinite length.* If $\omega < \infty$, then we say that $\mathcal{S}$ has *finite length.*

Remark . 1. The length of a reduction sequence is either a positive integer or ∞. The length of a semi-reduction sequence is either a positive integer or 0.

2. For simplicity, sometimes we identify an element in the quotient field of B_μ and the image of it in the quotient field of B_ν for any pair μ, ν of integers with $0 \le \mu \le \nu < \omega + 1$.
3. Under the identification in 2, $h = h_\nu$, $g = g_\nu$ and $t_\nu = t + H_1 + H_2 + \cdots + H_\nu$ for any $0 \le \nu < \omega + 1$.
4. By condition 2 of Definition 9.1 we have $\mathrm{w}(g_{\nu-1},\, t_{\nu-1}) < \infty$. Thus for every $0 \le \nu < \omega$ we have $\mathrm{w}(g_\nu,\, t_\nu) < \infty$. If $\mathcal{S}$ is a semi-reduction sequence, then by 4* of Definition 9.1 we have $\mathrm{w}(g_\nu,\, t_\nu) < \infty$ even for $\nu = \omega$. In particular, if either (h, g, t) has a reduction sequence, or (h, g, t) has a semi-reduction sequence of positive length, then $\mathrm{w}(g,\, t) < \infty$.
5. Assume that $\mathcal{S}$ ia a reduction sequence of (h, g, t) with finite length ω. The following four conditions are equivalent:
   (a) $\mathrm{w}(g_\omega,\, t_\omega) < \infty$.
   (b) $H_\omega = 0$
   (c) $(h_\omega, g_\omega, t_\omega) = (h'_{\omega-1}, g'_{\omega-1}, t'_{\omega-1})$
   (d) $(h_\omega, g_\omega, t_\omega)$ satisfies the Abhyankar condition, and is not reductive.
6. By condition 3 of Definition 9.1, for every $1 \le \nu < \omega$ $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ is reductive. If $\mathcal{S}$ is a semi-reduction sequence, then by 4* of Definition 9.1 $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ is reductive even for $\nu = \omega$.
7. If $\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \le \nu < \omega + 1\}$ is a reduction sequence, then for every integer λ with $1 \le \lambda < \omega$, the subsequence $\mathcal{S}_\lambda = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \le \nu \le \lambda\}$ of $\mathcal{S}$ is a semi-reduction sequence.

   For a given infinite sequence of quadruples $\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \le \nu < \infty\}$, $\mathcal{S}$ is a reduction sequence of (h, g, t) with infinite length, if and only if, for any positive integer λ the subsequence $\mathcal{S}_\lambda = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \le \nu \le \lambda\}$ of $\mathcal{S}$ is a semi-reduction sequence of (h, g, t).

Recall the statement $RW(B, W, P)$ we gave before Lemma 5.12.

Lemma 9.2. *Let $\mathcal{C}$ be an admissible class of framed space germs over k, and r be a positive integer.*

Assume that $RW(B, W, P)$ is true for every framed space germ (B, W, P) such that $(B, W, P) \in \mathcal{C}$ and $\sharp P \le r$.

Let (B, W, P) be a framed space germ over k such that $(B, W, P) \in \mathcal{C}$ and $\sharp P \le r$, and (h, g, t) be a Weierstrass representation over (B, W, P).

Then, there exists a reduction sequence of (h, g, t), if and only if, $\mathrm{w}(g,\, t) < \infty$.

Proof. By definition, if there exists a reduction sequence of (h, g, t), then $\mathrm{w}(g,\, t) < \infty$.

Assume $\mathrm{w}(g,\, t) < \infty$. The empty set $\mathcal{S}_0 = \{\}$ is a semi-reduction sequence of (h, g, t) with length 0. Thus by induction, we can assume that a semi-reduction sequence $\mathcal{S}_\lambda = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \le \nu \le \lambda\}$ of (h, g, t) with length λ has been already constructed for some non-negative integer λ. Let $(B_0, W_0, P_0) = (B, W, P)$, $(h_0, g_0, t_0) = (h, g, t)$, and $\{(h_\nu, g_\nu, t_\nu) \mid 0 \le \nu \le \lambda\}$ be the transform Weierstrass representation sequence along $\mathcal{S}_\lambda$. If $\lambda > 0$, then by condition 4^* of Definition 9.1 we have $\mathrm{w}(g_\lambda,\, t_\lambda) < \infty$. If $\lambda = 0$, then by assumption we have $\mathrm{w}(g_\lambda,\, t_\lambda) = \mathrm{w}(g,\, t) < \infty$. Since $(B_\lambda, W_\lambda, P_\lambda)$ is an IAMT of $(B_0, W_0, P_0) = (B, W, P) \in \mathcal{C}$, and since $\mathcal{C}$ is an admissible class, we have $(B_\lambda, W_\lambda, P_\lambda) \in \mathcal{C}$ and $\sharp P_\lambda = \sharp P \leq r$. Thus $RW(B_\lambda, W_\lambda, P_\lambda)$ is true by assumption.

By Corollary 6.9 there exists an IAMT $(B_{\lambda+1}, W_{\lambda+1}, P_{\lambda+1})$ with a normal crossing of $(B_\lambda, W_\lambda, P_\lambda)$ such that the pull-back $(h'_\lambda, g'_\lambda, t'_\lambda)$ of $(h_\lambda, g_\lambda, t_\lambda)$ by the associated morphism $(B_\lambda, W_\lambda, P_\lambda) \to (B_{\lambda+1}, W_{\lambda+1}, P_{\lambda+1})$ satisfies the Abhyankar condition.

In case where $(h'_\lambda, g'_\lambda, t'_\lambda)$ is not reductive, letting $H_{\lambda+1} = 0$, $\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \leq \nu \leq \lambda + 1\}$, we have a reduction sequence $\mathcal{S}$ of (h, g, t) with length $\lambda + 1$.

In case where $(h'_\lambda, g'_\lambda, t'_\lambda)$ is reductive, there exists an element $H_{\lambda+1} \in k[[P_{\lambda+1}]]$ satisfying $\mathrm{w}(g'_\lambda,\, t'_\lambda) < \mathrm{w}(g'_\lambda,\, t'_\lambda + H_{\lambda+1})$.

If $\mathrm{w}(g'_\lambda,\, t'_\lambda + H_{\lambda+1}) = \infty$, letting $\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \leq \nu \leq \lambda + 1\}$, we have a reduction sequence $\mathcal{S}$ of (h, g, t) with length $\lambda + 1$.

In case where $\mathrm{w}(g'_\lambda,\, t'_\lambda + H_{\lambda+1}) < \infty$, $\mathcal{S}_{\lambda+1} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \leq \nu \leq \lambda+1\}$ is a semi-reduction sequence of (h, g, t) with length $\lambda + 1$, and $\mathcal{S}_{\lambda+1}$ is an extension of $\mathcal{S}_\lambda$. We can ascend an induction step.

Assume that for any non-negative integer λ a reduction sequence $\mathcal{S}$ of (h, g, t) with length $\lambda + 1$ is not constructed, when we repeat the above procedure. Then, for any non-negative integer λ a semi-reduction sequence $\mathcal{S}_\lambda$ of (h, g, t) with length λ is constructed. For any pair μ, λ of integers with $0 \leq \mu < \lambda$ $\mathcal{S}_\lambda$ is an extension of $\mathcal{S}_\mu$. In this case $\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \infty\}$ is a reduction sequence of (h, g, t) with infinite length. □

We here extend the Abhyankar condition for m-tuples of functions.

Definition 9.3. Let (B, W, P) be a framed space germ over k, m be a non-negative integer, and $G_1, G_2, \ldots, G_m \in Z = k[[P]]$ be m numbered elements. We write $P = \{x_1, x_2, \ldots, x_n\}$.

1. We say that a m-tuple $(G_1, G_2, \ldots, G_m)$ satisfies the *Abhyankar condition*, if the following two conditions are simultaneously satisfied:
   (a) $m \geq 1$.
   (b) There exist n non-negative integers $b_1, b_2, \ldots, b_n$ such that
      (i)
$$\frac{G_j}{(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j} \in Z$$
      for every $1 \leq j \leq m$, and
      (ii)
$$\min\{\frac{1}{j}\,\mathrm{ord}(\frac{G_j}{(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j}) \mid 1 \leq j \leq m\} < 1.$$
   The non-negative integers $b_1, b_2, \ldots, b_n$ in (b) are called the *characteristic numbers* of $(G_1, G_2, \ldots, G_m)$.
2. If $m = 0$, we put $\mathrm{w}() = \infty$. If $m > 0$, we put $\mathrm{w}(G_1, G_2, \ldots, G_m) = \min\{\mathrm{ord}(G_j)/j \mid 1 \leq j \leq m\}$. We call $\mathrm{w}(G_1, G_2, \ldots, G_m) \in \mathbb{Q} \cup \{\infty\}$ the *weight* of $(G_1, G_2, \ldots, G_m)$.

Lemma 9.4. 1. *If* $(G_1, G_2, \ldots, G_m)$ *satisfies the Abhyankar condition, then characteristic numbers* $b_1, b_2, \ldots, b_n$ *are uniquely defined depending on* $(G_1, G_2, \ldots, G_m)$.

2. *Let* $t \in B$ *be an element such that* $t \notin P$ *and* $P \cup \{t\}$ *is a partial parameter system of* B*. Let* $g = t^m + \sum_{j=1}^{m} G_j t^{m-j}$.
   (a) $\mathrm{w}(G_1, G_2, \ldots, G_m) = \mathrm{w}(g, t)$.
   (b) *The* m*-tuple* $(G_1, G_2, \ldots, G_m)$ *satisfies the Abhyankar condition, if and only if, the semi-Weierstrass pair* (g, t) *satisfies the Abhyankar condition.*
   (c) *If the equivalent conditions in* (b) *are satisfied, then the characteristic numbers of* $(G_1, G_2, \ldots, G_m)$ *and the characteristic numbers of* (g, t) *coincide.*
3. *Let* $v \in Z^*$.
   (a) $\mathrm{w}(G_1, G_2, \ldots, G_m) = \mathrm{w}(vG_1, v^2G_2, \ldots, v^mG_m)$.
   (b) *An* m*-tuple* $(G_1, G_2, \ldots, G_m)$ *satisfies the Abhyankar condition, if and only if,* $(vG_1, v^2G_2, \ldots, v^mG_m)$ *satisfies the Abhyankar condition.*
   (c) *If the equivalent conditions in* (b) *are satisfied, then the characteristic numbers of* $(G_1, G_2, \ldots, G_m)$ *and the characteristic numbers of* $(vG_1, v^2G_2, \ldots, v^mG_m)$ *coincide.*

Proof. 1. See Lemma 6.5.1.
2. Refer to Definition 6.4.1. □

Lemma 9.5. *Let* (B, W, P) *be a framed space germ over* k*, and* (h, g, t) *be a Weierstrass representation over* (B, W, P)*. Assume that* (h, g, t) *satisfies the Abhyankar condition, and* (h, g, t) *is reductive. We denote* $P = \{x_1, x_2, \ldots, x_n\}$, $Z = k[[P]] = k[[x_1, x_2, \ldots, x_n]]$, $m = \deg(g, t)$*. Let* $G_1, G_2, \ldots, G_m \in Z$ *be the coefficient function sequence of* (g, t)*, and* $b_1, b_2, \ldots, b_n$ *be the characteristic numbers of* (g, t)*. Note that* $m \geq 1$ *by the Abhyankar condition. Let* $m = lq$ *be the* p*-decomposition of* m.

The following claims 1–9 hold:

1. $\mathrm{w}(g, t) = \sum_{i=1}^{n} b_i$.
2. *At least one of the characteristic numbers* $b_1, b_2, \ldots, b_n$ *is not zero.*
3. *There exists a unique element* u_j *satisfying*
$$G_j = u_j(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j, \quad u_j \in Z$$
for every $1 \leq j \leq m$.
4. *There exists a unique element* $a \in k$ *satisfying*
$$\mathrm{in}(g, t) = (t + a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^m, \quad a \neq 0.$$
5. $u_j(0) = \binom{m}{j} a^j$ *for every* $1 \leq j \leq m$*, where we denote* $u_j(0) = u_j \bmod M(Z) \in Z/M(Z) \cong k$*. Besides,* $u_q \in Z^*$*, and* $u_m \in Z^*$.
6. *The following three conditions are equivalent for* $H \in Z$*:*
   (a) H *is a reducing element of* (g, t)*. In other words,* $\mathrm{w}(g, t) < \mathrm{w}(g, t + H)$.
   (b) $\mathrm{in}(G_q) = l\, \mathrm{in}(H)^q$.
   (c) $\mathrm{in}(H) = a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$.
7. *There exists a reducing element* $H \in Z$ *of* (g, t) *satisfying*
$$\frac{H}{x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}} \in Z^*.$$

8. *Let $H \in Z$ be a reducing element of $(g,\, t)$ satisfying the condition in 7. Then, there exist m numbered elements $F_1, F_2, \ldots, F_m \in M(Z)$ satisfying*
$$g = (t+H)^m + \sum_{i=1}^{m} (x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j F_j (t+H)^{m-j}.$$
9. *Assume that non-negative integers $c_1, c_2, \ldots, c_n$ and elements $F_1', F_2', \ldots, F_m' \in M(Z)$ satisfy*
$$G_j = (x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n})^j F_j'$$
for every $1 \le j \le m$. Then, $(F_1', F_2', \ldots, F_m')$ satisfies the Abhyankar condition. Let $d_1, d_2, \ldots, d_n$ be the characteristic numbers of $(F_1', F_2', \ldots, F_m')$. The following three claims hold:
(a) *$b_i = c_i + d_i$ for every $1 \le i \le n$.*
(b) *$\mathrm{w}(F_1', F_2', \ldots, F_m') = \sum_{i=1}^{n} d_i$.*
(c) *At least one of $d_1, d_2, \ldots, d_n$ is not zero.*

Proof. 1. By Lemma 6.12.5 $\mathrm{w}(g,\, t)$ is an integer. By the Abhyankar condition for $(g,\, t)$ we have
$$0 \le \mathrm{w}(g,\, t) - \sum_{i=1}^{n} b_i = \min\{\frac{1}{j}\operatorname{ord}(\frac{G_j}{(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j})\} < 1.$$
2. It follows from 1 and Lemma 2.20.4.
3. It follows from that $(G_1, G_2, \ldots, G_m)$ satisfies the Abhyankar condition.
4 and 5. By 1 $\min\{\operatorname{ord}(u_j)/j \mid 1 \le j \le m\} = \min\{\operatorname{ord}(G_j/(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j)/j \mid 1 \le j \le m\} = 0$. Thus we have $\mathrm{in}(g,\, t) = t^m + \sum_{i=1}^{m} u_j(0)(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j t^{m-j}$. On the other hand, by Lemma 6.12.3 there is a unique element $K \in Z$ with $\mathrm{in}(g,\, t) = (t+K)^m$, and $K \neq 0$. Thus $u_j(0)(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j = \binom{m}{j} K^j$ for every $1 \le j \le m$. By Lemma 2.4.3 $\binom{m}{q} = \binom{l}{1} = l \neq 0$. Thus we have $u_q(0)(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^q = l\,K^q \neq 0$. Let $a \in k$ be an element satisfying $u_q(0) = la^q$. We have $a \neq 0$, since $u_q(0) \neq 0$. We have $K = a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$, since $l(K - a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^q = lK^q - l(a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^q = 0$. The uniqueness of a follows from the uniqueness of K. Also we obtain the former half of 5. The latter half of 5 is easy.
6. It follows from Lemma 6.12.6 and above 4.
7. By 6 one knows that $H = a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ satisfies the condition.
8. Assume that $H \in Z$, $h = H/(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}) \in Z^*$ and $\mathrm{w}(g,\, t) < \mathrm{w}(g,\, t+H)$. We have $h(0) x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} = \mathrm{in}(H) = a x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ by 6. Thus $h(0) = a$.

Now, we can write
$$g = t^m + \sum_{j=1}^{m} u_j (x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j t^{m-j} = (t+H)^m + \sum_{j=1}^{m} \Phi_j (t+H)^{m-j}$$
with $\Phi_1, \Phi_2, \ldots, \Phi_m \in Z$. Let $u_0 = 1$. By Lemma 6.10.2 we have
$$\Phi_j = \sum_{i=0}^{j} \binom{m-i}{m-j} (-H)^{j-i} u_i (x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^i$$
$$= (x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^j \sum_{i=0}^{j} \binom{m-i}{m-j} (-h)^{j-i} u_i$$

for every $1 \le j \le m$. Putting $F_j = \sum_{i=0}^{j} \binom{m-i}{m-j}(-h)^{j-i}u_i \in Z$, we have $\Phi_j = (x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^j F_j$ for $1 \le j \le m$. For the constant term $F_j(0)$ of F_j we have

$$\begin{aligned} F_j(0) &= \sum_{i=0}^{j} \binom{m-i}{m-j}(-h(0))^{j-i}u_i(0) = \sum_{i=0}^{j} \binom{m-i}{m-j}(-a)^{j-i}\binom{m}{i}a^i \\ &= a^j \sum_{i=0}^{j} \binom{m-i}{j-i}\binom{m}{i}(-1)^{j-i} = a^j \sum_{i=0}^{j} \binom{j}{i}\binom{m}{j}(-1)^{j-i} \\ &= a^j \binom{m}{j}\{1+(-1)\}^j = 0. \end{aligned}$$

We used 5 and Lemma 2.3.7. Thus $F_j \in M(Z)$.
9. Easy. □

Definition 9.6. Let (B, W, P) be a framed space germ over k, and δ be a non-negative integer. Let $(Z, U, P) = \underline{(B, W, P)}$, $\mathfrak{p}$ be a prime ideal of the valuation ring U, and $q = p^\delta$. If an element $u \in Z$ has an element $h \in Z$ satisfying $u - h^q \in \mathfrak{p}$, then we say that u is the *q-th power modulo* $\mathfrak{p}$.

Lemma 9.7. *Let (B, W, P) be a framed space germ over k, and δ be a non-negative integer. Let $(Z, U, P) = \underline{(B, W, P)}$, and $u \in Z$. Let $q = p^\delta$ and $\mathcal{P} = \{\mathfrak{p} \in \operatorname{Spec}(U) \mid u$ is the q-th power modulo $\mathfrak{p}\}$. Then $\mathcal{P} \neq \emptyset$ and $\mathcal{P}$ has the minimum element.*

Proof. Since $Z/M(U) \cap Z \cong Z/M(Z) \cong k$, and since k is algebraically closed, one has $M(U) \in \mathcal{P} \neq \emptyset$. One knows that $\mathcal{P}$ is a finite set, since $\mathcal{P} \subset \operatorname{Spec}(U)$ and since $\sharp \operatorname{Spec}(U) - 1 = \operatorname{rank} U \le \dim Z$ by Corollary 4.15.1 and Corollary 4.22.1. By Corollary 4.15.2 $\mathcal{P}$ has the minimum element. □

Definition 9.8. Let (B, W, P) be a framed space germ over k, and (h, g, t) be a Weierstrass representation over (B, W, P). We denote $P = \{x_1, x_2, \dots, x_n\}$, $(Z, U, P) = \underline{(B, W, P)}$, $m = \deg(g, t)$. Let $G_1, G_2, \dots, G_m \in Z$ be the coefficient function sequence of (g, t).

1. Assume that (h, g, t) satisfies the Abhyankar condition, and (h, g, t) is reductive. Let $b_1, b_2, \dots, b_n$ be the characteristic numbers of (g, t), and $m = lq$ be the p-decomposition of m. Note that $u_q = G_q/(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n})^q \in Z^*$ by Lemma 9.5.5.

   A reducing element $H \in Z$ of (g, t) satisfying $H/(x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}) \in Z^*$ is called an *effective* reducing element of (g, t).

   Let $\mathcal{P} = \{\mathfrak{p} \in \operatorname{Spec}(U) \mid u_q \in Z^*$ is the q-th power modulo $\mathfrak{p}\}$, and $\mathfrak{p}_0 \in \mathcal{P}$ be the minimum element of $\mathcal{P}$.

   An effective reducing element $H \in Z$ of (g, t) such that
   $$u_q - l\left(\frac{H}{x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}}\right)^q \in \mathfrak{p}_0$$
   is called a *very effective* reducing element of (g, t).
2. Consider a reduction sequence $\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \le \nu < \omega + 1\}$ of (h, g, t). Let $(B_0, W_0, P_0) = (B, W, P)$ and $(h_0, g_0, t_0) = (h, g, t)$. Let $\{(h_\nu, g_\nu, t_\nu) \mid 0 \le \nu < \omega + 1\}$ be the transform Weierstrass representation sequence of (h, g, t). For each ν (h_ν, g_ν, t_ν) is a Weierstrass representation over

(B_ν, W_ν, P_ν). Let $(h'_{\nu-1}, g'_{\nu-1}, t'_{\nu-1})$ be the pull-back of $(h_{\nu-1}, g_{\nu-1}, t_{\nu-1})$ by the associated morphism $(B_{\nu-1}, W_{\nu-1}, P_{\nu-1}) \to (B_\nu, W_\nu, P_\nu)$.

We say that the reduction sequence $\mathcal{S}$ is *effective*, if for every integer ν with $1 \leq \nu < \omega$ the element $H_\nu \in k[[P_\nu]]$ is an effective reducing element of $(g'_{\nu-1}, t'_{\nu-1})$.

We say that the reduction sequence $\mathcal{S}$ is *very effective*, if for every integer ν with $1 \leq \nu < \omega$ the element $H_\nu \in k[[P_\nu]]$ is a very effective reducing element of $(g'_{\nu-1}, t'_{\nu-1})$.

Remark . According to the above condition in 2, in the case that ω is a positive integer, H_ω is not necessarily an effective reducing element or a very effective reducing element of $(g'_{\omega-1}, t'_{\omega-1})$.

If $H_\omega = 0$, then H_ω is never an effective reducing element of $(g'_{\omega-1}, t'_{\omega-1})$.

Lemma 9.9. *Let (B, W, P) be a framed space germ over k, and (h, g, t) be a Weierstrass representation over (B, W, P). Assume that (h, g, t) satisfies the Abhyankar condition, and (h, g, t) is reductive. Then, there exists a very effective reducing element $H \in k[[P]]$ of the semi-Weierstrass pair (g, t).*

Proof. We use notations in Definition 9.8. In particular, $Z = k[[P]]$. By definition of $\mathfrak{p}_0$ there is an element $h_1 \in Z$ with $u_q - h_1^q \in \mathfrak{p}_0$. There is an element $s \in k$ such that $s \neq 0$ and $l = s^q$. Let $h = h_1/s \in Z$. We have $u_q - lh^q = u_q - h_1^q \in \mathfrak{p}_0$.

Let $H = hx_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \in Z$. We will show that H is a very effective reducing element of (g, t).

We have $u_q - lh^q \in M(U) \cap Z = M(Z)$, since $u_q - lh^q \in \mathfrak{p}_0 \subset M(U)$. Thus $u_q(0) = lh(0)^q$. By Lemma 9.5.5 we have $u_q(0) \neq 0$. Thus $h(0) \neq 0$, and we have $\mathrm{in}(G_q) = u_q(0)(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^q = l(h(0)x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})^q = l\,\mathrm{in}(H)^q$. By Lemma 9.5.6 one knows that H is a reducing element of (g, t).

We have $H/(x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}) = h \in Z^*$, since $h(0) \neq 0$. Thus H is an effective reducing element of (g, t).

Since
$$u_q - l(\frac{H}{x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}})^q = u_q - lh^q \in \mathfrak{p}_0,$$
H is a very effective reducing element of (g, t). □

Lemma 9.10. *Let $\mathcal{C}$ be an admissible class of framed space germs over k, and r be a positive integer.*

Assume that $RW(B, W, P)$ is true for every framed space germ (B, W, P) such that $(B, W, P) \in \mathcal{C}$ and $\sharp P \leq r$.

Let (B, W, P) be a framed space germ over k such that $(B, W, P) \in \mathcal{C}$ and $\sharp P \leq r$, and (h, g, t) be a Weierstrass representation over (B, W, P).

Then, the following four conditions are equivalent:

1. $\mathrm{w}(g, t) < \infty$.
2. *There exists a reduction sequence of (h, g, t).*
3. *There exists an effective reduction sequence of (h, g, t).*
4. *There exists a very effective reduction sequence of (h, g, t).*

Proof. The implications 4⇒3⇒2⇒1 are trivial. We have only to show 1⇒4.

Now, we have shown 1⇒2 in Lemma 9.2. Reading this proof again, we notice that if we define the concept of "very effective" also for semi-reduction sequences, and if we apply Lemma 9.9 when we choose a reducing element $H_{\lambda+1} \in k[[P_{\lambda+1}]]$, then the same proof shows 1⇒4. □

In the remaining part of this section we consider properties of effective reduction sequences.

Proposition 9.11. *Let (B, W, P) be a framed space germ over k, (h, g, t) be a Weierstrass representation over (B, W, P), and $\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \infty\}$ be an effective reduction sequence of (h, g, t) with infinite length. Let $(B_0, W_0, P_0) = (B, W, P)$ and $(h_0, g_0, t_0) = (h, g, t)$. Let $\{(h_\nu, g_\nu, t_\nu) \mid 0 \leq \nu < \omega + 1\}$ be the transform Weierstrass representation sequence of (h, g, t). We denote $m = \deg(g,\, t)$, $P_\nu = \{x_{\nu,1}, x_{\nu,2}, \dots, x_{\nu,n}\}$, $Z_\nu = k[[P_\nu]] = k[[x_{\nu,1}, x_{\nu,2}, \dots, x_{\nu,n}]]$, and by F_ν the quotient field of B_ν for every $0 \leq \nu < \infty$. By the associated homomorphism $F_\nu \to F_{\nu+1}$ we define an inductive system $\{F_\nu \mid \nu \geq 0\}$, and denote $\mathcal{F} = \varinjlim_\nu F_\nu$. For simplicity, we identify F_ν and the image of the canonical homomorphism $F_\nu \to \mathcal{F}$. Under this identification, we have $h = h_\nu$, $g = g_\nu$ and $t_\nu = t + \sum_{\mu=1}^{\nu} H_\mu$ for every non-negative integer ν. We understand $\sum_{\mu=1}^{0} H_\mu = 0$ here as a convention. Also we denote $\mathcal{Z} = \varinjlim_\nu Z_\nu = \cup_{\nu=0}^{\infty} Z_\nu$. Note that $\mathrm{w}(g,\, t) < \infty$ and $m \geq 1$, since S exists.*

Then, there exist an infinite sequence

$$\{(F_{\nu,1}, F_{\nu,2}, \dots, F_{\nu,m}) \mid 0 \leq \nu < \infty\}$$

of m-tuples $(F_{\nu,1}, F_{\nu,2}, \dots, F_{\nu,m})$ of elements $F_{\nu,j} \in \mathcal{Z}$, an infinite sequence

$$\{(b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n}) \mid 1 \leq \nu < \infty\}$$

of n-tuples $(b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n})$ of non-negative integers $b_{\nu,i}$, and an infinite sequence

$$\{h_\nu \mid 1 \leq \nu < \infty\}$$

of elements $h_\nu \in \mathcal{Z}$ satisfying the following conditions (1.0) and (1.ν), (2.ν), (3.ν) (4.ν) for every positive integer ν. For simplicity, we write

$$\langle \nu \rangle = x_{\nu,1}^{b_{\nu,1}} x_{\nu,2}^{b_{\nu,2}} \cdots x_{\nu,n}^{b_{\nu,n}} = \prod_{i=1}^{n} x_{\nu,i}^{b_{\nu,i}}.$$

(1.ν) *$F_{\nu,j} \in M(Z_\nu)$ for every $1 \leq j \leq m$.*

$$g = t_\nu^m + \sum_{j=1}^{m} \left(\prod_{\mu=1}^{\nu} \langle \mu \rangle \right)^j F_{\nu,j} t_\nu^{m-j}.$$

We understand $\prod_{\mu=1}^{0} \langle \mu \rangle = 1$ here as a convention.

(2.ν) *Note that $F_{\nu-1,j} \in Z_{\nu-1} \subset Z_\nu$ for every $1 \leq j \leq m$. The m-tuple $(F_{\nu-1,1}, F_{\nu-1,2}, \dots, F_{\nu-1,m})$ of elements in Z_ν satisfy the Abhyankar condition over the framed space germ (B_ν, W_ν, P_ν). The characteristic numbers of $(F_{\nu-1,1}, F_{\nu-1,2}, \dots, F_{\nu-1,m})$ are equal to $(b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n})$. $\mathrm{w}(F_{\nu-1,1}, F_{\nu-1,2}, \dots, F_{\nu-1,m}) = \sum_{i=1}^{n} b_{\nu,i}$.*

(3.ν) *At least one of $b_{\nu,1}, b_{\nu,2}, \dots, b_{\nu,n}$ is not zero. $\langle \nu \rangle \in M(Z_\nu)$.*

(4.ν)

$$H_\nu = h_\nu \prod_{\mu=1}^{\nu} \langle \mu \rangle, \quad h_\nu \in Z_\nu^*.$$

Proof. By induction on ν we define $F_{\nu,j}$ $(1 \le j \le m)$, $b_{\nu,i}$ $(1 \le i \le n)$, and h_ν.

First, let $F_{0,1}, F_{0,2}, \ldots, F_{0,m} \in M(Z_0) = M(Z)$ be the coefficient function sequence of $(g_0, t_0) = (g, t)$. The condition (1.0) holds.

Let ν be a positive integer. Assume that in addition to $F_{0,j}$ $(1 \le j \le m)$ which we have just defined, we have already defined $F_{\mu,j}$, $b_{\mu,i}$, h_μ for every integers μ, j, i satisfying $1 \le \mu < \nu$, $1 \le j \le m$, $1 \le i \le n$, and they satisfy (1.0) and (1.μ), (2.μ), (3.μ) (4.μ) for every positive integer μ with $1 \le \mu < \nu$.

We will define $b_{\nu,1}, b_{\nu,2}, \ldots, b_{\nu,n}$, h_ν, and $(F_{\nu,1}, F_{\nu,2}, \ldots, F_{\nu,m})$ so that they will satisfy (2.ν), (3.ν) (4.ν) and (1.ν).

By definition of the reduction sequence, the Weierstrass representation $(h, g, t_{\nu-1})$ over (B_ν, W_ν, P_ν) satisfies the Abhyankar condition and is reductive. By (1.$\nu - 1$) the coefficient function sequence of $(g, t_{\nu-1})$ is equal to $(\prod_{\mu=1}^{\nu-1}\langle\mu\rangle)^j F_{\nu-1,j}$ $1 \le j \le m$. Since the associated morphism $(B_\mu, W_\mu, P_\mu) \to (B_\nu, W_\nu, P_\nu)$ for any $0 \le \mu \le \nu$ preserves normal crossings, we have non-negative integers $c_1, c_2, \ldots, c_n$ and an element $v \in Z_\nu^*$ with

$$\prod_{\mu=1}^{\nu-1}\langle\mu\rangle = v x_{\nu,1}^{c_1} x_{\nu,2}^{c_2} \cdots x_{\nu,n}^{c_n}.$$

By Lemma 9.5.9 the m-tuple $v^j F_{\nu-1,j}$ $1 \le j \le m$ satisfies the Abhyankar condition, and the weight of it is equal to the sum of the characteristic numbers of it.

By Lemma 9.4.3 also the m-tuple $F_{\nu-1,j}$ $1 \le j \le m$ satisfies the Abhyankar condition. Besides, the m-tuple $v^j F_{\nu-1,j}$ $1 \le j \le m$ and the m-tuple $F_{\nu-1,j}$ $1 \le j \le m$ have common weight and common characteristic numbers.

Let $b_{\nu,1}, b_{\nu,2}, \ldots, b_{\nu,n}$ be the characteristic numbers of the m-tuple $F_{\nu-1,j}$ $1 \le j \le m$. By Lemma 9.5.9 one knows that they satisfy (2.ν) and (3.ν).

By Lemma 9.5.9 the characteristic numbers of $(g, t_{\nu-1})$ are equal to $c_i + b_{\nu,i}$ $1 \le i \le n$. By definition of $\langle\nu\rangle$ we have

$$x_{\nu,1}^{c_1+b_{\nu,1}} x_{\nu,2}^{c_2+b_{\nu,2}} \cdots x_{\nu,n}^{c_n+b_{\nu,n}} = \frac{1}{v}\prod_{\mu=1}^{\nu}\langle\mu\rangle.$$

Since H_ν is an effective reducing element of $(g, t_{\nu-1})$, we have

$$h = \frac{H_\nu}{x_{\nu,1}^{c_1+b_{\nu,1}} x_{\nu,2}^{c_2+b_{\nu,2}} \cdots x_{\nu,n}^{c_n+b_{\nu,n}}} = \frac{vH_\nu}{\prod_{\mu=1}^{\nu}\langle\mu\rangle} \in Z_\nu^*,$$

and $\mathrm{w}(g,\, t_{\nu-1}) < \mathrm{w}(g,\, t_{\nu-1} + H_\nu)$. Thus $h_\nu = h/v$ satisfies (4.ν).

Since $t_\nu = t_{\nu-1} + H_\nu$, by Lemma 9.5.8 there exists an m-tuple $F_j \in M(Z_\nu)$ $1 \le j \le m$ satisfying

$$g = t_\nu^m + \sum_{j=1}^{m}(x_{\nu,1}^{c_1+b_{\nu,1}} x_{\nu,2}^{c_2+b_{\nu,2}} \cdots x_{\nu,n}^{c_n+b_{\nu,n}})^j F_j t_\nu^{m-j}.$$

Let $F_{\nu,j} = F_j/v^j$ for $1 \le j \le m$. The m-tuple $F_{\nu,j}$ $1 \le j \le m$ satisfies (1.ν). ☐

Remark . By the equality in (1.ν) it is intuitively quite plausible that

$$g = \lim_{\nu\to\infty}\left\{t_\nu^m + \sum_{j=1}^{m}\left(\prod_{\mu=1}^{\nu}\langle\mu\rangle\right)^j F_{\nu,j} t_\nu^{m-j}\right\} = (\lim_{\nu\to\infty} t_\nu)^m = (t + \sum_{\nu=1}^{\infty} H_\nu)^m.$$

If this is true, then we can conclude that $g = (t+H)^m$ for some $H \in Z_0$ by Remark after Proposition 2.18, and we obtain a stronger version of our main theorem Theorem 1.1 without any assumption on valuation rings. (Note that $g = (t+H)^m \in B$ has a normal crossing.) However, it is very hard to give a meaning to the above limit, since the ring $\mathcal{Z}$ is not necessarily noetherian.

Below we apply the valuation theory to overcome this difficulty.

Corollary 9.12. 1. *For every positive integer ν also the following* $(5.\nu)$ *holds:*
$(5.\nu)$ *For every* $1 \le j \le m$

$$u_{\nu,j} = \frac{F_{\nu-1,j}}{\langle\nu\rangle^j} \in Z_\nu, \ u_{\nu,j}(0) = \binom{m}{j} h_\nu(0)^j, \ \text{and}$$

$$g = t_{\nu-1}^m + \sum_{j=1}^m \left(\prod_{\mu=1}^{\nu} \langle\mu\rangle\right)^j u_{\nu,j} t_{\nu-1}^{m-j}.$$

2. *For every positive integer ν and for every* $1 \le j \le m$

$$\langle\nu\rangle^j F_{\nu,j} = \binom{m}{m-j}(-h_\nu\langle\nu\rangle)^j + \sum_{i=1}^j \binom{m-i}{m-j}(-h_\nu\langle\nu\rangle)^{j-i} F_{\nu-1,i}.$$

Proof. 1. By $(2.\nu)$ we have $u_{\nu,j} \in Z_\nu$. Substituting $F_{\nu-1,j} = \langle\nu\rangle^j u_{\nu,j}$ into $(1.\nu-1)$, we obtain the equality for g.

For simplicity, we write $X = x_{\nu,1}^{c_1+b_{\nu,1}} x_{\nu,2}^{c_2+b_{\nu,2}} \cdots x_{\nu,n}^{c_n+b_{\nu,n}}$. By the equality we just obtained, one knows

$$\operatorname{in}(g, t_{\nu-1}) = t_{\nu-1}^m + \sum_{j=1}^m (v(0)X)^j u_{\nu,j}(0) t_{\nu-1}^{m-j}.$$

By $(4.\nu)$ $\operatorname{in}(H_\nu) = h_\nu(0)v(0)X$. By Lemma 6.12.2 $\operatorname{in}(g, t_{\nu-1}) = (t_{\nu-1} + \operatorname{in}(H_\nu))^m$. Thus for every $1 \le j \le m$

$$(v(0)X)^j u_{\nu,j}(0) = \binom{m}{j}(v(0)X)^j h_\nu(0)^j.$$

Since $v(0)X \neq 0$, we obtain the desired equality.
2. It follows from Lemma 6.10.2. □

Corollary 9.13. *Let $m = lq$ be the p-decomposition of m, ν_0 be a non-negative integer, and $\mathfrak{i}$ be an ideal of the ring $\mathcal{Z}$. The image of an element $\Phi \in \mathcal{Z}$ by the canonical surjective homomorphism $\mathcal{Z} \to \mathcal{Z}/\mathfrak{i}$ is denoted by $\overline{\Phi} \in \mathcal{Z}/\mathfrak{i}$.*

1. *We assume the following two conditions for an integer j_0:*
   (a) $1 \le j_0 < m$, $j_0 \not\equiv 0 \mod q$.
   (b) *If* $1 \le j < j_0$, $j \not\equiv 0 \mod q$, *then* $\overline{F}_{\nu_0,j} = 0$.
   For every integer ν with $\nu_0 \le \nu$ and for every integer j satisfying $1 \le j < j_0$ and $j \not\equiv 0 \mod q$ we have

$$\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^j \overline{F}_{\nu,j} = 0, \ \text{and} \quad \overline{F}_{\nu_0,j_0} = \left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{j_0} \overline{F}_{\nu,j_0}.$$

   We understand $\prod_{\mu=\nu_0+1}^{\nu_0} \overline{\langle\mu\rangle} = 1$ here as a convention.

2. *Assume that* $\overline{F}_{\nu_0,j} = 0$ *for every integer* j *with* $1 \le j < q$. *For every integer* ν *with* $\nu_0 \le \nu$
$$\overline{F}_{\nu_0,q} = l\left(\sum_{\mu=\nu_0+1}^{\nu} \overline{h}_\mu \prod_{\lambda=\nu_0+1}^{\mu} \overline{\langle\lambda\rangle}\right)^q + \left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^q \overline{F}_{\nu,q}.$$
We understand $\sum_{\mu=\nu_0+1}^{\nu_0} \overline{h}_\mu \prod_{\lambda=\nu_0+1}^{\mu} \overline{\langle\lambda\rangle} = 0$ *here as a convention.*
3. *We assume the following three conditions for an integer* i_0*:*
   (a) *If* $1 \le j \le m$, $j \not\equiv 0 \mod q$, *then* $\overline{F}_{\nu_0,j} = 0$.
   (b) $2 \le i_0 \le l$.
   (c) *If* $1 \le i < i_0$, *then*
$$\overline{F}_{\nu_0,iq} - \binom{l}{i}\left(\frac{\overline{F}_{\nu_0,q}}{l}\right)^i = 0.$$
For every integer ν *with* $\nu_0 \le \nu$ *and for every integer* i *with* $1 \le i < i_0$ *we have*
$$\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{iq} \left\{\overline{F}_{\nu,iq} - \binom{l}{i}\left(\frac{\overline{F}_{\nu,q}}{l}\right)^i\right\} = 0, \text{ and}$$
$$\overline{F}_{\nu_0,i_0q} - \binom{l}{i_0}\left(\frac{\overline{F}_{\nu_0,q}}{l}\right)^{i_0} = \left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{i_0q} \left\{\overline{F}_{\nu,i_0q} - \binom{l}{i_0}\left(\frac{\overline{F}_{\nu,q}}{l}\right)^{i_0}\right\}.$$

Proof. 1. If $q = 1$, then there is no integer j_0 satisfying (a), and thus the claim is true. We assume $q > 1$ below.

First, note that by Lemma 2.4.3, $\binom{m-i}{m-j} = 0$ for any pair of integers i, j satisfying $0 \le i \le j \le m$, $i \equiv 0 \mod q$, and $j \not\equiv 0 \mod q$.

Let j_1 be an arbitrary integer satisfying $1 \le j_1 \le j_0$ and $j_1 \not\equiv 0 \mod q$. By induction on j_1 we will show the following claim $[j_1]$.

$[\mathbf{j_1}]$ For every integer ν with $\nu_0 \le \nu$ and for every integer j satisfying $1 \le j < j_1$ and $j \not\equiv 0 \mod q$, $\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^j \overline{F}_{\nu,j} = 0$.

Claim [1] is true by the trivial reason. Assume $j_1 > 1$. Let j_2 be the maximum element of the set of integers j satisfying $1 \le j < j_1$ and $j \not\equiv 0 \mod q$. We assume that $[j_2]$ is true. Under these assumptions $[j_1]$ is true, if we can show $\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{j_2} \overline{F}_{\nu,j_2} = 0$.

By Corollary 9.12.2 for every integer ν with $\nu_0 \le \nu$ we have
$$\begin{aligned}&\left(\prod_{\mu=\nu_0+1}^{\nu+1} \overline{\langle\mu\rangle}\right)^{j_2} \overline{F}_{\nu+1,j_2}\\ &= \sum_{\substack{0\le i\le j_2,\\ i\not\equiv 0 \bmod q}} \binom{m-i}{m-j_2} (-\overline{h}_{\nu+1}\overline{\langle\nu+1\rangle})^{j_2-i} \left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{j_2} \overline{F}_{\nu,i}\\ &= \left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{j_2} \overline{F}_{\nu,j_2}.\end{aligned}$$

Thus, for every integer ν with $\nu_0 \leq \nu$ we have $\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{j_2} \overline{F}_{\nu,j_2} = \overline{F}_{\nu_0,j_2} = 0$ by assumption (b), since $1 \leq j_2 < j_1 \leq j_0$. It follows that $[j_1]$ is true.

In particular, $[j_0]$ is true. We can repeat the above argument until the middle, after replacing j_2 by j_0. We obtain the last claim of 1.

2. By 1 for every integer ν with $\nu_0 \leq \nu$ and for every integer j with $1 \leq j < q$ $\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{j} \overline{F}_{\nu,j} = 0$.

By Corollary 9.12.2 for every integer ν with $\nu_0 \leq \nu$ we have

$$
\begin{aligned}
&\left(\prod_{\mu=\nu_0+1}^{\nu+1} \overline{\langle\mu\rangle}\right)^{q} \overline{F}_{\nu+1,q} \\
=&\binom{m}{m-q}\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1} \overline{\langle\mu\rangle}\right)^{q} \\
&\qquad + \sum_{j=1}^{q}\binom{m-j}{m-q}(-\overline{h}_{\nu+1}\overline{\langle\nu+1\rangle})^{q-j}\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{q} \overline{F}_{\nu,j} \\
=&\, l\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1} \overline{\langle\mu\rangle}\right)^{q} + \left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{q} \overline{F}_{\nu,q}.
\end{aligned}
$$

Claim 2 follows from the above equality and induction on ν.

3. Let i_1 be an arbitrary integer with $2 \leq i_1 \leq i_0$. By induction on i_1 we will show the following claim $[i_1]$.

$[\mathbf{i_1}]$ For every integer ν with $\nu_0 \leq \nu$ and for every integer i with $1 \leq i < i_1$

$$
\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{iq}\left\{\overline{F}_{\nu,iq} - \binom{l}{i}\left(\frac{\overline{F}_{\nu,q}}{l}\right)^{i}\right\} = 0.
$$

Claim [2] is true, since $\binom{l}{1} = l$. Assume $i_1 > 2$ and $[i_1 - 1]$ is true. Let $i_2 = i_1 - 1$.

By Corollary 9.12.2 for every integer ν with $\nu_0 \leq \nu$ we have

$$\begin{aligned}
&\left(\prod_{\mu=\nu_0+1}^{\nu+1} \overline{\langle\mu\rangle}\right)^{i_2 q} \overline{F}_{\nu+1,i_2 q} \\
=&\binom{lq}{lq-i_2 q}\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1}\overline{\langle\mu\rangle}\right)^{i_2 q} \\
&\qquad + \sum_{r=1}^{i_2 q}\binom{lq-r}{lq-i_2 q}(-\overline{h}_{\nu+1}\overline{\langle\nu+1\rangle})^{i_2 q-r}\left(\prod_{\mu=\nu_0+1}^{\nu}\overline{\langle\mu\rangle}\right)^{i_2 q}\overline{F}_{\nu,r} \\
=&\binom{l}{i_2}\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1}\overline{\langle\mu\rangle}\right)^{i_2 q} \\
&\qquad + \sum_{i=1}^{i_2}\binom{l-i}{l-i_2}\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1}\overline{\langle\mu\rangle}\right)^{(i_2-i)q}\left(\prod_{\mu=\nu_0+1}^{\nu}\overline{\langle\mu\rangle}\right)^{iq}\overline{F}_{\nu,iq}.
\end{aligned}$$

We used assumption (a) and Lemma 2.4.3 above.

On the other hand, by the equality in the proof of 2 and by the binomial theorem, we have

$$\begin{aligned}
&-\left(\prod_{\mu=\nu_0+1}^{\nu}\overline{\langle\mu\rangle}\right)^{i_2 q}\binom{l}{i_2}\left(\frac{\overline{F}_{\nu+1,q}}{l}\right)^{i_2} \\
=&-\binom{l}{i_2}\left\{\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1}\overline{\langle\mu\rangle}\right)^{q} + \left(\prod_{\mu=\nu_0+1}^{\nu}\overline{\langle\mu\rangle}\right)^{q}\left(\frac{\overline{F}_{\nu,q}}{l}\right)\right\}^{i_2} \\
=&-\binom{l}{i_2}\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1}\overline{\langle\mu\rangle}\right)^{i_2 q} \\
&\qquad -\sum_{i=1}^{i_2}\binom{l}{i_2}\binom{i_2}{i}\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1}\overline{\langle\mu\rangle}\right)^{(i_2-i)q}\left(\prod_{\mu=\nu_0+1}^{\nu}\overline{\langle\mu\rangle}\right)^{iq}\left(\frac{\overline{F}_{\nu,q}}{l}\right)^{i}
\end{aligned}$$

Note here that by Lemma 2.3.3 and 7 $\binom{l}{i_2}\binom{i_2}{i} = \binom{l}{i}\binom{l-i}{i_2-i} = \binom{l}{i}\binom{l-i}{l-i_2}$. Thus we have

$$\begin{aligned}
&\left(\prod_{\mu=\nu_0+1}^{\nu+1}\overline{\langle\mu\rangle}\right)^{i_2 q}\left\{\overline{F}_{\nu+1,iq} - \binom{l}{i_2}\left(\frac{\overline{F}_{\nu+1,q}}{l}\right)^{i_2}\right\} \\
=&\sum_{i=1}^{i_2}\binom{l-i}{l-i_2}\left(-\overline{h}_{\nu+1}\prod_{\mu=\nu_0+1}^{\nu+1}\overline{\langle\mu\rangle}\right)^{(i_2-i)q}\left(\prod_{\mu=\nu_0+1}^{\nu}\overline{\langle\mu\rangle}\right)^{iq}\left\{\overline{F}_{\nu,iq} - \binom{l}{i}\left(\frac{\overline{F}_{\nu,q}}{l}\right)^{i}\right\} \\
=&\left(\prod_{\mu=\nu_0+1}^{\nu}\overline{\langle\mu\rangle}\right)^{i_2 q}\left\{\overline{F}_{\nu,i_2 q} - \binom{l}{i_2}\left(\frac{\overline{F}_{\nu,q}}{l}\right)^{i_2}\right\}
\end{aligned}$$

Thus, for every integer ν with $\nu_0 \leq \nu$ we have

$$\left(\prod_{\mu=\nu_0+1}^{\nu} \overline{\langle\mu\rangle}\right)^{i_2 q} \left\{\overline{F}_{\nu,i_2 q} - \binom{l}{i_2}\left(\frac{\overline{F}_{\nu,q}}{l}\right)^{i_2}\right\} = \overline{F}_{\nu_0,i_2 q} - \binom{l}{i_2}\left(\frac{\overline{F}_{\nu_0,q}}{l}\right)^{i_2} = 0$$

by assumption (c), since $1 \leq i_2 = i_1 - 1 < i_1 \leq i_0$. It follows that $[i_1]$ is true.

In particular, $[i_0]$ is true. We can repeat the above argument until the middle, after replacing i_2 by i_0. We obtain the last claim of 3. □

10. Ring theory, Part two

In this section we show several technical results in the ring theory. The theory of excellent rings (Theorem 2.2) and Theorem 4.31.3 play the essential role. Theorem 10.8.4 seems to be a key of this paper. It will be applied to avoid purely inseparable extensions of the function field.

Lemma 10.1. *Let (Z, U) be a space germ over k. Assume that finite elements $a_1, a_2, \ldots, a_r \in U$ are given. Denote $n = \dim Z$.*

1. *The subdomain $Z[a_1, a_2, \ldots, a_r]$ of the quotient field of Z is an excellent domain contained in U. The intersection $\mathfrak{m} = M(U) \cap Z[a_1, a_2, \ldots, a_r]$ is a prime ideal of $Z[a_1, a_2, \ldots, a_r]$. $\dim Z[a_1, a_2, \ldots, a_r] = n$.*
2. *Let $R = Z[a_1, a_2, \ldots, a_r]_{\mathfrak{m}}$ be the localization of $Z[a_1, a_2, \ldots, a_r]$ with respect to the prime ideal $\mathfrak{m}$. Ring R is an excellent local domain dominated by U. $\dim R = n$.*

Proof. 1. By definition of space germs we have $Z \subset U$. Since U is a ring, and since $a_1, a_2, \ldots, a_r \in U$, we have $Z[a_1, a_2, \ldots, a_r] \subset U$. By Theorem 2.2.3 Z is an excellent ring. By Theorem 2.2.4 also $Z[a_1, a_2, \ldots, a_r]$ is an excellent ring. By Theorem 2.2.6 Z is universally catenary. Obviously $Z[a_1, a_2, \ldots, a_r]$ is a domain, and $\mathfrak{m}$ is a prime ideal of $Z[a_1, a_2, \ldots, a_r]$.

Since $k \subset Z/\mathfrak{m} \cap Z \subset Z[a_1, a_2, \ldots, a_r]/\mathfrak{m} \subset U/M(U) \cong k$, we have $Z/\mathfrak{m} \cap Z \cong Z[a_1, a_2, \ldots, a_r]/\mathfrak{m} \cong k$ and $\mathfrak{m} \cap Z = M(Z)$.

By Theorem 2.1 we have $\mathrm{ht}(\mathfrak{m}) = \mathrm{ht}(\mathfrak{m} \cap Z) + \mathrm{tr.deg}_Z Z[a_1, a_2, \ldots, a_r] - \mathrm{tr.deg}_{(Z/\mathfrak{m}\cap Z)}(Z[a_1, a_2, \ldots, a_r]/\mathfrak{m}) = \mathrm{ht}(\mathfrak{m} \cap Z) = \mathrm{ht}(M(Z)) = \dim Z = n$. Note here that $\mathrm{tr.deg}_Z Z[a_1, a_2, \ldots, a_r] = 0$, since Z and $Z[a_1, a_2, \ldots, a_r]$ have the same quotient field.

Let $\mathfrak{p}$ be an arbitrary prime ideal of $Z[a_1, a_2, \ldots, a_r]$. By Theorem 2.1 we have $\mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p} \cap Z) \leq \dim Z = n$. We conclude $\dim Z[a_1, a_2, \ldots, a_r] = n$.

2. By definition of the localization, every element $x \in R$ has an element $z \in Z[a_1, a_2, \ldots, a_r] \subset V$ and an element $y \in Z[a_1, a_2, \ldots, a_r] - \mathfrak{m} \subset U - M(U) = U^*$ with $x = z/y$. Thus $z \in U$ and $R \subset U$. Obviously R is a local domain. $M(R) = \mathfrak{m}R \subset M(U)U = M(U)$. Thus U dominates R. $\dim R = \mathrm{ht}(\mathfrak{m}) = n$. □

Lemma 10.2. *Let V be a regular local ring of dimension one containing the field k, E be the quotient field of V, $E' = V/M(V)$ be the residue field of V, and δ be a positive integer. Let $q = p^\delta$ and let $F_\delta : E \to E$, $F_\delta : E' \to E'$ and $F_\delta : V \to V$ be the Frobenius homomorphisms of degree δ. We denote their images by E^q, $(E')^q$ and V^q respectively. Assume that the relative degree $[E : E^q]$ is finite.*

1. *$q[E' : (E')^q] \leq [E : E^q]$. In particular, the relative degree $[E' : (E')^q]$ is finite.*
2. *If $q[E' : (E')^q] = [E : E^q]$, then V is a finitely generated V^q-module.*

Proof. 1. Ring V^q is isomorphic to V, and is a regular local ring of dimension one. By Proposition 4.25 V^q is a valuation ring of the quotient field E^q. The residue field of V^q is $(E')^q$ by Lemma 2.6.2.

By the same reason V is a valuation ring of E. We can check that V dominates V^q. Thus the valuation ring $V \cap E^q$ of the field E^q dominates V^q. By Corollary 4.19 we have $V \cap E^q = V^q$.

Let π be a generator of $M(V)$. Obviously π^q is a generator of $M(V^q)$. By Lemma 4.26 one knows that the ramification index of V over V^q is equal to q.

By Theorem 4.31.1 we obtain 1.

2. Since $x^q \in V^q$ for every $x \in V$, V is integral over V^q. By Proposition 4.25 V is normal. Thus V coincides with the integral closure of V^q in E. By Theorem 4.31.3 we obtain 2. $\square$

Lemma 10.3. *Let (Z, U) be a space germ over k of iterated divisor type. Denote $n = \dim Z = \operatorname{rank} U$. Let R be an excellent ring with $Z \subset R \subset U$. By*

$$\{0\} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{i-1} \subset \mathfrak{p}_i \subset \cdots \subset \mathfrak{p}_n = M(U)$$

we denote the sequence of prime ideals of the valuation ring U. We assume $\mathfrak{p}_{i-1} \neq \mathfrak{p}_i$ for every $1 \leq i \leq n$. For each integer i with $1 \leq i \leq n$ we choose an element a_i with $a_i \in \mathfrak{p}_i - \mathfrak{p}_{i-1}$, and we consider the subring

$$S = R[a_1, a_2, \dots, a_n]$$

of the quotient field of Z.

1. *$S \subset U$. S is an excellent domain.*
2. *Let $\mathfrak{q}_i = \mathfrak{p}_i \cap S$ for every $0 \leq i \leq n$. Each $\mathfrak{q}_i$ is a prime ideal of S, and they satisfy*
$$\{0\} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_{i-1} \subset \mathfrak{q}_i \subset \cdots \subset \mathfrak{q}_n \subset S.$$
For every $0 \leq i \leq n$, $\operatorname{ht}(\mathfrak{q}_i) = i$, and $\operatorname{ht}(\mathfrak{q}_n/\mathfrak{q}_i) = n - i$.
3. *Let $U_i = U/\mathfrak{p}_i$, D_i denote the quotient field of U_i, and $\overline{S}_i$ denote the integral closure of the domain $S/\mathfrak{q}_i$ in the field D_i for every $0 \leq i \leq n$. For every $0 \leq i \leq n$, $S/\mathfrak{q}_i$ is an excellent ring, and $\overline{S}_i \subset U_i$.*
4. *For every $0 \leq i \leq n$, D_i coincides with the quotient field of $\overline{S}_i$, and the field D_i is a finite algebraic extension of the quotient field of $S/\mathfrak{q}_i$.*
5. *For every $0 \leq i \leq n$, $\overline{S}_i$ is a finitely generated $S/\mathfrak{q}_i$-module, and $\overline{S}_i$ is an excellent ring.*
6. *For every $0 \leq i \leq n-1$, by $\overline{\mathfrak{p}}_{i+1}$ we denote the prime ideal $\mathfrak{p}_{i+1}/\mathfrak{p}_i$ of U_i, and $\overline{\mathfrak{q}}_{i+1} = \overline{\mathfrak{p}}_{i+1} \cap \overline{S}_i$. We have*
$$(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}} = (U_i)_{\overline{\mathfrak{p}}_{i+1}}.$$
7. *For every $0 \leq i \leq n-1$, D_{i+1} coincides with the quotient field of $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$, and the integral closure of $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$ in the field D_{i+1} coincides with $\overline{S}_{i+1}$.*
8. *For every $0 \leq i \leq n-1$, $U_{\mathfrak{p}_{i+1}}/\mathfrak{p}_i U_{\mathfrak{p}_{i+1}}$ is an excellent regular local ring of dimension one.*
9. *The value group Γ of the valuation defined by U is isomorphic to $\mathbb{Z}^n$ with the lexicographic order as totally ordered abelian groups.*
10. *For every $0 \leq i \leq n$ and for every positive integer δ*
$$[D_i : D_i^{p^\delta}] = p^{(n-i)\delta}.$$

By $D_i^{p^\delta}$ we denoted here the image of the Frobenius homomorphism $F_\delta : D_i \to D_i$ of degree δ.

Proof. 1. It follows from Theorem 2.2.4.
2. Obviously $\mathfrak{q}_i$ is a prime ideal of S, and $\mathfrak{q}_{i-1} \subset \mathfrak{q}_i$. $a_i \in \mathfrak{p}_i \cap S = \mathfrak{q}_i$. On the other hand, $a_i \notin \mathfrak{q}_{i-1}$, since $\mathfrak{q}_{i-1} \subset \mathfrak{p}_{i-1}$ and $a_i \notin \mathfrak{p}_{i-1}$. Thus $\mathfrak{q}_{i-1} \neq \mathfrak{q}_i$.

One concludes $\mathrm{ht}(\mathfrak{q}_i) \geq i$, and $\mathrm{ht}(\mathfrak{q}_n/\mathfrak{q}_i) \geq n - i$ for every $0 \leq i \leq n$.

By Theorem 2.1 we have $\mathrm{ht}(\mathfrak{q}_n) \leq \mathrm{ht}(\mathfrak{q}_n \cap R) + \mathrm{tr.deg}_R S = \mathrm{ht}(\mathfrak{q}_n \cap R) \leq \dim R = n$. Note here that $\mathrm{tr.deg}_R S = 0$, since R and S have the same quotient field.

We have $n \leq \mathrm{ht}(\mathfrak{q}_i) + \mathrm{ht}(\mathfrak{q}_n/\mathfrak{q}_i) \leq \mathrm{ht}(\mathfrak{q}_n) \leq n$. We obtain 2.
3. We fix an integer i with $0 \leq i \leq n$.

By Theorem 2.2.4 $S/\mathfrak{q}_i$ is an excellent ring.

By Lemma 4.16.1 $U_i = U/\mathfrak{p}_i$ is a valuation ring of the field D_i. By Corollary 4.10.2 U_i is normal.

Since $S/\mathfrak{q}_i \subset U_i$, we have $\overline{S}_i \subset U_i$ by definition of $\overline{S}_i$.
4, 5, 6 and 7. We will show 4, 5, 6 and 7 together by induction on i. Fixing the integer i, by $4.i$, $5.i$, $6.i$ and $7.i$ we denote the corresponding claims of 4, 5, 6 and 7 with the value i fixed respectively.

First, note that 4.0 is true. By definition of space germs the quotient field of $S = S/\mathfrak{q}_0$ and the quotient field D_0 of $U = U_0 = U/\mathfrak{p}_0$ coincide. By definition $\overline{S}_0$ is the integral closure of S in the common quotient field D_0. Thus three fields, the quotient field of $S/\mathfrak{q}_0$, the quotient field of $\overline{S}_0$ and D_0 coincide, and 4.0 is true.

Second, note that the implication $4.i \Rightarrow 5.i$ holds. By $4.i$ the quotient field D_i of U_i is a finite algebraic extension of the quotient field of $S/\mathfrak{q}_i$. By definition $\overline{S}_i$ is the integral closure of $S/\mathfrak{q}_i$ in the field of D_i. By 3 $S/\mathfrak{q}_i$ is an excellent domain. Thus, by Theorem 2.2.7 $\overline{S}_i$ is a finitely generated $S/\mathfrak{q}_i$-module. In particular, $\overline{S}_i$ is a finitely generated $S/\mathfrak{q}_i$-algebra, and by Theorem 2.2.4 $\overline{S}_i$ is an excellent ring.

Third, we will show $4.i$ and $5.i \Rightarrow 6.i$.

By definition $\overline{\mathfrak{q}}_{i+1} \cap (S/\mathfrak{q}_i) = \overline{\mathfrak{p}}_{i+1} \cap (S/\mathfrak{q}_i) = \mathfrak{q}_{i+1}/\mathfrak{q}_i$. By 2 $\mathrm{ht}(\mathfrak{q}_{i+1}/\mathfrak{q}_i) = 1$. By $5.i$ $\overline{S}_i$ is integral over $S/\mathfrak{q}_i$. Thus by Matsumura [7], p.33–p.34, (5.E), THEOREM 5, ii), one knows $\mathrm{ht}(\overline{\mathfrak{q}}_{i+1}) = 1$. Thus $\dim(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}} = 1$. On the other hand, by $5.i$ $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}}$ is a noetherian normal local ring of dimension one. By Proposition 4.25 $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}}$ is a valuation ring of the quotient field of $\overline{S}_i$. By $4.i$ the quotient field of $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}}$ coincides with D_i. By definition $(U_i)_{\overline{\mathfrak{p}}_{i+1}}$ dominates $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}}$. By Corollary 4.19 we obtain $6.i$.

Fourth, we will show $5.i$ and $6.i \Rightarrow 7.i$. Considering the residue fields of local rings at the both-hand sides of the equality in $6.i$, we have

$$(*) \qquad (\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}}/\overline{\mathfrak{q}}_{i+1}(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}} \cong (U_i)_{\overline{\mathfrak{p}}_{i+1}}/\overline{\mathfrak{p}}_{i+1}(U_i)_{\overline{\mathfrak{p}}_{i+1}}.$$

The left-hand side of $(*)$ coincides with the quotient field of $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$, and the right-hand side of $(*)$ coincides with the quotient field D_{i+1} of $U_{i+1} = U_i/\overline{\mathfrak{p}}_{i+1}$. Thus we obtain the former half of $7.i$.

Note that $\overline{\mathfrak{q}}_{i+1} \cap (S/\mathfrak{q}_i) = \mathfrak{q}_{i+1}/\mathfrak{q}_i$, $(S/\mathfrak{q}_i)/(\overline{\mathfrak{q}}_{i+1} \cap (S/\mathfrak{q}_i)) \cong S/\mathfrak{q}_{i+1}$. Thus, by $5.i$ $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$ is a finitely generated $S/\mathfrak{q}_{i+1}$-module. In particular, $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$ is integral over $S/\mathfrak{q}_{i+1}$. By the former half of $7.i$ $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$ and U_{i+1} have the same quotient field D_{i+1}. By definition $\overline{S}_{i+1}$ is the integral closure of $S/\mathfrak{q}_{i+1}$ in the field D_{i+1}. Thus $S/\mathfrak{q}_i \subset \overline{S}_i/\overline{\mathfrak{q}}_{i+1} \subset \overline{S}_{i+1}$, and we obtain the latter half of $7.i$.

Fifth, we show $5.i$ and $7.i \Rightarrow 4.i+1$.

By 7.i the three fields, D_{i+1}, the quotient field of $\overline{S}_{i+1}$ and the quotient field of $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$, coincide. Thus, in particular, the former half of 4.$i+1$ is true.

By 5.i $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$ is a finitely generated $S/\mathfrak{q}_{i+1}$-module. Let E denote the quotient field of $S/\mathfrak{q}_{i+1}$. Let D' be the set of elements x of the quotient field D_{i+1} of $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$ such that there are elements $y \in \overline{S}_i/\overline{q}_{i+1}$ and $z \in S/\mathfrak{q}_{i+1}$ satisfying $z \neq 0$ and $x = y/z$. D' is a subdomain of D_{i+1}. D' is a finitely generated E-module. D' contains $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$. Thus D' is a field, and $D' = D_{i+1}$. D_{i+1} is a finite algebraic extension of E.

8. $U_{\mathfrak{p}_{i+1}}/\mathfrak{p}_i U_{\mathfrak{p}_{i+1}} \cong (U_i)_{\overline{\mathfrak{p}}_{i+1}}$. Thus by 6, the claim of 8 is equivalent to that $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}}$ is an excellent regular local ring of dimension one. We have already seen that $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}}$ is a regular local ring of dimension one in the proof of 4.i and 5.$i \Rightarrow$ 6.i. By 5 and by Theorem 2.2.5 $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}}$ is excellent.

9. It follows from Corollary 4.22.2.

10. We write $q = p^\delta$, and let $P = \{x_1, x_2, \dots, x_n\}$ be a parameter system of Z. The field D_0 is the quotient field of the ring $Z = k[[x_1, x_2, \dots, x_n]]$. Since k is an algebraically closed field, the Frobenius homomorphism $F_\delta : k \to k$ of degree δ over k is surjective. Thus D_0^q is the quotient field of $Z^q = k[[x_1^q, x_2^q, \dots, x_n^q]]$. Obviously Z is a free Z^q-module, and q^n elements $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ $0 \le \alpha_1, \alpha_2, \dots, \alpha_n \le q-1$ are a basis of Z over Z^q.

Let D' be the set of elements x of the field D_0 such that there are elements $y \in Z$ and $z \in Z^q$ satisfying $z \neq 0$ and $x = y/z$. D' is a subdomain of D_0. D' is a finitely generated D_0^q-module. D' contains Z. Thus D' is a field, and $D' = D_0$. There exists a surjective homomorphism $Z \otimes_{Z^q} D_0^q \to D' = D_0$ of D_0^q-modules, and one knows $[D_0 : D_0^q] \le q^n$.

By 8 $U_{\mathfrak{p}_{i+1}}/\mathfrak{p}_i U_{\mathfrak{p}_{i+1}}$ is a regular local ring of dimension one, the quotient field of it is D_i, and the residue field of it is D_{i+1}. By Lemma 10.2.1 we have $[D_{i+1} : D_{i+1}^q] \le [D_i : D_i^q]/q$, if $[D_i : D_i^q]$ is finite. We have above shown that $[D_0 : D_0^q]$ is finite. Thus, by induction, one knows that $[D_i : D_i^q]$ is finite for every $0 \le i \le n$. Besides, the following $(**)$ holds:

$$(**)\qquad \begin{aligned} [D_n : D_n^q] \le & \frac{[D_{n-1} : D_{n-1}^q]}{q} \le \cdots \le \frac{[D_{i+1} : D_{i+1}^q]}{q^{n-i-1}} \le \frac{[D_i : D_i^q]}{q^{n-i}} \le \\ & \cdots \le \frac{[D_0 : D_0^q]}{q^n} \le \frac{q^n}{q^n} = 1 \end{aligned}$$

By definition $[D_n : D_n^q]$ is a positive integer. Thus by $(**)$ one knows $[D_i : D_i^q] = q^{n-i}$ for every $0 \le i \le n$. □

Proposition 10.4. *Let (Z, U) be a space germ over k of iterated divisor type, and (Z', U') be an elementary monoidal transform of (Z, U). Denote $n = \dim Z = \operatorname{rank} U$.*

1. *A space germ (Z', U') is of iterated divisor type. Let*
$$\{0\} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_{i-1} \subset \mathfrak{p}_i \subset \cdots \subset \mathfrak{p}_n = M(U),$$
$$\{0\} = \mathfrak{p}'_0 \subset \mathfrak{p}'_1 \subset \cdots \subset \mathfrak{p}'_{i-1} \subset \mathfrak{p}'_i \subset \cdots \subset \mathfrak{p}'_n = M(U')$$
be the sequences of prime ideals of U and U' respectively. For every $0 \le i \le n$ $\mathfrak{p}_i = \mathfrak{p}'_i \cap U$.
2. *Let i be an arbitrary integer with $0 \le i \le n$. By D_i and D'_i denote the quotient field of $U/\mathfrak{p}_i$ and $U'/\mathfrak{p}'_i$ respectively. Let δ be an arbitrary positive*

integer. The field D'_i is generated by D_i and $(D'_i)^{p^\delta}$, where the image of the Frobenius homomorphism $F_\delta : D'_i \to D'_i$ of degree δ is denoted by $(D'_i)^{p^\delta}$.

3. *For every $0 \leq i \leq n-1$ the homomorphism of regular local rings of dimension one*
$$U_{\mathfrak{p}_{i+1}}/\mathfrak{p}_i U_{\mathfrak{p}_{i+1}} \to U'_{\mathfrak{p}'_{i+1}}/\mathfrak{p}'_i U'_{\mathfrak{p}'_{i+1}}$$
induced by the associated homomorphism $U \to U'$ sends a generator of the maximal ideal to a generator of the maximal ideal.
4. *Let Γ and Γ' denote the value group of the valuation ring U and U' respectively. The induced homomorphism $\Gamma \to \Gamma'$ by $U \to U'$ is an isomorphism.*

Proof. 1. It follows from Corollary 4.22.1 and Corollary 4.15.
2. Let $U_i = U/\mathfrak{p}_i$ and $U'_i = U'/\mathfrak{p}'_i$. The injective homomorphism $U_i = U/\mathfrak{p}_i = U/\mathfrak{p}'_i \cap U \to U'_i = U'/\mathfrak{p}'_i$ induces an injective homomorphism $D_i \to D'_i$. We regard D_i as a subfield of D'_i by this homomorphism.

By definition of an elementary monoidal transform, there exists a partial parameter system $x_0, x_1, \dots, x_m$ of Z with the following properties:

(a) $m \geq 1$.
(b) $b_i = x_i/x_0 \in M(U)$ for every $1 \leq i \leq m$.
(c) Let $\mathfrak{m} = M(U) \cap Z[b_1, b_2, \dots, b_m]$ and $R = Z[b_1, b_2, \dots, b_m]_{\mathfrak{m}}$. The valuation ring U dominates R.
(d) The completion $\hat{R}$ of the local ring R coincides with Z'.

By Lemma 10.1.2 R is an excellent local domain and $n = \dim R = \dim \hat{R}$. By Theorem 2.2.3 also $\hat{R} = Z'$ is excellent.

Choosing an element $a_i \in \mathfrak{p}_i - \mathfrak{p}_{i-1}$ for each integer i with $1 \leq i \leq n$, we consider a subdomain
$$S = R[a_1, a_2, \cdots, a_n]$$
of the quotient field D_0 of Z. Besides, regarding $a_i \in D_0 \subset D'_0$, we consider the subdomain
$$S' = \hat{R}[a_1, a_2, \cdots, a_n]$$
of the quotient field D'_0 of Z'. Note that $a_i \in \mathfrak{p}'_i - \mathfrak{p}'_{i-1}$ for every $1 \leq i \leq n$ by 1 above. Note also that $\hat{R} = Z' \subset U'$ by definition of an elementary monoidal transform.

Lemma 10.3.1–10 hold for both pairs (S, U) and (S', U').

Choosing an arbitrary integer i with $0 \leq i \leq n$, and we fix i below. Let $\overline{S}_i$ and $\overline{S}'_i$ denote the integral closure of the domain $S/\mathfrak{p}_i \cap S$ in the field D_i and the integral closure of the domain $S'/\mathfrak{p}'_i \cap S'$ in the field D'_i respectively. By Lemma 10.3.4 the quotient field of $\overline{S}_i$ coincides with D_i, and the quotient field of $\overline{S}'_i$ coincides with D'_i.

By definition, $\overline{S}_i$ is integral over $S/\mathfrak{p}_i \cap S$. Since $S/\mathfrak{p}_i \cap S \subset S'/\mathfrak{p}'_i \cap S'$, $\overline{S}_i$ is integral over $S'/\mathfrak{p}'_i \cap S'$. Thus we have $\overline{S}_i \subset \overline{S}'_i$.

$\overline{S}'_i$ contains $\hat{R}/\mathfrak{p}'_i \cap \hat{R}$ as a subring. The subring of $\overline{S}'_i$ generated by $\overline{S}_i$ and $\hat{R}/\mathfrak{p}'_i \cap \hat{R}$ is denoted by $\overline{S}_i[\hat{R}/\mathfrak{p}'_i \cap \hat{R}]$ or $\overline{S}_i[\hat{R}]$ in short. $\overline{S}_i[\hat{R}] \subset \overline{S}'_i \subset U'_i$. Let $\mathfrak{n}_i = M(U'_i) \cap \overline{S}_i[\hat{R}]$.

We consider the ring $\overline{S}_i \otimes_R \hat{R}$ and the maximal ideal $\mathfrak{m}_i = (M(U_i) \cap \overline{S}_i) \otimes \hat{R} + \overline{S}_i \otimes M(\hat{R})$ here. We need a lemma.

Lemma 10.5. 1. *The localization* $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i}$ *of the ring* $\overline{S}_i \otimes_R \hat{R}$ *with respect to* $\mathfrak{m}_i$ *is a normal local domain.*

2. *The canonical homomorphism* $\overline{S}_i \otimes_R \hat{R} \to \overline{S}_i[\hat{R}]$ *is surjective, and it induces an isomorphism* $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} \to \overline{S}_i[\hat{R}]_{\mathfrak{n}_i}$.
3. $L_i = \overline{S}_i[\hat{R}] - M(U_i') \cap \overline{S}_i[\hat{R}]$ *is a multiplicative subset of* $\overline{S}_i[\hat{R}]$. $\overline{S}_i[\hat{R}]_{\mathfrak{n}_i} = L_i^{-1}\overline{S}_i[\hat{R}]$.
4. $L_i^{-1}\overline{S}_i[\hat{R}] = L_i^{-1}\overline{S}_i'$. *The quotient field of* $\overline{S}_i[\hat{R}]$ *coincides with* D_i'.
5. $L_i^{-1}\overline{S}_i' = (\overline{S}_i')_{M(U_i')\cap\overline{S}_i'}$.

Proof. 1. For every positive integer ν we have $\mathfrak{m}_i^\nu \supset ((M(U_i) \cap \overline{S}_i) \otimes \hat{R})^\nu + (\overline{S}_i \otimes M(\hat{R}))^\nu \supset \mathfrak{m}_i^{2\nu-1}$, and $\varprojlim_\nu (\overline{S}_i \otimes_R \hat{R})/\mathfrak{m}_i^\nu \cong \varprojlim_\nu (\overline{S}_i/(M(U_i) \cap \overline{S}_i)^\nu) \otimes_R (\hat{R}/M(\hat{R})^\nu) \cong \varprojlim_\nu (\overline{S}_i/(M(U_i) \cap \overline{S}_i)^\nu)$. Thus the completion of the local ring $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i}$ coincides with the completion $\hat{\overline{S}}_i$ of the local ring $(\overline{S}_i)_{M(U_i)\cap\overline{S}_i}$.

Since $(\overline{S}_i)_{M(U_i)\cap\overline{S}_i}$ is an excellent normal local domain, $\hat{\overline{S}}_i$ is a noetherian complete normal local domain by Theorem 2.2.9. Since the completion $\hat{\overline{S}}_i$ of $(\overline{S}_i\otimes_R\hat{R})_{\mathfrak{m}_i}$ is faithfully flat over $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i}$, (Matsumura [7], p. 172, (24.A), THEOREM 56.) $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i}$ is a domain. By Matsumura [7], p. 176, (24.E), PROPOSITION ii), $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i}$ is normal.

2. Obviously the homomorphism $\overline{S}_i \otimes_R \hat{R} \to \overline{S}_i[\hat{R}]$ is surjective, and the inverse image of the prime ideal $\mathfrak{n}_i$ coincides with $\mathfrak{m}_i$. Thus we have an induced surjective homomorphism $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} \to \overline{S}_i[\hat{R}]_{\mathfrak{n}_i}$. In particular, $\dim(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} \geq \dim \overline{S}_i[\hat{R}]_{\mathfrak{n}_i}$.

The valuation ring U_i' has a sequence

$$\{0\} = \mathfrak{p}_i'/\mathfrak{p}_i' \subset \mathfrak{p}_{i+1}'/\mathfrak{p}_i' \subset \cdots \subset \mathfrak{p}_n'/\mathfrak{p}_i' = M(U_i')$$

of prime ideals. For every j with $i+1 \leq j \leq n$ we have

$$(*) \qquad (\mathfrak{p}_{j-1}'/\mathfrak{p}_i') \cap \overline{S}_i[\hat{R}] \subset (\mathfrak{p}_j'/\mathfrak{p}_i') \cap \overline{S}_i[\hat{R}].$$

$S'/\mathfrak{p}_i' \cap S' \subset \overline{S}_i[\hat{R}]$ and by Lemma 10.3.2 for every j with $i \leq j \leq n$ we have $(\mathfrak{p}_j'/\mathfrak{p}_i') \cap (S'/\mathfrak{p}_i' \cap S') = (\mathfrak{p}_j' \cap S')/(\mathfrak{p}_i' \cap S')$, $\mathrm{ht}((\mathfrak{p}_j' \cap S')/(\mathfrak{p}_i' \cap S')) = j - i$. Thus, the both-hand sides of $(*)$ never coincide, and one knows $\dim \overline{S}_i[\hat{R}]_{\mathfrak{n}_i} \geq n - i$.

We consider the homomorphism $\overline{S}_i \to \overline{S}_i \otimes_R \hat{R}$. The inverse image of $\mathfrak{m}_i$ by it coincides with $M(U_i) \cap \overline{S}_i$. By Matsumura [7], p. 79, (13.B), THEOREM 19 (1), we have $\dim(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} \leq \dim(\overline{S}_i)_{M(U_i)\cap\overline{S}_i} + \dim(\overline{S}_i/M(U_i) \cap \overline{S}_i) \otimes_R \hat{R} = \dim(\overline{S}_i)_{M(U_i)\cap\overline{S}_i}$. We used $(\overline{S}_i/M(U_i) \cap \overline{S}_i) \otimes_R \hat{R} \cong k \otimes_R \hat{R} \cong k$ here. Since $\overline{S}_i$ is integral over $S/\mathfrak{p}_i \cap S$, by Matsumura [7], p. 81, (13.C), THEOREM 20 (2) we have $\dim(\overline{S}_i)_{M(U_i)\cap\overline{S}_i} \leq \dim(S/\mathfrak{p}_i \cap S)_{M(U_i)\cap(S/\mathfrak{p}_i\cap S)} = \mathrm{ht}((\mathfrak{p}_n \cap S)/(\mathfrak{p}_i \cap S)) = n - i$. We used Lemma 10.3.2 here.

By the above we have $n - i \geq \dim(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} \geq \dim \overline{S}_i[\hat{R}]_{\mathfrak{n}_i} \geq n - i$, and in particular $\dim(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} = \dim \overline{S}_i[\hat{R}]_{\mathfrak{n}_i}$. The surjective ring homomorphism $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} \to \overline{S}_i[\hat{R}]_{\mathfrak{n}_i}$ between two local domains of the same dimension is an isomorphism.

3. Trivial.

4. Since $S'/\mathfrak{p}_i' \cap S' \subset \overline{S}_i[\hat{R}] \subset \overline{S}_i'$, and since $\overline{S}_i'$ is the integral closure of $S'/\mathfrak{p}_i' \cap S'$ in D_i', $\overline{S}_i'$ coincides with the integral closure of $\overline{S}_i[\hat{R}]$ in D_i'. Thus $L_i^{-1}\overline{S}_i'$ coincides with the integral closure of $L_i^{-1}\overline{S}_i[\hat{R}]$ in D_i'. Since $L_i^{-1}\overline{S}_i[\hat{R}] \cong (\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i}$ is normal

by 1, 2 and 3 above, the quotient field of $L_i^{-1}\overline{S}_i[\hat{R}]$ coincides with D_i', if and only if, $L_i^{-1}\overline{S}_i[\hat{R}] = L_i^{-1}\overline{S}_i'$.

By induction on i we will show that these equivalent conditions are satisfied for every $0 \leq i \leq n$.

In the case where $i = 0$, we have $D_0' \supset$ the quotient field of $L_0^{-1}\overline{S}_0[\hat{R}] \supset$ the quotient field of $\hat{R} = D_0'$.

Assume that $1 \leq i \leq n$ and $L_{i-1}^{-1}\overline{S}_{i-1}[\hat{R}] = L_{i-1}^{-1}\overline{S}_{i-1}'$. We have

$$
\begin{aligned}
&\text{the quotient field of } L_i^{-1}\overline{S}_i[\hat{R}] = \text{the quotient field of } \overline{S}_i[\hat{R}]\\
&= \text{the field generated by the quotient field of } \overline{S}_i\\
&\qquad\qquad \text{and the quotient field of } \hat{R}/\mathfrak{p}_i' \cap \hat{R}\\
&= \text{the field generated by the quotient field of } (\overline{S}_{i-1}/(\mathfrak{p}_i/\mathfrak{p}_{i-1}) \cap \overline{S}_{i-1})\\
&\qquad\qquad \text{and the quotient field of } \hat{R}/\mathfrak{p}_i' \cap \hat{R}\\
&= \text{the quotient field of } (\overline{S}_{i-1}[\hat{R}]/(\mathfrak{p}_i'/\mathfrak{p}_{i-1}') \cap \overline{S}_{i-1}[\hat{R}])\\
&= \text{the quotient field of } (L_{i-1}^{-1}\overline{S}_{i-1}[\hat{R}]/(\mathfrak{p}_i'/\mathfrak{p}_{i-1}') \cap L_{i-1}^{-1}\overline{S}_{i-1}[\hat{R}])\\
&= \text{the quotient field of } (L_{i-1}^{-1}\overline{S}_{i-1}'/(\mathfrak{p}_i'/\mathfrak{p}_{i-1}') \cap L_{i-1}^{-1}\overline{S}_{i-1}')\\
&= \text{the quotient field of } (\overline{S}_{i-1}'/(\mathfrak{p}_i'/\mathfrak{p}_{i-1}') \cap \overline{S}_{i-1}') = \text{the quotient field of } \overline{S}_i' = D_i'.
\end{aligned}
$$

We used Lemma 10.3.7 and 4.
5. Since $\overline{S}_i[\hat{R}] \subset \overline{S}_i'$, we have $L_i \subset \overline{S}_i' - M(U_i') \cap \overline{S}_i'$ and $L_i^{-1}\overline{S}_i' \subset (\overline{S}_i')_{M(U_i')\cap\overline{S}_i'}$. On the other hand, by 3 and 4 we have $L_i^{-1}\overline{S}_i' = \overline{S}_i[\hat{R}]_{\mathfrak{n}_i}$ and $\overline{S}_i[\hat{R}]_{\mathfrak{n}_i}$ is a local ring dominated by U_i'. Thus one has 5. □

Continuation of the proof of Proposition 10.4. By Lemma 10.5.4 the field D_i' coincides with the quotient field of $\overline{S}_i[\hat{R}/\mathfrak{p}_i' \cap \hat{R}]$. Thus D_i' coincides with the field generated by D_i and $\hat{R}/\mathfrak{p}_i' \cap \hat{R}$. Since $\hat{R}/\mathfrak{p}_i' \cap \hat{R}$ is a homomorphic image of the completion of the local ring $R/\mathfrak{p}_i \cap R$, $\hat{R}/\mathfrak{p}_i' \cap \hat{R}$ is generated by $R/\mathfrak{p}_i \cap R$ and $(\hat{R}/\mathfrak{p}_i' \cap \hat{R})^{p^\delta}$. Since $R/\mathfrak{p}_i \cap R \subset \overline{S}_i \subset D_i$, and since $(\hat{R}/\mathfrak{p}_i' \cap \hat{R})^{p^\delta} \subset (D_i')^{p^\delta}$, D_i' coincides with the field generated by D_i and $(D_i')^{p^\delta}$.
3. By Lemma 10.3.6 the homomorphism $U_{\mathfrak{p}_{i+1}}/\mathfrak{p}_i U_{\mathfrak{p}_{i+1}} \to U'_{\mathfrak{p}'_{i+1}}/\mathfrak{p}_i' U'_{\mathfrak{p}'_{i+1}}$ can be identified with the homomorphism $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}} \to (\overline{S}_i')_{\overline{\mathfrak{q}}'_{i+1}}$, where $\overline{\mathfrak{q}}_{i+1} = (\mathfrak{p}_{i+1}/\mathfrak{p}_i) \cap \overline{S}_i$ and $\overline{\mathfrak{q}}'_{i+1} = (\mathfrak{p}'_{i+1}/\mathfrak{p}_i') \cap \overline{S}_i'$.

On the other hand, by Lemma 10.5.1–5 there exists an isomorphism $(\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} \cong (\overline{S}_i')_{M(U_i')\cap\overline{S}_i'}$, and the homomorphism $(\overline{S}_i)_{M(U_i)\cap\overline{S}_i} \to (\overline{S}_i')_{M(U_i')\cap\overline{S}_i'}$ can be identified with the homomorphism $(\overline{S}_i)_{M(U_i)\cap\overline{S}_i} \to (\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i}$. Let $\overline{\mathfrak{q}}''_{i+1}$ be the inverse image of the prime ideal $\overline{\mathfrak{q}}'_{i+1}(\overline{S}_i')_{M(U_i')\cap\overline{S}_i'}$ by the homomorphism $\overline{S}_i \otimes_R \hat{R} \to (\overline{S}_i \otimes_R \hat{R})_{\mathfrak{m}_i} \cong (\overline{S}_i')_{M(U_i')\cap\overline{S}_i'}$. $\overline{\mathfrak{q}}''_{i+1}$ is a prime ideal of the ring $\overline{S}_i \otimes_R \hat{R}$, $\overline{\mathfrak{q}}''_{i+1} \subset \mathfrak{m}_i$, and the inverse image of $\overline{\mathfrak{q}}''_{i+1}$ by the homomorphism $\overline{S}_i \to \overline{S}_i \otimes_R \hat{R}$ coincides with $\overline{\mathfrak{q}}_{i+1}$. The homomorphism $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}} \to (\overline{S}_i')_{\overline{\mathfrak{q}}'_{i+1}}$ can be identified with the homomorphism $(\overline{S}_i)_{\overline{\mathfrak{q}}_{i+1}} \to (\overline{S}_i \otimes_R \hat{R})_{\overline{\mathfrak{q}}''_{i+1}}$.

We would like to apply Theorem 2.2.10 for the excellent ring R. Note that $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$ is a finitely generated R-algebra by Lemma 10.3.5. Let L be the quotient field of $\overline{S}_i/\overline{\mathfrak{q}}_{i+1}$ and $\mathfrak{Q} = \overline{\mathfrak{q}}''_{i+1}(L \otimes_R \hat{R})$. The field L contains the quotient field of $R/\overline{\mathfrak{q}}_{i+1} \cap R$, and L is finitely generated over the quotient field of $R/\overline{\mathfrak{q}}_{i+1} \cap R$. $\mathfrak{Q}$ is a prime ideal of $L \otimes_R \hat{R}$. By Theorem 2.2.10 the ring $(L \otimes_R \hat{R})_{\mathfrak{Q}}$ is a regular local ring, and has no nilpotent elements. Since $(L \otimes_R \hat{R})_{\mathfrak{Q}} \cong ((\overline{S}_i/\overline{\mathfrak{q}}_{i+1}) \otimes_R \hat{R})_{\overline{\mathfrak{q}}''_{i+1}} \cong (\overline{S}_i \otimes_R \hat{R})_{\overline{\mathfrak{q}}''_{i+1}}/\overline{\mathfrak{q}}_{i+1}(\overline{S}_i \otimes_R \hat{R})_{\overline{\mathfrak{q}}''_{i+1}}$, the ring $(\overline{S}_i \otimes_R \hat{R})_{\overline{\mathfrak{q}}''_{i+1}}/\overline{\mathfrak{q}}_{i+1}(\overline{S}_i \otimes_R \hat{R})_{\overline{\mathfrak{q}}''_{i+1}}$ has no nilpotent elements, which is equivalent to our claim 3.

4. Let $v : D_0 \to \Gamma \cup \{\infty\}$ and $v' : D'_0 \to \Gamma' \cup \{\infty\}$ denote the valuation defined by valuation rings U and U' respectively. Let $\Delta_i = v((U_{\mathfrak{p}_i})^*)$ and $\Delta'_i = v'((U'_{\mathfrak{p}'_i})^*)$ for every $0 \le i \le n$. By Lemma 4.16.2 each Δ_i is an isolated subgroup of Γ, and each Δ'_i is an isolated subgroup of Γ'. We have

$$\Gamma \supset \Delta_0 \supset \Delta_1 \supset \cdots \supset \Delta_i \supset \Delta_{i+1} \supset \cdots \supset \Delta_n = \{0\},$$

$$\Gamma' \supset \Delta'_0 \supset \Delta'_1 \supset \cdots \supset \Delta'_i \supset \Delta'_{i+1} \supset \cdots \supset \Delta'_n = \{0\}.$$

Since (Z', U') is an elementary monoidal transform of (Z, U) we have $U = U' \cap D_0$. By Lemma 4.13.2 we can regard Γ as a subgroup of Γ' and $v' \mid D_0 = v$. By Corollary 4.15.3 $\Delta_i = \Delta'_i \cap \Gamma$ for every $0 \le i \le n$.

We will show the following claim $4.i$ by induction on i.

$4.i$ The homomorphism $\Gamma/\Delta_i \to \Gamma'/\Delta'_i$ induced by the inclusion homomorphism $\Gamma \to \Gamma'$ is an isomorphism.

Obviously 4.0 is true. Assume that $0 \le i \le n-1$ and $4.i$ is true, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta_i/\Delta_{i+1} & \longrightarrow & \Gamma/\Delta_{i+1} & \longrightarrow & \Gamma/\Delta_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta'_i/\Delta'_{i+1} & \longrightarrow & \Gamma'/\Delta'_{i+1} & \longrightarrow & \Gamma'/\Delta'_i & \longrightarrow & 0
\end{array}$$

Two rows of the above diagram are exact, and by $4.i$ the right vertical arrow is an isomorphism. By Lemma 4.16 Δ_i/Δ_{i+1} coincides with the value group of the valuation ring $U_{\mathfrak{p}_{i+1}}/\mathfrak{p}_i U_{\mathfrak{p}_{i+1}}$, and Δ'_i/Δ'_{i+1} coincides with the value group of the valuation ring $U'_{\mathfrak{p}'_{i+1}}/\mathfrak{p}'_i U'_{\mathfrak{p}'_{i+1}}$. By Lemma 10.3.8 $\Delta_i/\Delta_{i+1} \cong \mathbb{Z}$ and $\Delta'_i/\Delta'_{i+1} \cong \mathbb{Z}$. By 3 above the image of the generator of Δ_i/Δ_{i+1} is a generator of Δ'_i/Δ'_{i+1}. Thus, the left vertical arrow is also an isomorphism. The middle vertical arrow is an isomorphism, and one knows that $4.i+1$ is true.

In particular, $4.n$ is true. We obtain 4. □

Lemma 10.6. *Let (Z, U) be a space germ over k of iterated divisor type, and (Z', U') be an IAMT of (Z, U). Denote $n = \dim Z = \operatorname{rank} U$. The same claims 1–4 as in Proposition 10.4 hold under this assumption.*

Proof. By definition there is a finite sequence $\{(Z_\nu, U_\nu) \mid 0 \le \nu \le \omega\}$ of space germs satisfying the following two conditions:

1. $(Z, U) = (Z_0, U_0)$, $(Z', U') = (Z_\omega, U_\omega)$.
2. For every $1 \le \nu \le \omega$ (Z_ν, U_ν) is an elementary monoidal transform of $(Z_{\nu-1}, U_{\nu-1})$.

If $\omega = 0$, then obviously our claims hold. If $\omega = 1$, then by Proposition 10.4 our claims holds. Thus by induction on ω we know that our claims hold.

We write only the induction step corresponding to claim 2 of Proposition 10.4, since other parts are trivial.

Assume $\omega \geq 2$. Let

$$\{0\} = \mathfrak{p}''_0 \subset \mathfrak{p}''_1 \subset \cdots \subset \mathfrak{p}''_{i-1} \subset \mathfrak{p}''_i \subset \cdots \subset \mathfrak{p}''_n = M(U_{\omega-1})$$

be the sequence of prime ideals of $U_{\omega-1}$. Let i be an integer with $0 \leq i \leq n$, and by D''_i we denote the quotient field of $U_{\omega-1}/\mathfrak{p}''_i$.

By induction hypothesis D''_i is generated by D_i and $(D''_i)^{p^\delta}$. By the case $\omega = 1$ D'_i is generated by D''_i and $(D'_i)^{p^\delta}$. Thus, D'_i is generated by D_i, $(D''_i)^{p^\delta}$ and $(D'_i)^{p^\delta}$. Since $D''_i \subset D'_i$, we have $(D''_i)^{p^\delta} \subset (D'_i)^{p^\delta}$. Thus D'_i is generated by D_i and $(D'_i)^{p^\delta}$. □

Lemma 10.7. *Let $\mathcal{D}$ be a field containing k, and δ be a positive integer. We consider the Frobenius homomorphism $F_\delta : \mathcal{D} \to \mathcal{D}$ of degree δ. We write $q = p^\delta$ and $\mathcal{D}^q = F_\delta(\mathcal{D})$. Assume that there exists an infinite sequence of subfields D_ν of $\mathcal{D}$ numbered by non-negative integers ν satisfying the following three conditions:*

1. *$D_\nu \subset D_{\nu+1}$ for every non-negative integer ν. $\mathcal{D} = \cup_{\nu=0}^{\infty} D_\nu$.*
2. *There exists a non-negative integer ϵ such that for every non-negative integer ν the relative degree $[D_\nu : D_\nu^q]$ is finite, and it is equal to p^ϵ, where $D_\nu^q = F_\delta(D_\nu)$.*
3. *D_ν is generated by subfields D_0 and $(D_\nu)^q$ for every non-negative integer ν.*

Then, the relative degree $[\mathcal{D} : \mathcal{D}^q]$ is finite, and it is equal to p^ϵ

Proof. Easy. □

Theorem 10.8. *Let $\{(Z_\nu, U_\nu) \mid \nu \geq 0\}$ be an infinite sequence of space germs over k numbered by non-negative integers ν. Assume the following two conditions:*

(a) $\dim Z_0 = \operatorname{rank} U_0 = n$.
(b) *$(Z_{\nu+1}, U_{\nu+1})$ is an IAMT of (Z_ν, U_ν) for every non-negative integer ν.*

We denote the quotient field of Z_ν by D_ν for every non-negative integer ν. By the associated homomorphism $U_\nu \to U_{\nu+1}$, $D_\nu \to D_{\nu+1}$ we define two inductive systems $\{U_\nu \mid \nu \geq 0\}$ and $\{D_\nu \mid \nu \geq 0\}$. Let $\mathcal{U} = \varinjlim_\nu U_\nu$ and $\mathcal{D} = \varinjlim_\nu D_\nu$.

1. *$\mathcal{U}$ is a valuation ring of the field $\mathcal{D}$ with $\operatorname{rank}\mathcal{U} = n$. Let ν be an arbitrary non-negative integer. $\operatorname{rank} U_\nu = n$. Let*
$$\{0\} = \mathfrak{p}_{\nu,0} \subset \mathfrak{p}_{\nu,1} \subset \cdots \subset \mathfrak{p}_{\nu,i-1} \subset \mathfrak{p}_{\nu,i} \subset \cdots \subset \mathfrak{p}_{\nu,n} = M(U_\nu),$$
$$\{0\} = \tilde{\mathfrak{p}}_0 \subset \tilde{\mathfrak{p}}_1 \subset \cdots \subset \tilde{\mathfrak{p}}_{i-1} \subset \tilde{\mathfrak{p}}_i \subset \cdots \subset \tilde{\mathfrak{p}}_n = M(\mathcal{U})$$
be the sequences of prime ideals of U_ν and $\mathcal{U}$ respectively. For every $0 \leq i \leq n$ we have $\mathfrak{p}_{\nu,i} = \tilde{\mathfrak{p}}_i \cap U_\nu$.
2. *Let ν be an arbitrary non-negative integer. For every $0 \leq i \leq n-1$*
$$V_{\nu,i} = (U_\nu)_{\mathfrak{p}_{\nu,i+1}}/\mathfrak{p}_{\nu,i}(U_\nu)_{\mathfrak{p}_{\nu,i+1}} \text{ and}$$
$$\mathcal{V}_i = \mathcal{U}_{\tilde{\mathfrak{p}}_{i+1}}/\tilde{\mathfrak{p}}_i\mathcal{U}_{\tilde{\mathfrak{p}}_{i+1}}$$
are regular local rings of dimension one. The induced homomorphism $V_{\nu,i} \to \mathcal{V}_i$ sends a generator of the maximal ideal to a generator of the maximal ideal. The induced homomorphism from the value group of the valuation ring U_ν to the value group of the valuation ring $\mathcal{U}$ is an isomorphism. In particular, the value group of $\mathcal{U}$ is isomorphic to $\mathbb{Z}^n$ with the lexicographic order as totally ordered abelian groups.

3. *Let δ be an arbitrary positive integer. We write $q = p^\delta$. Let $\mathcal{V}_i^q$ denote the image of the Frobenius homomorphism $F_\delta : \mathcal{V}_i \to \mathcal{V}_i$ of degree δ. For every $0 \le i \le n-1$ $\mathcal{V}_i$ is a finitely generated $\mathcal{V}_i^q$-module.*
4. *Let i be an arbitrary integer with $0 \le i \le n-1$. Let*

$$\hat{\mathcal{V}}_i = \varprojlim{}_\nu \mathcal{V}_i/M(\mathcal{V}_i)^\nu$$

be the completion of the local ring $\mathcal{V}_i$. By the canonical homomorphism $\mathcal{V}_i \to \hat{\mathcal{V}}_i$ we regard $\mathcal{V}_i$ as a subring of $\hat{\mathcal{V}}_i$. If elements $\overline{F} \in \mathcal{V}_i$ and $\overline{K} \in \hat{\mathcal{V}}_i$ satisfy $\overline{F} = \overline{K}^q$, then $\overline{K} \in \mathcal{V}_i$.

Proof. 1. It is easy to see that $\mathcal{U}$ is a valuation ring of the field $\mathcal{D}$. By Proposition 10.4.1 one has $\operatorname{rank} U_\nu = n$. For every integer μ with $\nu \le \mu$ and for every $0 \le i \le n$ we have $\mathfrak{p}_{\nu,i} = \mathfrak{p}_{\mu,i} \cap U_\nu$.

Let $\tilde{\mathfrak{p}}$ be an arbitrary prime ideal of $\mathcal{U}$. There exists a unique integer i with $0 \le i \le n$ and $\mathfrak{p}_{0,i} = \tilde{\mathfrak{p}} \cap U_0$. By the above we have $\mathfrak{p}_{\mu,i} = \tilde{\mathfrak{p}} \cap U_\mu$ for every non-negative integer μ. We have $\tilde{\mathfrak{p}} = \cup_{\mu=0}^\infty (\tilde{\mathfrak{p}} \cap U_\mu) = \cup_{\mu=0}^\infty \mathfrak{p}_{\mu,i}$. Thus, $\tilde{\mathfrak{p}}$ is uniquely determined by the number i. The morphism $\operatorname{Spec}(\mathcal{U}) \to \operatorname{Spec}(U_0)$ induced by the inclusion homomorphism $U_0 \to \mathcal{U}$ is injective. By Corollary 4.15.3 $\operatorname{Spec}(\mathcal{U}) \to \operatorname{Spec}(U_0)$ is surjective. Thus $\operatorname{Spec}(\mathcal{U}) \to \operatorname{Spec}(U_0)$ is bijective. One knows that $\operatorname{Spec}(\mathcal{U})$ is a finite set with $n+1$ elements. By Corollary 4.15.1 we have $\operatorname{rank}\mathcal{U} = n$. Since the morphism $\operatorname{Spec}(\mathcal{U}) \to \operatorname{Spec}(U_0)$ preserves inclusion relations, we obtain the last claim.

2. By Lemma 10.3.8 $V_{\nu,i}$ is a regular local ring of dimension one. Let $\Gamma_{\nu,i}$ denote the value group of the valuation ring $V_{\nu,i}$, and $D_{\nu,i}$ denote the quotient field of $V_{\nu,i}$. $\Gamma_{\nu,i} \cong D_{\nu,i}^*/V_{\nu,i}^* \cong \mathbb{Z}$.

By 1 we have $\mathcal{V}_i = \varinjlim{}_\nu V_{\nu,i}$. Let $\tilde{\Gamma}_i$ denote the value group of $\mathcal{V}_i$, and $\mathcal{D}_i$ denote the quotient field of $\mathcal{V}_i$. $\tilde{\Gamma}_i \cong \mathcal{D}_i^*/\mathcal{V}_i^* \cong \varinjlim{}_\nu D_{\nu,i}^*/\varinjlim{}_\nu V_{\nu,i}^* \cong \varinjlim{}_\nu (D_{\nu,i}^*/V_{\nu,i}^*) \cong \varinjlim{}_\nu \Gamma_{\nu,i}$. By Proposition 10.4.3 the homomorphism $\Gamma_{\nu,i} \to \Gamma_{\nu+1,i}$ is an isomorphism for every non-negative integer ν. Thus the homomorphism $\Gamma_{\nu,i} \to \tilde{\Gamma}_i$ is an isomorphism for every non-negative integer ν, and we have $\tilde{\Gamma}_i \cong \mathbb{Z}$.

By Proposition 4.25 $\mathcal{V}_i$ is a regular local ring of dimension one. The remaining claims follow from that the homomorphism $\Gamma_{\nu,i} \to \tilde{\Gamma}_i$ is an isomorphism.

3. We check that the field $\mathcal{D}$ and the sequence D_ν of subfields satisfy the three assumptions of Lemma 10.7. The first assumption is obviously satisfied. By Lemma 10.3.10 $[D_\nu : D_\nu^q] = q^n$ for every ν. Thus the second assumption is satisfied. By Proposition 10.4.2 and Lemma 10.6 also the third assumption is satisfied. By Lemma 10.7 we have $[\mathcal{D} : \mathcal{D}^q] = q^n$.

Let $\mathcal{D}_i$ denote the quotient field of $\mathcal{U}/\tilde{\mathfrak{p}}_i$ for every $0 \le i \le n$. $\mathcal{D}_0 = \mathcal{D}$. For every $0 \le i \le n-1$ the quotient field of the local ring $\mathcal{V}_i$ is $\mathcal{D}_i$, and the residue field of $\mathcal{V}_i$ is $\mathcal{D}_{i+1}$.

By applying Lemma 10.2.1 repeatedly as in the proof of Lemma 10.3.10, one knows that $[\mathcal{D}_i : \mathcal{D}_i^q]$ is finite for every $0 \le i \le n$. Besides,

$$[\mathcal{D}_n : \mathcal{D}_n^q] \le \frac{[\mathcal{D}_{n-1} : \mathcal{D}_{n-1}^q]}{q} \le \cdots \le \frac{[\mathcal{D}_{i+1} : \mathcal{D}_{i+1}^q]}{q^{n-i-1}} \le \frac{[\mathcal{D}_i : \mathcal{D}_i^q]}{q^{n-i}} \le \cdots \le \frac{[\mathcal{D}_0 : \mathcal{D}_0^q]}{q^n} \le \frac{q^n}{q^n} = 1$$

Since $[\mathcal{D}_n : \mathcal{D}_n^q] \geq 1$, we have

$$[\mathcal{D}_{i+1} : \mathcal{D}_{i+1}^q] = \frac{[\mathcal{D}_i : \mathcal{D}_i^q]}{q}$$

for every $0 \leq i \leq n-1$. By Lemma 10.2.2 one knows that $\mathcal{V}_i$ is a finitely generated $\mathcal{V}_i^q$-module.
4. We fix a number i. We write $\mathfrak{m} = M(\mathcal{V}_i^q)$ and $\mathcal{L} = \mathcal{V}_i/\mathcal{V}_i^q$. Note that $\mathcal{V}_i^q$ ($\cong \mathcal{V}_i$) is a noetherian local ring with the maximal ideal $\mathfrak{m}$. We have an exact sequence

$$0 \longrightarrow \mathcal{V}_i^q \longrightarrow \mathcal{V}_i \longrightarrow \mathcal{L} \longrightarrow 0$$

of finitely generated $\mathcal{V}_i^q$-modules. By Matsumura [7], p.168, (23.I), Theorem 54, the sequence

$$0 \longrightarrow \varprojlim{}_\nu \mathcal{V}_i^q/\mathfrak{m}^\nu\mathcal{V}_i^q \longrightarrow \varprojlim{}_\nu \mathcal{V}_i/\mathfrak{m}^\nu\mathcal{V}_i \longrightarrow \varprojlim{}_\nu \mathcal{L}/\mathfrak{m}^\nu\mathcal{L} \longrightarrow 0$$

is exact. By $\hat{\mathcal{V}}_i^q$ we denote the image of the Frobenius homomorphism $F_\delta : \hat{\mathcal{V}}_i \to \hat{\mathcal{V}}_i$. We have $\varprojlim{}_\nu \mathcal{V}_i^q/\mathfrak{m}^\nu\mathcal{V}_i^q \cong \hat{\mathcal{V}}_i^q$. Since $\mathfrak{m}^\nu\mathcal{V}_i = M(\mathcal{V}_i)^{\nu q}$, we have $\varprojlim{}_\nu \mathcal{V}_i/\mathfrak{m}^\nu\mathcal{V}_i \cong \hat{\mathcal{V}}_i$. Denoting $\hat{\mathcal{L}} = \varprojlim{}_\nu \mathcal{L}/\mathfrak{m}^\nu\mathcal{L}$, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{V}_i^q & \longrightarrow & \mathcal{V}_i & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \varphi & & \\
0 & \longrightarrow & \hat{\mathcal{V}}_i^q & \longrightarrow & \hat{\mathcal{V}}_i & \longrightarrow & \hat{\mathcal{L}} & \longrightarrow & 0
\end{array}$$

Two rows are exact. Thus our claim 4 is equivalent to that $\varphi : \mathcal{L} \to \hat{\mathcal{L}}$ is injective.
By Matsumura [7], p.172, (24.A), Theorem 56 (2)$\Leftrightarrow$(3), φ is injective. $\square$

11. Main results

The theorem below is the most essential result of this paper.

Theorem 11.1. *Let (B, W, P) be a framed space germ over k, and (h, g, t) be a Weierstrass representation over (B, W, P). Let $(Z, U, P) = \underline{(B, W, P)}$. Assume that the basic valuation ring U is of normal crossing type. Then, any very effective reduction sequence of (h, g, t) has finite length.*

Proof. Assume that there exists a very effective reduction sequence $\mathcal{S} = \{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \leq \nu < \infty\}$ of (h, g, t) with infinite length. We will deduce a contradiction.

Let $(B_0, W_0, P_0) = (B, W, P)$, and $(h_0, g_0, t_0) = (h, g, t)$. Let $\{(h_\nu, g_\nu, t_\nu) \mid 0 \leq \nu < \infty\}$ denote the transform Weierstrass representation sequence of (h, g, t) along $\mathcal{S}$. For every non-negative integer ν, we denote $(Z_\nu, U_\nu, P_\nu) = \underline{(B_\nu, W_\nu, P_\nu)}$, by F_ν we denote the quotient field of B_ν, by D_ν the quotient field of Z_ν, and $P_\nu = \{x_{\nu,1}, x_{\nu,2}, \ldots, x_{\nu,n}\}$. Let $\mathcal{Z} = \varinjlim{}_\nu Z_\nu$, $\mathcal{U} = \varinjlim{}_\nu U_\nu$, $\mathcal{D} = \varinjlim{}_\nu D_\nu$, and $\mathcal{F} = \varinjlim{}_\nu F_\nu$. For simplicity, we identify F_ν and the image of the canonical homomorphism $F_\nu \to \mathcal{F}$. Note that under this identification we have $h = h_\nu$, $g = g_\nu$ and $t_\nu = t + \sum_{\mu=1}^{\nu} H_\mu$ for every non-negative integer ν.

By Lemma 8.7.3 for every ν, (B_ν, W_ν, P_ν) has the basic valuation ring of normal crossing type, and in particular, $n = \dim Z_\nu = \operatorname{rank} U_\nu$. By Theorem 10.8.1 $\mathcal{U}$ is a valuation ring of the field $\mathcal{D}$, and $\operatorname{rank}\mathcal{U} = n$. By Corollary 4.19 $\mathcal{U} \cap D_\nu = U_\nu$ for every ν. Ring $\mathcal{Z}$ is a local ring, and $\mathcal{U}$ dominates $\mathcal{Z}$. Let

$$v : \mathcal{D} \to \Gamma \cup \{\infty\}$$

denote the valuation defined by $\mathcal{U}$. Let

$$\{0\} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_i \subset \mathfrak{p}_{i+1} \subset \cdots \subset \mathfrak{p}_n = M(\mathcal{U})$$

be the sequence of prime ideals of $\mathcal{U}$.

By Theorem 10.8.2 Γ is isomorphic to $\mathbb{Z}^n$ with the lexicographic order as totally ordered abelian groups. Fixing an appropriate isomorphism $\Gamma \cong \mathbb{Z}^n$, we identify Γ and $\mathbb{Z}^n$. For every integer i with $0 \leq i \leq n$, $\Delta_i = v(\mathcal{U}^*_{\mathfrak{p}_i})$ is an isolated subgroup of Γ, $\mathcal{U}^*_{\mathfrak{p}_i} = v^{-1}(\Delta_i)$, and $\Delta_i = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n \mid \alpha_j = 0 \text{ for every integer } j \text{ with } 1 \leq j \leq i.\}$ under the above identification.

We write

$$v(\Phi) = (v_1(\Phi), v_2(\Phi), \ldots, v_n(\Phi)) \in \mathbb{Z}^n$$

for $\Phi \in \mathcal{D}^*$. Each $v_i(\Phi)$ is an integer.

For every $0 \leq i \leq n-1$ we denote

$$\mathcal{V}_i = \mathcal{U}_{\mathfrak{p}_{i+1}}/\mathfrak{p}_i\mathcal{U}_{\mathfrak{p}_{i+1}}.$$

By Theorem 10.8.2 $\mathcal{V}_i$ is a regular local ring of dimension one. Regarding $\mathcal{V}_i$ as a valuation ring of the quotient field, by $\overline{v}_{i+1}$ we denote the valuation defined by $\mathcal{V}_i$. The value group of $\overline{v}_{i+1}$ is $\mathbb{Z}$. Fixing an integer i with $0 \leq i \leq n-1$, by $\overline{\Phi} \in \mathcal{V}_i$ we denote the image of an element $\Phi \in \mathcal{U}$ by the canonical homomorphism $\mathcal{U} \to \mathcal{V}_i$. If $\overline{\Phi} \neq 0$, then we have $v_{i+1}(\Phi) = \overline{v}_{i+1}(\overline{\Phi})$ by Lemma 4.16.

Let $m = \deg(g, t)$. Since $\mathcal{S}$ exists, we have $m \geq 1$. Let $m = lq$ denote the p-decomposition of m.

Since $\mathcal{S}$ is an effective reduction sequence with infinite length, we have three infinite sequences $\{(F_{\nu,1}, F_{\nu,2}, \ldots, F_{\nu,m}) \mid 0 \leq \nu < \infty\}$, $\{(b_{\nu,1}, b_{\nu,2}, \ldots, b_{\nu,n}) \mid 1 \leq \nu < \infty\}$, and $\{h_\nu \mid 1 \leq \nu < \infty\}$ satisfying the conditions in Proposition 9.11.

For simplicity, we write

$$\langle \nu \rangle = x_{\nu,1}^{b_{\nu,1}} x_{\nu,2}^{b_{\nu,2}} \cdots x_{\nu,n}^{b_{\nu,n}} = \prod_{i=1}^{n} x_{\nu,i}^{b_{\nu,i}} \in Z_\nu$$

for every positive integer ν. Besides, for every pair ν, μ of integers with $0 \leq \nu \leq \mu$ we write

$$\langle\langle \nu, \mu \rangle\rangle = \prod_{\lambda=\nu+1}^{\mu} \langle \lambda \rangle \in Z_\mu.$$

We understand $\langle\langle \nu, \nu \rangle\rangle = 1$ here. For $0 \leq \nu \leq \mu \leq \lambda$ we have $\langle\langle \nu, \lambda \rangle\rangle = \langle\langle \nu, \mu \rangle\rangle \langle\langle \mu, \lambda \rangle\rangle$.

By Proposition 9.11 (3.ν) we have $v(\langle \nu \rangle) > 0$ for every positive integer ν. Thus $v(\langle\langle 0, \nu \rangle\rangle) < v(\langle\langle 0, \nu+1 \rangle\rangle) = v(\langle\langle 0, \nu \rangle\rangle) + v(\langle \nu+1 \rangle)$ for every non-negative integer ν.

Since $\mathbb{Z}$ is discrete, by definition of the lexicographic order, we have two cases.

$\langle A \rangle$ $v_1(\langle\langle 0, \nu \rangle\rangle) \leq v_1(\langle\langle 0, \nu+1 \rangle\rangle)$ for every non-negative integer ν, and $\lim_{\nu \to +\infty} v_1(\langle\langle 0, \nu \rangle\rangle) = +\infty$.

$\langle B \rangle$ For some integer i with $1 \leq i \leq n-1$ and for some non-negative integer ν_0 the following two conditions are satisfied:

- $\langle B.1 \rangle$ $v_j(\langle\langle 0, \nu_0 \rangle\rangle) = v_j(\langle\langle 0, \nu \rangle\rangle)$ for every integer ν with $\nu \geq \nu_0$ and for every integer j with $1 \leq j \leq i$.
- $\langle B.2 \rangle$ $v_{i+1}(\langle\langle 0, \nu \rangle\rangle) \leq v_{i+1}(\langle\langle 0, \nu+1 \rangle\rangle)$ for every non-negative integer ν with $\nu \geq \nu_0$, and $\lim_{\nu \to +\infty} v_{i+1}(\langle\langle 0, \nu \rangle\rangle) = +\infty$.

In case $\langle A\rangle$, we put $i=0$ and $\nu_0=0$. In case $\langle B\rangle$, we choose integers i and ν_0 satisfying the conditions, and we fix them.

Consider the canonical homomorphism $\Phi\in\mathcal{U}\mapsto\overline{\Phi}\in\mathcal{V}_i$.

We consider the case $\langle B\rangle$. Let j be an arbitrary integer with $1\le j\le i$. If $\nu\ge\nu_0$, then by $\langle B.1\rangle$ we have $v_j(\langle\langle 0,\nu\rangle\rangle)=v_j(\langle\langle 0,\nu+1\rangle\rangle)=v_j(\langle\langle 0,\nu\rangle\rangle)+v_j(\langle\nu+1\rangle)$, and thus $v_j(\langle\nu+1\rangle)=0$. One knows that $v_1(\langle\nu\rangle)=v_2(\langle\nu\rangle)=\cdots=v_i(\langle\nu\rangle)=0$ if $\nu>\nu_0$. In particular, $\langle\nu\rangle\in\mathcal{U}^*_{\mathfrak{p}_i}$ and $\overline{\langle\nu\rangle}\ne 0$ if $\nu>\nu_0$.

In the case $\langle A\rangle$, the homomorphism $\mathcal{U}\to\mathcal{V}_i$ is injective, and $\Phi=\overline{\Phi}$. Thus $\overline{\langle\nu\rangle}\ne 0$ for $\nu>\nu_0$ also in this case.

In both cases

$$\overline{\langle\langle\nu_0,\nu\rangle\rangle}=\prod_{\mu=\nu_0+1}^{\nu}\overline{\langle\mu\rangle}\ne 0$$

for every integer ν with $\nu\ge\nu_0$.

Below we treat both cases together for a while.

We claim that $\overline{F}_{\nu_0,j}=0$ for every integer j satisfying $1\le j\le m$ and $j\not\equiv 0 \mod q$.

Assume otherwise. There exists an integer j_0 satisfying the following three conditions:

(a) $1\le j_0<m$, $j_0\not\equiv 0 \mod q$.
(b) If $1\le j<j_0$, $j\not\equiv 0 \mod q$, then $\overline{F}_{\nu_0,j}=0$.
(c) $\overline{F}_{\nu_0,j_0}\ne 0$.

By Corollary 9.13.1

$$\overline{F}_{\nu_0,j_0}=\overline{\langle\langle\nu_0,\nu\rangle\rangle}^{j_0}\overline{F}_{\nu,j_0}$$

for every integer ν with $\nu\ge\nu_0$. Since $F_{\nu,j_0}\in Z_\nu\subset\mathcal{Z}\subset\mathcal{U}$, $\overline{F}_{\nu,j_0}\in\mathcal{V}_i$ and $\overline{v}_{i+1}(\overline{F}_{\nu,j_0})\ge 0$. Thus

$$\overline{v}_{i+1}(\overline{F}_{\nu_0,j_0})\ge j_0\overline{v}_{i+1}(\overline{\langle\langle\nu_0,\nu\rangle\rangle})$$

for every integer ν with $\nu\ge\nu_0$. By (c) above the left-hand side of this equality is an integer independent of the integer ν. Consider the right-hand side. Since $\overline{\langle\langle\nu_0,\nu\rangle\rangle}\ne 0$, we have

$$\overline{v}_{i+1}(\overline{\langle\langle\nu_0,\nu\rangle\rangle})=v_{i+1}(\langle\langle\nu_0,\nu\rangle\rangle)=v_{i+1}(\langle\langle 0,\nu\rangle\rangle)-v_{i+1}(\langle\langle 0,\nu_0\rangle\rangle).$$

As $\nu\to+\infty$, $\overline{v}_{i+1}(\overline{\langle\langle\nu_0,\nu\rangle\rangle})\to+\infty$, since $v_{i+1}(\langle\langle 0,\nu\rangle\rangle)\to+\infty$. Thus, there is an integer ν satisfying $\nu\ge\nu_0$ and

$$\overline{v}_{i+1}(\overline{F}_{\nu_0,j_0})<j_0\overline{v}_{i+1}(\overline{\langle\langle\nu_0,\nu\rangle\rangle}),$$

which is a contradiction.

Applying Corollary 9.13.1 again, one has

$$\overline{\langle\langle\nu_0,\nu\rangle\rangle}^{j}\overline{F}_{\nu,j}=0$$

for every integer j satisfying $1\le j\le m$ and $j\not\equiv 0 \mod q$ and for every integer ν with $\nu\ge\nu_0$. Since $\mathcal{V}_i$ is a domain, and since $\overline{\langle\langle\nu_0,\nu\rangle\rangle}\ne 0$, we have $\overline{F}_{\nu,j}=0$.

One knows that $\overline{F}_{\nu,j}=0$ for every integer j satisfying $1\le j\le m$ and $j\not\equiv 0 \mod q$ and for every integer ν with $\nu\ge\nu_0$.

By the same argument applying Corollary 9.13.3 instead of Corollary 9.13.1, one knows

$$\overline{F}_{\nu,jq} = \binom{l}{j}\left(\frac{\overline{F}_{\nu,q}}{l}\right)^j$$

for every integer j with $2 \leq j \leq l$ and for every integer ν with $\nu \geq \nu_0$. Thus for $\nu \geq \nu_0$ we can write

$$T^m + \sum_{j=1}^{m} \overline{F}_{\nu,j} T^{m-j} = \left(T^q + \frac{\overline{F}_{\nu,q}}{l}\right)^l,$$

where T is an algebraically independent element. The problem has been reduced to considering only $\overline{F}_{\nu,q}$ for $\nu \geq \nu_0$.

By Corollary 9.13.2

$$\overline{F}_{\nu_0,q} = l\left(\sum_{\mu=\nu_0+1}^{\nu} \overline{h}_\mu \overline{\langle\langle \nu_0, \mu\rangle\rangle}\right)^q + \overline{\langle\langle \nu_0, \nu\rangle\rangle}^q \overline{F}_{\nu,q}$$

for every integer ν with $\nu \geq \nu_0$.

We consider the completion $\hat{\mathcal{V}}_i = \varprojlim_\alpha \mathcal{V}_i/M(\mathcal{V}_i)^\alpha$ of a regular local ring $\mathcal{V}_i$ of dimension one. Note that $\mathcal{V}_i$ is noetherian, and the canonical homomorphism $\mathcal{V}_i \to \hat{\mathcal{V}}_i$ is injective.

As $\nu \to +\infty$, $\overline{v}_{i+1}(\overline{\langle\langle \nu_0, \nu\rangle\rangle}) \to +\infty$. $\overline{F}_{\nu,q} \in \hat{\mathcal{V}}_i$ for every integer ν with $\nu \geq \nu_0$. Thus, as $\nu \to +\infty$, $\overline{\langle\langle \nu_0, \nu\rangle\rangle}^q \overline{F}_{\nu,q} \to 0$ in the ring $\hat{\mathcal{V}}_i$. Besides, an element

$$\overline{K}_{\nu_0} = \sum_{\mu=\nu_0+1}^{\infty} \overline{h}_\mu \overline{\langle\langle \nu_0, \mu\rangle\rangle} \in \hat{\mathcal{V}}_i$$

is defined, and it satisfies

$$\overline{F}_{\nu_0,q} = s^q \overline{K}_{\nu_0}^q,$$

where $s \in k$ is an element with $s \neq 0$ and $s^q = l$.

By Theorem 10.8.4

$$\overline{K}_{\nu_0} \in \mathcal{V}_i.$$

By Theorem 10.8.1 $\mathrm{Spec}(U_\nu) = \{\mathfrak{p}_\alpha \cap U_\nu \mid 0 \leq \alpha \leq n\}$ for every non-negative integer ν. By Lemma 8.7.3 for every ν, (B_ν, W_ν, P_ν) has the basic valuation ring of normal crossing type. By Lemma 8.7.2

$$(Z_\nu)_{(\mathfrak{p}_{i+1}\cap Z_\nu)}/(\mathfrak{p}_i \cap Z_\nu)(Z_\nu)_{(\mathfrak{p}_{i+1}\cap Z_\nu)} \cong (U_\nu)_{(\mathfrak{p}_{i+1}\cap U_\nu)}/(\mathfrak{p}_i \cap U_\nu)(U_\nu)_{(\mathfrak{p}_{i+1}\cap U_\nu)}$$

for every non-negative integer ν. Thus, one knows

$$\begin{aligned}\mathcal{V}_i &\cong \varinjlim_\nu \left[(U_\nu)_{(\mathfrak{p}_{i+1}\cap U_\nu)}/(\mathfrak{p}_i \cap U_\nu)(U_\nu)_{(\mathfrak{p}_{i+1}\cap U_\nu)}\right] \\ &\cong \varinjlim_\nu \left[(Z_\nu)_{(\mathfrak{p}_{i+1}\cap Z_\nu)}/(\mathfrak{p}_i \cap Z_\nu)(Z_\nu)_{(\mathfrak{p}_{i+1}\cap Z_\nu)}\right].\end{aligned}$$

By the above there exists an integer ν_1 satisfying $\nu_1 \geq \nu_0$ and

$$\overline{K}_{\nu_0} \in (Z_{\nu_1})_{(\mathfrak{p}_{i+1}\cap Z_{\nu_1})}/(\mathfrak{p}_i \cap Z_{\nu_1})(Z_{\nu_1})_{(\mathfrak{p}_{i+1}\cap Z_{\nu_1})}.$$

Similarly, defining an element

$$\overline{K}_{\nu_1} = \sum_{\mu=\nu_1+1}^{\infty} \overline{h}_\mu \overline{\langle\langle \nu_1, \mu\rangle\rangle} \in \hat{\mathcal{V}}_i,$$

one knows $\overline{K}_{\nu_1} \in \mathcal{V}_i$ and $\overline{F}_{\nu_1,q} = s^q \overline{K}_{\nu_1}^q$.

$$\overline{K}_{\nu_1} = \frac{\overline{K}_{\nu_0} - \sum_{\mu=\nu_0+1}^{\nu_1} \overline{h}_\mu \overline{\langle\langle \nu_0, \mu \rangle\rangle}}{\overline{\langle\langle \nu_0, \nu_1 \rangle\rangle}}.$$

By the right-hand side, $\overline{K}_{\nu_1}$ belongs to the quotient field of $Z_{\nu_1}/\mathfrak{p}_i \cap Z_{\nu_1}$.

Let $\nu_2 = \nu_1 + 1$. $\nu_2 > \nu_0$. $\overline{K}_{\nu_1}$ belongs to the quotient field of $Z_{\nu_2}/\mathfrak{p}_i \cap Z_{\nu_2}$. Besides, there exist an element $w_{\nu_2} \in Z_{\nu_2}^*$ and non-negative integers $c_{\nu_2,1}, c_{\nu_2,2}, \dots,$ $c_{\nu_2,n}$ satisfying

$$\langle\langle 0, \nu_2 - 1 \rangle\rangle = \prod_{\mu=1}^{\nu_2-1} \langle \mu \rangle = w_{\nu_2} x_{\nu_2,1}^{c_{\nu_2,1}} x_{\nu_2,2}^{c_{\nu_2,2}} \cdots x_{\nu_2,n}^{c_{\nu_2,n}}.$$

The Weierstrass representation (h, g, t_{ν_2-1}) over $(B_{\nu_2}, W_{\nu_2}, P_{\nu_2})$ satisfies the Abhyankar condition, (h, g, t_{ν_2-1}) is reductive, and we can write

$$g = t_{\nu_2-1}^m + \sum_{j=1}^m \langle\langle 0, \nu_2 - 1 \rangle\rangle^j F_{\nu_2-1,j} t_{\nu_2-1}^{m-j}.$$

Let $d_{\nu_2,1}, d_{\nu_2,2}, \dots, d_{\nu_2,n}$ denote the characteristic numbers of (h, g, t_{ν_2-1}). By Lemma 9.5.9 $d_{\nu_2,\alpha} = c_{\nu_2,\alpha} + b_{\nu_2,\alpha}$ for every $1 \le \alpha \le n$. By definition of characteristic numbers

$$Z_{\nu_2} \ni \frac{\langle\langle 0, \nu_2 - 1 \rangle\rangle^q F_{\nu_2-1,q}}{\left(x_{\nu_2,1}^{d_{\nu_2,1}} x_{\nu_2,2}^{d_{\nu_2,2}} \cdots x_{\nu_2,n}^{d_{\nu_2,n}} \right)^q} = \frac{w_{\nu_2}^q F_{\nu_2-1,q}}{\langle \nu_2 \rangle^q}.$$

Thus, we have

$$Z_{\nu_2}/\mathfrak{p}_i \cap Z_{\nu_2} \ni \overline{\left(\frac{w_{\nu_2}^q F_{\nu_2-1,q}}{\langle \nu_2 \rangle^q} \right)} = \left(\frac{\overline{w}_{\nu_2} s \overline{K}_{\nu_1}}{\overline{\langle \nu_2 \rangle}} \right)^q.$$

$\overline{\langle \nu_2 \rangle} \neq 0$, $\overline{\langle \nu_2 \rangle} \in Z_{\nu_2}/\mathfrak{p}_i \cap Z_{\nu_2}$, and $\overline{w}_{\nu_2} \in Z_{\nu_2}/\mathfrak{p}_i \cap Z_{\nu_2}$. Thus, the element $\overline{w}_{\nu_2} s \overline{K}_{\nu_1} / \overline{\langle \nu_2 \rangle}$ belongs to the quotient field of $Z_{\nu_2}/\mathfrak{p}_i \cap Z_{\nu_2}$, and it is integral over $Z_{\nu_2}/\mathfrak{p}_i \cap Z_{\nu_2}$. On the other hand, by Lemma 8.7.1 $Z_{\nu_2}/\mathfrak{p}_i \cap Z_{\nu_2}$ is a normal ring. We conclude that the element $\overline{w}_{\nu_2} s \overline{K}_{\nu_1} / \overline{\langle \nu_2 \rangle}$ belongs to $Z_{\nu_2}/\mathfrak{p}_i \cap Z_{\nu_2}$. One knows that the element $\overline{(w_{\nu_2}^q F_{\nu_2-1,q}/\langle \nu_2 \rangle^q)}$ is the q-th power modulo $\mathfrak{p}_i$.

By Proposition 9.11 $(4.\nu_2)$ we have

$$\frac{H_{\nu_2}}{x_{\nu_2,1}^{d_{\nu_2,1}} x_{\nu_2,2}^{d_{\nu_2,2}} \cdots x_{\nu_2,n}^{d_{\nu_2,n}}} = w_{\nu_2} h_{\nu_2}.$$

Since H_{ν_2} is a very effective reducing element, we have

$$\overline{\left(\frac{w_{\nu_2}^q F_{\nu_2-1,q}}{\langle \nu_2 \rangle^q} \right)} = l(\overline{w_{\nu_2} h_{\nu_2}})^q,$$

and

$$\left(\frac{\overline{w}_{\nu_2} \overline{K}_{\nu_1}}{\overline{\langle \nu_2 \rangle}} \right)^q = (\overline{w_{\nu_2} h_{\nu_2}})^q.$$

Thus, we obtain $\overline{K}_{\nu_1} = \overline{\langle \nu_2 \rangle h_{\nu_2}}$ and $\overline{F}_{\nu_1,q} = l(\overline{\langle \nu_2 \rangle h_{\nu_2}})^q$.

On the other hand, by Corollary 9.13.2

$$\overline{F}_{\nu_1,q} = l(\overline{\langle \nu_2 \rangle h_{\nu_2}})^q + \overline{\langle \nu_2 \rangle}^q \overline{F}_{\nu_2,q}.$$

We obtain $\overline{\langle\nu_2\rangle}^q\overline{F}_{\nu_2,q} = 0$ and $\overline{F}_{\nu_2,q} = 0$. By the above results we conclude that $\overline{F}_{\nu_2,j} = 0$ for every $1 \le j \le m$.

We consider the case $\langle A\rangle$. In this case $F_{\nu_2,j} = 0$ for every $1 \le j \le m$. Thus $g = t_{\nu_2}^m$ and $\mathrm{w}(g, t_{\nu_2}) = \infty$. On the other hand, since $\mathcal{S}$ is a reduction sequence with infinite length, we have $\mathrm{w}(g, t_{\nu_2}) < \infty$, which is a contradiction.

We proceed to case $\langle B\rangle$. By definition of characteristic numbers,

$$\frac{F_{\nu_2,j}}{\langle\nu_2+1\rangle^j} \in Z_{\nu_2+1}$$

for every $1 \le j \le m$. By Proposition 9.11($2.\nu_2+1$) $\mathrm{w}(F_{\nu_2,1}, F_{\nu_2,2}, \dots, F_{\nu_2,m}) = \sum_{\alpha=1}^n b_{\nu_2+1,\alpha}$. By definition of the weight, there is an integer j satisfying $1 \le j \le m$ and $\mathrm{ord}(F_{\nu_2,j})/j = \sum_{\alpha=1}^n b_{\nu_2+1,\alpha}$. For this number j, we have $\mathrm{ord}(F_{\nu_2,j}/\langle\nu_2+1\rangle^j) = \mathrm{ord}(F_{\nu_2,j}) - j\sum_{\alpha=1}^n b_{\nu_2+1,\alpha} = 0$. Thus, $F_{\nu_2,j}/\langle\nu_2+1\rangle^j \in Z_{\nu_2+1}^*$, and $v_\alpha(F_{\nu_2,j}) = jv_\alpha(\langle\nu_2+1\rangle)$ for every $1 \le \alpha \le n$.

On the other hand, $\overline{F}_{\nu_2,j} = 0$, and $F_{\nu_2,j} \in \mathfrak{p}_i \cap Z_{\nu_2+1}$. Thus, there is an integer α satisfying $1 \le \alpha \le i$ and $v_\alpha(F_{\nu_2,j}) \ne 0$. For this number α we have $v_\alpha(\langle\nu_2+1\rangle) \ne 0$. Note that $\nu_2+1 = \nu_1+2 > \nu_0$. We obtain a contradiction, since for every integer ν with $\nu > \nu_0$ we have $v_1(\langle\nu\rangle) = v_2(\langle\nu\rangle) = \cdots = v_i(\langle\nu\rangle) = 0$. □

The lemma below solves Problem($\mathcal{C}_0$) we gave at the last part of Section 7 for the admissible class $\mathcal{C}_0$. Recall the statement $RW(B,W,P)$ we gave before Lemma 5.12 and the statement $AN(B,W,P)$ we gave before Theorem 7.4.

Lemma 11.2. *Let $\mathcal{C}_0$ be the collection of framed space germs (B,W,P) such that the basic framed space germ $(A,V,P) = \underline{(B,W,P)}$ satisfies $\dim A = \operatorname{rank} V$. Let r be a positive integer. Assume that $RW(B,W,P)$ is true for every framed space germ (B,W,P) such that $(B,W,P) \in \mathcal{C}_0$ and $\sharp P \le r$. Then, $AN(B,W,P)$ is true for every framed space germ (B,W,P) such that $(B,W,P) \in \mathcal{C}_0$ and $\sharp P \le r$.*

Proof. By Corollary 8.3 $\mathcal{C}_0$ is an admissible class.

Let (B,W,P) be a framed space germ over k. We assume that $(B,W,P) \in \mathcal{C}_0$ and $\sharp P \le r$. Let (h,g,t) be a Weierstrass representation over (B,W,P).

By Corollary 8.6 there exists an IAMT (B',W',P') with a normal crossing of (B,W,P) such that (B',W',P') has the basic valuation ring of normal crossing type. Let (h',g',t') denote the pull-back of (h,g,t) by the associated morphism $(B,W,P) \to (B',W',P')$. If $\mathrm{w}(g', t') = \infty$, then putting $H' = 0 \in k[[P']]$ one knows that $AN(B,W,P)$ is true. We assume $\mathrm{w}(g', t') < \infty$ below. Since $\mathcal{C}_0$ is an admissible class, we have $(B',W',P') \in \mathcal{C}_0$. We have $\sharp P' = \sharp P \le r$.

By Lemma 9.10 there exists a very effective reduction sequence of (h',g',t'). By Theorem 11.1 any very effective reduction sequence of (h',g',t') has finite length. Thus, there exists a positive integer ω and a very effective reduction sequence $\{(B_\nu, W_\nu, P_\nu, H_\nu) \mid 1 \le \nu \le \omega\}$ with length ω. Let $(B'',W'',P'') = (B_\omega, W_\omega, P_\omega)$ and (h'',g'',t'') denote the pull-back of (h',g',t') by the associated morphism $(B',W',P') \to (B'',W'',P'')$. Obviously (B'',W'',P'') is an IAMT with a normal crossing of (B,W,P), and (h'',g'',t'') is the pull-back of (h,g,t) by the associated morphism $(B,W,P) \to (B'',W'',P'')$. Let $H'' = H_1 + H_2 + \cdots + H_\omega \in k[[P'']]$. By definition of reduction sequences (Definition 9.1.), if $\mathrm{w}(g'', t''+H'') < \infty$, then $(h'',g'',t''+H'')$ satisfies the Abhyankar condition, and moreover $(h'',g'',t''+H'')$ is not reductive.

Thus $AN(B,W,P)$ is true. □

Corollary 11.3. *The statement $RW(B,W,Q)$ is true for every framed space germ (B,W,Q) such that $(B,W,Q) \in \mathcal{C}_0$.*

Proof. Let r be an arbitrary positive integer. By induction on r we will show that $RW(B,W,Q)$ is true for every framed space germ (B,W,Q) such that $(B,W,Q) \in \mathcal{C}_0$ and $\sharp Q \leq r$.

If $r = 1$, then by Lemma 5.14 the claim is true.

Assume that the claim is true for r. By Lemma 11.2 and by Theorem 7.4 the claim is true also for $r+1$. $\square$

Our main result below has already cited in the introduction as Theorem 1.1.

Theorem 11.4. *Let (B,W) be a space germ over k. Assume that the valuation ring W is of iterated divisor type. Let $f_1, f_2, \ldots, f_\ell$ be a finite number of arbitrary non-zero elements in the ring B. Then, there exists an iterated analytic monoidal transform (B',W') of (B,W) such that the product $f_1'f_2'\cdots f_\ell' \in B'$ has a normal crossing, and the ideal in B' generated by any non-empty subset of $\{f_1', f_2', \ldots, f_\ell'\}$ is a principal ideal, where for every $1 \leq i \leq \ell$, $f_i' \in B'$ denotes the image of $f_i \in B$ by the associated homomorphism $B \to B'$.*

Proof. By definition, W is of iterated divisor type, if and only if, $\dim B = \operatorname{rank} W$.

Let Q be a parameter system of B. Our theorem follows from the statement $RS(B,W,Q)$ we gave after Lemma 5.11. By Corollary 5.13 $RS(B,W,Q)$ and $RW(B,W,Q)$ are equivalent. By Corollary 11.3 $RW(B,W,Q)$ is true. Thus $RS(B,W,Q)$ is true, and our theorem follows. $\square$

Problem: Remove the assumption that W is of iterated divisor type from Theorem 11.4.

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