Differential calculus in noncommutative algebraic geometry II.
D-calculus in the braided case.
The localization of quantized enveloping algebras.

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(A preliminary version)

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Contents:

Introduction.

This paper is a continuation of [LR1]. One of our main purposes here is to introduce a noncommutative D-calculus (i.e. calculus and differential operators on noncommutative 'spaces') rich enough to obtain a quantized version of Bernstein-Beilinson localization construction and to initiate D-module theory related to systems of q-differential equations.

In [LR1] we studied D-calculus over noncommutative associative algebras (and in abelian categories) without any additional structure. The D-calculus on associative algebras, interesting on its own right, involves most of geometric ideas and (prototypes of) facts needed here. It requires, however, two more steps to obtain an adequate version of differential operators on 'quantized spaces'. The reason is that 'quantized spaces' live in certain, naturally related to them, monoidal categories. A choice of a quasi-symmetry (=braiding) β in any monoidal category $\mathcal{C}$ determines calculus and differential operators on 'spaces' in this monoidal category. Thus, a quantized enveloping algebra $U_q(\mathfrak{g})$ is an algebra in the monoidal category of $\mathbb{Z}^r$-graded k-modules, where r is the rank of the Lie

algebra $\mathfrak{g}$, and k is a base ring or field (say $\mathbb{Q}(q)$, or $\mathbb{Z}[t,t^{-1}]$). And there is a natural choice of a braiding in $\mathfrak{gr}_{\mathbf{Z}^r}k - mod$ determined by the Cartan matrix of the Lie algebra $\mathfrak{g}$.

In the Section 1.1 of [LR1] we have presented, using the ring-theoretical language some of the main facts of the work, of this part too. (We suggest the reader to take a look at this Section.) Now we shall do something complementary: in order to explain the mentioned above 'two steps', we will sketch here the main 'geometrical' ideas.

Recall in a few words our approach to D-calculus in [LR1]. First we identify 'spaces' with categories (of quasi-coherent sheaves). For instance, the affine scheme corresponding to an algebra R is identified with the category $R - mod$ of left R-modules. Then we single out topologizing coreflective subcategories for the role of subschemes in the noncommutative setting and develop some notions of algebraic (or rather differential) geometry such as intersection of subschemes, formal neighborhoods of a subscheme etc.. We define what is the product of two 'spaces'. If they are (presented by) resp. categories $\mathcal{A}$ and $\mathcal{B}$, then the product, $\mathcal{A} \times \mathcal{B}$, is the category $\mathfrak{Hom}(\mathcal{A}, \mathcal{B})$ of 'continuous' functors from $\mathcal{A}$ to $\mathcal{B}$ ('continuous' means having left adjoint). The bridge between these notions and differential operators is *the diagonal*. Once the diagonal is chosen, we obtain the notion of differential operators by applying X2general 'calculus' to the diagonal: differential operators of order $\leq n$ are elements of the n plus first neighborhood of the diagonal.

On the other hand, different choices of a diagonal lead to different notions of differential operators.

In [LR1] the diagonal of $\mathcal{A} \times \mathcal{A}$ is defined as the minimal subscheme Δ of this 'space' containing the identical functor. It is the most natural choice, if no additional structure is involved.

Given a class Ξ of continuous functors from $\mathcal{A}$ to $\mathcal{A}$, we take as a new diagonal the minimal subscheme, Δ_Ξ, of $\mathcal{A} \times \mathcal{A} = \mathfrak{End}(\mathcal{A})$ containing all objects (functors) of Ξ.

0.1. Example. Let $\mathcal{A}$ be the category of quasi-coherent sheaves on a scheme $\mathbf{X} = (X, \mathcal{O})$. Take as Ξ' a set of invertible sheaves such that every invertible sheaf on $\mathbf{X}$ is isomoprhic to a sheaf from Ξ'. And let Ξ denote the set of auto-equivalences $\{L \otimes_{\mathcal{O}} \mid L \in \Xi'\}$. The differential operators corresponding to the diagonal Δ_Ξ are *twisted differential operators* in the conventional sense. ∎

Clearly any subscheme of $\mathcal{A} \times \mathcal{A}$ is Δ_Ξ for an appropriate choice of Ξ. And there are canonical choices.

0.2. Example. Let $\mathcal{C}^\sim$ be a monoidal category, and let R be an algebra in $\mathcal{C}^\sim$ (these and other notions of categorical algebra we need are recalled in Section 1.0). Any choice of a quasi-symmetry (otherwise called braiding) β determines an embedding of the category $\mathcal{C}^\sim$ into the category of continuous endofunctors of the category $R - mod$ of left R-modules. This embedding assigns to any object X of $\mathcal{C}^\sim$ the functor $X \otimes_\beta : R - mod \longrightarrow R - mod$ of tensoring by X (the braiding β appeares in the definition of the action of R on $X \otimes M$). We denote by Δ_β the diagonal Δ_Ξ, where Ξ is the image of the category $\mathcal{C}^\sim$ under the embedding $\mathcal{C}^\sim \longrightarrow \mathfrak{End}(\mathfrak{R} - \mathfrak{mod})$, $X \mapsto X \otimes$. ∎

The diagonal Δ_β is one of the principal characters of this work. And our first step towards a 'right' notion of differential operators is the replacing the 'minimal' diagonal Δ

by the diagonal Δ_β and recovering (following the pattern [LR1]) the corresponding to this choice β-differential calculus.

The second (and the last) step is the taking into consideration certain natural group actions which are explained below.

There are two groups attached to any monoidal category:

— The *Picard group* $Pic(\mathcal{C}^{\tilde{}})$ of isomorphy classes of invertible objects of the monoidal category $\mathcal{C}^{\tilde{}}$. Recall that an object P of $\mathcal{C}^{\tilde{}}$ is *invertible* if the functor $P\otimes : \mathcal{C}^{\tilde{}} \longrightarrow \mathcal{C}^{\tilde{}}$ is an auto-equivalence.

— The *fundamental group*, $\pi_1(\mathcal{C}^{\tilde{}})$, of the monoidal category $\mathcal{C}^{\tilde{}}$ which is by definition the group of automorphisms of the identical monoidal functor (cf. 8.3). Note that our monoidal category $\mathcal{C}^{\tilde{}}$ is naturally realized as a subcategory of the monoidal category of representations of $\pi_1(\mathcal{C}^{\tilde{}})$.

Suppose a symmetry of the monoidal category $\mathcal{C}^{\tilde{}}$ is fixed. Then every braiding of $\mathcal{C}^{\tilde{}}$ determines a group homomorphism from $Pic(\mathcal{C}^{\tilde{}})$ to the fundamental group $\pi_1(\mathcal{C}^{\tilde{}})$. This homomorphism allows to regard $\mathcal{C}^{\tilde{}}$ as a monoidal subcategory of the category of representations of $Pic(\mathcal{C}^{\tilde{}})$. In particular, for any algebra R in $\mathcal{C}^{\tilde{}}$, the category $R-mod$ is naturally embedded into the category $R\#Pic(\mathcal{C}^{\tilde{}})-mod$ (the crossed products in monoidal categories are defined in Section 5). We take this embedding and consider the diagonal in the category $R\#Pic(\mathcal{C}^{\tilde{}})-mod$. In practical terms, this choice of the diagonal means that the action of of each element of $Pic(\mathcal{C}^{\tilde{}})$ becomes a differential operator. Note that, in the classical (i.e. commutative) situation, the only invertible differential operators are multiplications by invertible elements of the algebra R.

Although the main examples of this work 'live' in categories of $\mathbb{Z}^r$-graded modules, it is appropriate to develop generalities in a natural for the differential calculus setting, i.e. in monoidal categories.

In Section 1, after a short recollection of some necessary facts on monoidal categories and some basics of linear algebra (module theory) in a convenient for us form, we outline a differential calculus and introduce differential operators in abelian monoidal categories following the pattern of [LR1].

In Section 2 we extend to the algebras and modules in monoidal categories localization theorems of [LR1].

In Section 3, we describe the algebra of differential operators on a 'symmetric affine space' which is the algebra of differential operators of a skew polynomial algebra R determined by a matrix $\mathbf{q} = (q_{ij} \mid 1 \leq i,j \leq r)$ with invertible entrees. Recall that R is a k-algebra generated by elements x_i, $1 \leq i \leq r$, subject to the relations

$$x_i x_j = q_{ij} x_j x_i$$

for all i,j. 'Symmetric' means that the matrix $\mathbf{q}$ defines a symmetry in the monoidal category of $\mathbb{Z}^r$-graded modules; or, equivalently, $q_{ij}q_{ji} = 1$ for all i,j. The algebra R is regarded as a 'commutative' algebra in the symmetric monoidal category of $\mathbb{Z}^r$-graded modules. We call the algebra of differential operators on symmetric affine space defined by the matrix $\mathbf{q}$ *the algebra of* $\mathbf{q}$*-differential operators.*

We show that if k is a field of characteristic zero, the algebra of $\mathbf{q}$-differential operators is generated by R (i.e. by operators of multiplication by elements of R) and by $\mathbf{q}$-derivations.

In Section 4 we define (the algebra of functions on) an 'affine space' in the case when the matrix $\mathbf{q}$ determines only a quasi-symmetry. In other words, we do not require the relations $q_{ij}q_{ji} = 1$. This notion involves naturally that of *a Weyl algebra of a ring* (in particular, of an 'affine space') which is, in general, a proper subalgebra of the algebra of differential operators on this ring. We study the simplest, but already curious, example – the quantum line – the Weyl algebra coincides with the algebra of q-differential operators.

In Section 5, we define Hopf algebras in a quasi-symmetric category and discuss some of relevant examples.

In Section 6, we study Hopf actions and crossed products in monoidal categories. A special case of this construction is the 'affine base space' for any (quantized) enveloping algebra of a reductive or Kac-Moody Lie algebra.

In Section 7, we construct, in an arbitrary monoidal category, *a Weyl algebra* associated with a bilinear form and a quasi-symmetry β. The Weyl algebras of Section 3 are special cases of this construction.

The Weyl algebras happen to be β-Hopf algebras. Our goal is to 'extend' them naturally to σ-Hopf algebras for a given symmetry σ. This is done in the next two sections.

Section 8 contains facts about relations between quasi-symmetries and the Picard group (– the group of isomorphy classes of invertible objects) of our monoidal category.

In Section 9 we obtain a family of Hopf algebras with a quasi-symmetry playing the role of a parameter. The quantized enveloping algebras by Drinfeld and Jimbo are particular cases of this construction.

Note that already at this early stage the generality of our approach pays back: one of the special cases of the construction are quantized enveloping algebras of Lie superalgebras.

In Section 10 we introduce 'differential calculus with actions' (which incorporates the graded and, therefore, nongraded D-calculus) on affine, quasi-affine, and projective noncommutative spaces. After defining the base affine space and the flag variety of a quantized enveloping algebra, we conclude with a short presentation of a quantized version of the Beilinson-Bernstein localization construction. The latter realizes the quantized enveloping algebra $U_q(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ as an algebra of differential operators on the base affine space of $\mathfrak{g}$. As in the classical case, this realization provides canonical algebra homomorphisms from $U_q(\mathfrak{g})$ to the algebras of twisted differential operators on the flag variety of $U_q(\mathfrak{g})$. In the classical case, we recover the conventional Beilinson-Bernstein localization construction.

Here we outline only some of its general properties. A more detailed study is in [LR3].

In 'Complementary facts', we explain what are twisted differential operators corresponding to integer and to arbitrary weights, and give a ring-theoretic construction of Hopf algebras corresponding to skew derivations.

Refering to the first part of this work, [LR1], we shall write I.reference in [LR1].

Part I. Differential calculus in the graded case.

1.0. Preliminaries on monoidal categories. Consider *categories with multiplication*, i.e. pairs $(\mathcal{C}, \otimes)$, where $\mathcal{C}$ is a category and $\otimes$ is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$. We define a morphism from $(\mathcal{C}, \otimes)$ to $(\mathcal{C}', \otimes')$ as a pair (F, f), where F is a functor $\mathcal{C} \longrightarrow \mathcal{C}'$ and f is a functor morphism $f = \{f_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y)\}$. The composition is defined naturally.

1.0.1. Strict monoidal categories. *A strict monoidal category* is a category with multiplication $(\mathcal{C}, \otimes)$ such that,

(i) For any objects X, Y, Z of $\mathcal{C}$, $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$.

(ii) There is an object $\mathbf{1}$ of $\mathcal{C}$ such that, for any $X \in Ob\mathcal{C}$, $\mathbf{1} \otimes X = X = X \otimes \mathbf{1}$.

Note that the object $\mathbf{1}$ of the condition (ii) is defined uniquely and is called *the identical object of* $\mathcal{C}$.

1.0.1.1. Example. Let $\mathcal{A}$ be any category equivalent to a small category. Then the pair $(End\mathcal{A}, \circ)$, where $\circ$ denotes the composition of endofunctors, is a strict monoidal category with $\mathbf{1} = Id_{\mathcal{A}}$. ∎

1.0.2. Strict morphisms. A morphism (F, f) between categories with multiplication is called *strict* if f is identical. In other words, a strict morphism from $(\mathcal{C}, \otimes)$ to $(\mathcal{C}', \otimes')$ is any functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ such that, for any $X, Y \in \mathcal{C}$, $F(X \otimes Y) = F(X) \otimes F(Y)$.

1.0.2.1. Lemma. *The following conditions on a category with multiplication $(\mathcal{C}, \otimes)$ are equivalent:*

(a) $(\mathcal{C}, \otimes)$ is a strict monoidal category.

(b) The canonical functors

$$\mathfrak{L} : \mathcal{C} \longrightarrow End\mathcal{C},\ X \mapsto X\otimes, \quad and \quad \mathfrak{R} : \mathcal{C} \longrightarrow End\mathcal{C},\ X \mapsto \ \otimes X, \tag{1}$$

are strict and the image of each of them contains the identical functor.

Proof is left to a reader. ∎

1.0.3. Monoidal categories. Consider now categories with multiplication $(\mathcal{C}, \otimes)$ together with a morphism $(\mathfrak{L}, a)$ to $(End\mathcal{C}, \circ)$ such that a is an isomorphism. Here $\mathfrak{L}$ is the functor of 'tensoring from the left' (cf. (1)). In other words, we are considering triples $(\mathcal{C}, \otimes, a)$, where $a = \{a_{X,Y,Z} : X \otimes (Y \otimes Z) \longrightarrow (X \otimes Y) \otimes Z)$ is a functorial isomorphism. Morphisms from $(\mathcal{C}, \otimes, a)$ to $(\mathcal{C}', \otimes', a')$ are those morphisms $(F, f) : (\mathcal{C}, \otimes) \longrightarrow (\mathcal{C}', \otimes')$ for which the following diagram commute for all $X, Y, Z \in Ob\mathcal{C}$:

$$\begin{array}{ccccc} F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{id\otimes f} & F(X) \otimes F(Y \otimes Z) & \xrightarrow{f} & F(X \otimes (Y \otimes Z)) \\ a' \downarrow & & & & \downarrow Fa \\ (F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{f\otimes id} & F(X \otimes Y) \otimes F(Z) & \xrightarrow{f} & F((X \otimes Y) \otimes Z) \end{array} \tag{1}$$

Given a triple $(\mathcal{C}, \otimes, a)$, we call a *an associativity constraint* if $(\mathfrak{L}, a)$ is a morphism from $(\mathcal{C}, \otimes, a)$ to $(End\mathcal{C}, \circ, id)$.

One can see that being an associativity constraint is equivalent to the commutativity of the so called 'pentagon diagram' which is nothing else, but the diagram (1) for $(\mathfrak{L}, a)$ with the left vertical arrow omitted, since it is identical in this case.

1.0.3.1. Identity objects. An identity object of a triple $(C, \otimes, a)$ is an object $\mathbf{1}$ of C together with functor isomorphisms

$$\lambda : \mathbf{1}\otimes \longrightarrow id_{\mathcal{C}},\ \lambda_X : \mathbf{1}\otimes X \longrightarrow X,\ \text{and}\ \rho : \otimes\mathbf{1} \longrightarrow id_{\mathcal{C}},\ \rho_X : X \otimes \mathbf{1} \longrightarrow X, \qquad (2)$$

such that the diagram

$$\begin{array}{ccc} X \otimes (\mathbf{1} \otimes Y) & \xrightarrow{a} & (X \otimes \mathbf{1}) \otimes Y \\ & \searrow \quad \swarrow & \\ & X \otimes Y & \end{array} \qquad (3)$$

is commutative for all X, Y.

1.0.3.1.1. Note. The existence of an identity object implies among other things that the functor

$$\mathfrak{L}, a) : (C, \otimes, a) \longrightarrow (EndC, \circ, id),\ X \mapsto X\otimes$$

is faithful. ∎

1.0.3.2. Monoidal categories. *A monoidal category* is a data $(\mathcal{C}, \otimes, a, \mathbf{1}, \lambda, \rho)$, where a is an associativity constraint and $(\mathbf{1}, \lambda, \rho)$ is an identity object.

A *monoidal functor* $F^{\tilde{}}$ from a monoidal category $\mathcal{B}^{\tilde{}} = (\mathcal{B}, \otimes, a, \mathbf{1})$ to a monoidal category $\mathcal{C}^{\tilde{}} = (\mathcal{C}, \otimes, a', \mathbf{1}')$ is a triple $(F, f; \phi)$, where (F, f) is a morphism $(\mathcal{B}, \otimes, a) \longrightarrow (\mathcal{C}, \otimes', a')$ and ϕ is an isomorphism $F(\mathbf{1}) \longrightarrow \mathbf{1}'$ such that the following diagrams commute for all $X, Y, Z \in Ob\mathcal{B}$.

$$\begin{array}{ccc} F(\mathbf{1}) \otimes F(X) & \xrightarrow{f} & F(\mathbf{1} \otimes X) \\ {\scriptstyle \phi \otimes id}\downarrow & & \downarrow{\scriptstyle F\lambda} \\ F(\mathbf{1}') \otimes F(X) & \xrightarrow{\lambda'} & F(X) \end{array} \qquad \begin{array}{ccc} F(X) \otimes F(\mathbf{1}) & \xrightarrow{id\otimes f} & F(X \otimes \mathbf{1}) \\ {\scriptstyle id \otimes \phi}\downarrow & & \downarrow{\scriptstyle F\rho} \\ F(X) \otimes \mathbf{1}' & \xrightarrow{\rho'} & F(X) \end{array} \qquad (4)$$

1.0.4. Examples of monoidal categories.

1.0.4.1. A standart example is $k - mod^{\tilde{}} = (k - mod, \otimes_k, a, k)$ for a commutative ring k with the usual associativity constraint. ∎

1.0.4.2. The category of graded modules $\mathfrak{gr}_\Gamma k - mod^{\tilde{}} = (\mathfrak{gr}_\Gamma k - mod^{\tilde{}}, \otimes, k)$, where Γ is a commutative (semi)group, k a commutative Γ-graded ring, and $\otimes$ is a graded tensor product over k. The simplest nontrivial case is $\Gamma = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, and k is a field. Then the category $\mathfrak{gr}_\Gamma k - mod^{\tilde{}}$ is called the (tensor) *category of super vector spaces.* In this work, we are interested mostly in the case $\Gamma = \mathbb{Z}^r$ - the free abelian group of a finite rank - and k is a ring between $\mathbb{Z}[t, t^{-1}]$ and $\mathbb{Q}(t)$. ∎

1.0.4.3. The category of representations of a bialgebra. Recall that a bialgebra over a commutative ring k is a triple (δ, H, m), where (H, m) is a k-algebra, (δ, H) is a k-coalgebra such that the comultiplication $\delta : H \longrightarrow H \otimes_k H$ and the coidentity $\epsilon : H \longrightarrow k$ are k-algebra morphisms. The comultiplication δ determines a tensor product of (H, m)-modules: $(V, \xi)_{\mathbf{H}} \otimes (V', \xi') = (V \otimes_k V', \xi \odot \xi')$, where the action $\xi \odot \xi'$ is the composition $\xi \otimes_k \xi' \circ H\sigma_{H,V} V' \circ \delta V \otimes_k V'$. ■

1.0.4.4. Strict monoidal categories and categories of endofunctors. Any strict monoidal category, in particular any category of endofunctors (cf. Example 1.0.1.1) is a monoidal category. ■

1.0.4.5. The category of continous endofunctors. We denote by $\mathfrak{End}\mathcal{A}$ the full subcategory of the category $End\mathcal{A}$ of Example 1.0.1.1 generated by all endofunctors having a right adjoint. Clearly $\mathfrak{End}\mathcal{A}$ is a monoidal subcategory of $\tilde{End}\,\mathcal{A}$. ■

1.0.4.6. Remark. For any monoidal category $\tilde{\mathcal{C}} = (\mathcal{C}, \otimes, a, \mathbf{1})$, we have a canonical monoidal functor $(\mathfrak{R}, a^{-1}, \rho) : \tilde{\mathcal{C}} \longrightarrow \tilde{End}(\mathcal{C})$, where the functor $\mathfrak{R}$ from $\mathcal{C}$ to $End\mathcal{C}$ assigning to each object X of $\mathcal{C}$ the functor $\otimes X$ of tensoring by X, and to any $f \in \mathrm{Hom}\mathcal{A}$ the functor morphism $\otimes$f. Thanks to the existence of the identical object, $\mathfrak{R}$ is a faithful functor.

If, for any $X \in Ob\mathcal{C}$, the functor $\otimes X$ has a right adjoint, the monoidal functor $\tilde{\mathfrak{R}} : \tilde{\mathcal{C}} \longrightarrow \tilde{End}\,\mathcal{C}$ takes values in the subcategory $\tilde{\mathfrak{End}}\,\mathcal{C}$ of Example 1.0.4.5.

Note that the monoidal categories of Examples 1.0.4.1–1.0.4.3 have this property. Hence they can be canonically embedded into $\tilde{\mathfrak{End}}\,\mathcal{C}$ for respective categories $\mathcal{C}$. ■

1.0.4.7. Remark on the subcategory of right exact endofunctors. Denote by $End'\mathcal{A}$ the full subcategory of $End\mathcal{A}$ generated by right exact functors. Clearly $End'\mathcal{A}$ is a monoidal subcategory of $\tilde{End}\,\mathcal{A}$; and $\mathfrak{End}\mathcal{A} \subset End'\mathcal{A}$. And if $\otimes X$ is a right exact functor for any $X \in Ob\mathcal{C}$, the canonical monoidal functor $\tilde{\mathfrak{F}}$ (cf. Remark 1.0.4.6) takes values in the subcategory $End'\mathcal{A}$.

In order to simplify the exposition, we usually require that the functor $\otimes X$ should have a left adjoint; i.e. the functor $\tilde{\mathfrak{F}}$ realizes $\tilde{\mathcal{C}}$ as a monoidal subcategory of $\tilde{\mathfrak{End}}\,\mathcal{A}$. An attentive reader can see that in many constructions of this work it suffices to assume that the functors $\otimes X$ are right exact for all $X \in Ob\mathcal{C}$. ■

1.0.5. Remark. Monoidal categories are a natural framework for important constructions and theories of mathematics and, in the recent time, of theoretical physics. The price to pay is dealing with nontrivial associativity constraints which lead even in relatively simple cases to rather complicated diagrams. This problem does not exist in the strict monoidal categories. This is the reason why we have introduced monoidal categories the way we did: as categories with multiplication together with a faithful canonical monoidal functor into a strict category (of endofunctors). We shall use this realization through the whole work. ■

1.0.6. Algebras and modules in monoidal categories. Most of general module theory can be naturally extended to monoidal categories. Below we sketch notions and elementary facts used in the main body of the work.

Fix a monoidal category $\mathcal{C}^{\tilde{}}$. An algebra in $\mathcal{C}^{\tilde{}}$ is a pair (R, μ), where R is an object of $\mathcal{C}$ and μ is a morphism $R \otimes R \longrightarrow R$ such that

(i) $\mu \circ id_R \otimes \mu = \mu \circ \mu \otimes id_R \circ a$;

(ii) there exists a morphism $\eta : \mathbf{1} \longrightarrow R$ such that $\mu \circ id_R \otimes \eta = \rho_R$ and $\mu \circ \eta \otimes id_R = \lambda_R$.

One can check that the *identity morphism* η is uniquely defined.

Algebras form a category: morphisms from (R, μ) to (R', μ') are morphisms f from R to R' such that $\mu' \circ f \otimes f = f \circ \mu$.

1.0.6.1. Examples. An algebra in the category $k - mod^{\tilde{}}$ (cf. Example 1.0.4.1) is a k-algebra in the conventional sense.

An algebra in the category $End^{\tilde{}}\mathcal{A}$ of endofunctors (cf. Example 1.0.1.1) is a monad (cf. 1.4.3). ∎

Fix an algebra $\mathcal{R} = (R, \mu)$ in $\mathcal{C}^{\tilde{}}$ with the identity element η. A left $\mathcal{R}$-module is a pair (M, m), where $M \in Ob\mathcal{C}$ and m is a morphism $R \otimes M \longrightarrow M$ such that

(i) $m \circ \mu \otimes id_M \circ a = m \circ id_R \otimes m$;

(ii) $m \circ \eta \otimes id_M = \lambda_M$.

An $\mathcal{R}$-module morphism $(M, m) \longrightarrow (M', m')$ is a morphism $f : M \longrightarrow M'$ compatible with the actions: $f \circ m = m' \circ id_R \otimes$f. Thus defined category of left $\mathcal{R}$-modules will be denoted by $\mathcal{R} - mod$. The category mod$-\mathcal{R}$ of right $\mathcal{R}$-modules is defined similarly.

Let $\mathcal{R} = (R, \mu)$ and $\mathcal{R}' = (R', \mu')$ be algebras in the monoidal category $\mathcal{C}^{\tilde{}}$. A triple (m, M, m'), where (m, M) is a left $\mathcal{R}$-module and (M, m') is a right $\mathcal{R}'$-module, is called an $(\mathcal{R}, \mathcal{R}')$-bimodule, if the left and right actions (m and m') commute; i.e. $m \circ id_R \otimes m' = m' \circ m \otimes id_R \circ a$. We leave to a reader the definition of bimodule morphisms and their composition.

The category of $(\mathcal{R}, \mathcal{R}')$-bimodules will be denoted by $(\mathcal{R}, \mathcal{R}') - bi$. If $\mathcal{R} = \mathcal{R}'$, we shall write simply $\mathcal{R} - bi$.

For any left $\mathcal{R}$-module $\mathcal{M} = (m, M)$ and a right $\mathcal{R}$-module $\mathcal{N} = (N, \nu)$, their *tensor product over* $\mathcal{R}$, $\mathcal{N} \otimes_{\mathcal{R}} \mathcal{M}$ is defined as the coequalizer of the pair $\nu \otimes id_M \circ a$, $id_N \otimes m$ of morphisms from $N \otimes (R \otimes M)$ to N⊗M.

From now on we assume that the bifunctor $\otimes$ is right exact with respect to each argument. This assumption does not hold for the category $End^{\tilde{}}\mathcal{A}$ of endofunctors (cf. Example 1.0.1.1). But it does hold for its full monoidal subcategory generated by right exact functors (cf. Remark 1.0.4.7).

Note that in the remaining examples of Subsection 1.0.4 the tensor products preserve colimits of all ('small') diagrams.

If $\mathcal{N}$ is an $(\mathcal{R}', \mathcal{R})$-bimodule and $\mathcal{M}$ is a left $\mathcal{R}$-module, then $\mathcal{N} \otimes_{\mathcal{R}} \mathcal{M}$ has (thanks to the right exactness of $\otimes$) a natural structure of a left $\mathcal{R}'$-module. Thus $\mathcal{N} \otimes_{\mathcal{R}}$ is a functor from $\mathcal{R}$-mod to $\mathcal{R}' - mod$. And the map $\mathcal{N} \mapsto \mathcal{N} \otimes_{\mathcal{R}}$ is extended to a functor from $(\mathcal{R}', \mathcal{R}) - bi$ to the category of (right exact) functors from $\mathcal{R} - mod$ to $\mathcal{R}' - mod$. In particular, we have a faithful functor $\mathcal{F}$ from $\mathcal{R} - bi$ to the category $End(\mathcal{R} - mod)$ taking values in the subcategory $End'\mathcal{A}$ of right exact functors.

Note that $\mathcal{R} - bi$ has a natural structure of a monoidal category with the tensor product $\otimes_{\mathcal{R}}$ and the identity object $\mathcal{R}$. And the functor $\mathcal{F}$ is naturally extended to a monoidal functor from $\mathcal{R} - bi^{\tilde{}}$ to $End'^{\tilde{}}(\mathcal{R} - mod)$.

1.1. Subschemes of monoidal categories. Fix a monoidal category $\mathcal{C}^{\tilde{}} = (\mathcal{C}, \otimes, a, \mathbf{1})$. We assume that $\mathcal{C}$ is an abelian category with the property (sup). Fix an associative algebra R in $\mathcal{C}^{\tilde{}}$. Let $R - mod$ be the category of left R-modules. And let $R - bi$ denote the category of R-bimodules. One can check that the categories $R - mod$ and $R - bi$ are abelian and have too the property (sup).

We call a subcategory $\mathbb{T}$ of $\mathcal{C}$ *a subscheme of the monoidal category* $\mathcal{C}^{\tilde{}}$ if it is a subscheme (i.e. a coreflective topologizing subcategory) of $\mathcal{C}$ and a monoidal subcategory of $\mathcal{C}^{\tilde{}}$. The latter implies, in particular, that the identity object $\mathbf{1}$ in $\mathcal{C}^{\tilde{}}$ belongs to $\mathbb{T}$.

A subscheme $\mathbb{T}$ shall be called (*Zariski*) *closed* if $\mathbb{T}$ is Zariski closed in $\mathcal{C}$.

1.1.0. Example. Let $\mathbf{X} = (X, \mathcal{O})$ be a scheme. Then category of quasi-coherent sheaves on a (closed) subscheme of $\mathbf{X}$ is a (closed) subscheme of the monoidal category of quasi-coherent sheaves on $\mathbf{X}$. ∎

1.1.1. Lemma. *(a) The intersection of any set of subschemes of a monoidal category* $\mathcal{C}^{\tilde{}}$ *is a subscheme.*

(b) The intersection of any set of Zariski closed subschemes of $\mathcal{C}^{\tilde{}}$ *is a Zariski closed subscheme.*

Proof. Clearly, the intersection of any set of monoidal subcategories of $\mathcal{C}^{\tilde{}}$ is a monoidal subcategory of $\mathcal{C}^{\tilde{}}$. The assertion follows now from Lemma I.2.7.1. ∎

In particular, the intersection of all subschemes of $\mathcal{C}^{\tilde{}}$ is (the smallest) subscheme of $\mathcal{C}^{\tilde{}}$. We denote it by $\Delta_{\mathcal{C}}$, or simply by Δ, if this does not cause any confusion, and call it *the diagonal of* $\mathcal{C}^{\tilde{}}$.

1.1.2. Example. If $\mathcal{C}^{\tilde{}} = End^{\tilde{}}\mathcal{A}$ (cf. Example 1.0.1.1) for some abelian category $\mathcal{A}$, then $\Delta_{\mathcal{C}}$ coincides with the minimal subscheme Δ of $End\mathcal{A}$ containing $Id_{\mathcal{A}}$ - the diagonal in the sense of Section I.4. In fact, it follows from Lemma I.4.1 that Δ is a monoidal subcategory of $End^{\tilde{}}\mathcal{A}$. ∎

1.1.3. Example. Let $\mathcal{C}^{\tilde{}}$ be the monoidal category of R-bimodules for some associative ring $R : \mathcal{C}^{\tilde{}} = R - bi^{\tilde{}} = (R - bi, \otimes_R, a, R)$. Then $\Delta_{\mathcal{C}}$ coincides with the subcategory $[K_\mu]$ defined by the kernel K_μ of the multiplication $\mu : R \otimes R \longrightarrow R$ (cf. Lemma I.5.2). ∎

1.1.4. Proposition. *Let* $\mathbb{T}$ *be a subscheme of a monoidal category* $\mathcal{B}^{\tilde{}} = (\mathcal{B}, \odot, a, \mathbf{1})$.

(a) Suppose that $\odot$ *is right exact with respect to each argument; i.e. the functors* $X\odot$ *and* $\odot X$ *are right exact for any* $X \in Ob\mathcal{B}$. *Then* $\mathbb{T}^{(n)} \odot \mathbb{T}^{(m)} \subseteq \mathbb{T}^{(nm)}$.

(b) If $\odot$ *respects countable direct colimits, then* $\mathbb{T}$*-objects of* $\mathcal{B}$ *generate a monoidal subcategory, hence a subscheme, of* $\mathcal{B}^{\tilde{}}$.

Proof. (a) Note first that $\mathbb{T} \odot \mathbb{T}^{(m)} \subseteq \mathbb{T}^{(m)}$. This is by assumption when $m = 1$. Let $X \in Ob\mathbb{T}, Y \in Ob\mathbb{T}^{(m)}$, and $m \geq 2$. Then we have an exact sequence

$$0 \longrightarrow M \longrightarrow Y \longrightarrow L \longrightarrow 0 \tag{1}$$

with $M \in Ob\mathbb{T}$, $L \in Ob\mathbb{T}^{(m-1)}$. Since $\odot$ is right exact, the sequence

$$X \odot M \longrightarrow X \odot Y \longrightarrow X \odot L \longrightarrow 0 \tag{2}$$

is exact. By the induction hypothesis, $X \odot L \in Ob\mathbb{T}^{(m-1)}$. And $X \odot M \in Ob\mathbb{T}$. Therefore the product $X \odot Y$ belongs $\mathbb{T}^{(m)}$.

Let now $X \in Ob\mathbb{T}^{(n)}, Y \in Ob\mathbb{T}^{(m)}$, and $n \geq 2$. Then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow X \longrightarrow L \longrightarrow 0 \tag{3}$$

with $M \in Ob\mathbb{T}, L \in Ob\mathbb{T}^{(n-1)}$. Since $\odot$ is right exact, the sequence

$$M \odot Y \longrightarrow X \odot Y \longrightarrow L \odot Y \longrightarrow 0$$

is exact. According to 1), $M \odot Y \in Ob\mathbb{T}^{(m)}$. By the induction hypothesis, $L \odot Y$ belongs to $\mathbb{T}^{(mn-m)}$. Therefore $X \odot Y \in Ob\mathbb{T}^{(mn)}$.

(b) Thus, the bifunctor $\odot$ is compatible with the canonical $\mathbb{T}$-filtration. If $\odot$ respects countable directed colimits (of subobjects), the product of $\mathbb{T}$-objects is a $\mathbb{T}$-object. ∎

1.1.5. Differential operators. Let $\mathcal{M} = (M, m)$ and $\mathcal{M}' = (M', m')$ be R-modules. Suppose that there exists an inner hom, $\mathfrak{End}(M, M')$; i.e.

$$\mathcal{C}(? \otimes M, M') \simeq \mathcal{C}(?, \mathfrak{End}(M, M')).$$

Note that $\mathfrak{End}(M, M')$ is an R-bimodule.

In fact, the left action of R on $\mathfrak{End}(M, M')$ is the image of $id_{M'}$ under the composition of canonical maps:

$$\begin{array}{ccccc} \mathcal{C}(M', M') & \longrightarrow & \mathcal{C}(R \otimes M', M') & \longrightarrow & \mathcal{C}(R \otimes (\mathfrak{Hom}(M, M') \otimes M), M') \\ & & & & \downarrow \\ & & & & \mathcal{C}(R \otimes \mathfrak{Hom}(M, M'), \mathfrak{Hom}(M, M')) \end{array}$$

Here the first map is induced from the action $R \otimes M' \longrightarrow M'$ of R, the second one by the canonical morphism $\mathfrak{End}(M, M') \otimes M \longrightarrow M'$. The right hand side isomorphism is induced from the associativity isomorphism

$$(R \otimes \mathfrak{End}(M, M')) \otimes M \longrightarrow R \otimes (\mathfrak{End}(M, M') \otimes M)$$

and the bijection

$$\mathcal{C}((R \otimes (\mathfrak{End}(M, M')) \otimes M, M') \longrightarrow \mathcal{C}(R \otimes \mathfrak{End}(M, M'), \mathfrak{End}(M, M')).$$

Similarly, the right action of R on $\mathfrak{End}(M, M')$ is the image of $id_{M'}$ under the composition

$$\begin{array}{ccccc} \mathcal{C}(M', M') & \longrightarrow & \mathcal{C}(\mathfrak{Hom}(M, M') \otimes M, M') & \longrightarrow & \mathcal{C}(\mathfrak{Hom}(M, M') \otimes (R \otimes M), M') \\ & & & & \downarrow \\ & & & & \mathcal{C}(\mathfrak{Hom}(M, M') \otimes R, \mathfrak{Hom}(M, M')) \end{array}.$$

We leave the routine checking that this is really a bimodule structure to a reader.

Fix a subscheme $\mathbb{T}$ of the monoidal category $(R-bi, \otimes_R, R)$ of R-bimodules.

We call the $\mathbb{T}$-part of the R-bimodule $\mathcal{H}om(M, M')$ *the object of* $\mathbb{T}$*-differential operators from* $\mathcal{M}$ *to* $\mathcal{M}'$. We denote it by $Diff_{\mathbb{T}}(\mathcal{M}, \mathcal{M}')$.

Note that, being a subscheme (hence a monoidal subcategory) of $R-bi, \otimes_R, R)$, $\mathbb{T}$ contains the bimodule $\mathcal{R}$. Let $\mathcal{M} = \mathcal{M}' = (M, m)$. And suppose that $\mathcal{E}nd(M, M)$ exists. Note that the action $m : R \otimes M \longrightarrow M$ provides a canonical morphism from R to $\mathcal{E}nd(M, M)$ which is, as one can check, an algebra morphism. This implies that $Diff_{\mathbb{T}}(\mathcal{M}, \mathcal{M})$ is a D-algebra.

Suppose that $\mathcal{E}nd(R, R)$ exists. Then, regarding R as a left R-module, we have the object (a $\mathbb{T}$-algebra) *of* $\mathbb{T}$*-differential operators on* R. We shall write $D_{\mathbb{T}}(R)$ instead of $Diff_{\mathbb{T}}(R, R)$.

1.1.6. The subscheme $\Delta^{\sim}$. Denote by $\Delta^{\sim}$ the minimal subscheme of $R-bi$ containing the bimodule R. We call objects of the category $\Delta^{\sim}$ *differential bimodules*, or simply *D-bimodules*. For any R-bimodule $\mathcal{M}$, the $\Delta^{\sim}$-torsion of $\mathcal{M}$ shall be called *the differential part of* $\mathcal{M}$.

Suppose that, for all $X \in Ob\mathcal{C}$, the functors $X\otimes$ and $\otimes X$ respect colimits. Then one can show that $\Delta^{\sim}$ is a monoidal subcategory of $R-bi^{\sim}$; i.e. $\Delta^{\sim}$ is a subscheme (the diagonal) of $R-bi^{\sim}$.

The subcategory $\Delta^{\sim}$ seems to be the most natural choice, when no additional structure is given. There is another, more natural possibility, if the monoidal category $\mathcal{C}^{\sim}$ is quasi-symmetric.

1.2. Quasi-symmetric categories. To simplify the calculations, we shall realize the monoidal category $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, a, \mathbf{1})$ as a subcategory of the monoidal category of endofunctors $End^{\sim}(\mathcal{C}) = (End\mathcal{C}, \circ)$ assigning to each object X of $\mathcal{C}$ the functor $\otimes X$ of tensoring by X, and to any arrow f the functor morphism $\otimes f$ (cf. Examples 1.0.1.1 and 1.0.4.5 and Remark 1.0.4.6). This way algebras in $\mathcal{C}^{\sim}$ become monads, $\mathbf{1}$ can be assumed to be equal to $Id_{\mathcal{C}}$; and the associativity and isomorphisms $\mathbf{1}\odot X \simeq X \simeq X \odot \mathbf{1}$ can be chosen to be identical.

To underline the fact that $\mathcal{C}^{\sim}$ is a monoidal subcategory of the category of endofunctors, we shall write $\odot$ instead of $\otimes$.

Suppose our monoidal category $\mathcal{C}^{\sim}$ is *quasi-symmetric*; i.e. there is a functor isomorphism $\beta = \{\beta_{X,Y} : X \odot Y \longrightarrow Y \odot X \mid X, Y \in Ob\mathcal{C}\}$ from $\odot$ to $\odot \circ \sigma$, where σ is the functor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}, (X, Y) \mapsto (Y, X)$, which satisfies the following requirements:

$$\beta_{X\odot Y,Z} = \beta_{X,Z}Y \circ X\beta_{Y,Z}, \quad \beta_{X,Y\odot Z} = Y\beta_{X,Z} \circ \beta_{X,Y}Z \tag{1}$$

$$\beta_{X,\mathbf{1}} = id_X = \beta_{\mathbf{1},X}. \tag{2}$$

The isomorphism β with the properties (1) and (2) is called *quasi-symmetry* or *braiding*.

Note that the two equalities (1) are equivalent to each other if β is a symmetry; i.e. if $\beta_{X,Y} \circ \beta_{Y,X} = id_{Y\odot X}$ for all $X, Y \in Ob\mathcal{C}$.

1.2.1. Example. Let Γ be an abelian group. And let $\mathcal{C}^{\sim}$ be the monoidal category of Γ-graded k-modules (cf. Example 1.0.4.2). Then any bicharacter (– a group homomorphism) $\chi : \Gamma \times \Gamma \longrightarrow k^*$ determines a quasi-symmetry and vice versa.

If $\Gamma = \mathbb{Z}^r$, then any bicharacter χ is determined by its values on the canonical generators, i.e. by a matrix $(q_{ij} \mid 1 \leq i,j \leq r)$ with $q_{ij} \in k^*$ for all i,j. ∎

The quasi-symmetry β provides a functor $^{\beta}$ from $Alg\mathcal{C}^{\sim}$ to $Alg\mathcal{C}^{\sim}$ which assigns to any algebra $\mathcal{R} = (R, \mu)$ the *β-opposite* algebra $\mathcal{R}^{\beta} := (R, \mu \circ \beta_{R,R})$, and acts identically on algebra morphisms. The algebra $\mathcal{R}$ is called β-commutative if $\mathcal{R}^{\beta} = \mathcal{R}$.

Another structure determined by β is the tensoring of algebras: the product of algebras $\mathcal{R} = (R, \mu)$ and $\mathcal{S} = (S, \nu)$ is given by

$$(R, \mu) \odot_{\beta} (S, \nu) = (R \odot S, \mu \odot \nu \circ R\beta_{S,R}S) \tag{3}$$

and $f \odot_{\beta} g = f \odot g$ for any pair f, g of algebra morphisms.

Finally, there is a canonical category isomorphism from $(\mathcal{R}, \mathcal{S}) - bi$ to the category $\mathcal{R} \odot_{\beta} \mathcal{S}^{\beta} - mod$ sending any $(\mathcal{R}, \mathcal{S})$-bimodule (u, M, v) into $(u \odot_{\beta} v, M)$, where $u \odot_{\beta} v$ is $u \odot v \circ R\beta_{S,M}S : R \odot S \odot M \longrightarrow M$.

In particular, we shall identify $\mathcal{R} - bi$ with the category $\mathcal{R} \odot_{\beta} \mathcal{R}^{\beta} - mod$.

Note that the multiplication $\mu : R \odot R \longrightarrow R$ is an $\mathcal{R}$-bimodule morphism; hence an $\mathcal{R} \odot_{\beta} \mathcal{R}^{\beta}$-module morphism from $\mathcal{R} \odot_{\beta} \mathcal{R}^{\beta}$ to $\mathcal{R}$. In particular, the kernel $\mathcal{J}_{\mu}$ of μ is a left ideal in $\mathcal{R} \odot_{\beta} \mathcal{R}^{\beta}$.

1.3. The β-diagonal Δ_{β}. We assume that, for any $X \in Ob\mathcal{C}$, the functor $X\odot$ respects colimits.

Call an $\mathcal{R}$-bimodule $\mathcal{M} = (m, M, \nu)$ *β-artinian*, if there is a morphism $f : X \longrightarrow M$ in $\mathcal{C}$ such that

a) the composition of $fR : X \odot R \longrightarrow M \odot R$ and the action ν is an epimorphism;

b) $\nu \circ fR = m \circ \beta_{M,R} \circ fR (= m \circ Rf \circ \beta_{X,R})$.

We call the morphism f in this definition *a generating morphism of $\mathcal{M}$*.

Let $Art_{\beta}\mathcal{R}$ denote the full subcategory of $\mathcal{R} - bi$ generated by β-artinian bimodules.

1.3.1. Proposition. *The subcategory $Art_{\beta}\mathcal{R}$ of β-artinian bimodules is closed with respect to colimits (taken in $\mathcal{R} - bi$) and $\otimes_{\mathcal{R}}$.*

Proof. 1) Let $\mathcal{M} = (m, M, \nu)$ be a β-artinian bimodule with a generating morphism $f : X \longrightarrow$M. And let φ be a bimodule epimorphism from $\mathcal{M}$ to some $\mathcal{M}' = (m', M', \nu')$. Then the composition $\varphi \circ f : X \longrightarrow M'$ is a generating morphism for $\mathcal{M}'$.

In fact, $\nu' \circ (\varphi \circ f)R = \varphi \circ (\nu \circ fR)$; and the right hand side is the composition of two epimorphisms. Hence $\nu' \circ (\varphi \circ f)R$ is an epimorphism. As to the second property of a generating morphism, we have:

$$\nu' \circ (\varphi \circ f)R = \varphi \circ \nu \circ fR = \varphi \circ m \circ \beta_{M,R} \circ fR = m' \circ R\varphi \circ \beta_{M,R} \circ fR = m' \circ R(\varphi \circ f) \circ \beta_{X,R}.$$

For any family $\{\mathcal{M}_i\}$ of β-artinian bimodules with generating morphisms f_i, the direct sum $\oplus\mathcal{M}_i$ is a β-artinian bimodule with a generating morphism $\oplus f_i$.

Altogether, this shows that the subcategory Art_{β} is closed with respect to any colimits in $\mathcal{R} - bi$.

2) Let now $\mathcal{M} = (m, M, \nu)$ and $\mathcal{M}' = (m', M', \nu')$ be β-artinian bimodules with generating morphisms resp. $f : X \longrightarrow M$ and $f' : X' \longrightarrow M'$. Then the composition of $f \odot f'$ and the canonical epimorphism $M \odot M' \longrightarrow M \odot_{\mathcal{R}} M'$ is a generating morphism.

Note that, for any $X \in Ob\mathcal{C}$, $X \odot \mathcal{R} = (X\mu \circ \hat{\beta}_{R,X\odot R}, X\mu)$ is a β-artinian bimodule.

In fact, we can take as f the natural morphism $X \longrightarrow X \odot R$. Then $fR = id_{X\odot R}$. And the obvious equality $X\mu \circ \hat{\beta}_{R,X\odot R} \circ \beta_{X\odot R,R} = X\mu$ is exactly the second property we need to check. Here $\hat{\beta}_{Y,X} := \beta^{-1}_{Y,X}$ for all $X, Y \in Ob\mathcal{C}$.

The conditions b) and a) in the definition of a β-artinian bimodule above mean exactly that $\nu \circ fR$ is a bimodule epimorphism.

One can check that $(X \odot \mathcal{R}) \odot_{\mathcal{R}} (X' \odot \mathcal{R}) \simeq (X \odot X') \odot \mathcal{R}$; hence the bimodule $(X \odot \mathcal{R}) \odot_{\mathcal{R}} (X' \odot \mathcal{R})$ is β-artinian. The epimorphisms

$$\nu \circ fR : X \odot \mathcal{R} \longrightarrow \mathcal{M} \quad \text{and} \quad \nu' \circ f'R : X' \odot \mathcal{R} \longrightarrow \mathcal{M}'$$

induce a bimodule epimorphism from $(X \odot \mathcal{R}) \odot_{\mathcal{R}} (X' \odot \mathcal{R}) \simeq (X \odot X') \odot \mathcal{R}$ to $\mathcal{M} \odot_{\mathcal{R}} \mathcal{M}'$. According to 1), this implies that $\mathcal{M} \odot_{\mathcal{R}} \mathcal{M}'$ is a β-artinian bimodule. ∎

In general, the subcategory $Art_\beta\mathcal{R}$ of β-artinian bimodules does not contain with each object all its subobjects. In other words, it is not topologizing.

We denote by Δ_β the minimal coreflective topologizing monoidal subcategory (i.e. a subscheme) of $\mathcal{R} - bi$ containing the subcategory $Art_\beta\mathcal{R}$ of β-artinian bimodules. We call Δ_β *the β-diagonal of* $\mathcal{R} - bi$.

1.3.2. Remarks. (a) It follows from the proof of Proposition 1.3.1 that Δ_β is the minimal subscheme of $\mathcal{R} - bi$ containing all $\mathcal{R}$-bimodules $X \odot \mathcal{R}$, $X \in Ob\mathcal{C}$.

(b) It follows from Lemma I.5.10.4.1 and Proposition 1.3.1 that $Ob\Delta_\beta$ consists of subobjects of β-artinian bimodules. ∎

1.3.3. The β-commutative case. Suppose that $\mathcal{R} = (R, \mu)$ is a β-commutative algebra in $\mathcal{C}^{\sim}$; i.e. $\mu \circ \beta_{R,R} = \mu$.

1.3.3.1. Proposition. *Let $\mathcal{R}$ be β-commutative. Then $\Delta_\beta = Art_\beta\mathcal{R}$ and it is a Zariski closed subscheme of $\mathcal{R}-bi$. The category $R-mod$ of left R-modules is naturally isomorphic to Δ_β.*

Proof. (a) It follows from β-commutativity of $\mathcal{R}$ that, for any R-module (M, m), the triple $(m, M, m \circ \beta_{M,R})$ is an R-bimodule; and the map $\mathcal{I}_\beta$ which assigns to each (M, m) the $\mathcal{R}$-bimodule $(m, M, m \circ \beta_{M,R})$ and acts 'identically' on morphisms, is an exact and fully faithful functor from $R - mod$ to $R - bi$. Clearly $\mathcal{I}_\beta$ takes values in the subcategory $Art_\beta\mathcal{R}$.

Moreover, $\mathcal{I}_\beta$ respects and reflects both limits and colimits which implies in particular, that the image of $\mathcal{I}_\beta$ is a topologizing subcategory.

Note that all bimodules of the form $X \odot \mathcal{R}, X \in Ob\mathcal{C}$, are images of the corresponding $\mathcal{R}$-modules $X \odot \mathcal{R} := (X\mu \circ \hat{\beta}_{R,X\odot R}, X \odot R)$. Since every β-artinian bimodule is an epimorphic image of a bimodule of the form $X \odot \mathcal{R}$ (cf. the second part of the argument of Proposition 1.3.1), it follows that the image of the functor $\mathcal{I}_\beta$ coincides with the subcategory $Art_\beta\mathcal{R}$ of β-artinian bimodules. By Proposition 1.3.1, the subcategory $Art_\beta\mathcal{R}$ is coreflective for any $\mathcal{R}$. Since it is topologizing, we have the equality $Art_\beta\mathcal{R} = \Delta_\beta$. It remains to show that the subcategory Δ_β is reflective.

(b) Since $\mathcal{R}$ is β-commutative, $\mathcal{R} \odot \mathcal{R}^\beta$ is β-commutative. In particular, $\mathcal{J}_\mu$ is a two-sided ideal, and $\mathcal{R} \odot \mathcal{R}^\beta/\mathcal{J}_\mu \simeq \mathcal{R}$. The image of $R - mod$ in $R - bi$ consists exactly of

bimodules annihilated by $\mathcal{J}_\mu$. So that the composition of the tensoring by $\mathcal{R}$ over $\mathcal{R}\odot\mathcal{R}^\beta$ with the 'forgetting' functor $\mathcal{R}\odot\mathcal{R}^\beta - mod = R - bi \longrightarrow R - mod$ is a right adjoint to the functor $\mathcal{I}_\beta$.

Another way to spell it:

For any bimodule $\mathcal{M} = (m, M, \nu)$, the coequalizer M' of $\nu, m\circ\beta_{M,R} : M\odot R \longrightarrow M$ has a unique $\mathcal{R}$-bimodule structure such that the canonical epimorphism $M \longrightarrow M'$ is a bimodule morphism. One can check that the bimodule $\mathcal{M}' = (m', M, \nu')$ belongs to Δ_β. The map $\mathcal{M} \mapsto \mathcal{M}'$ determines a functor which takes values in Δ_β and is left adjoint to the inclusion functor $J_\beta : \Delta_\beta \longrightarrow \mathcal{R} - bi$; i.e. the subcategory Δ_β is reflective. ∎

1.3.3.2. Corollary. *Let $\mathcal{R}$ be β-commutative. Then the defining ideal of Δ_β is the tensoring by $\mathcal{J}_\mu$ over $\mathcal{R}\odot\mathcal{R}^\beta$. The conormal bundle is the tensoring by the bimodule $\Omega_\mathcal{R} := \mathcal{J}_\mu/(\mathcal{J}_\mu^2)$ of 'differential 1-forms' over $\mathcal{R}\odot\mathcal{R}^\beta$.*

1.3.3.3. Note. In the classical case, when $\mathcal{C}^-$ is $(\mathbb{Z} - mod, \otimes)$, or, more generally, $\mathcal{C}^-$ is $(k - mod, \otimes_k, k)$ for some commutative ring k, any algebra $\mathcal{R}$ in $\mathcal{C}^-$ is a generator of the category $\mathcal{R} - mod$. This implies, in the case when $\mathcal{R}$ is commutative, that the minimal subscheme Δ containing the bimodule $\mathcal{R}$ coincides with the image of the functor $\mathcal{I}_\beta$. Usually, this is not the case. Take as an example the category of $\mathbb{Z}/2\mathbb{Z}$-graded spaces over a field k, and $R = R_0 = k$. ∎

1.4. Differential and β-differential actions. We begin with introducing *differential actions* on objects of an arbitrary abelian category.

1.4.1. Differential actions. Fix a monoidal subcategory $\mathcal{C}^-$ of $\mathfrak{End}\mathcal{A}$ and a monoidal subcategory $\mathbb{T}$ of $\mathcal{C}^-$. Let M, N be objects of $\mathcal{A}, X \in Ob\mathcal{C}$; and let f be any morphism (an 'action') $X(M) \longrightarrow N$. We call the morphism f a $\mathbb{T}$-*morphism* if there exists an object X' of $\mathbb{T}$ and morphisms $\phi : X \longrightarrow X'$, $f' : X'(M) \longrightarrow N$ such that f is the composition of $\phi(M)$ and f' : $f = f'\circ\phi(M)$.

1.4.2. Example. Every morphism $f : M \longrightarrow N$ of $\mathcal{A}$ is regarded as a morphism $1(M) \longrightarrow N$. Since $1{=}Id_\mathcal{A} \in Ob\Delta$, any morphism of the category $\mathcal{A}$ is a Δ^--morphism. ∎

Fix objects M and N of the category $\mathcal{A}$. Let $\mathfrak{G}_\mathbb{T}(M, N)$ denote the category objects of which are pairs (X, f), where $X \in Ob\mathbb{T}$, f is a morphism $X(M) \longrightarrow N$; morphisms from (X, f) to (X', f') are morphisms $\phi : X \longrightarrow X'$ such that $f = f'\circ\phi(M)$.

Denote the final object of the category $\mathfrak{G}_\mathbb{T}(M, N)$ (if any) by $\mathfrak{H}_\mathbf{T}(M, N)$. For any three objects, L, M, N, we have a natural 'composition' functor

$$\mathfrak{c}_{N,M,L} : \mathfrak{H}_\mathbb{T}(M, N)\times\mathfrak{H}_\mathbf{T}(L, M) \longrightarrow \mathfrak{H}_\mathbf{T}(L, N) \tag{1}$$

which assigns to a pair of objects (X, f), (Y, g) of resp. $\mathfrak{G}_\mathbb{T}(M, N)$ and $\mathfrak{G}_\mathbf{T}(L, M)$ the object $(X\odot Y, f\circ Xg)$ and sends a pair of morphisms, (ϕ, ψ), into $\phi\odot\psi$.

Thus, with any abelian category $\mathcal{A}$ and a monoidal coreflective subcategory $\mathbb{T}$ of $\mathfrak{End}\mathcal{A}$, we associate *the bicategory of* $\mathbb{T}$*-actions* $\mathbb{T}\mathcal{A}$.

The composition functor $\mathfrak{c}_{N,M,L}$ (cf. (1)) induces a composition map

$$c_{N,M,L} : \mathfrak{H}_\mathbf{T}(M, N)\odot\mathfrak{H}_\mathbf{T}(L, M) \longrightarrow \mathfrak{H}_\mathbb{T}(L, N) \tag{2}$$

whenever the objects $\mathfrak{H}_{\mathbf{T}}(M,N)$, $\mathfrak{H}_{\mathbf{T}}(L,M)$, and $\mathfrak{H}_{\mathbf{T}}(L,N)$ exist.

In particular, if $\mathfrak{H}_{\mathbf{T}}(M,N)$ exists for any $M,N \in Ob\mathcal{A}$, the composition morphisms $\{c_{N,M,L} \mid L,M,N \in Ob\mathcal{A}\}$ determine a structure of a $\mathbb{T}$-category on $\mathcal{A}$. We denote this $\mathbb{T}$-category by $\mathfrak{H}_{\mathbf{T}}\mathcal{A}$.

Let $\mathbb{T} = \mathbb{S}^{\infty}$ for some subscheme $\mathbb{S}$ of $\mathcal{C}^{-}$. In this case we call the bicategory $\mathfrak{G}_{\mathbb{T}}\mathcal{A}$ *the bicategory of* $\mathbb{S}$*-differential actions* and the category $\mathfrak{H}_{\mathbf{T}}\mathcal{A}$ *the category of* $\mathbb{S}$*-differential operators* on $\mathcal{A}$. And we shall write $\mathfrak{Diff}_{\mathbf{S}}\mathcal{A}$ instead of $\mathfrak{G}_{\mathbb{T}}\mathcal{A}$ and $Diff_{\mathbf{S}}\mathcal{A}$ instead of $\mathfrak{H}_{\mathbf{T}}\mathcal{A}$.

If $\mathbb{S} = \Delta$, then $\mathbb{S}$ shall be omitted; i.e. we shall write $\mathfrak{Diff}\mathcal{A}$ and $Diff\mathcal{A}$ and call them resp. the *bicategory of differential actions* and the *category of differential operators* on $\mathcal{A}$.

1.4.3. β-Differential bimodules and operators. If $\mathbb{S}$ is the β-diagonal of $\mathcal{C}^{-}$ for some quasi-symmetry β, we replace '$\mathbb{S}$-differential' by 'β-differential'. In particular, we will talk about *β-differential* actions and operators. The bicategory of β-differential actions and the corresponding $(\Delta_{\beta}^{\infty}-)$category will be denoted respectively by $\mathfrak{Diff}_{\beta}^{\#}\mathcal{A}$ and by $Diff_{\beta}^{\#}\mathcal{A}$.

1.4.4. Strongly β-Differential bimodules and operators. Now let $\mathbb{S}$ be the subcategory $Art_{\beta}\mathcal{R}$ of β-artinian $\mathcal{R}$-bimodules. In this case, we call the $\mathbb{S}$-differential bimodules and actions *strongly β-differential.* The bicategory of strongly β-differential actions and the corresponding $(\Delta_{\beta}^{\infty}-)$category will be denoted respectively by $\mathfrak{Diff}_{\beta}^{\#_s}\mathcal{A}$ and by $Diff_{\beta}^{\#_s}\mathcal{A}$.

1.4.4.1. Lemma. *Fix two $\mathcal{R}$-modules – $\mathfrak{M} = (m,M)$ and $\mathfrak{N} = (\nu,N)$. Assume that there exists an inner* hom, $\mathfrak{Hom}(M,N)$. *Then there exists the inner $\mathcal{R}$-module* hom, $\mathfrak{Hom}(\mathfrak{M},\mathfrak{N}) = \mathfrak{Hom}_{\mathcal{R}}(\mathfrak{M},\mathfrak{N})$ *and the object $Diff_{\beta}^{\#_s}(\mathfrak{M},\mathfrak{N})$ of strongly β-differential operators from $\mathfrak{M}$ to $\mathfrak{N}$.*

Proof. The fact follows from the assumption that $\mathcal{C}$ is a category with the property (sup) and from Proposition 1.3.1. ∎

1.4.4.2. Upper β-central series and strongly β-differential bimodules. Thanks to Proposition 1.3.1, we can define, for any $\mathcal{R}$-bimodule $\mathcal{M}$ in $\mathcal{C}^{-}$, the *upper β-central series of* $\mathcal{M}$ as the filtration $\{\mathfrak{z}_{\beta,n}\mathcal{M} \mid n \geq -1\}$, where $\mathfrak{z}_{\beta,-1}\mathcal{M} = 0$, and $\mathfrak{z}_{\beta,n}\mathcal{M}$ is the $Art_{\beta}\mathcal{R}^{(n)}$-torsion of $\mathcal{M}$. We define $\mathfrak{z}_{\beta,\infty}\mathcal{M}$ being the supremum of $\mathfrak{z}_{\beta,n}\mathcal{M}$ for all n. Since $Art_{\beta}\mathcal{R} \subseteq \Delta_{\beta}$, the subcategory $Art_{\beta}\mathcal{R}^{(n)}$ is contained in $\Delta_{\beta}^{(n)}$ for all n. Therefore, for any $\mathcal{M} \in Ob\mathcal{R}-bi$, $\mathfrak{z}_{\beta,\infty}\mathcal{M}$ is a subobject of the β-differential part of $\mathcal{M}$. We shall call $\mathfrak{z}_{\beta,\infty}\mathcal{M}$ *the strongly β-differential part of* $\mathcal{M}$. The reader can check that the facts of Section I.5.10 can be adapted for quasi-symmetric categories.

1.4.4.3. Strongly differential operators of zero order. The subbimodule $\mathfrak{z}_{\beta,0}\mathcal{M}$ of $\mathcal{M}$ can be defined the same way as in the classical case. Namely, thanks to the property (sup), we can define *the center* $\mathfrak{z}_{\beta,0}\mathcal{M}$ of the bimodule $\mathcal{M} = (m,M,\nu)$ as the supremum of subobjects $u : X \to M$ such that the diagram

$$\begin{array}{ccc} X \odot R & \xrightarrow{u\odot id_R \circ \beta_{X,R}} & R \odot M \\ {\scriptstyle u\odot id_R}\downarrow & & \downarrow{\scriptstyle m} \\ M \odot R & \xrightarrow{\nu} & M \end{array}$$

is commutative. Now, since the functor $R\odot$ is compatible with colimits, $\mathfrak{z}_{\beta,0}M$ is the image $R \cdot \mathfrak{z}_{\beta,o}M$ of $R \odot \mathfrak{z}_{\beta,0}M \longrightarrow M$.

Let now $\mathcal{L} = (u, L)$ and $\mathcal{N} = (v, N)$ be $\mathcal{R}$-modules such that there exists the inner hom from L to N: $\mathfrak{Hom}(L, N)$. Then $\mathfrak{Z}_{\beta,0}\mathfrak{Hom}(L, N)$ coincides with the inner hom $\mathfrak{Hom}_{\mathcal{R}}(\mathcal{L}, \mathcal{N})$. Therefore $\mathfrak{Z}_{\beta,n}\mathfrak{Hom}(L, N)$ is generated by the object of $\mathcal{R}$-module morphims from $\mathcal{L}$ to $\mathcal{N}$ and the left action of $\mathcal{R}$.

In particular, if $\mathcal{L} = \mathcal{N} = \mathcal{R}$, then the object differential operators on $\mathcal{R}$ of order zero is generated by left and right multiplications by $\mathcal{R}$.

The next term of the canonical filtration of the bimodule $\mathfrak{Hom}(L, N)$, the object of strongly β-differential operators of order ≤ 1 is generated by operators of order zero and by *β-derivations.*

1.5. β-derivations. Fix a quasi-symmetry β in the monoidal category $\mathcal{C}^{\tilde{}}$ and an associative algebra $\mathcal{R} = (R, \mu)$ in $\mathcal{C}^{\tilde{}}$. Let $\mathcal{M} = (m, M, \nu)$ be an $\mathcal{R}$-bimodule. Here m and ν denote resp. left and right action of $\mathcal{R}$.*A β-derivation* of $\mathcal{R}$ in the bimodule $\mathcal{M}$ is a pair (X, d), where X is an object of $\mathcal{C}$ and d is a morphism $X \odot R \longrightarrow M$ such that

$$d \circ X\mu = \nu \circ dR + m \circ Rd \circ \beta_{X,R}R \tag{3}$$

1.5.1. Lemma. *For any $\mathcal{R}$-bimodule (m, M, ν) in $\mathcal{C}^{\tilde{}}$, the 'β-bracket'*

$$ad_\beta = \nu - m \circ \beta_{M,R} : M \odot R \longrightarrow M$$

is a β-derivation.

Proof. We have to show that

$$ad_\beta \circ R\mu = \nu \circ ad_\beta R + m \circ Rad_\beta \circ \beta_{M,R}\mathrm{R}. \tag{1}$$

The left part of this equality is

$$(\nu - m \circ \beta_{M,R}) \circ R\mu = \nu \circ M\mu - m \circ \mu M \circ \beta_{M,R\odot R} \tag{2}$$

Expanding the right part of (1), we obtain:

$$\nu \circ ((\nu - m \circ \beta_{M,R})R + m \circ R(\nu - m \circ \beta_{M,R}) \circ \beta_{M,R}R) =$$

$$\nu \circ \nu R - (\nu \circ mR - m \circ R\nu) \circ \beta_{M,R}R - m \circ Rm \circ R\beta_{M,R} \circ \beta_{M,R}R = \nu \circ \nu R - m \circ Rm \circ \beta_{M,R\odot R}$$

since $R\beta_{M,R} \circ \beta_{M,R}R = \beta_{M,R\odot R}$ and $\mathcal{M}$ is a bimodule; i.e. $\nu \circ mR = m \circ R\nu$. Finally, the fact that ν and m are resp. right and left $\mathcal{R}$-module structures implies the equalities $\nu \circ \nu R = \nu \circ M\mu$ and $m \circ Rm = m \circ \mu\mathrm{M}$. This establishes (1). ∎

1.5.2. Corollary. *For any associative algebra $\mathcal{R} = (R, \mu)$ in $\mathcal{C}^{\tilde{}}$, the 'β-bracket'*

$$ad_\beta = \mu - \mu \circ \beta_{R,R} : R \odot R \longrightarrow R$$

is a β-derivation.

The derivation ad_β of Lemma 1.5.1 (and Corollary 1.5.2) is called the *inner β-derivation* of the $\mathcal{R}$-bimodule $\mathcal{M}$ (resp. of the algebra $\mathcal{R}$).

A morphism from a β-derivation (X, d) to a β-derivation (X', d') is any arrow $f : X \longrightarrow X'$ such that $d = d'$ofR. The composition is defined in a standart way. Denote thus defined category of derivations in $\mathcal{M}$ by $\tilde{}Der_{\mathcal{R},\beta}(\mathcal{M})$. And let $Der_{\mathcal{R},\beta}(\mathcal{M})$ denote a final object of the category $\tilde{}Der_{\mathcal{R},\beta}(\mathcal{M})$ (if any).

For any left $\mathcal{R}$-module $\mathcal{M} = (m, M)$ and any object X of $\mathcal{C}$, $X \odot \mathcal{M}$ will denote the module $(Xm \circ \beta\hat{}_{R,X} M, X \odot M)$, where $\beta\hat{}_{X,R} = \beta^{-1}_{X,R}$. For any pair $\mathcal{N}$, $\mathcal{M}$ of $\mathcal{R}$-bimodules, let $\tilde{}\mathfrak{Hom}_{\mathcal{R}-bi}(\mathcal{N}, \mathcal{M})$ denote the category objects of which are pairs (X, f), where f is an $\mathcal{R}$-bimodule morphism $X \odot \mathcal{N} \longrightarrow \mathcal{M}$.

Finally, let $\mathcal{J}_\mu$ be the kernel of the multiplication $\mu : R \odot R \longrightarrow R$.

1.5.3. Proposition. *The category of derivations $\tilde{}Der_{\mathcal{R},\beta}(\mathcal{M})$ is isomorphic to the category $\tilde{}\mathfrak{Hom}_{\mathcal{R}-bi}(\mathcal{J}_\mu, \mathcal{M})$.*

In particular, the category $\tilde{}Der_{\mathcal{R},\beta}(\mathcal{M})$ has a final object, $Der_{\mathcal{R},\beta}(\mathcal{M})$, iff there exists a final object $\mathfrak{Hom}_{\mathcal{R}-bi}(\mathcal{J}_\mu, \mathcal{M})$ of the category $\tilde{}\mathfrak{Hom}_{\mathcal{R}-bi}(\mathcal{J}_\mu, \mathcal{M})$.

Proof. (i) Note that if $d : R \longrightarrow M$ is a derivation, then $Xd : X \odot R \longrightarrow X \odot M$ is a β-derivation in $X \odot \mathcal{M} := (Xm \circ \beta\hat{}_{X,R}, X \odot M, X\nu)$, where $\beta\hat{}_{X,R} = \beta^{-1}_{X,R}$.

In fact,

$$Xd \circ X\mu = X(m \circ Rd + \nu \circ dR) = (Xm \circ \beta\hat{}_{X,R}M) \circ R(Xd) \circ \beta_{X,R}R + X\nu \circ XdR.$$

(ii) If $d : XR \longrightarrow M$ is a β-derivation in $\mathcal{M} = (m, M, \nu)$ and $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$ is a bimodule morphism, then $\varphi \circ d$ is a β-derivation in $\mathcal{M}'$.

(iii) The morphism $R\eta - \eta R : R \longrightarrow R \odot R$ (where η is the unity of (R, μ)) is a derivation in $(\mu R, R \odot R, R\mu)$ which takes values in $\mathcal{J}_\mu := Ker(\mu)$. Therefore it induces a canonical derivation $\nabla : R \longrightarrow \mathcal{J}_\mu$.

(iv) Consider the functor, F, which assigns to any object (a $\mathcal{R}$-bimodule morphism) $\varphi : X \odot \mathcal{J}_\mu \longrightarrow \mathcal{M}$ of the category $\tilde{}\mathfrak{Hom}_{\mathcal{R}-bi}(\mathcal{J}_\mu, \mathcal{M})$ the β-derivation $\varphi \circ X\nabla$ and maps arrows identically.

We claim that F is an isomorphism of $\tilde{}\mathfrak{Hom}_{\mathcal{R}-bi}(\mathcal{J}_\mu, \mathcal{M})$ onto $\tilde{}Der_{\mathcal{R},\beta}(\mathcal{M})$.

Let $d : XR \longrightarrow M$ be a β-derivation in $\mathcal{M} = (m, M, \nu)$. Set

$$F\hat{}d = (-m \circ Rd \circ \beta_{X,R}) \circ X\iota_\mu \tag{4}$$

where ι_μ is the embedding $\mathcal{J}_\mu \longrightarrow R \odot R$. The morphism $F\hat{}d$ is a bimodule morphism from $X \odot \mathcal{J}_\mu$ to $\mathcal{M}$.

Indeed, one can check that $F\hat{}d$ is a morphism of left R-modules. Since d is a β-derivation, $F\hat{}d = \nu \circ dR \circ X\iota_\mu$ which implies that $F\hat{}d$ is also a morphism of right $\mathcal{R}$-modules.

The map $F\hat{}$ extends to a functor $F\hat{} : \tilde{}\, Der_{\mathcal{R},\beta}(\mathcal{M}) \longrightarrow \tilde{}\, \mathfrak{Hom}_{\mathcal{R}-bi}(\mathcal{J}_\mu, \mathcal{M})$ mapping arrows identically.

We have: $F\hat{}(\varphi \circ X\nabla) = \varphi$ for any bimodule morphism $\varphi : X \odot \mathcal{J}_\mu \longrightarrow \mathcal{M}$. And $F(F\hat{}d)$=d. ∎

1.5.4. Corollary. *If, for any two objects, X, Y of $\mathcal{C}$, there exists an inner* Hom, $\mathcal{H}om(X,Y)$, *from X to Y, then $Der_{\mathcal{R},\beta}(\mathcal{M})$ exists for any $\mathcal{R}$-bimodule $\mathcal{M}$.*

Proof. Recall that $\mathcal{H}om(X,Y)$ is an object of $\mathcal{C}$ representing the functor $\mathcal{C}(? \otimes X, Y)$; i.e. $\mathcal{C}(? \otimes X, Y) \simeq \mathcal{C}(?, \mathcal{H}om(X,Y))$.

One can show that the existence of $\mathcal{H}om(\mathcal{J}_\mu, M)$ implies the existence of the inner hom $\mathcal{H}om_{\mathcal{R}-bi}(\mathcal{J}_\mu, \mathcal{M})$, $\mathcal{M} = (m, M, \nu)$. ∎

1.6. Note on D-calculus for β-commutative algebras. If $\mathcal{R}$ is a β-commutative algebra, one can imitate the classical approach (outlined in Section I.1) to define differential bimodules and algebras, and recover analogs of structures (like de Rham and Koszul complexes etc.) used in the conventional situation. ∎

1.7. Digression: the subcategory $\Delta^{\sim}$. The minimal subscheme $\Delta^{\sim}$ of $\mathcal{R}-bi$ containing the bimodule $\mathcal{R}$ is, usually, a small part of the β-diagonal Δ_β. If $\mathcal{R}$ is β-commutative, the subcategory Δ_β is reflective ('Zariski closed'). It is not clear (actually, doubtful) if the subcategory $\Delta^{\sim}$ is also reflective whenever $\mathcal{R}$ is β-commutative. We can prove the reflectiveness of $\Delta^{\sim}$ only under certain additional assumptions:

1.7.1. Lemma. *Suppose that $\mathbf{1}$ is a projective object of the category $\mathcal{C}$. Assume that either $\mathcal{C}$ has small direct sums, or $\mathbf{1}$ is an object of finite type And let $\mathcal{R} = (R, \mu)$ be a β-commutative algebra in $\mathcal{C}^{\sim}$ such that, for any nonzero ideal J of R, $\mathcal{C}(\mathbf{1}, J) \neq 0$. Then $\Delta^{\sim}$ is a reflective subcategory of $\mathcal{R} - bi$.*

Proof. *b*) Under the conditions on $\mathcal{R}$, the 'diagonal' subcategory $\Delta^{\sim}$ is generated by all $\mathcal{R}$-bimodules $\mathcal{M}$ such that, for any nonzero subobject $\mathcal{N}$ of $\mathcal{M}$, there exists a nonzero bimodule morphism from $\mathcal{R}$ to $\mathcal{N}$. Or, equivalently, $Ob\Delta^{\sim}$ consists of all bimodules $\mathcal{M} = (m, M, \nu)$ such that, for any nonzero subobject X of M, there exists a nonzero morphism from $\mathbf{1}$ to X.

Denote by $\mathcal{C}_\mathbf{1}$ the full subcategory of $\mathcal{C}$ generated by all objects X of $\mathcal{C}$ such that $\mathcal{C}(\mathbf{1}, X) = 0$.

It follows from the projectivity of $\mathbf{1}$ that the subcategory $Ob\mathcal{C}_\mathbf{1}$ is topologizing.

In fact, any nonzero morphism f from $\mathbf{1}$ to a subquotient Y of an object X can be lifted to a nonzero morphism from $\mathbf{1}$ to X. So that if $X \in Ob\mathcal{C}_\mathbf{1}$, then Y is an object of $\mathcal{C}_\mathbf{1}$ as well. Clearly $\mathcal{C}_\mathbf{1}$ is closed under $\oplus$.

Suppose that Y is the supremum of an increasing chain $\{Y_\alpha\}$ of subobjects of an object X. Suppose that all Y_α are objects of $\mathcal{C}_\mathbf{1}$.

If $\mathbf{1}$ is of finite type, then $\mathcal{C}(\mathbf{1}, Y) \simeq colim\mathcal{C}(\mathbf{1}, Y_\alpha) = 0$.

If there exists a direct sum $\oplus Y_\alpha$, then $\mathcal{C}(\mathbf{1}, \oplus Y_\alpha) \subseteq \mathcal{C}(\mathbf{1}, Y_\alpha) = 0$; i.e. $\oplus Y_\alpha$ is an object of $\mathcal{C}_\mathbf{1}$. the monomorphisms $Y_\alpha \longrightarrow X$ induce an epimorphism $\oplus Y_\alpha \longrightarrow Y = \sup\{Y_\alpha\}$. Therefore, thanks to the lifting property, the existence of a nonzero arrow from $\mathbf{1}$ to Y would imply that from $\mathbf{1}$ to $\oplus Y_\alpha$.

By Zorn's Lemma, each object X of $\mathcal{C}$ has the biggest subobject, $X_\mathbf{1}$, from $\mathcal{C}_\mathbf{1}$. In other words, the subcategory $\mathcal{C}_\mathbf{1}$ is coreflective.

Denote by $\mathcal{C}'$ the full subcategory of the category $\mathcal{C}$ generated by all $X \in Ob\mathcal{C}$ such that, for any subobject Y of X, $\mathcal{C}(\mathbf{1}, Y) \neq 0$.

We claim that, for any $X \in Ob\mathcal{C}$, the object $X' = X/X_{\mathbf{1}}$ belongs to $\mathcal{C}'$. Suppose that Y' is a subobject of X' such that $\mathcal{C}(\mathbf{1}, Y') = 0$. And let Y be the preimage of Y' in X. The composition of any morphism $f : \mathbf{1} \longrightarrow Y$ with the projection $Y \longrightarrow Y'$ is zero by assumption; i.e. all morphisms from $\mathbf{1}$ to Y factor through the subobject $X_{\mathbf{1}}$ of Y. Hence $\mathcal{C}(\mathbf{1}, Y) = 0$. Due to the maximality of $X_{\mathbf{1}}$, this means that the canonical morphism $X_{\mathbf{1}} \longrightarrow Y$ is an isomorphism; i.e. $Y' = 0$.

(b) Let $\mathcal{M} = (m, M, \nu)$ be an $\mathcal{R}$-bimodule. One can see that $M_{\mathbf{1}}$ is a subbimodule of $\mathcal{M}$; and the quotient bimodule belongs to Δ. The functor assigning to each bimodule $\mathcal{M}$ this quotient is right adjoint to the embedding of $\Delta\tilde{}$ into $\mathcal{R} - bi$. ∎

1.7.2. Note. It follows from the projectivity of $\mathbf{1}$ in $\mathcal{C}$ that $\mathcal{R}$ is a projective $\mathcal{R}$-module.

In fact, any $\mathcal{R}$-module morphism $f : \mathcal{R} \longrightarrow \mathcal{M} = (M, m)$ is uniquely determined by the composition of f and the unity $e : \mathbf{1} \longrightarrow R$. If $g : (N, \nu) \longrightarrow (M, m)$ is an $\mathcal{R}$-module epimorphism, then, by hypothesis, there exists an arrow $u : \mathbf{1} \longrightarrow N$ such that $g \circ u = f \circ e$. Clearly the map $f' = \nu \circ Ru : R \longrightarrow N$ is an $\mathcal{R}$-module morphism which lifts f : $g \circ f' = f$. ∎

1.7.3. Example. If $\mathcal{C}\tilde{}$ is the monoidal category of Γ-graded k-modules (cf. Example 1.0.4.2), the condition of Lemma 1.7.1 holds. ∎

2. β-Differential monads and localizations.

Now we will discuss the compatibility of β-differential monads with localizations. We begin with a general observation.

2.1. Proposition. *(a) Let $\mathbb{S}$ be a coreflective subcategory of $\mathfrak{End}\mathcal{A}$. And let $F : \mathcal{A} \longrightarrow \mathcal{A}$ be an $\mathbb{S}$-object (i.e. $F \in Ob\mathbb{S}^{\infty}$). Suppose that F is an exact functor. Then, for any Serre subcategory $\mathbb{T}$ stable under all functors from $\mathbb{S}$, the functor F induces an endofunctor $F_{\mathbf{T}}$ in the quotient category $\mathcal{A}/\mathbb{T}$.*

If the localization $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ has a right adjoint, then $F_{\mathbb{T}} \in \mathfrak{End}\mathcal{A}_{\mathbf{T}}$, i.e. $F_{\mathbf{T}}$ has a right adjoint.

(b) Let $\mathbb{F} = (F, \mu)$ be an $\mathbb{S}$-monad such that the functor F is exact.

Then, for any Serre subcategory $\mathbb{T}$ stable under all functors from $\mathbb{S}$, the monad $\mathbb{F}$ induces a monad $\mathbb{F}_{\mathbf{T}}$ in the quotient category $\mathcal{A}/\mathbb{T}$ and a canonical exact and faithful functor $\Psi_{\mathbb{T}} : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbb{T}) \longrightarrow \mathbb{F}_{\mathbb{T}} - mod$.

If the subcategory $\mathbb{T}$ is 'localizable' (i.e. the localization $\mathcal{A} \longrightarrow \mathcal{A}/\mathbb{T}$ has a right adjoint), then the functor $\Psi_{\mathbf{T}}$ is an equivalence of categories.

Proof. The fact follows from Propositions I.6.1. ∎

We shall analyze the stability conditions of Proposition 2.1 in the case when $\mathcal{A}$ is the category $\mathcal{R} - mod$ for some associative algebra $\mathcal{R}$ in a monoidal category $\mathcal{C}\tilde{} = (\mathcal{C}, \odot, \mathbf{1})$ and $\mathbb{S}$ is the subcategory $Art_{\beta}\mathcal{R}$ of β-artinian $\mathcal{R}$-bimodules (cf. the end of Section 1.3.1 and 1.3.3).

2.2. Lemma. *Let $\mathfrak{X}$ be a set of generators of $\mathcal{C}$. And let $\mathbb{T}$ be a full coreflective subcategory of $\mathcal{R} - mod$ containing with each object all its quotients (in $\mathcal{R} - mod$) and stable under the functors*

$$X\odot : \mathcal{R} - mod \longrightarrow \mathcal{R} - mod, \quad (m, M) \mapsto (X \odot m \circ \beta\hat{}_{R,X} M, X \odot M)$$

for all $X \in \mathfrak{X}$. *Then* $\mathbb{T}$ *is* Art_β*-stable.*

Proof. For any $Y \in Ob\mathcal{C}$, there is a natural functor isomorphism: $(Y \odot \mathcal{R})\odot_{\mathcal{R}} \simeq Y\odot$. Therefore [$\mathbb{T}$ is $\mathfrak{X}$-stable] $\Leftrightarrow$ [$\mathbb{T}$ is stable under the functors $(X\odot\mathcal{R})\odot_{\mathcal{R}}$ for all $X \in \mathfrak{X}$]. But then, being a coreflective subcategory of $\mathcal{R} - mod$, $\mathbb{T}$ is stable under colimits of functors $(X \odot \mathcal{R})\odot_{\mathcal{R}}$, $X \in \mathfrak{X}$. Since $\mathfrak{X}$ is a set of generators on $\mathcal{C}$, any β-artinian $\mathcal{R}$-bimodule is a colimit of bimodules $\{X \odot \mathcal{R},\ X \in \mathfrak{X}\}$. Therefore, $\mathbb{T}$ is stable under $\mathcal{M}\odot_{\mathcal{R}}$ for any $\mathcal{M} \in Ob\Delta_\beta$. ∎

2.3. Example. Let Γ be an abelian group, k a commutative ring. And let R be a Γ-graded associative k-algebra. In other words, R is an algebra in the monoidal category $\mathcal{C}^{\tilde{}}$ of Γ-graded k-modules. Suppose that β is the standart symmetry.

For any Γ-graded R-module $M = \oplus_{\gamma\in\Gamma} M_\gamma$ and any $\nu \in \Gamma$, denote by $M(\nu)$ the translation of M by ν : $M(\nu)_\gamma := M(\nu + \gamma)$ for all $\gamma \in \Gamma$. It follows from Lemma 2.2 that a Serre subcategory $\mathbb{T}$ of $R - mod$ is Δ_β-stable iff it is stable with respect to translations; i.e. if $M \in Ob\mathbb{T}$, then $M(\nu) \in Ob\mathbb{T}$ for all $\nu \in \Gamma$. ∎

We have the following analog of Proposition I.6.3.1:

2.4. Proposition. *Let* $\mathfrak{X}$ *be a set of generators of* $\mathcal{C}$.

(a) Let M *be a strongly differential* R*-bimodule. If* M *is flat as a right* R*-module, then, for any Serre subcategory* $\mathbb{T}$ *of* $R - mod$ *stable under the functors* $X\odot$, $X \in Ob\mathcal{C}$, *the functor* $M\odot_R$ *induces a functor* $M_{\mathbb{T}} : R - mod/\mathbb{T} \longrightarrow R - mod/\mathbb{T}$.

(b) Let $R \longrightarrow A$ *be an algebra morphism such that* A *is a strongly differential* R*-bimodule flat as a right* R*-module. Then, for any Serre subcategory* $\mathbb{T}$ *of the category* $R - mod$, *stable under the functors* $X\odot$, $X \in \mathfrak{X}$, *the algebra* A *induces a monad,* $A_{\mathbb{T}}$, *on* $R - mod/\mathbb{T}$.

Proof. The fact follows from Proposition 2.1 and Lemma 2.2. ∎

There is also a direct generalization of Proposition I.6.3.2:

2.5. Proposition. *Let* $R \longrightarrow R'$ *be an algebra morphism such that the functor* $Q = R'\odot_R$ *is an exact localization. Then*

(a) Any strongly differential R*-bimodule* M *which is flat as a right* R*-module determines a strongly differential* R'*-bimodule* $M' = R' \odot_R M \odot_R R'$. *And* M' *is isomorphic to* $R' \odot_R M$ *as* (R', R)*-bimodules.*

(b) If $M \in ObArt^{(n)}_{\beta,R}$, *i.e. if* M *is a strongly differential* R*-module of the order* $\leq n$, *then the* R'*-module* M' *has the order* $\leq n$: $M' \in ObArt^{(n)}_{\beta,R'}$.

(c) Let $R \longrightarrow A$ *be an algebra morphism such that* A *is a strongly differential* R*-bimodule flat as a right* R*-module. Then* $R' \odot_R A$ *has a unique algebra structure such that the canonical maps* $A \longrightarrow R' \odot_R A \longleftarrow R'$ *are algebra morphisms. And* $R' \odot_R A$ *is a strongly differential* R'*-bimodule.*

Proof. 1) Let M be an R-bimodule. By Lemma I.6.2.3, the functor $M\odot_R$ is compatible with the localization $Q : R - mod \longrightarrow R - mod/\mathbb{S}$ iff the canonical morphism

$$Q \circ (M\odot_R) \longrightarrow Q \circ (M\odot_R) \circ Q\hat{} \circ Q \qquad (1)$$

is an isomorphism. In the case when $R - mod/\mathbf{S} = R' - mod$ for some algebra R', hence Q can be taken equal to $R'\odot_R$, the isomorphness of (1) means that the canonical R', R-bimodule morphism

$$R' \odot_R M \longrightarrow R' \odot_R M \odot_R R' \tag{2}$$

is an isomorphism.

(a) Let M be a strongly differential R-bimodule. By Proposition 2.4, the functor $M\odot_R$ induces a functor $M_{\mathbb{T}}$, where $\mathbb{T}$ is the kernel of the localization Q. Since $Q = R'\odot_R$ for some k-algebra morphism $R \longrightarrow R'$, the canonical morphism (2) is an isomorphism. This proves the assertion *(a)*.

(b) The assertion *(b)* follows from the fact that the functor $R'\odot_R$, being a localization, is exact and, for any R'-module L, the natural R'-module morphism $R' \odot_R L \longrightarrow L$ is an isomorphism. In particular, we have: $R' \odot_R R' \simeq R' \simeq R'\odot_R$R. Therefore if $M \in Ob Art_R$, i.e. M is a colimit of a diagram of functors $X \odot R$, $X \in \mathfrak{X}$, then $R' \odot_R M$ is a colimit of the corresponding diagram of bimodules $X \odot R'$. The rest of the proof is a standart induction argument which goes through thanks to the exactness of of the localization $R'\odot_R$. Details are left to a reader.

(c) The fact follows from *(a)* and the assertion *(b)* of Proposition 2.4. ∎

2.6. The β-commutative case. Propositions 2.4 and 2.5 provide the following assertion.

2.6.1. Proposition. *Let $\mathfrak{X}$ be a set of generators of C. Let $\mathcal{R}$ be a β-commutative algebra. And let M be a differential $\mathcal{R}$-bimodule which is flat as a right $\mathcal{R}$-module. Then*

(a) For any Serre subcategory $\mathbb{T}$ of the category $\mathcal{A} := \mathcal{R} - mod$, stable with respect to the functors $X\odot$, $X \in \mathfrak{X}$ the functor $M\odot_{\mathcal{R}}$ induces a (unique up to isomorphism) differential functor $M_{\mathbb{T}} : \mathcal{A}/\mathbb{T} \longrightarrow \mathcal{A}/\mathbb{T}$.

(b) If the quotient category $\mathcal{A}/\mathbb{T}$ is equivalent to $\mathcal{R}' - mod$ for some $\mathcal{R}$-algebra $\mathcal{R}'$, then $M_{\mathbb{T}}$ is isomorphic to the functor $\mathcal{R}' \odot_{\mathcal{R}} M\odot_{\mathcal{R}}$.

Proof. The fact follows from Propositions 2.4 and 2.5 and the coincidence, for a β-commutative ring $\mathcal{R}$, of the subcategory $Art_{\beta,\mathcal{R}}$ of artinian $\mathcal{R}$-bimodules and the β-diagonal $\Delta_\beta\mathcal{R}$ (cf. Proposition 1.3.3.1). ∎

2.7. Localization of differential actions in derived categories of categories of modules. Let R be an algebra in the monoidal category $C^{\tilde{}}$. Let $\mathcal{A} = R - mod$ – the category of left R-modules; and let $\mathcal{B} = R - bi$ – the category of R-bimodules. The natural action

$$\mathcal{B} \times \mathcal{A} \longrightarrow \mathcal{A}, \quad (M, N) \mapsto M \odot_R N$$

is an action of the monoidal category of R-bimodules, $\mathcal{B}^{\tilde{}} = (\mathcal{B}, \odot_R, R)$, on $\mathcal{A}$. This action induces an action

$$\Phi : \mathfrak{D}^-(\mathcal{B}) \times \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})$$

of the monoidal derived category $\mathfrak{D}^-(\mathcal{B})^{\tilde{}}$ of the bounded from above complexes over $\mathcal{B}$ on $\mathfrak{D}^-(\mathcal{A})$.

Fix a Serre subcategory $\mathbf{S}$ of $\mathcal{A}$. And let $\mathcal{B}_{\mathbf{S}}$ denote the full subcategory of $\mathcal{B}$ generated by all R-bimodules M such that the functor $M\odot_R$ preserves $\mathbb{S}$. Denote by $\mathfrak{D}^-\mathbb{S}$ the full subcategory of $\mathfrak{D}^-(\mathcal{A})$ generated by all complexes X of R-modules such that $H^n(X) \in Ob\mathbb{S}$

for all n.. The category $\mathfrak{D}^-\mathbb{S}$ is a thick subcategory of $\mathfrak{D}^-(\mathcal{A})$. By a standart argument (using spectral sequence) one can show that the action of the subcategory $\mathfrak{D}^-\mathcal{B}_{\mathbf{S}}$ of $\mathfrak{D}^-(\mathcal{B})$ preserves $\mathfrak{D}^-\mathbb{S}$; i.e. the restriction to $\mathfrak{D}^-\mathcal{B}_{\mathbf{S}} \times \mathfrak{D}^-\mathbb{S}$ of the functor $\odot_R^L : \mathfrak{D}^-(\mathcal{B}) \times \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})$ takes values in $\mathfrak{D}^-\mathbf{S}$.

One of consequences of this fact is the following generalization of Proposition I.6.4.1:

2.7.1. Proposition. *For any Serre subcategory $\mathbb{S}$ of $\mathcal{A} = R - mod$, the action of $\mathfrak{D}^-\mathcal{B}_{\mathbf{S}}$ on $\mathfrak{D}^-(\mathcal{A})$ induces an action of $\mathfrak{D}^-\mathcal{B}_{\mathbf{S}}$ on the quotient triangulated category*

$$\mathfrak{D}^-\mathcal{B}_{\mathbf{S}} \times \mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbf{S} \longrightarrow \mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbf{S}.$$

2.7.2. Proposition. *Let $\mathfrak{X}$ be a set of generators of the category $\mathcal{C}$. And let $\mathbb{S}$ be any Serre subcategory of $\mathcal{A} = R - mod$. stable with respect to the functors $X\odot$ for all $X \in \mathfrak{X}$ Then there is a natural action of the category $Art^{\infty}_{\beta,R}$ of strongly β-differential R-bimodules on $\mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{S}$.*

Proof. Thanks to the stability of $\mathbf{S}$ with respect to the endofunctors $X\odot : \mathcal{A} \longrightarrow \mathcal{A}$, the subcategory $Art^{\infty}_{\beta,R}$ is contained in the subcategory $\mathcal{B}_{\mathbf{S}}$. ∎

2.7.3. Proposition. *Let $\mathbb{F} = (F, \mu)$ be an algebra in $\mathfrak{D}^-\mathcal{B}_{\mathbf{S}}$ (i.e. $\mathbb{F}$ is an algebra in the monoidal category $\mathfrak{D}^-(\mathcal{B})^{\sim}$ such that $F \in Ob\mathfrak{D}^-\mathcal{B}_{\mathbf{S}}$). Then $\mathbb{F}$ determines a monad $\mathbb{F}_{\mathbf{S}} = (F_{\mathbf{S}}, \mu_{\mathbf{S}})$ on $\mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{S}$.*

A localization $Q : \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{S}$ induces an equivalence of triangulated categories

$$\Psi : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathfrak{D}^-\mathbb{T}) \longrightarrow \mathbb{F}_{\mathbf{S}} - mod,$$

where $\mathfrak{F}$ is a forgetting functor $\mathbb{F} - mod \longrightarrow \mathfrak{D}^-(\mathcal{A})$.

Proof. The assertion can be proved by the argument used for a similar statement in Proposition I.6.2.2. ∎

Denote by ${}^{s}\mathfrak{D}^-(\mathcal{B})$ the full subcategory of $\mathfrak{D}^-(\mathcal{B})$ generated by all complexes X of R-bimodules such that $H^n(X)$ is a strongly differential bimodule for all n.

2.7.4. Corollary. *Let $\mathfrak{X}$ be a set of generators of the category $\mathcal{C}$. Let $\mathbb{F} = (F, \mu)$ be an algebra in $\mathcal{B} = R - bi$ such that F is a strongly differential R-bimodule. Then, for any Serre subcategory $\mathbb{T}$ of $\mathcal{A} = R - mod$ stable with respect to the endofunctors $X\odot$, $X \in \mathfrak{X}$, $\mathbb{F}$ induces a unique up to isomorphism monad $\mathbb{F}_{\mathbf{T}} = (F_{\mathbf{T}}, \mu_{\mathbb{T}})$ on the triangulated category $\mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{T}$.*

A localization $Q : \mathfrak{D}^-(\mathcal{A}) \longrightarrow \mathfrak{D}^-(\mathcal{A})/\mathfrak{D}^-\mathbb{T}$ induces an equivalence of triangulated categories

$$\Psi : \mathbb{F} - mod/\mathfrak{F}^{-1}(\mathbb{T}) \longrightarrow \mathbb{F}_{\mathbf{T}} - mod,$$

where $\mathfrak{F}$ is a forgetting functor $\mathbb{F} - mod \longrightarrow \mathfrak{D}^-(\mathcal{A})$.

2.8. Localization of differential operators. The (not necessarily strongly) β-differential bimodules are compatible with localizations given by $R'\odot_R$ for an algebra morphism $R \longrightarrow R'$ such that R' is a flat left R-module as well.

Fix a family $\mathfrak{X}$ of generators of the category C. We assume that R' satisfies the following property of $\mathfrak{X}$-stability:

2.8.0. Lemma. *Let $R \longrightarrow R'$ be an algebra morphism such that R' is a flat left R-module. And let $\mathfrak{X}$ be a family of generators of the category C. The following conditions are equivalent:*

(a) If $L \in ObR-mod$ is such that $R' \odot L = 0$, then $R' \odot (X \odot L) = 0$ for all $X \in \mathfrak{X}$.

(b) If $L \in ObR-mod$ is such that $R' \odot L = 0$, then $R' \odot (X \odot L) = 0$ for all $X \in ObC$.

Proof. This follows from the fact that the functor $R' \odot_R$ is compatible with colimits. The details are left to the reader. ∎

2.8.0.1. Example. Let Γ be an abelian group, k a commutative ring. Let $f : R \longrightarrow R'$ be a morphism of Γ-graded associative k-algebras. In other words, f is a morphism of algebras in the monoidal category $C^{\sim}$ of Γ-graded k-modules. For any graded R-module L and any $\gamma \in \Gamma$, denote by $L(\gamma)$ the Γ-graded R-module with the components $L(\gamma)_\nu = L(\gamma+\nu)$ for all $\nu \in \Gamma$. The equivalent conditions of Lemma 2.8.0 mean that $R' \odot_R L = 0 \Leftrightarrow R' \odot_R L(\gamma) = 0$ for all $\gamma \in \Gamma \Leftrightarrow R'(\gamma) \odot_R L = 0$ for all $\gamma \in \Gamma$. ∎

2.8.1. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor $Q = R' \odot_R$ is an exact localization and R' is flat as a left R-module too. And let R' satisfies the equivalent conditions of Lemma 2.8.0. Then*

(a) For any R-bimodule M which belongs to Δ^{-}_{β}, the functor $M \odot_R$ is compatible with the localization $Q = R' \odot_R$. Or, equivalently, the canonical (R', R)-bimodule morphism $R' \odot_R M \longrightarrow R' \odot_R M \odot_R R'$ is an isomorphism.

(b) If $M \in Ob\Delta^{(n)}_{\beta,R}$, i.e. if M is a β-differential R-bimodule of n-th order, then the R'-bimodule M' has the same order: $M' \in Ob\Delta^{(n)}_{\beta,R'}$.

(c) Let $\varphi : R \longrightarrow A$ be a differential algebra (i.e. φ is an algebra morphism turning A into a β-differential R-bimodule). Then $R' \odot_R A$ has a unique algebra structure such that the canonical arrows $A \longrightarrow R' \odot_R A \longleftarrow R'$ are algebra morphisms. And $R' \odot_R A$ is a differential R'-bimodule.

Proof. (a) Consider the full subcategory Ξ of $R-bi$ generated by all modules M such that the canonical (R', R)-bimodule morphism

$$R' \odot_R M \longrightarrow R' \odot_R M \odot_R R' \tag{1}$$

is an isomorphism. It follows from the exactness of the functors $R' \odot_R$ and $\odot_R R'$ that Ξ is a Serre subcategory of the category $R-bi$. Since Ξ contains the R-bimodule R, it contains the Serre subcategory $\Delta^{-}_{\beta,R}$. According to the part 1) of the proof of Proposition 2.5, the functor $M \odot_R$ is compatible with the localization $R' \odot_R$ if and only if the morphism (1) is an isomorphism. This proves the assertion *(a)*.

The assertions *(b)* and *(c)* are proved by the same argument as the corresponding assertions of Proposition I.6.3.2. ∎

2.8.2. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R' \odot_R : R-mod \longrightarrow R'-mod$$

is an exact localization.

(a) Let M' be an artinian R'-bimodule. And let $M := Art_{\beta,R}Q\hat{}(M')$ (i.e. M is the β-artinian part of the R-bimodule obtained from M' by restriction of scalars). Then the canonical morphism $\varphi : R' \odot_R M \longrightarrow M'$ is an isomorphism of R'-bimodules.

(b) Let $M' \in \Delta_{\beta,R'}$ And let $M := \Delta_R Q\hat{}(M')$ (i.e. M is the $\Delta_{\beta,R}$-part of the R-bimodule obtained from M' by restriction of scalars). Then the canonical morphism $\varphi : R' \odot_R M \longrightarrow M'$ is an isomorphism of R-bimodules.

Proof. The proof is an adaptation of the argument of Proposition I.6.5.2.

(a) Let M' be any artinian R'-bimodule; i.e. there exists an R'-bimodule epimorphism $X \odot R' \longrightarrow M'$ for some $X \in Ob\mathcal{C}$. We can include this epimorphism into a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & X \odot R' & \longrightarrow & M' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & K & \longrightarrow & X \odot R & \longrightarrow & M & \longrightarrow & 0 \end{array} \tag{1}$$

Here the upper row is regarded as a sequence of R-bimodule morphisms; K is the pullback of the corresponding morphisms. Thus M is an artinian R-bimodule, and in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K' & \longrightarrow & X \odot R' & \longrightarrow & M' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & R' \odot_R K & \longrightarrow & R' \odot_R X \odot R & \longrightarrow & R' \odot_R M & \longrightarrow & 0 \end{array} \tag{2}$$

the central vertical arrow is, obviously, an isomorphism. Since $R' \odot_R$ is an exact localization, the canonical epimorphism $R' \odot_R L \longrightarrow L$ is an isomorphism for any R'-module L; and $R' \odot_R$ sends universal squares into universal squares. Therefore the left vertical arrow is an isomorphism too. This implies, since both rows of (2) are exact, that the right vertical arrow is an isomorphism.

(b) Let now K' be any R'-bimodule from the diagonal $\Delta_{\beta,R'}$. According to Proposition I.5.11.4.1, K is a submodule of an artinian R'-bimodule M'. By (i), there exists an artinian R-bimodule M and an R-bimodule monomorphism $M \longrightarrow M'$ such that the canonical R'-bimodule morphism $R' \odot_R M \longrightarrow M'$ is an isomorphism. Let K be a pull-back of the R-bimodule morphisms $K' \longrightarrow M' \longleftarrow M$. Then K is an R-subbimodule of M, hence $K \in Ob\Delta_{\beta,R}$; and the canonical (R', R)-bimodule morphism $R' \odot_R K \longrightarrow K'$ is an isomorphism (cf. the argument in *(a)*). ∎

2.8.2.1. Corollary. *Let $R \to R'$ is as in Proposition 2.8.2. Then, for any R'-bimodule M', the canonical morphisms*

$$R' \odot_R Art_{\beta,R}Q\hat{}(M') \longrightarrow Art_{\beta,R'}(M') \quad \textit{and} \quad R' \odot_R \Delta_{\beta,R}Q\hat{}(M') \longrightarrow \Delta_{\beta,R'}(M')$$

are isomorphisms.

Proof. In fact, the canonical R'-module morphism $R' \odot Art_{\beta,R}Q\hat{}(M') \longrightarrow M'$ is a monomorphism (since $Q = R' \odot_R$ is a localization) and, by Proposition 6.3.2, its image is

contained in $Art_{\beta,R'}M'$. Now it follows from the assertion *(a)* of Proposition 2.8.2 that $R' \odot_R Art_{\beta,R}Q\hat{}(M') \longrightarrow Art_{\beta,R'}(M')$ is an isomorphism.

Similar argument (using the assertion *(b)* of Proposition 2.8.2) shows that $R' \odot \Delta_{\beta,R}Q\hat{}(M') \longrightarrow \Delta_{\beta,R'}(M')$ is an isomorphism. ∎

2.8.2.2. Note. If $f : R \longrightarrow R'$ is any morphism of β-commutative algebras, then, for any β-artinian (or β-differential) R'-bimodule M', the R-bimodule $M = f_\# M'$ obtained by restriction of scalars is β-artinian (resp. β-differential) too. More generally, for any R'-bimodule M' and for any nonnegative n, $f_\#(\Delta^{(n)}_{\beta,R'}M) \subseteq \Delta^{(n)}_{\beta,R}(f_\# M)$ and, therefore, $f_\#(M'_{diff}) \subseteq f_\#(M)_{diff}$. This follows from the observation that, as a set, $\Delta^{(n)}_{\beta,R'}M$ is the biggest subbimodule of M annihilated by $K^n_{R'}$, where $K_{R'}$ is the kernel of the multiplication $R' \odot R'^{\beta} \longrightarrow R'$, and $f \odot f(K_R) \subseteq K_{R'}$.

Clearly, in the case of a β-commutative R, Proposition 2.8.2 is a consequence of this fact. ∎

2.8.3. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R' \odot_R : R - mod \longrightarrow R' - mod$$

is an exact localization. Assume that R' is flat as a left R-module.

Let M be an R-bimodule, $M' := R' \odot_R M \odot_R R'$. If the natural morphism $M \longrightarrow M'$ is injective, then, for any $n \geq 0$,

(a) The morphism $R' \odot_R \Delta^{(n)}_{\beta,R}M \longrightarrow \Delta^{(n)}_{\beta,R'}M'$ is an isomorphism. In particular, $R' \odot_R M_{diff} \longrightarrow M'_{diff}$ is an R-bimodule isomorphism.

(b) The morphism $R' \odot_R Art^{(n)}_{\beta,R}M \longrightarrow Art^{(n)}_{\beta,R'}M'$ is an isomorphism. In particular, the map $R' \odot_R Art^{\infty}_{\beta,R}(M) \longrightarrow Art^{\infty}_{\beta,R'}(M')$ is an R'-bimodule isomorphism.

Proof. (i) By Proposition 2.8.2, $R' \odot_R \Delta_{\beta,R}(Q\hat{}(M')) \longrightarrow \Delta_{\beta,R'}M'$ is an isomorphism. Let $\mathcal{M}$ be the image of the canonical morphism $M \longrightarrow R' \odot_R M \odot_R R' = M'$. Clearly $\Delta_{\beta,R}(\mathcal{M}) = \mathcal{M} \cap \Delta_{\beta,R}(Q\hat{}(M'))$. The functor $\Omega : L \longmapsto R' \odot_R L \odot_R R'$, being exact, respects pull-backs. In particular, it respects intersections. Note that

$$R' \odot_R \mathcal{M} \odot_R R' \cap R' \odot_R \Delta_{\beta,R}(Q\hat{}(M')) = R' \odot_R \Delta_{\beta,R}(Q\hat{}(M'))$$

It follows that in the commutative diagram

$$\begin{array}{ccc} R' \odot_R \Delta_{\beta,R}(\mathcal{M}) \odot_R R' & \longrightarrow & R' \odot_R \Delta_{\beta,R}((Q\hat{}(M')) \\ \uparrow & & \downarrow \\ R' \odot_R \Delta_{\beta,R}(\mathcal{M}) & \xrightarrow{\phi} & \Delta_{\beta,R'}M' \end{array}$$

both vertical arrows and the upper horizontal arrow are isomorphisms. Therefore the morphism $\phi : R' \odot_R \Delta_{\beta,R}\mathcal{M} \longrightarrow \Delta_{\beta,R'}M'$ is an isomorphism.

(ii) Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R' \odot_R \Delta_{\beta,R}M & \longrightarrow & M' & \longrightarrow & R' \odot_R (M/\Delta_{\beta,R}M) \odot_R R' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow id & & \downarrow & & \\ 0 & \longrightarrow & \Delta_{\beta,R'}M' & \longrightarrow & M' & \longrightarrow & M'/\Delta_{\beta,R'}M' & \longrightarrow & 0 \end{array} \qquad (1)$$

The both rows in (1) are exact. The left vertical arrow is an isomorphism by (i). Therefore the right vertical arrow, $R' \odot_R (M/\Delta_{\beta,R}M) \odot_R R' \longrightarrow M'/\Delta_{\beta,R'}M'$, is an isomorphism. Besides, the injectivity of the morphism $M \longrightarrow M'$ implies that the morphism $M/\Delta_{\beta,R}M \longrightarrow M'/\Delta_{\beta,R'}M'$ is injective. Now it follows by an induction argument that $R' \odot_R \Delta^{(n)}_{\beta,R}M \longrightarrow \Delta^{(n)}_{\beta,R'}M'$ is an isomorphism for all n. The latter implies that $R' \odot_R M_{diff} \longrightarrow M'_{diff}$ is an R-bimodule isomorphism.

The similar argument proves the assertion *(b)*. ∎

2.8.4. Localization of differential operators and coherent modules. An object X of the monoidal category $\mathcal{C}^{\sim} = (calC, \odot, \mathbf{1})$ is called *finite* if the functor $\mathcal{H}om(X, -) : \mathcal{C} \longrightarrow \mathcal{C}$ is isomorphic to the functor $X' \odot -$ for some object X'. Note that the object X' is isomorphic to the dual object to X': $X' \simeq \mathcal{H}om(X, \mathbf{1})$; in particularly, it is defined uniquely up to isomorphism. We call an R-module L *finite* if $L \simeq R \odot X$ for some finite object X of $\mathcal{C}^{\sim}$.

We call an R-module L *coherent* if there exists an exact sequence $F_1 \longrightarrow F_0 \longrightarrow L$ of R-modules with F_0 and F_1 finite.

2.8.4.0. Note. In order to formulate the next proposition, we need a couple of observations.

1) Fix left R-modules L and N. If N is a $R \odot S^{\beta}$-module for some ring S, then $Hom(L, N)$, $Diff_n(L, N)$ $Diff^s_n(L, N)$, $Diff(L, N)$, and $Diff^s(L, N)$ have a natural structure of a $R \odot R^{\beta} \odot S^{\beta}$-module.

2) Suppose now that N is an artinian R-bimodule. Then, for any algebra morphism $R \to R'$, the (R', R)-bimodule $R' \odot_R N$ is an artinian R'-bimodule and $R' \odot_R N \simeq N \odot_R R'$. In particular, $Hom(L, N)$, $Diff_n(L, N)$, $Diff^s_n(L, N)$, $Diff(L, N)$, and $Diff^s(L, N)$ have a natural structure of a $R \odot R^{\beta} \odot R'^{\beta}$-module. ∎

2.8.4.1. Proposition. *Suppose that finite objects of $\mathcal{C}^{\sim}$ form a class of generators of $\mathcal{C}$. Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$Q = R' \odot_R : R - mod \longrightarrow R' - mod$$

is an exact localization and R' is a flat left R-module. Let L be a coherent R-module (i.e. there exists an exact sequence $F_1 \longrightarrow F_0 \longrightarrow L \longrightarrow 0$, where F_i are free modules of finite type). Then, for any artinian R-module N, the natural R-bimodule morphism

$$\mathcal{H}om(L, N) \longrightarrow \mathcal{H}om(R' \odot_R L, R' \odot_R N) \tag{1}$$

induces, for all $n \geq 0$, isomorphisms

$$R' \odot_R Diff^s_n(L, N) \odot_R R' \longrightarrow Diff^s_n(R' \odot_R L, R' \odot_R N). \tag{2}$$

In particular, we have an R'-bimodule isomorphism

$$R' \odot_R Diff^s(L, N) \odot_R R' \longrightarrow Diff^s(R' \odot_R L, R' \odot_R N).$$

Here $Diff^s$ (resp. $Diff^s_n$) denotes strongly differential operators (resp. strongly differential operators of order no greater than n).

Proof. Set for convenience $L' := R' \odot_R L$ and $N' := R' \odot_R N$.
For any R'-bimodule X, we have a canonical isomorphism

$$\mathcal{H}om_{R' \odot R'^{\beta}}(X, \mathcal{H}om(L', N') \longrightarrow \mathcal{H}om_{R'}(X \odot_{R'} L, N'). \tag{3}$$

1) Assume that $L = R \odot P$ for a finite object P. Then the right hand side of (1) is $\mathcal{H}om_{R'}(X \odot V, N')$.

(i) Suppose in addition that $X = R' \odot V$ for some finite V. Then the right hand side of (3) is isomorphic to $\mathcal{H}om_{R'}(R' \odot V \odot P, N') \simeq (V \odot P) \odot N'$. Here the second N' is regarded as a right R'-module. Recall that, since N is artinian, $N' \simeq N \odot_R R'$ (cf. Note 2.8.4.0). We have a canonical isomorphism

$$\mathcal{H}om_{R \odot R^{\beta}}(R \odot V, \mathcal{H}om(L, N')) \longrightarrow (V \odot P)^* \odot N' = (V \odot P)^* \odot N \odot_R R'$$

constructed in a similar way, and the diagram

$$\begin{array}{ccc} \mathcal{H}om_{R \odot R^{\beta}}(R \odot V, \mathcal{H}om(L, N')) & \longrightarrow & (V \odot P)^* \odot N' = (V \odot P)^* \odot N \odot_R R' \\ \phi \downarrow & & \downarrow id \\ \mathcal{H}om_{R' \odot R'^{\beta}}(R' \odot V, \mathcal{H}om(L', N')) & \longrightarrow & (V \odot P)^* \odot N' = (V \odot P)^* \odot N \odot_R R' \end{array} \tag{4}$$

is commutative. Here the right vertical arrow ϕ is a natural map. Since the horizontal arrows in (4) are isomorphisms, ϕ is an isomorphism. The map ϕ assigns to each R-bimodule morphism $f : X_0 := R \odot V \longrightarrow \mathcal{H}om(L, N')$ a unique R'-bimodule morphism $f' : R' \odot V \longrightarrow \mathcal{H}om(L', N')$ such that the diagram

$$\begin{array}{ccc} R' \odot_R X_0 & \xrightarrow{id_{R'} \odot_R f} & R' \odot \mathcal{H}om(L, N) \odot_R R' \\ id \downarrow & & \downarrow \\ R' \odot V & \xrightarrow{f} & \mathcal{H}om(L', N') \end{array} \tag{5}$$

Here we identify $R' \odot_R X_0 = R' \odot_R (m)R$ with $R' \odot V$.

This shows that, at least for a finite module L, the map (1) induces an isomorphism (2) for $n = 0$.

(ii) Assume now that the isomorphism (2) is established for a positive n. Let f be an R'-bimodule morphism $X \longrightarrow \mathcal{H}om(L', N')$ with $X \in ObArt_{\beta,R}^{(n+1)}$ being an R'-bimodule of finite type. There exists a short exact sequence

$$0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{e} X'' \longrightarrow 0$$

such that $X' \in ObArt_{\beta,R}^{(n)}$ and $X'' \in ObArt_{\beta,R}$. Note that we can assume that X'' is a finite artinian R'-bimodule; i.e. $X'' = R \odot V$ for some finite V.

In fact, since X'' is an artinian R'-bimodule of finite type, there exists a R'-bimodule epimorphism $\psi : R \odot V \longrightarrow X''$ for some finite m. Let Y denote the pullback of the arrows $X \xrightarrow{e} X'' \xleftarrow{\psi} R \odot V$. Note that the 'projection' $Y \longrightarrow R \odot V$ is an epimorphism and the

kernel of this projection is naturally isomorphic to X'. The other projection, $Y \longrightarrow X$, is an epimorphism too.

Thus we assume that $X'' = R \odot V$.

Consider the diagram with the exact row

$$\begin{array}{ccccccc} 0 \longrightarrow & X' \odot_R L & \xrightarrow{i'} & X \odot_R L & \xrightarrow{e'} & X \odot_R L & \longrightarrow 0 \\ & & & {\scriptstyle f'}\downarrow & & & \\ & & & N' & & & \end{array} \tag{6}$$

Here f' denotes the morphism dual to f; $i' := i \odot_R id_L$, $e' := e \odot_R id_L$. Since L is a finite R-module, $L = R \odot P$, $X'' \odot_R L$ is a finite module, $X'' \odot_R L \simeq R' \odot (V \odot P)$. Therefore the exact sequence in (6) splits: $X \odot_R L \simeq X' \odot_R L \oplus X'' \odot_R L$. The morphism f' is the product of morphisms $g' : X' \odot_R L \longrightarrow N' = N \odot_R R'$ and $h' : X'' \odot_R L \longrightarrow N' = N \odot_R R'$.

According to (i), the R'-bimodule morphism $h : X'' \longrightarrow \mathcal{H}om(L', N')$ dual to h' factors through

$$R' \odot_R Diff_0^s(L, N) \odot_R R' \longrightarrow \mathcal{H}om(L', N').$$

The morphism $g : X' \longrightarrow \mathcal{H}om(L', N')$ dual to g' factors through

$$R' \odot_R Diff_n^s(L, N) \odot_R R' \longrightarrow \mathcal{H}om(L', N')$$

by the induction hypothesis. Therefore the morphism f factors through

$$R' \odot_R Diff_{n+1}^s(L, N) \odot_R R' \longrightarrow \mathcal{H}om(L', N').$$

2) Assume now that L is an arbitrary finitely presented left R-module; i.e. there exists an exact sequence $F_1 \longrightarrow F_0 \longrightarrow L \longrightarrow 0$, where F_0 and F_1 are finite R-modules. Therefore we have an exact sequence of R'-bimodule morphisms

$$0 \longrightarrow \mathcal{H}om(L', N') \longrightarrow \mathcal{H}om(F_0', N') \longrightarrow \mathcal{H}om(F_1', N') \tag{7}$$

where $F_i' := R' \odot_R F_i$, $i = 0, 1$. Since the taking $Art_{\beta,R'}^{(n)}$-torsion is a left exact functor for all n (because it has a left adjoint), we obtain from (7) an exact sequence

$$0 \longrightarrow Diff_n^s(L', N') \longrightarrow Diff_n^s(F_0', N') \longrightarrow Diff_n^s(F_1', N') \tag{8}$$

for any nonnegative integer n. Since R' is a flat left and right R-module, we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & Diff_n^s(L',N') & \longrightarrow & Diff_n^s(F_0',N') & \longrightarrow & Diff_n^s(F_1',N') \\ & {\scriptstyle\phi}\uparrow & & {\scriptstyle\phi_0}\uparrow & & {\scriptstyle\phi_1}\uparrow \\ 0 \longrightarrow & R' \odot_R Diff_n^s(L,N) \odot_R R' & \longrightarrow & R' \odot_R Diff_n^s(F_0',N') \odot_R R' & \longrightarrow & R' \odot_R Diff_n^s(F_1',N') \odot_R R' \end{array} \tag{9}$$

The vertical arrows ϕ_i, $i = 0, 1$, are isomorphisms. Therefore ϕ is an isomorphism.

3) Since the functors $R'\odot_R$ and $\odot_R R'$ commute with colimits, it follows that the morphism $R' \odot_R Diff^s(L,N) \odot_R R' \longrightarrow Diff^s(R' \odot_R L, R' \odot_R N)$ is an isomorphism. ∎

2.8.4.2. Proposition. *Let $R \longrightarrow R'$ be an algebra morphism such that the functor*

$$R'\odot_R : R - mod \longrightarrow R' - mod$$

is an exact localization and R' is a flat left R-module (say the ring R' is the localization of R at a left Ore set). Then

(a) The action of $D^s(R)$ on R extends naturally to an action on R' giving a canonical ring homomorphism $D^s(R) \longrightarrow D^s(R')$ which induces an R'-bimodule isomorphism

$$R' \odot_R D^s(R) \odot_R R' \longrightarrow D^s(R').$$

(b) For any $D^s(R)$-module M, the R'-module $R' \odot_R M$ has a natural, in particular compatible with $D^s(R) \longrightarrow D^s(R')$, structure of a $D^s(R')$-module.

Proof. The assertion *(a)* follows from Proposition 2.8.4.1.
(b) The assertion *(b)* follows from *(a)*. ∎

2.8.4.3. Remark. If the ring R (hence R') in Proposition 2.8.4.2 is commutative, any R-module has a canonical structure of an artinian bimodule, and the canonical isomorphisms

$$R' \odot_R Diff_n^s(L,N) \odot_R R' \longrightarrow Diff_n^s(R' \odot_R L, R' \odot_R N).$$

and

$$R' \odot_R Diff^s(L,N) \odot_R R' \longrightarrow Diff^s(R' \odot_R L, R' \odot_R N).$$

can be replaced by left R'-module isomorphisms

$$R' \odot_R Diff_n^s(L,N) \longrightarrow Diff_n^s(R' \odot_R L, R' \odot_R N). \quad (1)$$

and

$$R' \odot_R Diff^s(L,N) \longrightarrow Diff^s(R' \odot_R L, R' \odot_R N). \quad (2)$$

In particular, we have a left R'-module isomorphism

$$R' \odot_R D^s(R) \longrightarrow D^s(R'). \quad (3)$$

The morphisms (1), (2) and (3) make sense in the noncommutative case. They are even R-bimodule morphisms. But, in general they are not isomorphisms (cf. Note 2.8.4.0). ∎

3. Differential operators on a symmetric affine space.

Let k be a commutative ring. A skew affine k-algebra is the k-algebra R generated by indeterminates $x_i, i \in J$, subject to the relations:

$$x_i x_j = q_{ij} x_j x_i \quad \text{for some } q_{ij} \in k^*,\ i,j \in J. \quad (1)$$

Here $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for all $i, j \in J$.

Let $\mathcal{C}$ be the monoidal category of $\mathbb{Z}^J$-graded k-modules with the product being the graded tensor product over k. And let β be the symmetry determined by the matrix $\mathbf{q} = (q_{ij})$ (cf. Example 1.2.1).

Note that the skew affine algebra R is a β-commutative algebra in the monoidal category $\mathcal{C}^{\tilde{}} = (\mathcal{C}, \otimes, k)$, where $\otimes = \otimes_k$. Our next objective is to describe the algebra $D_\beta(R)$ of β-differential operators.

3.1. Lemma. *The R-module Ω_β of 1-forms is a free module of the rank $\mid J \mid$.*

Proof. The claim is that $\Omega_\beta \simeq \oplus_{i\in J} Rd_i$, where d_i has the parity i for each $i \in J$. The isomorphism is given by

$$d_i \mapsto (x_i \otimes 1 - 1 \otimes x_i)\text{modulo}(\mathcal{J}_\mu^2), \tag{1}$$

for all $i \in J$. ∎

3.2. Lemma. *Suppose that J is finite. Then the R-module $Der_\beta(R)$ of β-derivations is a free R-module of the rank $\mid J \mid$. Explicitly, $Der_\beta(R) = \oplus_{i\in J} R\partial_i$, where ∂_i is a β-derivation of the parity -i uniquely defined by*

$$\partial_i(k) = \{0\}, \quad \partial_i(x_j r) = \delta_{ij} r + q_{ij} x_j \partial_i(r) \tag{2}$$

for all $j \in J$ and $r \in R$ (in particular, $\partial_i(x_j) = \delta_{ij}$).

Proof. By Proposition 1.4.1, $Der_\beta(R) \simeq \mathcal{H}\text{om}_R(\Omega_\beta, R)$; and

$$\mathcal{H}\text{om}_R(\Omega_\beta, R) \simeq \prod_{i\in J} \mathcal{H}\text{om}_R(Rd_i, R) \simeq \prod_{i\in J} R\partial_i',$$

where ∂_i' is a morphism such that $\partial_i'(d_j) = \delta_{ij}$. One can see that the corresponding to the morphism $\partial_i'\beta$-derivation ∂_i satisfies (is uniquely defined by) the conditions (2) of the lemma.

If J is finite, the product $\prod_{i\in J} R\partial_i'$ equals to the direct sum $\oplus_{i\in J} R\partial_i'$. ∎

3.3. Proposition. *Suppose that the base ring k is a field of zero characteristic and that J is finite. Then the algebra $D_\beta(R)$ of differential operators on R is generated by $Der_\beta(R)$ and R.*

Proof. Let $A_\beta(R)$ denote the subalgebra of $D_\beta(R)$ generated by R and $Der_\beta(R)$.

(a) Note that the $A_\beta(R)$-module R is simple.

In fact, since k is a field of zero characteristic, for any nonzero element (polynomial in $\mathbf{x}$) f, there exists a polynomial D in $\{\partial_i \mid i \in J\}$ such that Df is a nonzero element of k. Thus, for any $g \in k_\mathbf{q}[\mathbf{x}]$, we have: $(1/Df)gD(f)$=g.

(b) Denote by M_n the set of all monomials of degree $\leq$n. Since R is a simple $A_\beta(R)$-module, for any $B \in$End(R), there exists, by the Jacobson's density theorem, a $\partial \in D_\beta(R)$ such that the restrictions of B and ∂ to M_n coincide. Clearly, one can assume that ∂ is of order $\leq$n. If B is a differential operator of order $\leq n$, then the difference, $D = B - \partial$ is a differential operator of order $\leq n$ such that the restriction of D to M_n is zero.

(c) If D is a differential operator of order $\leq n$ such that the restriction of D to M_n is zero, then $D = 0$.

1) The fact is certainly true for $n = 0$, because the differential operators of degree 0 are multiplications by elements of R. And they are uniquely determined by their values at the identity element of R.

2) Let now D be a β-differential operator of order $n \geq 1$. And suppose that the restriction of D to M_m is zero, $m \geq$n. Let $b \in M_{m+1}$. Then $b = rc$, where $c \in M_m$ and $r = x_i$ for some i. We have:

$$D(rc) = (D(rc) - r\beta_r(D)(c)) + r\beta_r(D)(c) = [D, r]_\beta(c) + r\beta_r(D)(c) \tag{1}$$

where β_r is the automorphism acting as follows: if $r = x_i$, then $\beta_r(D_\mathbf{i}) = \prod_{j\in\mathbf{i}} q_{ji} D_\mathbf{i}$ for any multi-index $\mathbf{i}$. Note that $D(c) = 0$ for all $c \in M_m$ iff $D_\mathbf{i}(c) = 0$ for all $\mathbf{i} \in \mathbb{Z}^J$. Therefore, if $D(c) = 0$ for all $c \in M_m$, then $\beta_r(D)(c) = 0$ for all $c \in M_m$ and any $r = x_i$, $i \in J$. In particular, for any $c \in M_{m-1}, [D, r]_\beta(c) := D(rc) - r\beta_r(D)(c) = 0$. But the order of the differential operator $[D, r]_\beta$ is $\leq n - 1$. Since $m \geq n$, $[D, r]_\beta(c) = 0$ for all $c \in M_{n-1}$. By induction hypothesis, this implies that $[D, r]_\beta = 0$. Therefore it follows from (1) that $D(rc) = r\beta_r(D)(c) = 0$ for all $c \in M_m$ and $r \in M_1$; i.e. the restriction of D to M_{m+1} is zero. Hence $D = 0$. ∎

3.4. Note. One of the consequences of the proof of Proposition 3.3 is that any β-differential operator of order $\leq n$ is uniquely determined by its values on monomials in $\mathbf{x}$ of degree $\leq$n. ∎

3.5. Generators and relations in $D_\beta(R)$. The natural generators are x_i, ∂_j, $i, j \in J$. Here by x_i we mean the endomorphism of multiplication by x_i. We know the relations between different x_i, $i \in J$ (cf. (1)). The relations between x_i and ∂_j, $i, j \in J$, follow from Lemma 3.2:

$$\partial_i x_j - q_{ij} x_j \partial_i = \delta_{ij} \quad \text{for all } i, j \in J. \tag{3}$$

We shall prove (– the assertion (a) of Proposition 4.2.1) that the relations between ∂_i, $i \in J$, look as follows:

$$\partial_i \partial_j = q_{ji} \partial_j \partial_i, \quad \text{for all } i, j \in J. \tag{4}$$

Thus, $D_\beta(R)$ is generated by x_i, ∂_j, $i, j \in J$, subject to the relations:

$$x_i x_j = q_{ij} x_j x_i \tag{1}$$

$$\partial_i x_j - q_{ij} x_j \partial_i = \delta_{ij} \quad \text{for all } i, j \in J. \tag{3}$$

$$\partial_i \partial_j = q_{ji} \partial_j \partial_i \tag{4}$$

3.6. Playing with relations. For any $i \in J$, set $\xi_i = \partial_i x_i$. It follows from the relations of 3.5 that $\xi_i \xi_j = \xi_j \xi_i$ for all $i, j \in J$. Denote by A the algebra $k[(\xi_i)]$ of polynomials in $\xi_i, i \in J$. Define automorphisms $\theta_i, i \in J$, of the algebra A by the formulas:

$$\theta_i(\xi_j) = \xi_j \text{ if } i \neq j; \ \theta_i(\xi_i) = \xi_i + 1 \tag{1}$$

Then we can regard the algebra $D_\beta(R)$ of β-differential operators as a k-algebra generated by A and elements x_i, ∂_i subject to the relations:

$$x_i x_j = q_{ij} x_j x_i, \quad \partial_i \partial_j = q_{ji} \partial_j \partial_i \tag{2}$$

$$\partial_i x_i = \xi_i, \quad x_i \partial_i = \theta_i^{-1}(\xi_i) \tag{3}$$

$$\partial_i r = \theta_i(r) \partial_i, \quad r x_i = x_i \theta_i(r) \tag{4}$$

for all $i, j \in J$ and $r \in R$.

The relations (2), (3), (4) define an iterated hyperbolic ring in the sense of [R], Ch.IV. Fix an $m \in J$; and set $J_m := J - \{m\}$. Let $D_{\beta,m}$ be a subalgebra of D_β generated by $\{x_i, \partial_i \mid i \in J_m\}$ and ξ_m. Clearly $D_{\beta,m}$ is the algebra of β-differential operators on the $(q-)$subspace of R generated by $\{x_i \mid i \in J_m\}$.

We extend θ to an automorphism Θ of $D_{\beta,m}$ by setting

$$\Theta(x_i) = q_{mi} x_i, \quad \Theta(\partial_i) = q_{\mathrm{im}} \partial_i \quad \text{for all } i \in J_m.$$

Then ξ_m is a central element of $D_{\beta,m}$, and $D_\beta(R)$ is a hyperbolic ring over $D_{\beta,m}$ determined by the automorphism Θ_m and the central element ξ_m. This allows, in particular, to reduce (using the results of [R], Chapter IV) the study of $D_\beta(R)$-modules (more specifically, the left spectrum and irreducible representations of $D_\beta(R)$) to the study of $D_{\beta,m}$-modules.

3.7. Example: q-difference operators. Let, again, we have a symmetric matrix $\{q_{ij} \mid i, j \in J\}$ with entrees being invertible elements of a ring k. Consider the k-algebra $\mathcal{R} = k[\mathbf{x}]_{\mathbf{q}}[\mathbf{t}]$ generated by polynomial rings $k[(x_i)]$ and $k[(t_i)]$, where x_i and t_j satisfy the following relations:

$$x_i t_j = q_{ij} t_j x_i \quad \text{for all } i, j \in J. \tag{1}$$

Note that if $q_{ii} = q \in k^*$ for all $i \in J$ and $q_{ji} = 1$ when $i \neq j$, the algebra $\mathcal{R}$ coincides with the introduced by C. Sabbah *algebra of q-differences operators on the affine space* (cf. [Sa]).

Set $x_i t_i = \xi_i$. For any $i, j \in J$, we have:

$$\xi_i \xi_j := x_i t_i x_j t_j = q_{ji}^{-1} x_i x_j t_i t_j = q_{ji}^{-1} x_j x_i t_j t_i = q_{ji}^{-1} q_{ij} (x_j t_j)(x_i t_i) = \xi_j \xi_i.$$

Let A denote the k-algebra generated by commuting elements ξ_i, $i \in J$. There are no other relations between $\{\xi_i\}$; so that the algebra A is isomorphic to the algebra $k[\{\xi_i\}]$ of polynomials in $\{\xi_i \mid i \in J\}$ with coefficients in k.

For each $i \in J$, define the automorphism θ_i by the formula $\theta_i(\xi_j) = q_{ij} \xi_j$ for all $j \in J$. Then the ring $\mathcal{R}$ is a k-algebra generated by A, and elements $\{x_i, t_i \mid i \in J\}$ satisfying the relations:

$$x_i t_i = \xi_i, \quad t_i x_i = \theta_i^{-1}(\xi_i), \tag{2}$$

$$x_i a = \theta_i(a) x_i, \quad a t_i = t_i \theta_i(a) \tag{3}$$

for all $i \in J$ and $a \in$A. I.e. $\mathcal{R}$ is a hyperbolic ring over A.

Consider the k-algebra $\mathcal{R}^{\sim}$ generated by the algebras $k[\mathbf{x}, \mathbf{x}^{-1}]$ and $k[\mathbf{t}, \mathbf{t}^{-1}]$ of Laurent polynomials resp. in $\mathbf{x} = \{x_i\}$ and $\mathbf{t} = \{t_i\}$ with relations (1). This algebra is called (in [Sa]) *algebra of q-differences operators on the torus*. In terms of the elements $\{\xi_i\}$, the relations are (2) and (3). But this time the elements x_i and t_i (hence ξ_i) are invertible for all $i \in J$. So that the ring A of polynomials in ξ_i should be replaced by the algebra B of Laurent polynomials in ξ_i, and $t_i = x_i^{-1}\xi_i$ for all $i \in J$. This shows that the algebra $\mathcal{R}^{\sim}$ is isomorphic to the algebra of skew Laurent polynomials in $\mathbf{x} = \{x_i\}$ with coefficients in B. 'Skew' means the relation $x_i b = \theta_i(b)x_i$ for all $b \in B$ and $i \in J$.

3.7.1. A tensor-category viewpoint. Denote by $\mathcal{C}_2$ the category of $\mathbb{Z}^J \times \mathbb{Z}^J$-graded k-modules with the graded tensor product. Each object M of $\mathcal{C}$ can be regarded as a direct sum $M = M_0 \oplus M_1$ of $\mathbb{Z}^J$-graded k-modules. In other words, if $\mathcal{C}$ denotes the category of $\mathbb{Z}^J$-graded k-modules, $\mathcal{C}_2$ is the category of $\mathbb{Z}_2$-graded objects of $\mathcal{C}$.

We define the structure $\mathcal{C}_2^{\sim} = (\mathcal{C}_2, \otimes, \mathbf{1})$ of a monoidal category on $\mathcal{C}_2$, taking as $\otimes$ the graded tensor product and the module k (with zero grading) as the identity object $\mathbf{1}$. Now we define a symmetry β in $\mathcal{C}_2^{\sim}$ setting

$$\beta(x_i \otimes t_j) = q_{ij} t_j \otimes x_i; \quad \beta(x_i \otimes x_j) = x_j \otimes x_i, \quad \beta(t_i \otimes t_j) = t_j \otimes t_i \tag{1}$$

for all $i, j \in J$.

Note that the algebra $\mathcal{R} = k[\mathbf{x}]_{\mathbf{q}}[\mathbf{t}]$ of q-differences on the affine space is a particular case of a skew affine space.

4. Differential operators on quasi-symmetric affine spaces.

Fix a commutative ring k. Let $\mathbf{q} = (q_{ij})_{i,j \in J}$ be any matrix with entrees in k^*. Let $\mathcal{C}^{\sim} = (\mathcal{C}, \otimes, \mathbf{1})$ be the monoidal category of $\mathbb{Z}^J$-graded k-modules with the quasi-symmetry β defined by $\mathbf{q}$.

Denote by U_J, or simply by U, the free algebra generated by indeterminates $\{x_i \mid i \in J\}$ with the natural $\mathbb{Z}^J$-grading - the parity of x_i is the i-th generator of $\mathbb{Z}^J$.

4.1. Lemma. *(a) There is a natural isomorphism $Der_\beta(U) \simeq \mathrm{Maps}(J, U)$ of $\mathbb{Z}^J$-graded k-modules*

(b) $Der_\beta(U)_{-i} = k\partial_i$ for each $i \in J$, where the β-derivation ∂_i is (uniquely) defined by $\partial_i(x_j) = \delta_{ij}$ for all $j \in J$.

Proof is left to a reader. ∎

4.2. The algebra $\mathfrak{U}_{\mathbf{q}}$. The algebra $D_\beta(U)$ of β-differential operators on U (which contains U as a subalgebra and $Der_\beta(U)$ as a k-submodule) is huge. Denote by $\mathfrak{U}_{\mathbf{q}}$ the subalgebra of $D_\beta(U)$ generated by (multiplications by) x_i, $i \in J$, (hence containing the image of U in $D_\beta(U)$) and by the derivations ∂_i for all $i \in J$ (cf. Lemma 4.1).

On the other hand, consider the algebra $\mathfrak{A}_{\mathbf{q}}$ generated by indeterminates $x_i, y_i, i \in J$, subject to the relations

$$x_i y_j - q_{ij} y_j x_i = \delta_{ij} \quad \text{for all } i, j \in J. \tag{1}$$

There is a canonical epimorphism φ from $\mathfrak{A}_{\mathbf{q}}$ to $\mathfrak{U}_{\mathbf{q}}$ sending x_i into x_i and y_i into ∂_i. Moreover, φ is an epimorphism of $\mathbb{Z}^J$-graded algebras: we assign to each y_i, $i \in J$, the

parity $-i$. The relations (1) allow to express every element f of $\mathfrak{A}_q$ as a sum $\sum_{\mathbf{i}\in\mathbf{J}} f_{\mathbf{i}} y^{\mathbf{i}}$. Here $\mathbf{J}$ denotes the set of multi-indeces; and $f_{\mathbf{i}} \in U$ for any $\mathbf{i} \in \mathbf{J}$. One can see that the coefficients $f_{\mathbf{i}}$ are uniquely defined. By definition of φ, we have: $\varphi(f) = \sum_{\mathbf{i}\in\mathbf{J}} f_{\mathbf{i}} \partial^{\mathbf{i}}$.

Clearly $\varphi(\sum_{i\in J} f_i y^i) = \sum_{i\in J} f_i \partial^i = 0$ iff $f_i = 0$ for all $i \in J$, since $\sum_{i\in J} f_i \partial^i(x_j) = f_j$ for any $j \in J$. But, in general, the injectivity of φ might fail already at the next level as the following assertion shows.

4.2.1. Proposition. *(a) Suppose that $q_{ij}q_{ji} = 1$. Then $\partial_i\partial_j = q_{ji}\partial_j\partial_i$.*

(b) If $1 - q_{ij}q_{ji}$ is not a zero divisor, then the algebra $\mathfrak{A}_q$ has no quadratic relations involving ∂_i and ∂_j.

Proof. (a) The group homomorphism $\mathbb{Z}^J \longrightarrow \mathbb{Z}$, $(n_i) \mapsto \sum_{i\in J} n_i$, provides $\mathbb{Z}^J$-graded modules with $\mathbb{Z}$-grading. In particular, U becomes a $\mathbb{Z}_+$-graded algebra. And we have a $\mathbb{Z}_+$-filtration in U associated with the grading.

We shall prove the assertion *(a)* by induction on this filtration.

1) The equalities $\partial_i\partial_j(x_\nu) = 0 = q_{ij}\partial_j\partial_i(x_\nu)$ which hold for all $\nu \in J$ provide the first induction step.

2) Fix an $r \in U$. We have:

$$\partial_i\partial_j(x_\nu r) = \partial_i(\delta_{j\nu} r + q_{j\nu} x_\nu \partial_j(r)) = \delta_{j\nu}\partial_i(r) + q_{j\nu}(\delta_{i\nu}\partial_j(r) + q_{i\nu}\partial_i\partial_j(r)) \tag{2}$$

Hence

$$\partial_j\partial_i(x_\nu r) = \partial_j(\delta_{i\nu} r + q_{i\nu} x_j \partial_i(r)) = \delta_{i\nu}\partial_j(r) + q_{i\nu}(\delta_{j\nu}\partial_i(r) + q_{j\nu}\partial_j\partial_i(r)) \tag{3}$$

a) If $i \neq \nu \neq j$, then it follows from (2) and (3) that respectively

$$\partial_i\partial_j(x_\nu r) = q_{j\nu}q_{i\nu}\partial_i\partial_j(r) \text{ and } \partial_j\partial_i(x_\nu r) = q_{i\nu}q_{j\nu}\partial_j\partial_i(r) \tag{4}$$

So that if $\partial_i\partial_j(r) = q_{ji}\partial_j\partial_i(r)$, then $\partial_i\partial_j(x_\nu r) = q_{ji}\partial_j\partial_i(x_\nu r)$.

b) Suppose now that $\nu = i$. Then one obtains from (2) and (3) the equalities

$$\partial_i\partial_j(x_i r) = q_{ji}(\partial_j(r) + q_{ii}\partial_i\partial_j(r)) \tag{5}$$

$$\partial_j\partial_i(x_i r) = \partial_j(r) + q_{ii}q_{ji}\partial_j\partial_i(r) \tag{6}$$

If $\partial_i\partial_j(r) = q_{ji}\partial_j\partial_i(r)$, then if follows from (5) and (6) that

$$\partial_i\partial_j(x_i r) = q_{ji}(\partial_j(r) + q_{ii}q_{ji}\partial_j\partial_i(r)) = q_{ji}\partial_j\partial_i(x_i r).$$

c) Similarly, if $\nu = j$,

$$\partial_i\partial_j(x_j r) = \partial_i(r) + q_{jj}q_{ij}\partial_i\partial_j(r) \tag{7}$$

$$\partial_j\partial_i(x_j r) = q_{ij}(\partial_i(r) + q_{jj}\partial_j\partial_i(r)) \tag{8}$$

The equality $\partial_i\partial_j(r) = q_{ji}\partial_j\partial_i(r)$, together with (7) implies that

$$\partial_i\partial_j(x_j r) = \partial_i(r) + q_{jj}q_{ij}q_{ji}\partial_j\partial_i(r) \tag{9}$$

Since, by condition $q_{ji}q_{ij} = 1$,

$$\partial_i\partial_j(x_jr) = \partial_i(r) + q_{jj}\partial_j\partial_i(r).$$

Comparing with (8), we get $\partial_j\partial_i(x_jr) = q_{ij}\partial_i\partial_j(x_ir)$; or, equivalently,

$$\partial_i\partial_j(x_ir) = q_{ji}\partial_j\partial_i(x_ir).$$

This provides the second induction step.

(b) Let we have a relation $\sum_{i\in J} f_i\partial^i = 0$. As it was already observed, $f_i = 0$ for all $i \in J$. This implies that

$$0 = \sum_{i\in J} f_i\partial^i(x_ix_j) = f_{ij}\partial_i\partial_j(x_ix_j) + f_{ji}\partial_j\partial_i(x_ix_j) \tag{10}$$

and

$$0 = \sum_{i\in J} f_i\partial^i(x_jx_i) = f_{ij}\partial_i\partial_j(x_jx_i) + f_{ji}\partial_j\partial_i(x_jx_i). \tag{11}$$

It follows from (5)-(8) that

$$\partial_i\partial_j(x_ix_j) = q_{ji}, \quad \partial_j\partial_i(x_ix_j) = 1; \quad \partial_i\partial_j(x_jx_i) = 1, \quad \partial_j\partial_i(x_jx_i) = q_{ij}.$$

Thus (10) and (11) can be expressed as

$$f_{ij}q_{ji} + f_{ji} = 0 = f_{ij} + f_{ji}q_{ij}$$

which implies that

$$f_{ij}(q_{ji}q_{ij} - 1) = 0 = f_{ji}(q_{ij}q_{ji} - 1).$$

If $q_{ji}q_{ij} - 1$ is not a zero divisor, these equalities mean that $f_{ij} = 0 = f_{ji}$. ∎

4.3. Serre relations and a q-affine algebra. For any k-submodule W of U, denote by $\tilde{W}$ the k-submodule of W generated by all homogenious elements of W. Note that if W is stable with respect to a set $\mathfrak{X}$ of homogenious elements of $\mathfrak{End}(U_{\mathbf{q}})$, then such is $\tilde{W}$.

Consider the set Ξ of all homogenious two-sided ideals of U which are contained in the augmentation ideal (=the graded complement to $k = (U)_0$) and are stable under the derivations ∂_i for all $i \in J$. The sum S_+ of all ideals of Ξ is an ideal of Ξ which we call *the ideal of Serre relations* or simply *the Serre ideal.* Denote by $U_{\mathbf{q}}^+$, or simply U^+, the quotient algebra U/S^+. We call U^+ *the* **q***-affine algebra generated by* $\{x_i \mid i \in J\}$.

4.3.1. Example. Suppose that the matrix $\mathbf{q}$ defines a symmetry; i.e. $q_{ij}q_{ji} = 1$ for all $i, j \in J$. And let the base ring k be a field of zero characteristic. Then the two-sided ideal generated by $\{x_ix_j - q_{ij}x_jx_i \mid i, j \in J\}$ is ∂_j-stable for any $j \in J$. This means that the algebra $\mathfrak{U}_{\mathbf{q}}$ acts on the skew polynomial algebra $k_{\mathbf{q}}[\mathbf{x}]$. And according to the proof of Proposition 3.3, this action is irreducible.

Since the $\mathfrak{U}_{\mathbf{q}}$-module $k_{\mathbf{q}}[\mathbf{x}]$ is simple, the natural epimorphism $k_{\mathbf{q}}[\mathbf{x}] \longrightarrow U^+$ is an isomorphism. In other words, the **q**-affine algebra U^+ coincides in this case with the skew polynomial algebra $k_{\mathbf{q}}[\mathbf{x}]$. ∎

4.4. The quantum Weyl algebra. The ideal S^+ is stable with respect to the action of the algebra $\mathfrak{U}_{\mathbf{q}}$; hence $\mathfrak{U}_{\mathbf{q}}$ acts on the algebra U^+. We denote the image of $\mathfrak{U}_{\mathbf{q}}$ in $End(U^+)$ by $A_{\mathbf{q}}$ and call it *the quantum Weyl algebra* or *the* **q**-*Weyl algebra.*

It follows from the construction that the **q**-affine algebra U^+ is a simple left $A_{\mathbf{q}}$-module. Since the algebra $A_{\mathbf{q}}$ is generated by (multiplications by) elements of U^+ and derivations, it is a subalgebra of the algebra of β-differential operators on U^+ : $A_{\mathbf{q}} \subseteq D_\beta^\#(U^+)$.

4.5. Example: differential operators on the quantum line. The simplest possible example of a 'noncommutative space' is the 'quantum line'.

Let k be a field. The algebra of functions on a quantum line over k is the algebra $R = k[x]$ of polynomials in one variable regarded as an algebra in the category $\mathfrak{gr}_{\mathbf{Z}}Vec_k$ of $\mathbb{Z}$-graded k-vector spaces with the parity of x equal to 1. We define the quasi-symmetry β by (the necessary requirements) $\beta(1,0) = 1 = \beta(0,1)$, and $\beta(1,1) = q$ for some $q \in k^*$. Note that the algebra $k[x]$ is far from being β-commutative if $q \neq 1$ – the maximal β-commutative quotient algebra of $k[x]$ is the algebra $k[x]/(x^2)$ of double numbers.

The algebra $D_\beta^\#(R) = D_q^\#(R)$ is generated by (multiplications by elements of) R and the canonical β-derivation $\partial = \partial_q$ (having the parity -1). The latter happens to be the so called *q-derivation* - an operator acting on polinomials by the formula:

$$\partial = \partial_q : f(x) \mapsto (f(qx) - f(x))/x(q-1). \tag{1}$$

Thus $D_q^\#(R)$ is a k-algebra generated by x and ∂ subject to the relation:

$$\partial x - qx\partial = 1. \tag{2}$$

When $q = 1$, $D_q^\#(R)$ is the first Weyl algebra, i.e. $D_1^\#(R)$ is isomorphic to the algebra of differential operators on the one-dimensional affine space.

If the base field k is of zero characteristic, the Weyl algebras have a remarkable property – the Bernstein's Theorem (cf. [B]) which in the case of A_1 claims that any nonzero A_1-module is of infinite dimension over k. This property does not hold for $D_q^\#(R)$ if $q \neq 1$. To see this, it is convenient to switch to a different, 'hyperbolic' (in the sense of [R], Ch.2), description of the algebra $D_q^\#(R)$.

Let θ denote an automorphism of the polynomial ring $k[\xi]$ assigning to any $f(\xi) \in k[\xi]$ the polynomial $f(q\xi + 1)$. The algebra $D_q^\#(R)$ can be described as a ring generated by $k[\xi]$, x, and ∂ subject to the relations:

$$\partial x = \xi, \quad x\partial = \theta^{-1}(\xi); \quad \partial f = \theta(f)\partial, \quad fx = x\theta(f). \tag{3}$$

In other words, $D_q^\#(R)$ is a hyperbolic algebra ([R], Chapter II) with coefficients in the polynomial algebra $k[\xi]$.

Note that the element $\eta := \xi - 1/(1-q)$ has the property: $\theta(\eta) = q\eta$. This and the relations (3) imply that η is a normal element; hence the left (and right) ideal μ in $D_q^{\#}(R)$ generated by η is two-sided. One can see that the quotient algebra, $D_q^{\#}(R)/\mu$, is (isomorphic to) the commutative algebra of functions of the hyperbola given by the equation $\partial x = 1/(1-q)$. In other words, $D_q^{\#}(R)/\mu$ is isomorphic to the algebra of Laurent polynomials $k[x, x^{-1}]$ in one variable. In particular, the algebra $D_q^{\#}(R)$ has a parametrized by k^* family of one-dimensional representations. Note however that if M is a finite dimensional (over k) $D_q^{\#}(R)$-module, then it is annihilated by the element η (this can be easily deduced from the description of the left spectrum of $D_q^{\#}(R)$, cf. [R], II.4). It follows from the latter fact that the (Ore) localization of $D_q^{\#}(R)$ at the multiplicative set $(\eta) := \{\eta^n \mid n \in \mathbb{Z}_+\}$ possesses the Bernstein's property: every $(\eta)^{-1}D_q^{\#}(R)$-module is infinite-dimensional. Moreover, the algebra $D_q(R) := (\eta)^{-1}D_q^{\#}(R)$ seems to be a 'right' analog of the first Weyl algebra in all respects. For instance, $D_q(R)$ is simple, and its Krull, homological, and Gelfand-Kirillov's dimensions coincide and equal to 1.

We shall see in Section 9 of this work that the algebra $D_q(R)$ is a special case of a very natural, canonical construction of a 'right' algebra of differential operators.

Part II. Quasi-symmetries, Hopf algebras, and crossed products.

Fix a monoidal subcategory $\mathcal{C}^{\tilde{}} = (\mathcal{C}, \odot, \mathbf{1})$ of the category $End^{\tilde{}}(\mathcal{A})$ of endofunctors of an abelian category $\mathcal{A}$.

5. Hopf algebras in monoidal categories.

Fix a quasi-symmetry β in $\mathcal{C}^{\tilde{}}$. A *β-bialgebra* in $\mathcal{C}^{\tilde{}}$ is a triple (δ, H, μ), where (H, μ) is an algebra and (δ, H) is a coalgebra in $\mathcal{C}^{\tilde{}}$ such that the comultiplication $\delta : H \longrightarrow H \odot H$ is an (unital) algebra morphism from (H, μ) to $(H, \mu) \odot_\beta (H, \mu)$ and the counity is an algebra morphism too.

One can check that, like in the classical case, one can switch the algebra and coalgebra structures in the latter requirement. In other words, (δ, H, μ) is a bialgebra in $\mathcal{C}^{\tilde{}}$ iff it is a bialgebra in the dual monoidal category.

We denote by η the unity $\mathbf{1} \to H$ and by e the counity $H \to \mathbf{1}$ of $\mathfrak{H} = (\delta, H, \mu)$.

5.1. Lemma. *Let $\mathcal{B} = (\delta, B)$ be a coalgebra in $\mathcal{C}^{\tilde{}}$ with the counity $\mathfrak{e}$; and let $\mathcal{R} = (R, m)$ be an algebra in $\mathcal{C}^{\tilde{}}$ with the unity η. Then the map*

$$* : \mathcal{C}(B, R) \otimes \mathcal{C}(B, R) \longrightarrow \mathcal{C}(B, R), \quad f*g = m \circ f \otimes g \circ \delta, \tag{1}$$

is an associative multiplication with the identity element $\eta \circ \mathfrak{e}$.

Proof. In fact,

$$f*(\eta \circ e) = m \circ f \odot (\eta \circ e) \circ \delta = m \circ fR \odot (B\eta \circ Be) \circ \delta = m \circ fR \odot B\eta = f.$$

And similarly, $(\eta \circ e)*f = f$ for any $f \in \mathcal{C}(B, R)$.

We leave the verifying the associativity to the reader. ∎

In particular, for any β-bialgebra $\mathbf{H} = (\delta, H, m)$ with the counity e and unity η, the construction of Lemma 5.1 defines the *convolution algebra* $\mathfrak{C}\mathbf{H} = (\mathfrak{C}H, *)$ of $\mathbf{H}$.

An antipode in $\mathbf{H}$ is a morphism $\vartheta : H \longrightarrow H$ such that $m \circ \vartheta H \circ \delta = m \circ H\vartheta \circ \delta = \eta \circ e$.

Since ϑH and $H\vartheta$ are notations for resp. $\vartheta \odot id_H$ and $id_H \odot \vartheta$, it follows from the definition of the antipode that it is the inverse element (of the convolution algebra) to the identity morphism id_H. In particular, the antipode is unique.

A bialgebra in $\mathcal{C}^{\tilde{}}$ equipped with an antipode is called a *β-Hopf algebra*.

Let $\mathcal{A} = (\delta, A, m)$ and $\mathcal{B} = (\Delta, B, \mu)$ be β-bialgebras in $\mathcal{C}^{\tilde{}}$. Set $\mathcal{A} \odot_\beta \mathcal{B} := (\delta', A \odot B, \mu')$, where $\mu' := m \odot \mu \circ A\beta_{A,B}B$ and $\delta' := A\beta_{A,B}B \circ \delta \odot \Delta$.

5.2. Lemma. *The triple $\mathcal{A} \odot_\beta \mathcal{B} = (\delta', A \odot B, \mu')$ is a bialgebra with the unity $\eta \odot \eta'$ and the counity $e \odot e'$. Here η (resp. η') is the unity of $\mathcal{A}$ (resp. $\mathcal{B}$) and e (resp. e') is the counity of $\mathcal{A}$ (resp. $\mathcal{B}$).*

If ϑ and ϑ' are antipodes of resp. $\mathcal{A}$ and $\mathcal{B}$, then $\vartheta \odot \vartheta'$ is an antipode of $\mathcal{A} \odot_\beta \mathcal{B}$.

Proof is a straitforward checking left to the reader. ∎

5.2. Example: free β-Hopf algebras. Let β be a quasi-symmetry in a monoidal subcategory $\mathcal{C}^{\tilde{}} = (\mathcal{C}, \odot, \mathbf{1})$ of $\mathfrak{End}\mathcal{A}$.

For any $W \in Ob\mathcal{C}$, denote by $T(W)$ the free algebra of W, $T(W) = (\oplus_{n \geq 0} W^{\odot n}, m)$.

5.2.1. Lemma. *(a) The map $W \mapsto T(W)$ extends to a functor T from the category $\mathcal{C}$ to the category $Alg\mathcal{C}^{\tilde{}}$ of algebras in $\mathcal{C}^{\tilde{}}$ which is a left adjoint to the forgetting functor $Alg\mathcal{C}^{\tilde{}} \longrightarrow \mathcal{C}$.*

(b) For any $V, W \in Ob\mathcal{C}$, there is a natural epimorphism $\phi : T(V \oplus W) \longrightarrow T(V) \odot_\beta T(W)$; i.e. the pair $T_\beta = (T, \phi)$ is a monoidal functor (in the sense of [M]*) from the monoidal category $(\mathcal{C}, \oplus, 0)$ to the monoidal category $Alg_\beta\mathcal{C}^{\tilde{}} = (Alg_\beta\mathcal{C}^{\tilde{}}, \odot_\beta, \mathbf{1})$ of algebras in $\mathcal{C}^{\tilde{}}$.*

Proof is left to a reader. ∎

5.2.2. Proposition. *For any $W \in Ob\mathcal{C}$, the composition δ' of the 'diagonal' morphism $W \longrightarrow W \odot \mathbf{1} \oplus \mathbf{1} \odot W$ and the natural embedding $W \odot \mathbf{1} \oplus \mathbf{1} \odot W \subset T(W) \odot_\beta T(W)$ determines a coalgebra structure $\Delta : T(W) \longrightarrow T(W) \odot_\beta T(W)$ which is compatible with the multiplication m on $T(W)$; i.e. $(\Delta, T(W), m)$ is a bialgebra with the coidentity e uniquely defined by the fact that its restriction to W equals to zero.*

The automorphism $-id : W \longrightarrow W$ induces an automorphism ϑ of $T(W)$ which happens to be the antipode on the bialgebra $T(W)$; i.e.

$$m \circ \vartheta T(W) \circ \Delta = m \circ T(W)\vartheta \circ \Delta = \eta \circ e \tag{1}$$

where η is the unity of $T(W)$. Thus, $\mathbf{H}(W) := (e, \Delta, T(W), m, \eta; \vartheta)$ is a β-Hopf algebra.

Proof. The first assertion follows from the universal property of the functor $W \mapsto (T(W), m)$ (cf. Lemma 5.2.1).

The existence (and uniqueness) of ϑ follows from Lemma 5.2.1. Thanks to the universal property of the functor T (Lemma 5.2.1), it suffices to check the equality

$$m \circ \vartheta T(W) \circ \Delta \circ \iota_W = m \circ T(W)\vartheta \circ \Delta \circ \iota_W = \eta \circ e \circ \iota_W,$$

where ι_W is the embedding $W \longrightarrow T(W)$. But $\eta \circ e \circ \iota_W = 0$. And it follows from the definition of ϑ that $m \circ \vartheta T(W) \circ \Delta \circ \iota_W = 0 = m \circ T(W)\vartheta \circ \Delta \circ \iota_W$. ∎

5.3. Example: affine σ-spaces. Suppose now that σ is a symmetry in a monoidal category $\mathcal{C}^{\cdot} = (\mathcal{C}, \odot, \mathbf{1})$.

For any $W \in Ob\mathcal{C}$, denote by $S_\sigma(W)$ the symmetric algebra of W. Recall that $S_\sigma(W)$ is the quotient of the free algebra $T(W)$ of W by the two-sided ideal generated by the image of $i_{W\odot W} - i_{W\odot W} \circ \sigma_{W,W} : W \odot W \longrightarrow T(W)$. Here $i_{W\odot W}$ is the embedding $W \odot W \longrightarrow T(W)$.

The algebra $S_\sigma(W)$ shall be called sometimes the affine σ-algebra.

5.3.1. Lemma. *(a) The map $W \mapsto S_\sigma(W)$ extends to a functor S_σ from the category $\mathcal{C}$ to the category $An_\sigma\mathcal{C}^{\cdot}$ of σ-commutative algebras in $\mathcal{C}^{\cdot}$ which is a left adjoint to the forgetting functor $An_\sigma\mathcal{C}^{\cdot} \longrightarrow \mathcal{C}$.*

(b) For any $V, W \in Ob\mathcal{C}$, $S_\sigma(V \oplus W)$ is naturally isomorphic to $S_\sigma(V) \odot_\sigma S_\sigma(W)$.

More explicitly, S_σ is a monoidal functor from the symmetric monoidal category $(\mathcal{C}, \oplus, 0, s)$ to the quasi-symmetric monoidal category $Alg_\sigma\mathcal{C}^{\cdot} = (Alg_\sigma\mathcal{C}^{\cdot}, \odot_\sigma, \mathbf{1}, \sigma)$ of σ-commutative algebras in $\mathcal{C}^{\cdot}$.

Proof is left to a reader. ∎

5.3.2. Corollary. *For any $W \in Ob\mathcal{C}$, the diagonal morphism $W \longrightarrow W \oplus W$ induces* a *coalgebra structure $\Delta : S_\sigma(W) \longrightarrow S_\sigma(W) \odot_\sigma S_\sigma(W)$ which is compatible with the multiplication μ on $S_\sigma(W)$; i.e. $(\Delta, S_\sigma(W), m)$ is a bialgebra with the counity $e : S_\sigma(W) \longrightarrow \mathbf{1}$ the restriction of which to W equals to zero.*

The automorphism $-id : W \longrightarrow W$ induces an automorphism ϑ of $S_\sigma(W)$ which happens to be an antipode on the bialgebra $S_\sigma(W)$; i.e.

$$m \circ \vartheta S_\sigma(W) \circ \Delta = m \circ S_\sigma(W)\vartheta \circ \Delta = e \circ \eta \qquad (1)$$

where η is the unity of $S_\sigma(W)$. Thus, $(e, \Delta, S_\sigma(W), m, \eta; \vartheta)$ is a *σ-Hopf algebra.*

5.3.3. Note. Let W, V be objects of $\mathcal{C}$. One can check that the canonical isomorphism $S_\sigma(W \oplus V) \longrightarrow S_\sigma(W) \odot_\sigma S_\sigma(V)$ is a Hopf algebras isomorphism. ∎

5.4. Example: group algebras in a monoidal category. Let G be a group. We assume that $\mathcal{C}$ has direct sums of Card(G) objects. The group algebra $\mathbf{1}$(G) of the group G in $\mathcal{C}^{\cdot}$ is the pair $(\oplus_{s\in G}\mathbf{1}_s, m)$, where $\mathbf{1}_s = \mathbf{1}$ for all $s \in G$ and the multiplication m is determined by the identical morphisms $\mathbf{1}_s \odot \mathbf{1}_t \to \mathbf{1}_{st}$, $s, t \in G$. There is a natural comultiplication $\delta : \mathbf{1}(G) \longrightarrow \mathbf{1}(G)\odot\mathbf{1}(G)$ defined by the isomorphisms $\mathbf{1}_s \to \mathbf{1}_s\odot\mathbf{1}_s$, $s \in$G. And the set of identical isomorphisms $\mathbf{1}_s \to \mathbf{1}_{s^{-1}}$, $s \in G$, defines an antipode.

5.4.1. Note. We do not use any symmetry to define the group algebra in a monoidal category $\mathcal{C}^{\cdot}$. This is due to the fact that in order to define a comultiplication and antipode

on an algebra (H, m), we need a symmetry only on some (any) monoidal subcategory of $\mathcal{C}^-$ containing the object H and morphism m. In the case of $H = \oplus_{s\in G}\mathbf{1}_s$, we can take the full subcategory generated by direct sums of the identity object $\mathbf{1}$. This subcategory is monoidal and has a unique symmetry determined by morphisms λ and ρ; or ruther by the compatibility condition: $\sigma_{\mathbf{1},\mathbf{1}} = \lambda \circ \rho^{-1}$. ∎

6. Crossed products and basic constructions.

6.1. β-Hopf actions. Fix an abelian monoidal category $\mathcal{C}^- = (\mathcal{C}, \odot, \mathbf{1})$ with a quasi-symmetry β. Let $\mathcal{R} = (R, m)$ be an algebra in $\mathcal{C}^-$. And let $\mathcal{U} = (\delta, \mathcal{U}, \mu)$ be a bialgebra in $\mathcal{C}^-$. We call a $\mathcal{U}$-module structure $\tau : \mathcal{U} \odot \mathcal{R} \longrightarrow \mathcal{R}$ a *β-Hopf action* if the diagram

$$\begin{array}{ccccc} \mathcal{U}\odot\mathcal{R}\odot\mathcal{R} & \xrightarrow{\mathcal{U}m} & \mathcal{U}\odot\mathcal{R} & \xrightarrow{\tau} & \mathcal{R} \\ {\scriptstyle \delta\mathcal{R}\mathcal{R}}\downarrow & & & & \uparrow{\scriptstyle m} \\ \mathcal{U}\odot\mathcal{U}\odot\mathcal{R}\odot\mathcal{R} & \xrightarrow{\mathcal{U}\beta\mathcal{R}} & \mathcal{U}\odot\mathcal{R}\odot\mathcal{U}\odot\mathcal{R} & \xrightarrow{\tau\odot\tau} & \mathcal{R}\odot\mathcal{R} \end{array} \qquad (1)$$

is commutative.

6.1.1. Example: the adjoint action. Let $\mathcal{R} = (\delta, R, \mu)$ be a Hopf algebra in the monoidal category $\mathcal{C}^-$; and let ϑ denote the antipode of $\mathcal{R}$. The *adjoint action*, $ad_{\beta,R}$, on $\mathcal{R}$ is the composition

$$R\odot R \xrightarrow{\delta R} R\odot R\odot R \xrightarrow{R\beta_{R,R}} R\odot R\odot R \xrightarrow{R\odot R\vartheta_{R,R}} R\odot R\odot R \xrightarrow{\mu\circ\mu R} R \qquad (1)$$

One can check that the adjoint action is a β-Hopf action. ∎

For any $\mathcal{U}$-module (M, ξ_M), define an action $\varphi : \mathcal{U}\odot\mathcal{R}(M) \longrightarrow \mathcal{R}(M)$ by the formula:

$$\varphi = \varphi_M = \tau \odot \xi_M \circ \mathcal{U}\beta(M) \circ \delta\mathcal{R}(M). \qquad (2)$$

In particular, we have a morphism $\phi : \mathcal{U} \odot (\mathcal{R} \odot \mathcal{U}) \longrightarrow \mathcal{R} \odot \mathcal{U}$ defined by

$$\phi = \tau \odot \mu \circ \mathcal{U}\beta\mathcal{U} \circ \delta\mathcal{R} \odot \mathcal{U}. \qquad (3)$$

The action ϕ defines an action $m_\tau := m\mathcal{U} \circ \mathcal{R}\phi : (\mathcal{R} \odot \mathcal{U}) \odot (\mathcal{R} \odot \mathcal{U}) \longrightarrow \mathcal{R} \odot \mathcal{U}$.

6.2. Lemma. *(a) The action m_τ is an algebra structure on $\mathcal{R} \odot \mathcal{U}$.*

(b) For any $\mathcal{U}$-module M, the $\mathcal{R}$-module structure on $\mathcal{R}(M)$ extends to a structure of an $\mathcal{R}\#\mathcal{U}$-module $\psi_M : (\mathcal{R} \odot \mathcal{U})(\mathcal{R}(M)) \longrightarrow \mathcal{R}(M)$, where $\mathcal{R}\#\mathcal{U} = (\mathcal{R} \odot \mathcal{U}, m_\tau)$.

Proof. (a) It suffices to show that the action (3) is associative and unitale. We begin with the latter property.

Let $\eta : Id \longrightarrow \mathcal{U}$ be the unity of $\mathcal{U}$. Since $\delta \circ \eta = \mathcal{U}\eta \circ \eta = \eta\mathcal{U} \circ \eta$,

$$\phi \circ \eta\mathcal{R}\mathcal{U} = \tau \odot \mu \circ \mathcal{U}\beta\mathcal{U} \circ \delta\mathcal{R}\mathcal{U} \circ \eta\mathcal{R}\mathcal{U} = \tau \odot \mu \circ \mathcal{U}\beta\mathcal{U} \circ (\mathcal{U}\eta)\mathcal{R}\mathcal{U} \circ \eta\mathcal{R}\mathcal{U} =$$

$$\tau \odot \mu \circ \mathcal{U}\mathcal{R}\eta\mathcal{U} \circ \mathcal{U}\beta\mathcal{U} \circ \eta\mathcal{R}\mathcal{U} =$$

$$\tau\mathcal{U}\circ\mathcal{U}\mathcal{R}(\mu\circ\eta\mathcal{U})\circ\mathcal{U}\beta\mathcal{U}\circ\eta\mathcal{R}\mathcal{U}=\tau\mathcal{U}\circ\mathcal{U}\beta\mathcal{U}\circ\eta\mathcal{R}\mathcal{U}=(\tau\circ\eta\mathcal{R})\mathcal{U}=id_{\mathcal{R}\mathcal{U}}$$

since $\tau\circ\eta\mathcal{R}=id_{\mathcal{R}}$.

Similarly, with the associativity:

$$\tau\odot\mu\circ\mathcal{U}\beta\mathcal{U}\circ\delta\mathcal{R}\mathcal{U}\circ\mathcal{U}(\tau\odot\mu\circ\mathcal{U}\beta\mathcal{U}\circ\delta\mathcal{R}\mathcal{U})=\tau\odot\mu\circ\mathcal{U}\beta\mathcal{U}\circ\mathcal{U}\mathcal{U}(\tau\odot\mu)\circ\delta\mathcal{U}\mathcal{R}\mathcal{U}\mathcal{U}\circ\mathcal{U}(\mathcal{U}\beta\mathcal{U}\circ\delta\mathcal{R}\mathcal{U})=$$

$$\tau\mathcal{U}\circ\mathcal{U}\mathcal{R}\mu\circ\mathcal{U}(\tau\mathcal{U}\circ\mathcal{U}\mathcal{R}\mu)\circ\mathcal{U}\mathcal{U}\beta\mathcal{U}\circ\delta\mathcal{U}\mathcal{R}\mathcal{U}\mathcal{U}\circ\mathcal{U}(\mathcal{U}\beta\mathcal{U}\circ\delta\mathcal{R}\mathcal{U}).$$

We leave the finishing of this checking to a reader.

(b) Define the action ψ_M by the formula: $m(M)\circ\mathcal{R}\varphi_M$.

We leave to a reader to verify that $\mathcal{R}\#M:=(R(M),\psi_M)$ is an $\mathcal{R}\#\mathcal{U}$-module. ∎

Following the classical example, we call the algebra $\mathcal{R}\#\mathcal{U}=(\mathcal{R}\odot\mathcal{U},m_\tau)$ *the crossed product of* $\mathcal{R}$ *and* $\mathcal{U}$.

6.3. Note. The map assigning to any $\mathcal{U}$-module M the $\mathcal{R}\#\mathcal{U}$-module $\mathcal{R}\#M$ of Lemma 6.2 is functorial. And the corresponding functor, which we denote by $\mathcal{R}\#$, from $\mathcal{U}-mod$ to $\mathfrak{gr}\mathcal{R}\#\mathcal{U}-mod$ is isomorphic to the tensoring $\mathcal{R}\#\mathcal{U}\odot_{\mathcal{U}}$ over $\mathcal{U}$. This implies, in particular, that $\mathcal{R}\#$ is a left adjoint to the functor $\mathfrak{F}_0:\mathcal{R}\#\mathcal{U}-mod\longrightarrow\mathcal{U}-mod$ which forgets about the action of $\mathcal{R}$. ∎

6.4. Lemma. *The functor* $\mathcal{R}\#\mathcal{U}\odot_{\mathcal{R}}$ *is isomorphic to the functor* $\mathcal{U}*$ *from* $\mathcal{R}-mod$ *to* $\mathcal{R}\#\mathcal{U}-mod$ *which assigns to any* $\mathcal{R}$*-module* $\mathcal{M}=(M,\xi_M)$ *the* $\mathcal{R}\#\mathcal{U}$*-module* $(\mathcal{U}(M),\nu)$. *The action of* $\mathcal{U}$ *here is natural and the action of* $\mathcal{R}$ *is the composition*

$$R\odot\mathcal{U}(M)\xrightarrow{\beta_{R,\mathcal{U}}}\mathcal{U}\odot R(M)\xrightarrow{\delta R(M)}\mathcal{U}\odot\mathcal{U}\odot R(M)\xrightarrow{\mathcal{U}\tau R(M)}\mathcal{U}\odot R(M)\xrightarrow{\mathcal{U}\xi_M}\mathcal{U}(M)\quad(1)$$

Proof is left to a reader. ∎

6.5. Note. The formula (1) defines a functor from $\mathcal{R}-mod$ to $\mathcal{R}-mod$ (- the composition of $\mathcal{U}*$ and the functor $\mathcal{R}\#\mathcal{U}-mod\longrightarrow\mathcal{R}-mod$ forgetting the action of $\mathcal{U}$) which can be interpreted as an action of the bialgebra $\mathcal{U}$ on the category $\mathcal{R}-mod$.

Similarly, the composition of the functor $\mathcal{R}\#:\mathcal{U}-mod\longrightarrow\mathcal{R}\#\mathcal{U}-mod$ with the functor $\mathcal{R}\#\mathcal{U}-mod\longrightarrow\mathcal{U}-mod$ forgetting the action of $\mathcal{R}$ is an action of the algebra $\mathcal{R}$ on the category $\mathcal{U}-mod$. ∎

6.6. Example. Suppose that $\mathcal{U}$ is the group algebra of a group G, $\mathcal{U}=\mathbf{1}(G)$. So that the action τ is determined by a group morphism $G\longrightarrow\mathrm{Aut}(\mathcal{R})$. Note that, since $\mathbf{1}(G)=\oplus_{s\in G}\mathbf{1}_s$ (cf. 5.1.5), $\mathbf{1}$(G)(M) is the direct sum $\oplus_{s\in G}M_s$ of copies of M and the action of $\mathbf{1}$(G) is determined by the identical morphisms $\mathbf{1}_t(M_s)\longrightarrow M_{ts}$, $s,t\in G$. And the action of $\mathcal{R}$ on M_s is the composition $\xi_M\circ\tau(s)(M)$.

Thus, the action of $\mathcal{U}$ on $\mathcal{R}-mod$ (cf. Note 6.5) is the functor assigning to each $\mathcal{R}$-module $\mathcal{M}=(M,\xi_M)$ the G-graded $\mathcal{R}$-module $\oplus_{s\in G}\mathcal{M}_s$, where $\mathcal{M}_s=(M,\xi_M\circ\tau(s)(M))$ for any $s\in G$. ∎

6.7. The algebra $\mathcal{R}*\mathcal{U}$. We denote this way the quotient of the algebra $\mathcal{R}\#\mathcal{U}$ by the annihilator of $\mathcal{R}$ in the canonical action $(\mathcal{R}\#\mathcal{U})\odot\mathcal{R}\longrightarrow\mathcal{R}$.

for all $t \in G$ and $b \in R$.

In particular, if the group G is commutative, the (left) action of $\mathcal{R}\#G$ upon itself respects the grading. ∎

6.9. The Hopf algebra $\mathcal{R}^+ * \mathcal{U}_-$. Let $\mathcal{U}$ and $\mathcal{R}$ be Hopf algebras, and let $\tau : \mathcal{U} \odot \mathcal{R} \longrightarrow \mathcal{R}$ be a Hopf algebra action compatible with the coproduct. So that we have a well defined Hopf action of $\mathcal{U}\#\mathcal{R}$ on $\mathcal{R}$ (cf. Lemma 6.8.1).

Let $\mathcal{R}_+$ be the augmentation ideal of $\mathcal{R}$; i.e. $\mathcal{R}_+$ is the kernel of the coidentity morphism $\epsilon : \mathcal{R} \to \mathbf{1}$. Denote by L_+ the largest $\mathcal{U}$-stable ideal in $\mathcal{R}$ contained in $\mathcal{R}_+$. Set $\mathcal{R}^+ := \mathcal{R}/L_+$. By construction, $\mathcal{U}$ acts on $\mathcal{R}^+$.

6.9.1. Lemma. *The ideal L_+ is a Hopf ideal; so that $\mathcal{R}^+$ is a Hopf algebra. The action of $\mathcal{U}$ on $\mathcal{R}^+$ is compatible with the comultiplication on $\mathcal{R}^+$.*

Proof. The ideal L_+ is a Hopf ideal, because $\mathcal{R}_+$ is a Hopf ideal, and the action of $\mathcal{U}$ is compatible with the comultiplication. The second assertion is a consequence of the first one. ∎

Thus $\mathcal{R}^+\#\mathcal{U}$ acts on $\mathcal{R}^+$. We denote by $\mathcal{R}^+ * \mathcal{U}_-$ the quotient of the Hopf algebra $\mathcal{R}^+\#\mathcal{U}$ by the annihilator of $\mathcal{R}^+$, $\mathcal{U}_-$ being the image of $\mathcal{U}$ in $\mathcal{R}^+\#\mathcal{U}_-$.

We shall call the kernel K_- of the canonical (Hopf algebra) epimorphism from $\mathcal{U}$ to $\mathcal{U}_-$ *the (Hopf) ideal of Serre relations.*

6.9.2. Remark: the form ψ. Consider the bilinear form $\epsilon_\tau := \epsilon \circ \tau \circ \beta_{\mathcal{R},\mathcal{U}} : \mathcal{R} \odot \mathcal{U} \longrightarrow \mathbf{1}$, where ϵ is the counity. The form ϵ_τ is invariant with respect to the action of $\mathcal{R}\#\mathcal{U}$. So that its kernel, L_-, is a Hopf ideal in $\mathcal{R}\#\mathcal{U}$. Let $\mathcal{U}_-$ and $\mathcal{R}^+$ denote the images of resp. $\mathcal{U}$ and $\mathcal{R}$ in the quotient Hopf algebra $\mathcal{R}\#\mathcal{U}/L_-$. Both $\mathcal{U}_-$ and $\mathcal{R}^+$ are Hopf algebras, and the form ϵ_τ induces a nondegenerate invariant form ψ_τ on $\mathcal{R}^+ \odot \mathcal{U}_-$. ∎

6.10. Crossed products and differential actions. For the notion of a differential action see Section 1.4.

6.10.1. Lemma. *Let a β-Hopf action $\mathcal{U} \odot \mathcal{R} \longrightarrow \mathcal{R}$ be β-differential. Then the action of $\mathcal{R}\#\mathcal{U}$ on $\mathcal{R}$ is β-differential.*

Proof. This follows from the fact that the action of $\mathcal{R}$ on $\mathcal{R}$ by the left multiplication is differential, hence β-differential. And, for any $\mathbb{S}$, the composition of $\mathbb{S}$-differential actions is a $\mathbb{S}$-differential action. ∎

6.10.2. Example: β-Hopf actions of free algebras are differential. Fix a quasi-symmetry β. Let $W \in Ob\mathcal{C}$; and let $T_\beta(W)$ be the free β-Hopf algebra of W (cf. Example 5.2). Any β-Hopf action of $T_\beta(W)$ on an algebra $\mathcal{R}$ in $\mathcal{C}^-$ is β-differential.

It follows from the definition of the comultiplication on $T_\beta(W)$ and the defining a β-Hopf action diagram (1) in 6.0 that the restriction of any β-Hopf action of $T_\beta(W)$ to W, $d : W \odot R \longrightarrow R$, is a β-derivation. In particular, the action d is β-differential. Since W generates the algebra $T_\beta(W)$, the action of the whole $T_\beta(W)$ is β-differential. ∎

7. The generalized Weyl algebras.

6.8. Hopf algebra structure. Suppose that, in addition, $\mathcal{R}$ has a coalgebra structure, $\Delta : R \longrightarrow R \odot R$.

6.8.1. Lemma. *(a) The morphism* $\Delta \odot_\beta \delta := \beta_{2,3} \circ \Delta \odot \delta : R \odot \mathcal{U} \longrightarrow (R \odot \mathcal{U}) \odot (R \odot \mathcal{U})$ *is a coalgebra structure on* $\mathcal{R}\#\mathcal{U}$ *iff the diagram*

$$\begin{array}{ccc} \mathcal{U} \odot R & \xrightarrow{\delta \odot_\beta \Delta} & (\mathcal{U} \odot R) \odot (\mathcal{U} \odot R) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau \odot \tau} \\ R & \xrightarrow{\Delta} & R \odot R \end{array} \qquad (1)$$

is commutative.

(b) If the diagram (1) *is commutative and* (δ', R, μ) *is a bialgebra, then* $\delta' \odot_\beta \delta$ *is a bialgebra structure on* $\mathcal{R}\#\mathcal{U}$.

(c) The action $\tau : \mathcal{U} \odot \mathcal{R} \longrightarrow \mathcal{R}$ *and the adjoint action* $\mathrm{ad}_{\mathcal{R}} : \mathcal{R} \odot \mathcal{R} \longrightarrow \mathcal{R}$ *define a bialgebra action of* $\mathcal{R}\#\mathcal{U}$ *on* $\mathcal{R}$.

(d) If, under the conditions (b), both $\mathcal{U}$ *and* $\mathcal{R}$ *are Hopf algebras, then* $\mathcal{R}\#\mathcal{U}$ *is a Hopf algebra with a naturally defined antipode.*

Proof is left to a reader. ∎

6.8.2. Example. Let $\mathcal{R} = (\delta', R, \mu)$ be a Hopf k-algebra with an antipode ϑ. And let ϕ be a group morphism from G to $\mathrm{Aut}_K(R, \mu)$. Take as $\mathcal{U}$ the group (Hopf) algebra of G. The group morphism from G to $\mathrm{Aut}_K(R, \mu)$ induces a bialgebra action of $\mathcal{U}$ on (R, μ). In this case, $\mathcal{R}\#\mathcal{U} = \mathcal{R}\#G$; and the commutativity of the diagram (1) of Lemma 6.8.1 means exactly that the image of ϕ is contained in $\mathrm{Aut}_K(\delta', R, \mu)$.

Suppose that ϕ takes values in $\mathrm{Aut}_K(\delta', R, \mu)$.

The coalgebra structure $\delta' = \delta' \odot_\beta \delta$ on $\mathcal{R}\#G$ sends every homogenious element $x_s r$ of $\mathcal{R}\#G, r \in R, s \in G$, into $\sum_{i,j} x_s r_i \otimes x_s r_j$, where $\sum_{i,j} r_i \otimes r_j = \delta'(r)$.

Let $\mathcal{R}$ have an antipode ϑ. Then the antipode on $\mathcal{R}\#G$ maps the element $x_s r$ into $\vartheta(r) x_{1/s} = x_{1/s} s\vartheta(r)$.

Finally, the adjoint action of an element $x_s r$ sends any element $x_t b$ of $\mathcal{R}\#G$ into

$$\sum_{i,j} x_s r_i x_t b x_{1/s} s\vartheta(r_j) = \sum_{i,j} x_{st/s} st^{-1}(r_i) s(b) s\vartheta(r_j) = s(x \sum_{t_{i,j}} t^{-1}(r_i) b \vartheta(r_j)) \qquad (1)$$

where $\sum_{i,j} r_i \otimes r_j = \delta'(r)$.

Here, as before, we denote by the same letter the automorphism s and its canonical extension to an automorphism of $\mathcal{R}\#G$ sending, for all $t \in G, x_t$ into $x_{st/s}$ (cf. Lemma 6.8.1).

In particular, since $\delta'(1) = 1 \otimes 1$, $ad_{x_s}(x_t b) = s(x_t b)$ for any $t \in G$ and $b \in R$; i.e., for any $s \in G$, the automorphism $ad_{x_s} : R\#G \longrightarrow R\#G$ coincides with s.

If s belongs to the center of G (for example, the group G is commutative), then $s(x_t) = x_t$ for all $t \in G$; hence in this case

$$ad_{x_s r}(x_t b) = x_t s(\sum_{i,j} t^{-1}(r_i) b \vartheta(r_j)) \qquad (2)$$

Fix a quasi-symmetry β in the monoidal category $\mathcal{C}^{\sim} = (\mathcal{C}, \odot, \mathbf{1})$. Let V, W be objects of $\mathcal{C}$; and let $\epsilon : W \odot V \longrightarrow \mathbf{1}$ be a morphism.

The morphism ϵ determines (uniquely) a β-derivation $\partial_\epsilon = \partial_{\epsilon,\beta} : W \odot T(V) \longrightarrow T(V)$. The derivation ∂_ϵ induces a β-differential action $\mathfrak{d}_\epsilon : T(W) \odot T(V) \longrightarrow T(V)$.

7.1. Lemma. *The action $\mathfrak{d}_\epsilon$ is a β-Hopf action which respects the coproduct in the β-Hopf algebra $T(V)$.*

Proof is left to the reader. ∎

In particular, we can take the determined by the β-Hopf action $\mathfrak{d} = \mathfrak{d}_\epsilon$ crossed product $T(V)\#_{\mathfrak{d}}T(W) = (\delta, T(V) \odot T(W), \mu)$.

7.2. Note. One can see that the algebra structure of $T(V)\#_{\mathfrak{d}}T(W)$ is determined by the requirement that $T(V)$ and $T(W)$ are subalgebras, and by the morphism

$$\beta_{W,V} + \epsilon : W \odot V \longrightarrow V \odot W \oplus \mathbf{1}. \tag{1}$$

In particular, if $\epsilon = 0$, the β-Hopf algebra $T(V)\#_{\mathfrak{d}}T(W)$ coincides with the product $T(V) \odot_\beta T(W)$ of $T(V)$ and $T(W)$ (cf. Lemma 5.2). ∎

7.3. The 'affine space' and the Weyl algebra associated with a pairing. Given a pairing $\epsilon : W \odot V \longrightarrow \mathbf{1}$ in $\mathcal{C}$, we can apply the construction of Subsection 6.9 to the β-Hopf action $d_\epsilon = d_{\epsilon,\beta}$. This way we obtain

1) The quotient $T(V)^+ = T(V)^+_\epsilon$ of $T(V)$ by the largest d_ϵ-stable ideal L_+ contained in the augmetation ideal $T(V)_+ := \oplus_{n\geq 1} V^n$ (– the *ideal of Serre relations*). We shall call $T(V)^+_\epsilon$ *the affine algebra associated with* ϵ.

2) The β-Hopf algebra $A_{\epsilon,\beta} = T(V)^+ * T(W)_-$. We shall call the algebra $A_{\epsilon,\beta}$ *the Weyl β-Hopf algebra associated with* the *pairing* ϵ (and β).

7.4. Note. If $\epsilon = 0$, $T(V)\#_{\mathfrak{d}}T(W)$ is isomorphic to $T(V) \odot_\beta T(W)$ (cf. Note 7.2) which implies that the affine space $T(V)^+_\epsilon$ and the (β, ϵ)-Weyl algebra $A_{\epsilon,\beta}$ are isomorphic to $\mathbf{1}$. So, the algebra $A_{\epsilon,\beta}$ is meaningful only when the form ϵ is nontrivial. We are interested in the case when ϵ is nondegenerate. Say, W is an object dual to V and ϵ is the evaluation morphism. Or $V = \oplus_{\alpha\in\Gamma} V_\alpha$ is a graded object, and $W = \oplus_{\alpha\in\Gamma} W_\alpha$ is the direct sum of dual to V_α objects, $\alpha \in \Gamma$. The pairing ϵ is determined by evaluation morphisms $\epsilon_\alpha : W_\alpha \odot V_\alpha \longrightarrow \mathbf{1}$. ∎

7.5. The case of a symmetry. Suppose that β is a symmetry. And let $S_{\epsilon,\beta}$ denote the algebra which is $S_\beta(V) \odot S_\beta(W)$ as a $(S_\beta(V), S_\beta(W))$-bimodule with the rest of the multiplication determined by $\beta_{W,V} + \epsilon : W \odot V \longrightarrow V \odot W \oplus \mathbf{1}$ and the associativity. One can show that the canonical epimorphism $T(V) \odot_\beta T(W) \longrightarrow A_{\epsilon,\beta}$ factors through an epimorphism $S_{\epsilon,\beta} \longrightarrow A_{\epsilon,\beta}$.

7.6. A canonical pairing. Now we consider only the action of $T(W)$ on $T(V)^+$ which we denote for convenience by U^+. Denote by S^- the annihilator of this action (i.e. the supremum of all graded ideals $\mathcal{J}$ in $T(W)$ which act trivially on U^+). And set $U^- := T(W)/S^-$. Thus, we have a nondegenerate action $\varphi^+ : U^- \odot U^+ \longrightarrow U^+$.

We define a pairing $\varphi : U^- \odot U^+ \longrightarrow \mathbf{1}$ as the composition of φ^+ and the coidentity $e : U^+ \longrightarrow \mathbf{1}$.

7.6.1. Lemma. *The pairing φ is nondegenerate.*

Proof. 1) Clearly the restriction of φ to $\mathbf{1} = (U^+)_0$ is nondegenerate. Suppose that the restriction of φ to $U^{+m} := \overline{\underset{0\leq}{}} \leq_m (U^+)_i$ is nondegenerate for $1 \leq m \leq$n. The action of W on U^+ sends $(U^+)_{n+1}$ into U^{+n}. And, by the definition of U^+ and the action φ^+, W cannot act trivially on $(U^+)_{n+1}$. By induction hypothesis, the composition of φ with $U^-\varphi^+ : U^- \odot (W \odot (U^+)_{n+1}) \longrightarrow U^- \odot U^{+n}$ is nonzero. Since the image of this restriction coincides with the image of the action of $U^- \cdot W$ on $(U^+)_{n+1}$, we have showed that φ is nondegenerate with respect to U^+.

2) The nondegeneracy with respect to U^- can be argued in a similar fashion. We leave details to a reader. ∎

7.7. Proposition. *All β-Hopf actions of a β-Weyl algebra are β-differential.*

Proof. This follows from the fact that β-Hopf actions of any free β-Hopf algebra are differential (Example 6.10.2) and that a Weyl β-Hopf algebra is the quotient of the crossed product of free β-Hopf algebras. ∎

7.8. The β-Weyl algebra of an algebra with generators. It is convenient to have a notion of a β-Weyl algebra on a wider class of algebras than just β-affine algebras defined in 7.3.

Consider the category of pairs $(R, v : V \to R)$, where R is an algebra in $\tilde{\mathcal{C}}$ and $v : V \to R$ is a subobject such that the adjoint algebra morphism $v\hat{} : T_\beta(V) \longrightarrow R$ is an epimorphism. The pairs $(R, v : V \to R)$ generate a category, $Alg_g\tilde{\mathcal{C}}$ (here the second g means 'generators') with obviously defined morphisms.

We shall write, when it is convenient, (R, V) instead of $(R, v : V \to R)$.

For any pair $(R, v : V \to R)$, consider the subcategory $\mathfrak{Der}_\beta(R, V)$ of the category $\mathfrak{Der}_\beta(R)$ generated by all β-derivations $d : X \odot R \to R$ such that the composition of d with $X \odot v : X \odot V \to X \odot R$ factors through the identity 'element' $\mathbf{1} \to R$ of R; i.e. there exists a commutative diagram:

$$\begin{array}{ccc} X \odot R & \xrightarrow{d} & R \\ {\scriptstyle X\odot v}\uparrow & & \uparrow{\scriptstyle \mathfrak{e}} \\ X \odot V & \xrightarrow{\epsilon_d} & \mathbf{1} \end{array}$$

Since $V \to R$ and $\mathfrak{e}$ generate R, the β-derivations d is uniquely determined by the form ϵ_d. If the identity morphism $\mathfrak{e}$ is an epimorphism, ϵ_d is uniquely determined by d.

Denote by $\mathfrak{A}_\beta(R, V)$ the full monoidal subcategory of the monoidal category $\mathfrak{D}_\beta(R, V)$ of differential endomorphisms of the algebra R generated by $\mathfrak{Der}_\beta(R)$ and the left action of R, $R \odot R \longrightarrow R$. We denote a final object of $\mathfrak{A}_\beta(R, V)$, if any, by $A_\beta(R, V)$ and call it *the β-Weyl algebra of* (R, V). Thus the β-Weyl algebra is a (proper in general) subalgebra of the algebra $D_\beta(R)$ of β-differential operators on R (when the latter exists).

In 'algebraic' situations both the final objects of $\mathfrak{Der}_\beta(R, V)$ and $\mathfrak{A}_\beta(R, V)$ exist and are naturally related to each other.

In fact, suppose that there exists a dual to V object W; and let ϵ be the evaluation form, $\epsilon : W \odot V \longrightarrow \mathbf{1}$. Then (under the condition that $\odot$ is compatible with colimits), there exists a monomorphism $W' \longrightarrow W$ such that the composition ϵ' of ϵ and $W' \odot V \longrightarrow W \odot V$ determines a final object – a β-derivation $W' \odot R \longrightarrow R$ - of the category $\mathfrak{Der}_\beta(R, V)$. This β-derivation defines a β-Hopf action $T_\beta(W') \odot R \longrightarrow R$. The latter determines the action, φ, of $R\#T_\beta(W')$ on R. The quotient of $R\#T_\beta(W')$ by the kernel of the action φ is $A_\beta(R, V)$.

7.8.1. Note. If the monomorphism $W' \longrightarrow W$ above is an isomorphism, the β-affine algebra corresponding to the evaluation ϵ is the quotient of R. ■

7.8.2. Example. Suppose that $R = T_\beta(V)$ and V has a dual object, W. Then $A_\beta(R, V)$ is the crossed product $T_\beta(V)\#T_\beta(W)$. ■

7.8.3. Note. In known examples of interest the subobject V is uniquely defined. For instance, if R is a $\mathbb{Z}_+$-graded algebra, $R = \oplus_{n\geq 0} R_n$ with $R_0 = \mathbf{1}$ and the irrelevant ideal $R_+ := \oplus_{n\geq 1} R_n$ is generated by R_1, we take $V = R_1$. In such cases (one of them is discussed in detail in Section 7.9) we omit V in the notations and talk about the β-Weyl algebra of the algebra R. ■

Note that the β-Weyl algebra of a pair (R, V) is not, in general, a β-Hopf algebra.

7.9. The β-Weyl algebra of an affine quantum space. Let now R be the algebra of $\mathbf{q}$-polynomials, $k_\mathbf{q}[\mathbf{x}]$, in $\mathbf{x} = (x_i \mid i \in J)$ with coefficients in a commutative ring k. Here $\mathbf{q} = \{q_{ij} \mid i, j \in J\}$ is a matrix such that $q_{ij}q_{ji} = 1$ for any $i, j \in J$. We shall regard R as an algebra in the monoidal category $\mathcal{C}^-$ of $\mathbb{Z}^J$-graded k-vector spaces assuming that the parity of the generator x_i is i. Let β be a quasi-symmetry in $\mathcal{C}^-$ determined by a matrix $\mathbf{b} = \{b_{ij} \mid i, j \in J\}$ with entrees in k^*. Fix an $i \in J$.

7.9.1. Proposition. *(i) The following conditions are equivalent:*

(a) $q_{ij}b_{ij} = 1$ for any $i, j \in J$ such that $i \neq j$.

(b) For any $i \in J$, there exists a (unique) β-derivation ∂_i of the algebra R such that $\partial_i(x_j) = \delta_{ij}$ for all $j \in J$.

(ii) Suppose that b_{ii} is either 1 or not a root of one. And if $b_{ii} = 1$ for some i, then $char(k) = 0$. Then the conditions (a) and (b) are equivalent to

(c) The algebra $R = k_\mathbf{q}[\mathbf{x}]$ is β-affine.

Proof. (i) (b)⇔(a). Suppose that there is a β-derivation ∂_i (uniquely) determined by the requirement: $\partial_i(x_j) = \delta_{ij}$. For any $r \in R$ and any $j \in J$, we have:

$$\partial_i(x_j r) = \delta_{ij} r + b_{ij} x_j \partial_i(r).$$

Or, if we regard x_j as the operator of multiplication by x_j,

$$\partial_i x_j - b_{ij} x_j \partial_i = \delta_{ji}$$

for all $j \in J$. In particular, we have:

$$\partial_i x_i x_j = b_{ii} x_i \partial_i x_j + x_j = b_{ii} b_{ij} x_i x_j \partial_i + x_j. \tag{1}$$

On the other hand, if $i \neq j$, we have:

$$\partial_i x_i x_j = q_{ij}\partial_i x_j x_i = q_{ij}b_{ij}x_j\partial_i x_i = q_{ij}b_{ij}b_{ii}x_j x_i\partial_i + q_{ij}b_{ij}x_j = b_{ii}b_{ij}x_i x_j\partial_i + q_{ij}b_{ij}x_j$$

which implies that $q_{ij}b_{ij} = 1$ for all $j \neq$i.

(a)⇔(b). It follows from the above computations that if $q_{ij}b_{ij} = 1$ for all i and j such that $i \neq j$, then, for any $i \in J$, there exists a β-derivation ∂_i uniquely defined by the property: $\partial_i(x_j) = \delta_{ij}$ for all $j \in J$.

(ii) Suppose that the equivalent conditions *(a), (b)* hold. Then the β-affine algebra, $\mathfrak{S}_\beta$, generated by $\{x_i \mid i \in J\}$ is a quotient algebra of $R = k_{\mathbf{q}}[\mathbf{x}]$. Note, however, that under the conditions *(ii)* of Proposition 7.9.1, the canonical epimorphism from R to $\mathfrak{S}_\beta$ is an isomorphism. This follows from the fact that R is an irreducible $\mathfrak{S}_\beta$-module.

The argument is standart: it suffices to show that, for any nonzero polynomial $f \in R$, there exists a multi-index i such that $\partial^{\mathbf{i}}(f) \in k^*$.

In fact, set for convenience $b_i := b_{ii}$. Then

$$\partial_i^n(x_i^n) = \sum_{0 \leq m < n} b_i^m.$$

If $b_i = 1$, then $\partial_i^n(x_i^n) = n \neq 0$ by assumption.

If $b_i \neq 1$, then $\partial_i^n(x_i^n) = (1 - b_i^n)/(1 - b_i) \neq 0$ for any $n \neq 0$, since b_i is not a root of one. This implies that, for any multi-index $\mathbf{n}, \partial^{\mathbf{n}}(x^{\mathbf{n}}) \in k^*$.

We leave the finishing the argument to a reader. ∎

7.9.2. Note. Proposition 7.9.1 is valid in the case when k is not a field, but a domain. In this case we need to modify slightly the conditions of the part *(ii)*. The modified requirements in *(ii)* are:

If $b_{ii} = 1$ for some i *then $char(k) = 0$. If $b_{ii} \neq 1$, then $1 - b_{ii}$ is invertible and b_{ii} is not a root of one.*

For instance, k might be the localization of the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials at the multiplicative set generated by $\{1 - b_{ii} \mid b_{ii} \neq 1\}$ and by all entrees b_{ij}. ∎

7.9.3. The Weil algebra $A_\beta(R)$. Fix the setting of Proposition 7.9.1: $R = k_{\mathbf{q}}[\mathbf{x}]$, β is determined by a matrix (b_{ij}) with invertible entrees such that $b_{ij}q_{ij} = 1$ for all $i, j \in J$ such that $i \neq j$.

7.9.3.1. Lemma. *The β-Weyl algebra $A_\beta(R)$ of R is generated by x_i and ∂_i, $i \in J$, (each ∂_i has the parity* -i*) satisfying the relations:*

$$\partial_i x_j - b_{ij}x_j\partial_i = \delta_{ji} \tag{1}$$

for all $i, j \in J$ and

$$x_i x_j = q_{ij}x_j x_i, \quad \partial_i\partial_j = q_{ji}\partial_j\partial_i \tag{2}$$

for all $i, j \in J$ such that $i \neq j$.

Proof. The only thing to check here is that $\partial_i\partial_j = b_{ij}\partial_j\partial_i$ for all $i, j \in J$ such that $i \neq j$. But since $b_{ij}q_{ij} = 1 = q_{ij}q_{ji}$, the fact follows from the first assertion of Proposition 7.9.1. ∎

Set $x_i\partial_i = \xi_i$. If $i \neq j$, we have:

$$x_i\xi_j = x_ix_j\partial_j = q_{ij}x_jx_i\partial_j = q_{ij}b_{ji}^{-1}x_j\partial_jx_i = \xi_jx_i \tag{3}$$

$$\xi_j\partial_i = x_j\partial_j\partial_i = b_{ij}x_j\partial_i\partial_j = b_{ij}(b_{ij})^{-1}\partial_ix_j\partial_j = \partial_i\xi_j \tag{4}$$

In particular, $\xi_i\xi_j = \xi_j\xi_i$ for all $i, j \in J$.

Note that

$$\xi_ix_i = x_i\partial_ix_i = x_i(b_{ii}x_i\partial_i + 1) = x_i(b_{ii}\xi_i + 1) \tag{5}$$

and

$$\partial_i\xi_i = \partial_ix_i\partial_i = (b_{ii}x_i\partial_i + 1)\partial_i = (b_{ii}\xi_i + 1)\partial_i \tag{6}$$

Let A denote the k-algebra generated by the elements $\xi_i, i \in J$. There are no other relations between $\{\xi_i\}$; so that the algebra A is isomorphic to the algebra $k[\{\xi_i\}]$ of polynomials in $\{\xi_i \mid i \in J\}$ with coefficients in k.

For each $i \in J$, define the automorphism θ_i by the formula

$$\theta_i(\xi_i) = b_{ii}\xi_i + 1 \tag{7}$$

$$\theta_i(\xi_j) = \xi_j \text{ if } j \neq i.$$

Then the β-Weyl algebra $A_\beta(R)$ is a k-algebra generated by A and by the set of the elements $\{\partial_i, x_i \mid i \in J\}$ satisfying the relations:

$$x_ix_j = q_{ij}x_jx_i, \quad \partial_i\partial_j = q_{ij}\partial_j\partial_i \tag{8}$$

$$x_i\partial_i = \xi_i, \quad \partial_ix_i = \theta_i^{-1}(\xi_i), \tag{9}$$

$$\partial_ia = \theta_i(a)\partial_i, \quad ax_i = x_i\theta_i(a) \tag{10}$$

for all $i, j \in J$ such that $i \neq j$ and all $a \in A$.

7.9.4. Note. If $q_{ij} = 1$ for all $i, j \in J$ such that $i \neq j$, then the algebra $A_\beta(R)$ is a hyperbolic ring over A in the sense of [R], IV.1.3. In the general case, $A_\beta(R)$ is a PBW algebra (cf. [R], Ch.V) over a commutative (polynomial) ring. ∎

Fix an $m \in J$, and suppose that $b_{mm} \neq 1$. Set $\eta_m = \xi_m - 1/(1 - b_{mm})$. One can check that $\theta_m(\eta_m) = b_{mm}\eta_m$. Clearly $\theta_j(\eta_m) = \eta_m$ if $j \neq$m. It follows from the relations (8)-(10) that η_m is a normal element: the left (and right) ideal generated by η_m is two-sided. One can see that the quotient algebra $A_\beta(R)/A_\beta(R)\eta_m$ looses one dimension and the images of x_m and ∂_m become invertible and commute with each other.

More generally, if I is a subset of J such that $b_{ii} \neq 1$ for all $i \in I$, then the left ideal $\mathcal{I}$ generated by $\{\eta_i \mid i \in I\}$ is two-sided, and the quotient algebra $A_\beta/\mathcal{I}$ is given by

– the relations (8),

– the relations (9), (10) with J replaced by $J - I$ and A replaced by the algebra of polynomials in $\{\xi_i \mid i \in J - I\}$,

– the relations

$$x_i\partial_i = \partial_ix_i = 1/(1 - b_{ii}) \tag{11}$$

for all $i \in$I.

In particular, if $b_{ii} \neq 1$ for all $i \in J$, we can take $I = J$, and the corresponding quotient algebra, $A_\beta/\mathcal{I}$ is described by the relations (8) and (11). If $q_{ij} = 1$ for all $i, j \in J$, the algebra $A_\beta/\mathcal{I}$ is isomorphic to the ring of Laurent polynomials in x_i, $i \in J$. But even in the general case $A_\beta/\mathcal{I}$ has (families of) one-dimensional representations.

Similarly to what we did in the one-dimensional case, consider the localization $\tilde{A}_\beta(R)$ of the algebra $A_\beta(R)$ at the multiplicative set S_J generated by the elements $\{\eta_j \mid j \in J\}$. One can show that the algebra $\tilde{A}_\beta(R)$ has the properties analogous to those of the Weyl algebra of the same rank.

8. Quasi-symmetries and the Picard group.

8.1. Quasi-symmetries in a symmetric category. Suppose that the monoidal category $\tilde{\mathcal{C}} = (\mathcal{C}, \odot, \mathbf{1})$ has a (fixed) symmetry σ. Then every quasi-symmetry β in $\tilde{\mathcal{C}}$ is the composition $\sigma \circ \lambda$, where λ is an automorphism of the functor $\odot$ satisfying the following conditions:

$$\lambda_{X\odot Y,Z} = X\sigma_{Z,Y} \circ \lambda_{X,Z}Y \circ X\sigma_{Y,Z} \circ X\lambda_{Y,Z} \tag{1}$$

$$\lambda_{X,Y\odot Z} = \sigma_{Y,X}Z \circ Y\lambda_{X,Z} \circ \sigma_{X,Y}Z \circ \lambda_{X,Y}Z \tag{2}$$

Clearly

$$\hat{\beta}_{X,Y} := \beta^{-1}_{Y,X} = (\lambda_{Y,X})^{-1} \circ \sigma_{X,Y}.$$

8.2. The Picard group. An object P of the monoidal category $\tilde{\mathcal{C}} = (\mathcal{C}, \odot, \mathbf{1})$ is called *invertible* if the functor $P\odot$ from $\mathcal{C}$ to $\mathcal{C}$ is an auto-equivalence. Denote by $\mathbf{Pic}(\tilde{\mathcal{C}})$ the subcategory of $\mathcal{C}$ objects of which are all invertible objects P of $\tilde{\mathcal{C}}$ and morphisms are isomorphisms of $\mathcal{C}$. Clearly $\mathbf{Pic}(\tilde{\mathcal{C}})$ is a monoidal subcategory of $\tilde{\mathcal{C}}$.

The adjoint (i.e. quasi-inverse) to $P\odot$ functor is $\hat{P}\odot$, where $\hat{P}$ is a dual to P object; and the adjunction morphism

$$\epsilon_P : (P\odot)\hat{} \circ (P\odot) = (\hat{P} \odot P)\odot \longrightarrow \mathrm{Id}_{\mathcal{C}} = \mathbf{1}\odot$$

is determined by the evaluation map $ev_P : \hat{P} \odot P \longrightarrow \mathbf{1}$.

This shows that the semigroup $Pic(\tilde{\mathcal{C}})$ of the isomorphy classes of $\mathbf{Pic}(\tilde{\mathcal{C}})$ is a group. We shall call $Pic(\tilde{\mathcal{C}})$ *the Picard group of* $\tilde{\mathcal{C}}$. It is commutative, if $\tilde{\mathcal{C}}$ has a quasi-symmetry.

8.2.1. Example. Let $\tilde{\mathcal{C}}$ be the monoidal category of G-graded modules over a commutative ring k (with a trivial grading, $k = k_0$; cf. Example I.6.0.1.2), where G is a, not necessarily commutative, group. Then $Pic(\tilde{\mathcal{C}})$ is naturally isomorphic to G. In this case, $Pic(\tilde{\mathcal{C}})$ is commutative iff $\tilde{\mathcal{C}}$ has a (quasi-)symmetry. ■

8.3. Quasi-symmetries and the fundamental group of a monoidal category. The *fundamental group* $\pi_1(\tilde{\mathcal{C}})$ of the monoidal category $\tilde{\mathcal{C}}$ is the group of automorphisms of the identical monoidal functor $\tilde{Id} = (Id, id) : \tilde{\mathcal{C}} \longrightarrow \tilde{\mathcal{C}}$. In other words, $\pi_1(\tilde{\mathcal{C}})$ consists of all invertible elements γ of the center of $\mathcal{C}$ compatible with the tensor product; i.e. $\gamma(X \odot Y) = \gamma(X) \odot \gamma(Y)$ for all $X, Y \in Ob\mathcal{C}$.

Note that $\tilde{\mathcal{C}}$ can be viewed as a monoidal subcategory of the category of representations of the group $\pi_1(\tilde{\mathcal{C}})$ in the monoidal category $\tilde{\mathcal{C}}$.

Fix a quasi-symmetry $\beta = \sigma \circ \lambda$. For each $P \in \mathrm{Ob}\mathbf{Pic}(\mathcal{C})$, consider the action

$$\chi_P(X) := \epsilon_P \circ X \circ P\hat{}\lambda_{P,X} \circ \epsilon_P^{-1} X : X \longrightarrow X \tag{1}$$

Since $\lambda_{P,X}$ is an isomorphism for any $X \in Ob\mathcal{C}$, $\chi_P = \{\chi_P(X) \mid X \in Ob\mathcal{C}\}$ is an automorphism of the functor $Id_{\mathcal{C}}$. In other words, χ_P is an invertible element of the center of $\mathcal{C}$. And $\lambda_{P,X} = P\chi_P(X)$. In particular,

$$\beta_{P,X} = \sigma_{P,X} \circ P\chi_P(X) = \chi_P(X)P \circ \sigma_{P,X} \tag{2}$$

One can check that the automorphism χ_P depends only on the isomorphy class of P. If follows from (2) in 5.5 that

$$P\chi_P(Y \odot Z) = \lambda_{P,Y\odot Z} = \sigma_{Y,P}Z \circ Y\lambda_{P,Z} \circ \sigma_{P,Y}Z \circ \lambda_{P,Y}Z =$$

$$\sigma_{Y,P}Z \circ Y \odot P\chi_P(Z) \circ \sigma_{P,Y}Z \circ P\chi_P(Y)Z$$

$$PY\chi_P(Z) \circ P\chi_P(Y)Z = P(\chi_P(Y) \odot \chi_P(Z))$$

which implies the equality

$$\chi_P(Y \odot Z) = \chi_P(Y) \odot \chi_P(Z). \tag{3}$$

Let now Q be another object of $\mathbf{Pic}(\mathcal{C})$. It follows from the relation (1) in 5.5 that

$$P \odot Q\chi_{P\odot Q}(Z) = P\sigma_{Z,Q} \circ P\chi_P(Z)Q \circ P\sigma_{Q,Z} \circ P\chi_Q(Z) = P \odot Q\chi_P(Z)Q \circ P \odot Q\chi_Q(Z) =$$

$$P \odot Q(\chi_P(Z)\chi_Q(Z))$$

hence

$$\chi_{P\odot Q}(Z) = \chi_P(Z)\chi_Q(Z). \tag{4}$$

The equality (4) means that the map $P \mapsto \chi_P$ induces a homomorphism of the Picard group $Pic(\mathcal{C})$ of the category $\mathcal{C}^{\tilde{}}$ to the group $C(\mathcal{C})^*$ of invertible elements of the center $C(\mathcal{C})$ of $\mathcal{C}$. The equality (3) shows that this map is compatible with the 'tensor' product; i.e. $\chi_P \in \pi_1(\mathcal{C}^{\tilde{}})$. Thus, we have assigned to every quasi-symmetry β of $\mathcal{C}^{\tilde{}}$ a homomorphism $\chi = \chi^\beta$ from $Pic(\mathcal{C})$ to the fundamental group $\pi_1(\mathcal{C}^{\tilde{}})$ of the monoidal category $\mathcal{C}^{\tilde{}}$.

8.4. Note. Let $\mathcal{R} = (R, \mu)$ be an algebra in $\mathcal{C}^{\tilde{}}$. Then, for any $\omega \in \pi_1(\mathcal{C}^{\tilde{}})$, the morphism $\omega(R) : R \longrightarrow R$ is an algebra automorphism, since

$$\mu \circ \omega(R \odot R) = \omega(R) \circ \mu \text{ and } \omega(R \odot R) = \omega(R) \odot \omega(R).$$

In particular, for any $P \in \mathbf{Pic}(\mathcal{C})$, the morphism $\chi_P(R)$ is an algebra automorphism $\mathcal{R} \longrightarrow \mathcal{R}$. ∎

8.5. Lemma. *Suppose that $Ob\mathbf{Pic}(\mathcal{C})$ is an integral class of objects in $\mathcal{C}$. Then the map $\beta \mapsto \chi^\beta$ is bijective.*

Proof. The morphism χ^β defines β on the full subcategory $\mathcal{F}(\mathcal{C})$ the full subcategory of $\mathcal{C}$ generated by all direct sums of objects of $\mathbf{Pic}(\mathcal{C})$ (- *skew free* objects). Clearly $\mathcal{F}(\mathcal{C})$ is a monoidal subcategory of $\mathcal{C}^-$, and the morphism χ^β determines uniquely the restriction of β to $\mathcal{F}(\mathcal{C})$. It follows from the fact that $\odot X$ is a right exact functor for any X that it transfers any integral family of arrows to an object Y to an integral family of morphisms to $Y \odot X$. This implies the injectivity of the map $\beta \mapsto \chi^\beta$.

Fix an object M of $\mathcal{C}$. And let $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be an exact sequence such that F_1 and F_0 are skew free objects. Then, for any object X of $\mathcal{F}(\mathcal{C})$, there exists a unique morphism $\lambda_{M,X} : M \odot X \longrightarrow M \odot X$ such that the diagram

$$\begin{array}{ccccc} F_1 \odot X & \longrightarrow & F_0 \odot X & \longrightarrow & M \odot X \\ {\scriptstyle \lambda_{F_1,X}}\downarrow & & {\scriptstyle \lambda_{F_0,X}}\downarrow & & \downarrow{\scriptstyle \lambda_{M,X}} \\ F_1 \odot X & \longrightarrow & F_0 \odot X & \longrightarrow & M \odot X \end{array}$$

is commutative. The uniqueness follows from the epimorphness of $F_0 \odot X \longrightarrow M \odot X$. The isomorphness of $\lambda_{F_i,X}$, $i = 1, 2$, implies that $\lambda_{M,X}$ is an isomorphism. We leave to a reader the checking that $\lambda = \{\lambda_{M,X}\}$ is an automorphism of $\odot$ satisfying relations (1) and (2) in 5.5. ∎

8.6. Example. Let $\mathcal{C}^-$ be the category of $\mathbb{Z}^J$-graded k-modules with the graded tensor product over k and the standart symmetry $\sigma : v \otimes w \mapsto w \otimes v$.

One can see that $Pic(\mathcal{C}^-)$ is naturally isomorphic to the group $\mathbb{Z}^J$, and the group $C(\mathcal{C})^*$ of all invertible elements of the center $C(\mathcal{C})$ of $\mathcal{C}$ is (isomorphic to) the $|J|$-dimensional torus $(k^*)^J$.

Note by passing that, in this case, the embedding of $Pic(\mathcal{C}^-)$ into the group $\mathrm{Aut}_k(\mathcal{C})$ of k-linear auto-equivalences of the category $\mathcal{C}$ is an isomorphism.

Let β be the quasi-symmetry defined by a matrix $\mathbf{q} = [q_{ij}]_{i,j\in J}$ with entrees in k^* (cf. Section F). This means that $\beta = \sigma \circ \lambda$, where $\lambda = \lambda_{\mathbf{q}}$ is an automorphism of the $\mathbb{Z}^J$-graded tensor product uniquely defined by

$$\lambda_{X,Y}(x_i \otimes y_j) = q_{ij} x_i \otimes y_j$$

for any graded k-modules X and Y and any elements $x_i \in X_i$, $y_j \in Y_j, i, j \in J$.

We shall identify J with the set of generators of $Pic(\mathcal{C}) = \mathbb{Z}^J$. The associated with λ and $i \in Pic(\mathcal{C})$ automorphism χ_i acts on the j-th component of any graded k-module as the multiplication by q_{ij} and is uniquely determined by this property. The uniqueness follows from the equality

$$\chi_i(X \odot Y) = \chi_i(X) \odot \chi_i(Y)$$

for all k-modules X and Y (cf. (3) in P).

The conditions of Lemma 8.5 are, evidently, satisfied. This implies, in particular, that quasi-symmetries in $\mathcal{C}$ are in one-to-one correspondence with matrices $(q_{ij})_{i,j\in J}$ with entries from k^*. ∎

8.7. Skew derivations in monoidal categories. Let $\mathcal{R} = (R, \mu)$ be an algebra in $\mathcal{C}^-$. And let $\mathcal{M} = (m, M, \nu)$ be an R-bimodule. Let θ be an automorphism of R. We call a morphism $d : X \odot R \longrightarrow M$ is a *θ-derivation* in $\mathcal{M}$, if

$$d \circ X\mu = \nu \circ dR + m \circ Rd \circ \theta X \odot R \circ \sigma_{X,R} R = \nu \circ dR + m \circ \theta M \circ Rd \circ \sigma_{X,R} R \qquad (1)$$

We shall omit θ (or skew) if θ=id. Note that

(a) Any θ-derivation in $\mathcal{M}$ is a derivation in the bimodule $\mathcal{M}_\theta := (m \circ \theta M, M, \nu)$.

(b) There is a natural isomorphism $\mathcal{R}_\theta \odot_{\mathcal{R}} \mathcal{M} \longrightarrow \mathcal{M}_\theta$. In particular, if the bimodule $\mathcal{M}$ is $\mathcal{R}_\Phi$ for some automorphism Φ of the algebra $\mathcal{R}$, we have a natural bimodule isomorphism $\gamma_{\theta,\Phi} : \mathcal{R}_\theta \odot_{\mathcal{R}} \mathcal{R}_\Phi \longrightarrow \mathcal{R}_{\theta\circ\Phi}$. (This shows that the map $\theta \mapsto \mathcal{R}_\theta \odot_{\mathcal{R}}$ determines a group homomorphism from $\mathrm{Aut}(\mathcal{R})$ to the group $\mathrm{Aut}(R - mod)$ of all isomorphy classes of auto-equivalences of the category $\mathcal{R} - mod$ of left $\mathcal{R}$-modules.)

8.8. Picard group, quasi-symmetries, and skew derivations. Let $\beta = \sigma \circ \lambda$ be a quasi-symmetry. Fix an object P of $\mathbf{Pic}(\mathcal{C})$ and consider β-derivations of weight P in a bimodule $\mathcal{M} = (m, M, \nu)$. The defining property of a β-derivation $d : P \odot R \longrightarrow M$ (cf. C.2) can be rewritten as

$$d \circ P\mu = \nu \circ dR + m \circ Rd \circ \chi_P(R) P \odot R \circ \sigma_{P,R} R = \nu \circ dR + m \circ \chi_P(R) M \circ Rd \circ \sigma_{P,R} R \quad (2)$$

Since $\chi_P(R)$ is an automorphism of the algebra $\mathcal{R}$ (cf. Note 8.4), the formula (2) shows that d is a skew derivation.

8.8.1. Derivations, quasi-symmetries, and the Picard group. Fix a symmetric monoidal category $\mathcal{C}^{\tilde{}} = (\mathcal{C}, \odot, \mathbf{1}; \sigma)$ and a quasi-symmetry β. Let $\mathcal{R} = (R, m)$ be an algebra in $\mathcal{C}^{\tilde{}}$ and $\partial : W \odot R \longrightarrow R$ a β-derivation of the algebra $\mathcal{R} = (R, m)$.

Note that, for any morphism $f : V \longrightarrow W$, the morphism $\partial_f = \partial \circ fR : V \odot R \longrightarrow R$ is a β-derivation of $\mathcal{R}$.

Suppose that W is a *skew-free* object in $\mathcal{C}^{\tilde{}}$; i.e. $W = \oplus_{P \in X} P$ for some subset X of $\mathrm{Ob}\mathbf{Pic}\mathcal{C}$. Then the β-derivation ∂ can be regarded as the set $\{\partial_P : P \odot R \longrightarrow R \mid P \in X\}$ of β-derivations. And, according to 8.8, each of the derivations ∂_P, $P \in X$, is a skew σ-derivation. More exactly, ∂_P is a $\chi_P(R)$-derivation, where χ_P is the associated with β, σ, and P element of $\mathcal{C}^{\tilde{}}$ (cf. 8.3).

9. Bialgebras associated with skew derivations.

Fix a subset X of $\mathbf{Pic}\mathcal{C}^{\tilde{}}$. Let, for any $P \in X$, we have a β-derivation $\partial_P : P \odot R \longrightarrow R$ of a ring $\mathcal{R} = (R, m)$. By 8.8, ∂_P is a $\chi_P(R)$-derivation, where χ_P is an element of the fundamental group of $\mathcal{C}^{\tilde{}}$ associated with the quasi-symmetry β and (the equivalence class of) P. This means that

$$\partial_P \circ Pm = m \circ (\partial_P R + R\partial_P \circ \chi_P(R) P \odot R \circ \sigma_{P,R} R) = m \circ (\partial_P R + \chi_P(R) R \circ R\partial_P \circ \sigma_{P,R} R) \quad (1)$$

(cf. 8.8).

Denote by G the subgroup of $Pic\mathcal{C}^{\tilde{}}$ generated by the image of X in $Pic\mathcal{C}^{\tilde{}}$.

Let $W := \oplus_{P \in X} P$. The β-derivations $\partial_P, P \in X$, define a β-derivation $\partial : W \odot R \longrightarrow R$. The morphism ∂ determines, in turn, an action of the free algebra $\mathcal{T} = T(W)$ on $\mathcal{R}$. And we have actions of G on $\mathcal{R}$ and $\mathcal{T}$ defined by resp. $P \mapsto \chi_P(R)$ and $P \mapsto \chi_P(\mathcal{T})$, $P \in$G.

To these actions, there correspond the crossed products $\mathbf{R} = \mathcal{R}\#G$ and $\mathbf{T} = \mathcal{T}\#G$.

We define the coalgebra structure δ on $\mathbf{T} = \oplus_{s \in G} \mathcal{T}_s = \oplus_{s \in G} \mathcal{T} \odot \mathbf{1}_s$ by setting

$$\delta \circ \iota_P = \iota_P \odot \mathbf{1} + \mathbf{1}_{|P|} \odot \iota_P \quad (2)$$

where ι_P is the natural embedding $P \longrightarrow W$; $\mathbf{1}$ is the unity of $\mathbf{T}$; $|P|$ is the image of P in $Pic(\mathcal{C}^{\sim})$. The morphisms $\delta \circ \iota_P$ determine a unique comultiplication $\delta : \mathbf{T} \longrightarrow \mathbf{T} \odot \mathbf{T}$ compatible with the multiplication μ on $\mathbf{T}$, i.e. such that $(\delta, \mathbf{T}, \mu)$ is a bialgebra.

9.1. Lemma. *There is a unique anti-automorphism of* $\mathbf{T}$ *which maps* W *on itself identically and extends the antipode on* $\mathbf{1}(\mathrm{G})$ *(cf. Lemma 8.10.2).*

Proof. Consider the σ-opposite to $\mathcal{T} = T(W)$ algebra $(\mathcal{T}, \mu_\sigma)$. Here $\mu_\sigma := \mu \circ \sigma_{\mathcal{T},\mathcal{T}}$, μ denotes the multiplication in $\mathcal{T}$. The embedding $W \longrightarrow T(W)$ extends to uniquely defined algebra morphisms resp. ϑ from $(\mathcal{T}, \mu)$ to $(\mathcal{T}, \mu_\sigma)$ and ϑ' from $(\mathcal{T}, \mu_\sigma)$ to $(\mathcal{T}, \mu)$. It follows from the universal property of the tensor algebra, that the morphism ϑ is inverse to ϑ' : $\vartheta \circ \vartheta' = id$, $\vartheta' \circ \vartheta = id$.

On the other hand, there is the antipode ϑ_G of $\mathbf{1}(\mathrm{G})$ determined by the set of the identical morphisms $\mathbf{1}_s \longrightarrow \mathbf{1}_{1/s}$, $s \in$G. These two anti-automorphisms determine a unique anti-automorphism θ of $\mathbf{T}$. It follows from the definition of the coproduct δ on $\mathbf{T}$ that θ is an antipode in the bialgebra $\mathbf{T}$. ∎

9.2. Proposition. *There is an associative action* $\phi : \mathbf{T} \odot R \longrightarrow R$ *the composition of which with the embedding* $W \odot R \longrightarrow \mathbf{T} \odot R$ *coincides with* ∂, *and each* $\mathbf{1}_s$, $s \in G$, *acts as* $\chi_s(R)$ *is a Hopf action.*

Proof. We need to check that

$$m \circ \phi \odot \phi \circ \mathbf{T}\sigma_{\mathbf{T},R} R \circ \delta(R \odot R) = \phi \circ \mathbf{T}m \tag{1}$$

(cf. T.1). It suffices to check (1) on 'generators'. The restriction of (1) to P, $P \in X$, is:

$$m \circ \partial_P \odot \partial_P \circ P\sigma_{W,R} R \circ \delta(R \odot R) = \partial_P \circ Pm. \tag{2}$$

It follows from the definition of δ that the equality (2) expresses the fact that ∂_P is a $\chi_P(R)$-derivation which is, really, the case (cf. 8.9).

The restriction of (1) to $\mathbf{1}_s$, $s \in G$, holds iff $\mathbf{1}_s$ acts by an algebra automorphism. But, $\mathbf{1}_s$ acts, for all $s \in G$, as $\chi_s(R)$, and $\chi_s(R)$ is an algebra automorphism (cf. Note 8.4). ∎

9.3. Proposition. *The* β*-derivation* $\partial' = \partial\mathbf{1}(G) : W \odot R\#G \longrightarrow R\#G$ *and* $\chi_s(R\#G)$, $s \in G$, *define an associative action* $\phi : \mathbf{T} \odot R\#G \longrightarrow R\#G$ *which happens to be a Hopf action.*

Proof. The argument is similar to that of Proposition 9.2. ∎

9.3.1. Note. The action ϕ of Proposition 9.3 respects the natural grading on the crossed product $\mathcal{R}\#G$. ∎

9.4. Bialgebras and β-derivations. Denote for convenience the crossed product $\mathcal{R}\#G$ by $\mathcal{B}$, $\mathcal{B} = (B, m')$. Let $\mathcal{B}$ have a Hopf algebra structure; i.e. a comultiplication Δ and an antipode ϑ, which extend those on $\mathbf{1}(G)$: $\Delta(\mathbf{1}_s) = \mathbf{1}_s \otimes \mathbf{1}_s$, and $\vartheta(\mathbf{1}_s) = \mathbf{1}_{1/s}$ for all $s \in$G. Suppose that the action of $\mathbf{T}$ on $\mathcal{B}$ is compatible with the coalgebra structure Δ. Since,

for any $s \in G$, $\chi_s(B)$ is an automorphism of the Hopf algebra $\mathcal{B}$, the latter means that the derivation $\partial : W \odot B \longrightarrow B$ should be compatible with Δ; i.e. the diagram

$$\begin{array}{ccc} W \odot B & \xrightarrow{W\Delta} & W \odot (B \odot B) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ B & \xrightarrow{W\Delta} & B \odot B \end{array} \tag{1}$$

where $\partial' = \partial B + B\partial \circ \beta_{W,B}B$, is commutative.

Then $U' := \mathbf{T}\#\mathcal{B}$ is a Hopf algebra and the actions of $\mathbf{T}$ on $\mathcal{B}$ and the adjoint action of $\mathcal{B}$ determine a Hopf algebra action of U' on $\mathcal{B}$. Denote by U'' the quotient of U' by the annihilator of $\mathcal{B}$. This is a Hopf algebra.

It follows from the definition of the action of $\mathbf{T} = \mathcal{T}\#G$ on $\mathcal{B} = \mathcal{R}\#G$ that, for any $s \in G$, the actions of elements $\mathbf{1}_s \to \mathcal{B}$ and $\mathbf{1}_s \to \mathbf{T}$ on $\mathcal{B}$ coincide.

Applying the construction of 11.9 to the Hopf action of U'' on $\mathcal{B}$, we obtain a bialgebra $\mathcal{B}^+ * U''$. Recall that $\mathcal{B}^+$ is the quotient of $\mathcal{B}$ by the augmentation ideal $\mathcal{B}_+$ - the kernel of the coidentity $\epsilon : \mathcal{B} \to \mathbf{1}$.

9.5. 'Quantum groups' associated with a quasi-symmetry. Fix a set X of invertible objects of $\mathcal{C}^{\tilde{}}$. We assume for convenience that, for any $P, Q \in X$, either $P \ncong Q$, or $P = Q$. Let W^* denote the coproduct $\oplus_{P \in X} P^{\hat{}}$. And let ϵ be the canonical pairing $W^* \odot W \longrightarrow \mathbf{1}$.

According to Lemma 5.2, one can associate with this data a β-derivation

$$\partial : W^* \odot T(W) \longrightarrow T(W). \tag{1}$$

Since W^* is a (skew) free object, the β-derivation (1) is represented by the set of β-derivations $\partial_{P^{\hat{}}} : P^{\hat{}} \odot T(W) \longrightarrow T(W)$, $P \in X$, which are uniquely defined by the equalities

$$\partial_{P^{\hat{}}} \circ P^{\hat{}} \iota_Q = \delta_{P,Q} \epsilon_P \tag{2}$$

Here ι_Q denotes, for any $Q \in X$, the embedding $Q \longrightarrow T(W)$; ϵ_P is the evaluation isomorphism $P^{\hat{}} \odot P \longrightarrow \mathbf{1}$; and $\delta_{P,Q}$ is the Kroneker symbol: it equals to zero if $P \neq Q$ and to 1 if P=Q.

Let G be the subgroup of $Pic\mathcal{C}^{\tilde{}}$ generated by the image of X in $Pic\mathcal{C}^{\tilde{}}$. Denote by $\mathcal{R}$ or (R, m) the crossed product $T(W)\#G$ determined by the morphism $\chi = \chi^\beta$ from G to $\mathrm{Aut}(\mathcal{R})$ (cf. 8.10). Moreover, according to Lemma 9.1, $\mathcal{R}$ has a natural coalgebra structure $\Delta : R \longrightarrow R \odot R$ such that (Δ, R, m) is a Hopf algebra. By Proposition 9.3, the β-derivation ∂ induces a Hopf action of the Hopf algebra $\mathbf{T} := T(W^*)\#G$ on the algebra $\mathcal{R}$.

The action of $\mathbf{T}$ on $\mathcal{R}$ and the adjoint action of $\mathcal{R}$ on itself induce a Hopf action of $\mathcal{R}\#\mathbf{T}$ on $\mathcal{R}$. Let J_+ be the largest among $\mathcal{R}\#\mathbf{T}$-stable ideals in $\mathcal{R}$ contained in the augmentation ideal $\mathcal{R}_+$ (– the kernel of the coidentity $\mathcal{R} \longrightarrow \mathbf{1}$). Then $\mathcal{R}\#\mathbf{T}$ acts on $\mathcal{R}^+ := \mathcal{R}/J_+$.

9.5.1. Proposition. *The quotient $\mathcal{U}$ of the algebra $\mathcal{R}\#\mathbf{T}$ by the annihilator of $\mathcal{R}^+$ is a β-Hopf algebra.*

Proof. The assertion follows from Lemma 6.9.1. ■

9.5.2. Remark. The Hopf algebra $\mathcal{U}$ can be constructed in two steps. First, we take the largest among $\mathbf{T}$-invariant ideals $\mathcal{J}$ contained in R_+. Let $\mathbf{T}'$ be the quotient of $\mathbf{T}$ by the annihilator of $\mathcal{R}' := \mathcal{R}/\mathcal{J}$. Then $\mathbf{T}'$ and $\mathcal{R}'$ are Hopf algebras, and the action of $\mathbf{T}'$ on $\mathcal{R}'$ is a Hopf action. This action together with the adjoint action of $\mathcal{R}'$ determine a Hopf action of $\mathcal{R}'\#\mathbf{T}'$ on $\mathcal{R}'$. The quotient of $\mathcal{R}'\#\mathbf{T}'$ by the annihilator of $\mathcal{R}'$ is isomorphic to the Hopf algebra $\mathcal{U}$. We leave the checking of this fact to the reader. ■

9.5.3. Remark. The construction of $\mathcal{U}$ depends on the choice of the set of X of objects of $\mathbf{Pic}(\mathcal{C}^-)$. However, in the examples we are interested in there are canonical choices. For instance, if $\mathcal{C}^-$ is the monoidal category of $\mathbb{Z}^J$-graded k-modules, it is natural to take as X the set of invertible modules P_i, $i \in J$, corresponding to the generators of $\mathbb{Z}^J$. The group G coincides with $Pic(\mathcal{C}^-) = \mathbb{Z}^J$. Our construction assigns to each quasi-symmetry β of $\mathcal{C}^-$ (given by a matrix $(q_{ij})_{i,j\in J}$ with entrees from k^*) a Hopf algebra $\mathcal{U}_\beta$ provided with a natural $\mathbb{Z}^J$-grading.

In particular, taking as k the field of rational functions in q, and setting $q_{ij} = q^{<i,j>}$, where $< i, j >$ denotes the ij-entree of a Cartan matrix, we obtain a quantized enveloping algebra of Drinfeld and Jimbo. Taking k equal to the ring of Laurent polynomials in q with integer coefficients, we obtain a $\mathbb{Z}$-form of the corresponding quantized enveloping algebra. ■

10. Localization construction.

10.1. β-Hopf actions on graded algebras. Fix a monoidal subcategory $\mathcal{C}^- = (\mathcal{C}, \odot, \mathbf{1})$ of $End^-(\mathcal{A})$ with a quasi-symmetry β, and a commutative group Γ. Let $\mathcal{R} = (\oplus_{\lambda\in\Gamma}\mathcal{R}_\lambda, m)$ be an Γ-graded algebra in $\mathcal{C}^-$. Let $\mathcal{U} = (\delta, \mathcal{U}, \mu)$ be a bialgebra in $\mathcal{C}^-$; and let $\tau : \mathcal{U}\odot\mathcal{R} \longrightarrow \mathcal{R}$ be a β-Hopf action respecting the grading.

For any $\mathcal{U}$-module (M, ξ_M), define an action $\varphi : \mathcal{U}\odot\mathcal{R}(M) \longrightarrow \mathcal{R}(M)$ by the formula:

$$\varphi = \varphi_M = \tau \odot \xi_M \circ \mathcal{U}\beta(M) \circ \delta\mathcal{R}(M). \tag{2}$$

In particular (when $\mathcal{A} = \mathcal{C}$), we have a morphism $\phi : \mathcal{U} \odot (\mathcal{R} \odot \mathcal{U}) \longrightarrow \mathcal{R} \odot \mathcal{U}$ defined by

$$\phi = \tau \odot \mu \circ \mathcal{U}\beta\mathcal{U} \circ \delta\mathcal{R} \odot \mathcal{U}. \tag{3}$$

The action ϕ defines an action $m_\tau := m\mathcal{U} \circ \mathcal{R}\phi : (\mathcal{R} \odot \mathcal{U}) \odot (\mathcal{R} \odot \mathcal{U}) \longrightarrow \mathcal{R} \odot \mathcal{U}$.

10.1.1. Lemma. *(a) The action m_τ is an Γ-graded algebra structure on $\mathcal{R} \odot \mathcal{U}$.*

(b) For any $\mathcal{U}$-module M, the $\mathcal{R}$-module structure on $\mathcal{R}(M)$ extends to a structure of a Γ-graded $\mathcal{R}\#\mathcal{U}$-module $\psi_M : (\mathcal{R} \odot \mathcal{U})(\mathcal{R}(M)) \longrightarrow \mathcal{R}(M)$, where $\mathcal{R}\#\mathcal{U} = (\mathcal{R} \odot \mathcal{U}, m_\tau)$.

We call the algebra $\mathcal{R}\#\mathcal{U} = (\mathcal{R} \odot \mathcal{U}, m_\tau)$ *the crossed product of $\mathcal{R}$ and $\mathcal{U}$.*

10.1.2. Note. The map assigning to any $\mathcal{U}$-module M the $\mathcal{R}\#\mathcal{U}$-module $\mathcal{R}\#M$ (cf. Lemma 10.1.1) extends naturally to a functor $\mathcal{R}\# : \mathcal{U} - mod \longrightarrow \mathfrak{gr}\mathcal{R}\#\mathcal{U} - mod$ which is isomorphic to the tensoring by $\mathcal{R}\#\mathcal{U}$ over $\mathcal{U}$. This implies, in particular, that the functor $\mathcal{R}\#$ is a left adjoint to the functor $\mathfrak{F}_0$:$\mathfrak{gr}\mathcal{R}\#\mathcal{U} - mod \longrightarrow \mathcal{U} - mod$ which assigns to any graded $\mathcal{R}\#\mathcal{U}$-module its zero component and forgets about the action of $\mathcal{R}_0$. ■

10.1.3. Lemma. *Suppose that $\mathcal{R}_0 = \mathbf{1}$, and the action of $\mathcal{U}$ on $\mathcal{R}_0$ is trivial. Then the functor $\mathcal{R}\#$ is fully faithful.*

Proof. In fact, under the conditions, the adjunction morphism from $Id_{\mathcal{U}-mod}$ to $\mathfrak{F}_0 \circ \mathcal{R}\#$ can be chosen to be identical. ∎

10.1.4. Lemma. *The functor $\mathcal{R}\#\mathcal{U}\odot_{\mathcal{R}}$ is isomorphic to the functor $\mathcal{U}*$ from $\mathfrak{gr}\mathcal{R} - mod$ to $\mathfrak{gr}\mathcal{R}\#\mathcal{U} - mod$ which assigns to any graded $\mathcal{R}$-module $\mathcal{M} = (M, \xi_M)$ the graded $\mathcal{R}\#\mathcal{U}$-module $(\mathcal{U}(M), \nu)$, where action of $\mathcal{U}$ is natural and the action of $\mathcal{R}$ is the composition*

$$R \odot \mathcal{U}(M) \xrightarrow{\beta_{R,\mathcal{U}}} \mathcal{U} \odot R(M) \xrightarrow{\delta_R(M)} \mathcal{U} \odot \mathcal{U} \odot R(M) \xrightarrow{\mathcal{U}\tau R(M)} \mathcal{U} \odot R(M) \xrightarrow{\mathcal{U}\xi_M} \mathcal{U}(M) \quad (1)$$

Proof is left to a reader. ∎

10.1.5. Note. The formula (1) defines a functor from $\mathfrak{gr}_\Gamma\mathcal{R} - mod$ to $\mathfrak{gr}_\Gamma\mathcal{R} - mod$ (– the composition of $\mathcal{U}*$ and the forgetting the action of $\mathcal{U}$ functor from $\mathfrak{gr}\mathcal{R}\#\mathcal{U} - mod$ to $\mathfrak{gr}_\Gamma\mathcal{R} - mod$) which can be interpreted as an action of the bialgebra $\mathcal{U}$ on the category $\mathfrak{gr}_\Gamma\mathcal{R} - mod$.

Similarly, the composition of the functor $\mathcal{R}\# : \mathcal{U} - mod \longrightarrow \mathfrak{gr}_\Gamma\mathcal{R}\#\mathcal{U} - mod$ with the functor $\mathfrak{gr}_\Gamma\mathcal{R}\#\mathcal{U} - mod \longrightarrow \mathcal{U} - mod$ forgetting the action of $\mathcal{R}$ and the grading could be regarded as an action of the algebra $\mathcal{R}$ on the category $\mathcal{U} - mod$. ∎

10.1.6. The algebra $\mathcal{R}*\mathcal{U}$. We denote this way the quotient of the algebra $\mathcal{R}\#\mathcal{U}$ by the annihilator of $\mathcal{R}$ in the canonical action $(\mathcal{R}\#\mathcal{U}) \odot \mathcal{R} \longrightarrow \mathcal{R}$. Since this action respects the grading, the epimorphism from $\mathcal{R}\#\mathcal{U}$ to $\mathcal{R}*\mathcal{U}$ induces a grading on $\mathcal{R}*\mathcal{U}$.

10.1.7. Remark: the form ψ. Consider the bilinear form $\epsilon_\tau := \epsilon \circ \tau \circ \beta_{\mathcal{R},\mathcal{U}} : \mathcal{R} \odot \mathcal{U} \longrightarrow \mathbf{1}$, where ϵ is the counity. The form ϵ_τ is invariant (with respect to the action of $\mathcal{R}\#\mathcal{U}$. So that its kernel, L_-, is a Hopf ideal in $\mathcal{R}\#\mathcal{U}$. Let $\mathcal{U}_-$ and $\mathcal{R}^+$ denote the images of resp. $\mathcal{U}$ and $\mathcal{R}$ in the quotient Hopf algebra $\mathcal{R}\#\mathcal{U}/L_-$. Both $\mathcal{U}_-$ and $\mathcal{R}^+$ are Hopf algebras, and the form ϵ_τ induces a nondegenerate invariant form ψ_τ on $\mathcal{R}^+ \odot \mathcal{U}_-$. ∎

10.2. Projective spectrum and a quasi-affine space related to a graded algebra. Fix a monoidal category $\mathcal{C}^- = (\mathcal{C}, \odot, \mathbf{1})$ with a quasi-symmetry β. Let Γ be a commutative directly ordered group and $\mathcal{R}$ a Γ-graded algebra in $\mathcal{C}^-$. For any element γ in Γ, set $\mathcal{R}_{>\gamma} := \oplus_{\sigma>\gamma} R_\sigma$. Denote by $\mathcal{T}_+$ the full subcategory of $\mathcal{R} - mod$ generated by all modules $(M, m : \mathcal{R} \odot M \longrightarrow M)$ such that $M = \sup\{M_\gamma \mid \gamma \in \Gamma\}$, where each subobject M_γ is annihilated by $\mathcal{R}_{>\gamma}$. One can see that $\mathcal{T}_+$ is a subscheme of $\mathcal{R} - mod$. Let $\mathcal{T}_+^-$ be the minimal Serre subcategory containing $\mathcal{T}_+$.

Identifying (would be) spaces with categories of quasi-coherent sheaves on them, we shall call the quotient category $\mathcal{R} - mod/\mathcal{T}_+^-$ *the quasi-affine space of* $\mathcal{R}$, or, imitating the Grothendieck's terminology, *the affine cone of* $\mathcal{R}$.

Let $\mathcal{F}$ be a natural (exact and faithful) functor from $\mathfrak{gr}_\Gamma\mathcal{R} - mod$ to $\mathcal{R} - mod$. Denote by $\mathfrak{T}_+$ the preimage of $\mathcal{T}_+$ with respect to $\mathcal{F}$. And let $\mathfrak{T}_+^-$ be the minimal Serre subcategory of $\mathfrak{gr}_\Gamma\mathcal{R} - mod$ containing $\mathfrak{T}_+$. We call the quotient category $\mathfrak{gr}_\Gamma\mathcal{R} - mod/\mathfrak{T}_+^-$ *the projective spectrum of* $\mathcal{R}$ and denote it by $\mathbf{Proj}(\mathcal{R})$.

It follows from the definitions of the quasi-affine space and the projective spectrum of $\mathcal{R}$ that the natural functor $\mathfrak{gr}_\Gamma\mathcal{R} - mod \longrightarrow \mathcal{R} - mod$ induces a functor from $\mathbf{Proj}(\mathcal{R})$ to $\mathcal{R} - mod/\mathcal{T}_+^-$. The latter should be viewed as an inverse image functor of the projection from the quasi-affine space (the affine cone) of $\mathcal{R}$ onto the projective spectrum of $\mathcal{R}$ (cf. [R], Chapter VII).

10.3. Some examples of quasi-affine and projective spaces. We begin with most important for this work examples, leaving the simplest one – the projective space – to the end.

10.3.1. Example: the base affine space and the flag variety of a reductive Lie algebra. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ (or over any other algebraically closed field of characteristic zero). Let $\mathcal{U} = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{P}$ be the group of integral weights of $\mathfrak{g}$ (isomorphic to $\mathbb{Z}^r$, $r = rank(\mathfrak{g})$), and let $\mathcal{P}_+$ denote the semigroup of nonnegative integral weights. Let $\mathcal{R}$ the $\mathcal{P}$-graded algebra $(\oplus_{\lambda\in\mathcal{P}_+}\mathcal{R}_\lambda, \mu)$, where $\mathcal{R}_\lambda$ is the vector space of the (canonical) finite dimensional representation with the highest weight λ. The multiplication μ is determined by the projections

$$\mathcal{R}_\lambda \otimes \mathcal{R}_\nu \longrightarrow \mathcal{R}_{\lambda+\nu}, \quad \lambda, \nu \in \mathcal{P}_+.$$

Clearly the natural action of $\mathcal{U}$ on each $\mathcal{R}_\lambda$ is a respecting grading Hopf action of $\mathcal{U}$ on $\mathcal{R}$. Setting $\mathcal{P}_{>0} := \mathcal{P}_+$, we make $\mathcal{P}$ a directly ordered group: $\gamma > \sigma$ iff $\gamma - \sigma$ is an element of $\mathcal{P}_+$.

Note that the algebra $\mathcal{R}$ is commutative and is known to be isomorphic to the algebra of regular functions on the so called 'base affine space' Y of $\mathfrak{g}$ (which is, actually, quasi-affine). Recall that $Y = G/U$, where G is the simply connected connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup. The category $\mathcal{R} - mod/\mathcal{T}_+^-$ is equivalent to the category of quasi-coherent sheaves on the base affine space.

And $\mathbf{Proj}(\mathcal{R})$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$. Recall that the flag variety of $\mathfrak{g}$ is the homogenious space G/B, where G is the simply connected connected algebraic group with the Lie algebra $\mathfrak{g}$, and B is a Borel subgroup in G. ∎

10.3.2. The base affine space and the flag variety of a quantized enveloping algebra. The construction of Example 10.3.1 can be reproduced word by word in the case of the quantized enveloping algebra $\mathcal{U} = U_q(\mathfrak{g})$ of a simple Lie algebra over the field of zero characteristic, with q being generic (i.e. not a root of 1), or a formal parameter. This time, however, the algebra $\mathcal{R}$ is not commutative. Following the classical example, we shall call the quasi-affine space of $\mathcal{R}$ *the base affine space of* $U_q(\mathfrak{g})$ or simply *quantized base affine space.* And we call $\mathbf{Proj}(\mathcal{R})$ *the quantized flag variety of* $\mathfrak{g}$. ∎

10.3.3. Note on the base affine space and the flag variety of a Kac-Moody algebra. The same construction as in 10.3.1; only $\mathcal{R}_\lambda$ is an integral simple representation with the highest weight λ. Similarly to the finite dimensional case, this also can be extended to the case of a quantized enveloping algebra of a Kac-Moody Lie algebra. ∎

10.3.4. Projective spaces. Fix a monoidal category $\mathcal{C}^{\sim} = (\mathcal{C}, \odot, \mathbf{1})$ with a symmetry σ. Let $S_\sigma(W)$ be a σ-symmetric algebra of an object W of $\mathcal{C}$ (cf. Example 5.3). Being a quotient algebra of the $\mathbb{Z}_+$-graded 'tensor' algebra $T(W) = \oplus_{n\geq 0} W^{\odot n}$ by a homogenious ideal, the algebra $S_\sigma(W)$ is $\mathbb{Z}_+$-graded itself. Taking $\Gamma = \mathbb{Z}$ with the natural ordering, we define the *σ-affine cone*, $\mathfrak{C}_\sigma(W) := S_\sigma(W) - mod/\mathcal{T}_+^-$, and *$\sigma$-projective space*, $\mathbb{P}_\sigma(W) := \mathbf{Proj}(S_\sigma(W)) := \mathfrak{gr}_{\mathbf{Z}} S_\sigma(W) - mod/\mathfrak{T}_+^-$. Note that the Serre subcategory $\mathcal{T}_+^-$ (hence $\mathfrak{T}_+^-$) admits in this case the following discription: $\mathcal{T}_+^-$ is the minimal Serre subcategory of $S_\sigma(W) - mod$ containing all modules annihilated by W (cf. [R], VII.2).

For instance, we can take as $\mathcal{C}^{\sim}$ the category of $\mathbb{Z}^J$-graded k-modules for some commutative ring k with the symmetry σ defined by a matrix $\mathbf{q} = (q_{ij})_{i,j\in J}$ of invertible elements of k such that (since σ is a symmetry) $q_{ij}q_{ji} = 1$ for all $i, j \in J$. Let W be the canonical skew free object; i.e. W is the direct sum of J generators of $\mathbf{Pic}(\mathcal{C}^{\sim})$ (note that $Pic(\mathcal{C}^{\sim}) = \mathbb{Z}^J$). Then $S_\sigma(W)$ is the skew polynomial algebra of Section 3. If the matrix (q_{ij}) is identical, then the symmetry σ is standart, $S_\sigma(W)$ coincides with the polynomial ring in J indeterminates over k, and $\mathbf{Proj}(S_\sigma(W))$ is equivalent to the category of quasi-coherent sheaves on the usual projective space. In the generic case, when the matrix (q_{ij}) is nontrivial, $\mathbb{P}_\sigma(W) = \mathbf{Proj}(S_\sigma(W))$ has properties very similar to those of its commutative prototype. For instance $\mathbb{P}_\sigma(W)$ is canonically covered with skew affine spaces (cf. [R], Chapter I, Example 1.2.2.4). ∎

10.4. Differential calculus on non-affine 'schemes'. Our next step is to define a differential calculus on noncommutative projective spaces. In particular, on the quantized flag varieties. This means that we need to define the diagonal. This is already done in Part I for all 'noncommutative spaces' in the 'absolute, minimal, case': the 'absolute' (=minimal) diagonal of an abelian category $\mathcal{A}$ is the minimal subscheme of $\mathcal{A} \times \mathcal{A} := \mathfrak{End}\mathcal{A}$ (– the category of functors $\mathcal{A} \longrightarrow \mathcal{A}$ having a left adjoint) containing $Id_{\mathcal{A}}$. But we need to define a β-diagonal, where β is a fixed quasi-symmetry of the base monoidal category $\mathcal{C}^{\sim}$.

10.4.1. Actions of (monoidal) categories and associated diagonals. We begin with a slightly different interpretation of the β-diagonal in $R - mod$, where R is any algebra in $\mathcal{C}^{\sim}$. Note that the quasi-symmetry β defines an action of $\mathcal{C}^{\sim}$ on $R - mod$: the action F_X of $X \in Ob\mathcal{C}$ sends any R-module (M, m) into the module $(X \odot M, Xm \circ \beta^{\wedge}_{R,X} M)$, where $\beta^{\wedge}_{R,X} = \beta^{-1}_{X,R}$ and any module morphism f into $id_X \odot f$ (cf. Proposition I.6.4.3 and the preceeding discussion). Suppose that, for any $X \in Ob\mathcal{C}$, the functor $X\odot$ is continuous; i.e. it has a left adjoint. Then all functors F_X have left adjoint, and the β-diagonal coincides with the minimal subscheme of $\mathfrak{End}(R - mod)$ containing all functors F_X, $X \in Ob\mathcal{C}$.

Let $\mathcal{A}$ be an abelian category with a given 'continuous' action F of $\mathcal{C}$. Here 'continuous' means that, for any $X \in Ob\mathcal{C}$, the corresponding functor $F_X : \mathcal{A} \longrightarrow \mathcal{A}$ has a left adjoint. We call the minimal subscheme of $\mathfrak{End}\mathcal{A}$ containing all F_X, $X \in Ob\mathcal{C}$, the F-diagonal of $\mathfrak{End}\mathcal{A}$.

10.4.2. The β-diagonal and differential calculus on Proj. An object of the monoidal category $\mathcal{C}^{\sim}$ is *flat* if the functor $X\odot$ is exact. We say that the monoidal category $\mathcal{C}^{\sim}$ has enough flat objects if, for any object Y of $\mathcal{C}$, there exists an epimorphism $X \longrightarrow Y$ with X flat.

Fix an abelian group Γ. Let $\mathcal{R}$ be an Γ-graded algebra in $\mathcal{C}^{\sim}$. The category $\mathcal{C}$ acts

on the category $\mathfrak{gr}_\Gamma\mathcal{R}-mod$ of Γ-graded modules the same way as it acts on R-modules. This action respects the grading, stabilizes the subcategory $\mathfrak{T}_+$.

10.4.2.1. Lemma. *Suppose that $\mathcal{C}^-$ has enough flat objects. Then the action of $\mathcal{C}$ on $\mathfrak{gr}_\Gamma\mathcal{R}-mod$ defines an action, $\mathcal{F}$, of $\mathcal{C}$ on* $\mathbf{Proj}(\mathcal{R})$.

Proof. The action of any flat object X on $\mathfrak{gr}_\Gamma\mathcal{R}-mod$ determines, by Proposition I.6.1, an action, $\mathcal{F}_X$, of X on $\mathbf{Proj}(\mathcal{R})$. Let now Y be an arbitrary object of $\mathcal{C}$. Since $\mathcal{C}^-$ has enough flat objects, there exists an exact sequence

$$X' \longrightarrow X \longrightarrow Y \longrightarrow 0$$

with X' and X flat. Since the tensoring is a right exact functor, it follows that the action of Y on $\mathfrak{gr}_\Gamma\mathcal{R}-mod$ determines an action $\mathcal{F}_Y$ uniquely (up to isomorphism) determined by the exactness of the sequence

$$\mathcal{F}_{X'} \longrightarrow \mathcal{F}_X \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

The standart detailes of this argument are left to a reader. ∎

We call the $\mathcal{F}$-diagonal in $\mathfrak{End}(\mathbf{Proj}(\mathcal{R}))$ the *β-diagonal.*

Having a notion of a β-diagonal on $\mathbf{Proj}(\mathcal{R})$, we obtain the rest of the differential calculus on $\mathbf{Proj}(\mathcal{R})$ automatically. Thus we have *β-differential actions* (cf. Section 6.10). In particular, for any two objects L and M of $\mathbf{Proj}(\mathcal{R})$, we have the object of β-differential operators, $Diff_\beta(L,M)$. We denote by $D_\beta(\mathcal{R})$ the object of β-differential operators from $\mathcal{R}$ to $\mathcal{R}$. Here, as usual, we take the canonical realization of quotient categories; i.e. the localization

$$\mathfrak{gr}\mathcal{R}-mod \longrightarrow \mathbf{Proj}(\mathcal{R})$$

maps objects identically. In particular the Γ-graded left $\mathcal{R}$-module $\mathcal{R}$ is regarded as an object of $\mathbf{Proj}(\mathcal{R})$.

10.4.3. Remark: other versions of diagonals. It is more convenient, whenever it is possible, to deal with auto-equivalences. Then we don't need restrictions like in Lemma 10.4.2.1.

Fix a grouppoid $\mathfrak{G}$ (i.e. a category all morphisms of which are invertible). And consider pairs $(\mathcal{A},\Phi)$, where $\mathcal{A}$ is an abelian category, Φ is a functor (or, more conveniently, a diagram) from $\mathfrak{G}$ to the grouppoid $\mathrm{Aut}(\mathcal{A})$ of auto-equivalences of $\mathcal{A}$ – an action of $\mathfrak{G}$ by auto-equivalences. Then we have a notion of Φ-diagonal which is the minimal subscheme of $\mathfrak{End}\mathcal{A}$ containing all functors $\Phi(X)$, $X \in Ob\mathfrak{G}$. As 'morphisms' from a $(\mathcal{A},\Phi)$ to $(\mathcal{A}',\Phi')$, we allow only functors from $\mathcal{A}$ to $\mathcal{A}'$ compatible with the actions of $\mathfrak{G}$. Thus, subschemes (in particular Serre subcategories) of $(\mathcal{A},\Phi)$ are Φ-stable subschemes of $\mathcal{A}$. If $\mathbb{S}$ is any Φ-stable Serre subcategory of $\mathcal{A}$, then the quotient category, $\mathcal{A}/\mathbb{S}$, has the induced action of $\mathfrak{G}$, hence a diagonal and the rest of differential calculus.

For instance, we might restrict the action of $\mathcal{C}^-$ on $\mathfrak{gr}_\Gamma\mathcal{R}-mod$ to the action of the subcategory $\mathbf{Pic}(\mathcal{C}^-)$ of invertible objects of $\mathcal{C}^-$ on $\mathfrak{gr}_\Gamma\mathcal{R}-mod$. Since $\mathbf{Pic}(\mathcal{C}^-)$ acts by

auto-equivalences, it defines an action (by auto-equivalences) on any quotient category $\mathfrak{gr}_\Gamma\mathcal{R}-mod/\mathbb{S}$, provided that the Serre subcategory $\mathbb{S}$ is stable with respect to all these actions (which holds for $\mathbb{S}=\mathbb{T}_+$). Note that in the (important for this work) case when $\mathcal{C}^-$ is the monoidal category of graded modules, the diagonal obtained this way coincides with the β-diagonal. ∎

10.5. Differential calculus in 'spaces' with operators and crossed products. Fix a monoidal category $\mathcal{C}^-=(\mathcal{C},\odot,\mathbf{1})$ with a quasi-symmetry β. A β-bialgebra $\mathbf{H}=(\delta,H,m)$ is *β-cocommutative* if $\beta_{H,H}\circ\delta=\delta$.

10.5.1. Example. Let G be a group. Then the group algebra $\mathbf{1}(\mathrm{G})$ of G in $\mathcal{C}^-$ (cf. Example 5.4) is β-cocommutative β-Hopf algebra for any β. ∎

10.5.2. Lemma. *Let $\mathbf{H}=(\delta,H,m)$ be a β-cocommutative β-Hopf algebra. Then the category $\mathbf{H}-mod$ of $\mathbf{H}$-modules is a monoidal category with a quasi-symmetry β' canonically determined by β.*

Proof. The monoidal structure on $\mathbf{H}-mod$ is defined for any β-bialgebra $\mathbf{H}$. The quasi-symmetry β' assigns to any pair $\mathbf{V}=(V,m),\mathbf{W}=(W,\nu)$ of $\mathbf{H}$-modules the isomorphism $\beta_{V,W}$. We leave to the reader the verifying that, thanks to the β-cocommutativity of $\mathbf{H}$, $\beta_{V,W}$ is really an $\mathbf{H}$-module isomorphism. ∎

Thus, for any β-cocommutative β-Hopf algebra $\mathbf{H}$, we obtain, replacing $(\mathcal{C}^-,\beta)$ with the monoidal category $\mathbf{H}-mod^-=(\mathbf{H}-mod,\odot,\mathbf{1})$ and the quasi-symmetry β' of Lemma 10.5.2, an $\mathbf{H}$-*equivariant* differential calculus.

Fix a β-cocommutative β-Hopf algebra $\mathbf{H}$.

10.5.3. Lemma. *For any algebra $\mathcal{R}$ in $\mathbf{H}-mod^-$, the category $\mathcal{R}-mod$ is isomorphic to the category $\mathcal{R}\#\mathbf{H}-mod$.*

Proof. Let $(M,m:\mathcal{R}\#\mathbf{H}\odot M\longrightarrow M)$ be an object of $\mathcal{R}\#\mathbf{H}-mod$ (here $\mathcal{R}\#\mathbf{H}$ is regarded as an algebra in $\mathcal{C}^-$). The restriction m' of the action m to $\mathbf{H}\odot M$ is an $\mathbf{H}$-module structure; and one can see that the restriction m'' of m to $\mathcal{R}\odot M$ is an $\mathbf{H}$-module morphism. The map sending (M,m) into $((M,m'),m'')$ is, obviously, functorial. It is the claimed isomorphism from $\mathcal{R}\#\mathbf{H}-mod$ to $\mathcal{R}-mod$. The checking detailes is left to a reader. ∎

Lemma 10.5.3 implies that the category $\mathfrak{End}(\mathcal{R}-mod)$ is equivalent to the category $\mathcal{R}\#\mathbf{H}$-bimodules. And, for any $\mathcal{R}\#\mathbf{H}$-bimodule M, the β'-differential part of M is the $\mathbf{H}$-subbimodule of M generated by the β-differential part of M regarded as an $\mathcal{R}$-bimodule: $D_{\beta'}(M)=HD_\beta(M)\mathrm{H}$. In particular, for any $\mathcal{R}\#\mathbf{H}$-module L, $Diff_{\beta'}(L,L)$ is generated by $Diff_\beta(L,L)$ (where L is viewed as an $\mathcal{R}$-module) and by the image of H in End(L).

10.5.3.1. Example: equivariant differential calculus on affine spaces. Take as $\mathcal{R}$ the skew polynomial k-algebra of Section 5.3. defined by a matrix $\mathbf{q}=(q_{ij})_{i,j\in J}$ with entrees in k such that $q_{ij}q_{ji}=1$. The matrix $\mathbf{q}$ defines a symmetry β in the monoidal category $\mathcal{C}^-$ of Z^J-graded k-modules. As usual, the standart symmetry σ in $\mathcal{C}^-$ is determined by the identical matrix. The symmetry β (i.e. the matrix $\mathbf{q}$) defines an action of the

group $G = \mathbb{Z}^J$ on objects of the category $\mathcal{C}^-$. It follows from Lemma 10.5.3 and Proposition 3.3 that the equivariant β-differential operators on $\mathcal{R}$ are generated by β-derivatives, multiplications by elements of $\mathcal{R}$, and by the action of the group G. ∎

10.5.4. Differential calculus on projective spaces. Fix a commutative group Γ and a β-cocommutative β-Hopf algebra $\mathbf{H} = (\delta, H, m)$ in $\mathcal{C}^-$. Suppose now that $\mathcal{R}$ is a Γ-graded algebra in the monoidal category $\mathbf{H}-mod^-$. Or, equivalently, we are given a β-Hopf action of $\mathbf{H}$ on $\mathcal{R}$ which respects Γ-grading. The graded analog of Lemma 10.5.3 states that the category $\mathfrak{gr}_\Gamma\mathcal{R} - mod$ of Γ-graded $\mathcal{R}$-modules (everything over $\mathbf{H} - mod^-$) is isomorphic to the category $\mathfrak{gr}_\Gamma\mathcal{R}\#\mathbf{H} - mod$ (over $\mathcal{C}^-$). This isomorphism induces an equivalence of the category $\mathbf{Proj}(\mathcal{R})$ (over $\mathbf{H} - mod^-$) to $\mathbf{Proj}(\mathcal{R}\#\mathbf{H})$ (over $\mathcal{C}^-$). Therefore we have an $(\mathbf{H}, \beta)$-differential calculus on $\mathbf{Proj}(\mathcal{R})$. The corresponding differential actions and (objects of) differential operators will be called $(\mathbf{H}, \beta)$-differential. If $\mathbf{H} = \mathbf{1}(G)$ for some group G, we might replace $(\mathbf{H}, \beta)$ by (G, β).

10.6. The localization construction. Now we will apply the observations of 10.5 to the case when $\mathbf{H}$ is the group algebra of a subgroup G of $Pic(\mathcal{C}^-)$: $\mathbf{H} = \mathbf{1}(G)$. Note that $\mathbf{1}$(G) is a cocommutative β-Hopf algebra for any quasi-symmetry β (cf. Example 10.5.1). It is ($\beta-$)commutative too, since the existence of a quasi-symmetry implies that the group $Pic(\mathcal{C}^-)$ is commutative. Any quasi-symmetry β determines an action of G on all objects of the category $\mathcal{C}^-$. More exactly, β defines a monoidal fully faithful exact functor $\mathfrak{F}_\beta$ from $\mathcal{C}^-$ to $\mathbf{1}$(G)-mod$^-$ which realizes $\mathcal{C}^-$ as a subscheme of $\mathbf{1}$(G)-mod$^-$ and sends algebras in $\mathcal{C}^-$ into algebras in $\mathbf{1}$(G)-mod$^-$. The functor $\mathfrak{F}_\beta$ allows to transfer G-differential calculus onto affine (i.e. $\mathcal{R} - mod$) and projective (i.e. $\mathbf{Proj}(\mathcal{R})$) 'spaces' in $\mathcal{C}^-$. This way one can talk about (G, β)-differential actions and, for any two $\mathcal{R}$-modules (or objects of $\mathbf{Proj}(\mathcal{R})$), L and M, about (G, β)-differential operators from L to M.

Let X be a subset of Ob$\mathbf{Pic}(\mathcal{C}^-)$ such that if $P, P' \in X$, then either $P = P'$, or $P \ncong P'$. Let $W = \oplus_{P \in X} P$; and let G denote the subgroup of $Pic(\mathcal{C}^-)$ generated by the image of X in $Pic(\mathcal{C}^-)$. Finally, let $\mathcal{U} = \mathcal{U}_{\beta,W}$ be the β-Hopf algebra corresponding to this data (cf. 9.5). We call an action of $\mathcal{U}$ of an $\mathbf{1}$(G)-module M *natural* if the action of the subalgebra $\mathbf{1}$(G) of $\mathcal{U}$ coincides with the $\mathbf{1}$(G)-module structure on M, i.e. if the action $\mathcal{U} \odot M \longrightarrow M$ is a $\mathbf{1}(G)$-module morphism.

10.6.1. Proposition. *Let $\mathcal{R}$ be an algebra in $\mathbf{1}(G) - mod^-$. Any natural β-Hopf action of $\mathcal{U}_{\beta,W}$ on $\mathcal{R}$ is (G, β)-differential.*

Proof. By Proposition 7.7, any β-Hopf action of the affine (β-Hopf) subalgebras $\mathcal{U}_\beta^+$ and $\mathcal{U}_\beta^-$ of $\mathcal{U}_{\beta,W}$ are β-differential. This implies that natural β-Hopf actions of $\mathcal{U}_\beta^+\#G$ and $\mathcal{U}_\beta^-\#G$ are (G, β)-differential (cf. Lemma 10.5.3). ∎

10.6.2. Corollary. *Let τ be a β-Hopf action of $\mathcal{U}_{\beta,W}$ on a* $\mathbf{1}$(G)*-algebra $\mathcal{R}$. Then*

(a) The crossed product $\mathcal{R}\#\mathcal{U}_{\beta,W}$ is a (G, β)-differential $\mathcal{R}$-algebra (i.e. it is a (G, β)-differential $\mathcal{R}$-bimodule).

(b) Any action of $\mathcal{U}_{\beta,W}$ on an $\mathcal{R}$-module L compatible with the action τ is (G, β)-differential.

Proof. (a) The assertion follows from Propositions 10.6.1 and 6.10.4.

(b) 'Compatible with the action τ' means that the actions of $\mathcal{U}_{\beta,W}$ and $\mathcal{R}$ on L determine the action of $\mathcal{R}\#\mathcal{U}_{\beta,W}$ which is (G,β)-differential, because $\mathcal{R}\#\mathcal{U}_{\beta,W}$ is a (G,β)-differential over $\mathcal{R}$. ∎

10.6.3. Remark. If the quasi-symmetry β is trivial, i.e. it coincides with the fixed symmetry σ in $\mathcal{C}^-$, then the actions of G on objects of $\mathcal{C}$ is trivial which implies that (G,β)-differential actions on $\mathcal{R}$-modules (or on objects of $\mathbf{Proj}(\mathcal{R})$) are just β-differential. ∎

10.6.4. Note. Of course, in the assertions above, one can assume that $\mathcal{R}$ is a graded algebra and the action of $\mathcal{U}_{\beta,W}$ respects the grading. ∎

Complimentary facts.

C1. The category $\mathcal{O}$ and twisted differential operators. Fix a monoidal category $\mathcal{C}^-$ with a quasi-symmetry β. Let $\mathcal{R}$ be a Γ-graded algebra in $\mathcal{C}^-$. For any $\nu \in \Gamma$ and any Γ-graded $\mathcal{R}$-module M, denote by $M(\nu)$ the graded $\mathcal{R}$-module $\oplus_{\gamma\in\Gamma} M(\nu)_\gamma$, where $M(\nu)_\gamma$ is $M(\nu+\gamma)$ for all $\gamma \in \Gamma$. In particular, we have left $\mathcal{R}$-modules $\mathcal{R}(\nu), \nu \in \Gamma$.

Note that the left modules $\mathcal{R}(\nu)$ are, actually, $\mathcal{R}$-bimodules, and the functor $\mathcal{R}(\nu)\otimes_{\mathcal{R}}$ from $\mathcal{R}-mod$ to $\mathcal{R}-mod$ is isomorphic to the 'translation' functor $M \mapsto M(\nu)$.

We call the $\mathfrak{D}_\nu := Diff(\mathcal{R}(\nu),\mathcal{R}(\nu))$ *the algebra of ν-differential operators*, or *twisted differential operators*. If $\mathcal{R}(\nu)$ is viewed as an object of $\mathbf{Proj}(\mathcal{R})$, then we say that $\mathfrak{D}_\nu$ is the algebra of differential operators on the projective spectrum. If $\mathcal{R}(\nu)$ is regarded as an object of the quasi-affine space of $\mathcal{R}$ (cf. 10.2), then we call $\mathfrak{D}_\nu$ the algebra of twisted differential operators on that space.

Applying this to the algebras $\mathcal{R} = \oplus_{\lambda\in\mathcal{P}_+}\mathcal{R}_\lambda$ of 10.3.1 and 10.3.2, we obtain, for any integer weight ν, the algebras of differential operators on the corresponding base affine spaces and flag varieties - in the classical and quantized cases.

In the classical case, ν-differential operators are defined for any, not necessarily integral, weight. To see how this can be done for quantized enveloping algebras, we shall reproduce the construction of ν-differential operators of a reductive Lie algebra over a field of zero characteristic in a way which can be easily 'quantized'.

C1.1. The category $\mathcal{O}$ and the functor Φ. Fix a finite dimensional reductive Lie algebra $\mathfrak{g}$ over a field k of characteristic zero. As usual, we denote by $\mathfrak{h}$ and $\mathfrak{b}$ resp. a Cartan and a Borel Lie subalgebras of $\mathfrak{g}$ and by $\mathcal{Z}(\mathfrak{g})$ the center of the enveloping algebra $U(\mathfrak{g})$.

Recall that the category $\mathcal{O} = \mathcal{O}(\mathfrak{g})$ is the full subcategory of the category of $U(\mathfrak{g})$-modules generated by all $U(\mathfrak{g})$-modules of finite type which are semisimple as $\mathfrak{h}$-modules and are locally $U(\mathfrak{b})$-finite. Clearly $\mathcal{O}$ is a topologizing subcategory of $U(\mathfrak{g})$-mod. And the minimal subscheme, $\bar{\mathcal{O}}$, of $U(\mathfrak{g})$-mod containing $\mathcal{O}$ is obtained by dropping the condition of $U(\mathfrak{g})$-finiteness.

For any $\lambda \in \mathfrak{h}^*$, let $\mathcal{O}_\lambda$ denote the full subcategory of $\mathcal{O}$ generated by all modules on which operators $z - \chi_\lambda(z)1$ are locally nilpotent for all $z \in \mathcal{Z}(\mathfrak{g})$. Since $\chi_\lambda = \chi_{\lambda'}$ iff λ' and λ lie in the same W-orbit (where W denotes, as usually, the Weyl group of $\mathfrak{g}$), it is more appropriate to write $\mathcal{O}_{W\lambda}$ instead of $\mathcal{O}_\lambda$. Then $\mathcal{O}$ is a direct sum of the subcategories $\mathcal{O}_{W\lambda}$

in the sense that each module of the category $\mathcal{O}$ is uniquely represented as a direct sum of its submodules from the subcategories $\mathcal{O}_{W\lambda}$.

For any $\mu \in \mathfrak{h}^*$, denote by k_μ the one-dimensional $U(\mathfrak{b})$-module, where $\mathfrak{h}$ acts by μ and the nilpotent Lie subalgebra $\mathfrak{n}^+$ acts by zero. Recall that the Verma module $M(\lambda)$ is the induced module $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_{\lambda-\rho}$. Denote by $L(\lambda)$ the irreducible quotient of $M(\lambda)$ by the maximal submodule. Recall that, for any $\lambda \in \mathfrak{h}^*$, the modules $M(w\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_{w\lambda-\rho}$, $w \in W$, and $L(w\lambda)$, $w \in W$, form bases for the Grothendieck group of $\mathcal{O}_\lambda = \mathcal{O}_{W\lambda}$.

For any $\mu \in \mathfrak{h}^*$, let $\pi_\mu : \mathcal{O} \longrightarrow \mathcal{O}_\mu$ denote the natural 'projection' functor assigning to any object of $\mathcal{O}$ its $\mathcal{O}_\mu$-summand. Now define a functor $\Phi : \mathcal{O} \longrightarrow \mathfrak{gr}\mathcal{R} - mod$ by

$$\Phi(M) = \oplus_{\lambda \in P^+} (\oplus_{\mu \in \mathfrak{h}^*} \pi_{\mu+\lambda}(\mathcal{R}_\lambda \otimes \pi_\mu(M))) \tag{1}$$

Note that

$$\Phi(M(w\mu)) = \oplus_{\lambda \in P^+} M(w(\mu+\lambda)) \quad \text{and} \quad \Phi(L(w\mu)) = \oplus_{\lambda \in P^+} L(w(\mu+\lambda)) \tag{2}$$

for all $\mu \in \mathfrak{h}^*$ and $w \in W$.

If $\mu \in P^+$, then $L(\mu+\rho) \simeq \mathcal{R}_\mu$, and $\Phi(\mathcal{R}_\mu) = \oplus_{\lambda \in P^+} \mathcal{R}_{\lambda+\mu} = \mathcal{R}(\mu)$. In particular, $\Phi(\mathbf{1}) = \mathcal{R}$ as a left $\mathcal{R}$-module. Here $\mathbf{1}$ is the trivial one-dimensional $U(\mathfrak{g})$-module, $\mathbf{1} = L(\rho) = \mathcal{R}_0$.

C1.2. Twisted differential operators. Fix $\mu \in \mathfrak{h}^*$. Call $\mathfrak{D}_\mu := \mathfrak{Diff}(\Phi(M(\mu), \Phi(M(\mu))$ *the algebra of twisted differential* (or μ-*differential*) *operators* on the affine base space.

Denote by $\mathfrak{F}$ the composition of the functor $\Phi : \mathcal{O} \longrightarrow \mathfrak{gr}\mathcal{R} - mod$ with the localization functor $\mathfrak{gr}\mathcal{R} - mod \longrightarrow \mathbf{Proj}(\mathcal{R})$. For any $\mu \in \mathfrak{h}^*$, we set $\mathfrak{D}_\mu := \mathfrak{Diff}(\mathfrak{F}(M(\mu)), \mathfrak{F}(M(\mu))$. and call $\mathfrak{D}_\mu$ *the algebra of twisted differential* (or μ-*differential*) *operators* on the flag variety.

C1.3. Note. The construction above can be repeated for the quantized enveloping algebra of a semisimple Lie algebra. This allows to define μ-differential quantized operators for any, not necessarily integral, weight μ. The detailes are left to a reader. ∎

C2. Extension of β-derivations. Let $\mathcal{C} = (\mathcal{C}, \odot, \mathbf{1})$ be a monoidal category with a fixed quasi-symmetry β. Let $\mathcal{R} = (R, m)$ be an algebra in $\mathcal{C}$; and let $\partial : W \odot R \longrightarrow R$ be a β-derivation.

C2.1. Lemma. *Let $\mathcal{B} = (B, \mu)$ and $\mathcal{R} = (R, m)$ be algebras in $\mathcal{C}$. Then, for any β-derivation $\partial : W \odot R \longrightarrow R$ of $\mathcal{R}$, the morphism $\partial B : W \odot (R \odot B) \longrightarrow R \odot B$ is a β-derivation of $\mathcal{R} \odot_\beta \mathcal{B}$.*

Similarly, for any β-derivation $\partial' : W \odot B \longrightarrow B$ of $\mathcal{B}$, the morphism

$$R\partial' \circ \beta_{R,W} B : W \odot (R \odot B) \longrightarrow R \odot B$$

is a β-derivation of $\mathcal{R} \odot_\beta \mathcal{B}$.

Proof. Let f_∂ denote the corresponding to $\partial\mathcal{R}$-bimodule morphism $W \odot \mathcal{J}_m \longrightarrow \mathcal{R}$ (cf. Proposition I.6.4.1). Here $\mathcal{J}_m$:=Ker(m). The kernel $\mathcal{J}_{m \odot_\beta \mu}$ of the multiplication

$m \odot_\beta \mu$ coincides with $\mathcal{J}_m \odot B \odot B + R \odot R \odot \mathcal{J}_\mu$. The quotient of the bimodule $\mathcal{J}_{m\odot_\beta\mu}$ by $R \odot R \odot \mathcal{J}_\mu$ is isomorphic to $\mathcal{J}_m B$, since $B \odot B/\mathcal{J}_\mu \simeq$B. Let φ_∂ denote the composition of the projection $\mathcal{J}_{m\odot_\beta\mu} \longrightarrow \mathcal{J}_m B$ and the bimodule morphism $f_\partial B : \mathcal{J}_m B \longrightarrow \mathcal{R} \odot_\beta \mathcal{B}$. The corresponding to this morphism derivation is exactly ∂B.

We leave the checking of omitted detailes and the proof of the second assertion to a reader. ∎

C2.2. Lemma. *Let $\mathcal{A} = (A, m)$ and $\mathcal{B} = (B, \mu)$ be algebras in $\mathcal{C}$; and let $\mathcal{M} = (M, \xi), \mathcal{N} = (N, \nu)$ be modules resp. over $\mathcal{A}$ and $\mathcal{B}$. Then*

$$\mathcal{M} \odot_\beta \mathcal{N} := (M \odot N, \xi \odot \nu \circ A\beta_{B,M}N)$$

is an $\mathcal{A} \odot_\beta \mathcal{B}$-module.

Proof is a straightforward checking left to a reader. ∎

The derivation $\partial : W \odot R \longrightarrow R$ induces a left action $T\partial$ of the free algebra $\mathbf{T} = T(W)$ on $\mathcal{R}$. By Lemma C2.2, the morphism

$$T\partial' \odot \mu \circ \mathbf{T}\beta_{B,R}B : (\mathbf{T}\odot B) \odot (R \odot B) \longrightarrow R \odot B \tag{1}$$

is a left $\mathbf{T}\odot_\beta\mathcal{B}$-module structure on $\mathcal{R} \odot_\beta \mathcal{B}$.

C3. Skew derivations and Hopf algebras. Here we sketch a ring-theoretical construction of Hopf algebras related to skew derivations and crossed products.

C3.1. Skew derivations and crossed products. Let k be a commutative ring, R a k-algebra, G a subgroup of the group $\mathrm{Aut}_k(R)$ of automorphisms of R. This means that we have a Hopf action of the group algebra $k(G)$ of G on R. We shall write $R\#G$ instead of $R\#k(G)$ and call it *the crossed product of G and R*. Recall that $R\#G$ is a free right R-module with the basis $\{x_g \mid g \in G\}$ and the multiplication given by

$$g(r)x_g = x_g r \text{ for all } r \in R, \text{ and } x_g x_h = x_{gh} \text{ for all } g, h \in G.$$

In particular, $R\#G$ has a natural structure of a G-graded algebra.

C3.1.1. Lemma. *The action of G on R extends to an action on $R\#G$ by $s(x_t) = x_{st/s}$ for all $s, t \in G$.*

Proof. Clearly the action is well defined on the image of the group algebra of G in $R\#G$: for any $s, t, u \in G$, we have $s(x_t x_u) = s(x_t)s(x_u)$.

It remains to check that, for any $r \in R$ and $s, t \in G$,

$$s(x_t)s(r) := s(x_t r) = s(t(r)x_t) := st(r)s(x_t).$$

In fact, we have:

$$s(x_t)s(r) := x_{st/s}s(r) = (sts^{-1})(s(r))x_{st/s} = st(r)x_{st/s} = st(r)s(x_t).$$

■

C3.1.2. Note. The extension of $s \in G$ to an automorphism of $R\#G$ respects the natural G-grading of $R\#G$ only if s belongs to the center of G. ■

C3.1.3. Lemma. *Let h be an automorphism of the algebra R which commutes with all $g \in$G. Let d be an h-derivation of R such that, for any $g \in G$, there exists $\lambda_{d,g} \in k^*$ satisfying the conditions:*

(a) $\lambda_{d,id} = 1$, $\lambda_{d,s}\lambda_{d,t} = \lambda_{d,st}$ for all $s, t \in G$.

(b) $d \circ g = \lambda_{d,g} g \circ d$ for all $g \in G$.

Then there exists an extension $h' \in \mathrm{Aut}_k(R\#G)$ of h and the extension d' of d to an h'-derivation of $R\#G$ such that $d'(x_g) = 0$ for all $g \in$G.

Proof. The extension h' of h is defined by $h'(x_s) = \lambda_{d,s} x_s$. The conditions *(a)* imply that h' is an automorphism.

Set for convenience $\lambda_{d,s} = \lambda_s$. For any $s, t \in G$ and $a, b \in R$, we have:

$$d'(x_s t(a) \cdot x_t b) = d'(x_{st} ab) = \lambda_{st} x_{st} d(ab) = \lambda_{st} x_{st}(d(a)b + h(a)d(b)). \tag{1}$$

On the other hand,

$$d'(x_s t(a)) x_t b + h'(x_s t(a)) d'(x_t b) = \lambda_s x_s dt(a) x_t b + \lambda_s x_s h t(a) \lambda_t x_t d(b) =$$

$$\lambda_s x_{st} t^{-1} dt(a) b + \lambda_{st} x_{st} t^{-1} h t(a) d(b) = \lambda_s \lambda_t x_{st} d(a) b + \lambda_{st} x_{st} t^{-1} h t(a) d(b) =$$

$$\lambda_s \lambda_t x_{st}(d(a)b + t^{-1} h t(a) d(b)).$$

Since $t \circ h = h \circ t$ for all $t \in G$, the right part of the last equalities is the same as the right part of (1). ■

C3.2. The algebra U^-. Fix a map $\phi : J \longrightarrow G$ which takes values in the center of G; and let, for each $i \in J$, d_i be an $\phi(i)$-derivation of R. Suppose that, for any $i \in J$, there exists $\{\lambda_{h,s} \mid s \in G, h = \phi(i)\} \subset k^*$ such that

(a) $\lambda_{h,id} = 1$, $\lambda_{h,s}\lambda_{h,t} = \lambda_{h,st}$ for all $s, t \in G$;

(b) $d_i \circ g = \lambda_{\phi(i),g} g \circ d_i$ for all $g \in G$.

Let U^- be a free k-algebra generated by $\{x_i \mid i \in J\}$. The group G acts on U^- by

$$g(x_i) = \lambda_{\phi(i),g} x_i \text{ for all } i \in J \text{ and } g \in G \tag{2}$$

So that we can consider the crossed product $U^-\#G = \oplus_{s \in G} y_s U^-$.

C3.2.1. Proposition. *(a) Under the conditions above, the map which assigns to any x_i, $i \in J$, the $\phi(i)$-derivation d_i and to each generator y_s, $s \in G$, the automorphism s defines an action of $U^-\#G$ on R; i.e. an algebra morphism from $U^-\#G$ to $End(R)$.*

(b) The map which assigns to any $x_i, i \in J$, the $\phi(i)'$-derivation d_i' of Lemma C2.1.1 *and to each generator y_s, $s \in G$, the automorphism s (cf. Lemma* C2.1.1*), defines a ring morphism Φ from $U^-\#G$ to $End(R\#G)$.*

Proof. (a) Since U^- is a free algebra, it remains only to check that Φ is compatible with the relations between x_i and y_s for all $i \in H$ and $s \in G$, i.e. that $\Phi(y_s x_i) := s \circ d_i$ coincides

with $\Phi(g(x_i)y_s) = \lambda_{\phi(i),s}\Phi(x_iy_s) := \lambda_{\phi(i),s}d_i$os. Which is the case by the assumptions of the Proposition.

(b) The assertion *(b)* follows from Lemma C3.1.3. ∎

Define a comultiplication δ^- on the algebra $V^- := U^-\#G$ by

$$\delta^-(x_i) = x_i \otimes 1 + y_{\phi(i)} \otimes x_i, \quad \delta^-(y_s) = y_s \otimes y_s \tag{3}$$

for all $i \in J$ and $s \in G$ and by the requirement that $(\delta^-, U^-\#G, \mu^-)$ is a bialgebra.

Denote by ϑ^- the anti-automorphism of $U^-\#G$ which is identical on the generators x_i, $i \in J$, and sends y_s to $y_{1/s}$ for all $s \in G$.

C3.3. Lemma. *The bialgebra $(\delta^-, U^-\#G, \mu^-)$ is a Hopf algebra with the antipode ϑ^- .*

Proof follows from the fact that the comultiplication δ^- and the antipode ϑ^- are extensions of those on the group G, and the algebra U^- being free. ∎

C3.4. Proposition. *(a) The action of $U^-\#G$ on R (cf. Proposition C2.1.2) is a bialgebra action; i.e. it induces a bialgebra morphism from $(\delta^-, U^-\#G, \mu^-)$ to the bialgebra $\mathfrak{End}(R)$ of endomorphisms of R.*

(b) Similarly, the action of $U^-\#G$ on $R\#G$ is a bialgebra action. If the group G is commutative, the action of $U^-\#G$ respects the G-grading on $R\#G$.

Proof. (a) We need to check that $\mu \circ \delta^-(z)(a \otimes b) = z(ab)$ for all $z \in U^-\#G$ and any elements a, b of R. It suffices to check the fact for all generators of $U^-\#G$.

It is definitely true for all y_s, $s \in G$, since the action of y_s is s, and s is an automorphism.

For any $i \in J$, we have:

$$\mu \circ \delta^-(x_i)(a \otimes b) := \mu \circ (x_i \otimes 1 + y_{\phi(i)} \otimes x_i)(a \otimes b) = d_i(a)b + \phi(i)(a)d_i(b) = d_i(ab),$$

since d_i is an $\phi(i)$-derivation.

(b) A similar argument (together with Note C3.1.2) works for the second assertion. ∎

C3.5. Example. Let $\mathcal{R} = (R, \mu)$ be a k-algebra. And let G be a subgroup of $\mathrm{Aut}_k(R, \mu)$. Suppose we are given the data of C3.2; so that the algebra U^- with the action of G on it is defined (cf. C3.2). Take as $\mathcal{U}$ the Hopf algebra $U^-\#G$ and as τ its natural bialgebra action on $\mathcal{R}\#G$ (cf. Lemma C3.3 and Proposition C3.4).

Suppose that $R\#G = \oplus_{s\in G} x_s R$ has a Hopf algebra structure, i.e. a comultiplication Δ and an antipode ϑ, which extend those on $k(G)$: $\Delta(x_s) = x_s \otimes x_s$, and $\vartheta(x_s) = x_{1/s}$ for all $s \in$G. Suppose that the action of $U^-\#G$ on $\mathcal{R}\#G$ is compatible with the coalgebra structure Δ. The latter means that

(a) all elememts of G are automorphisms of the Hopf algebra $\mathcal{R}\#G$;

(b) the derivations d_i, $i \in J$, (cf. C3.2) are compatible with Δ; i.e., for any $r \in R$ and $i \in J$, we have:

$$\Delta \circ d_i(r) = \sum_\nu (d_i(r_\nu) \otimes r'_\nu + \phi(i)(r_\nu) \otimes d_i(r'_\nu)) \tag{1}$$

where $\sum_\nu r_\nu \otimes r'_\nu = \Delta(r)$ (cf. Lemma 6.8.1).

Then $U' := (U^-\#G)\#(\mathcal{R}\#G)$ is a Hopf algebra and the actions of $U^-\#G$ on $\mathcal{R}\#G$ and the adjoint action of $\mathcal{R}\#G$ determine a Hopf algebra action of U' on $\mathcal{R}\#$G. Denote by U'' the quotient of U' by the annihilator of $\mathcal{R}\#$G. This is a Hopf algebra.

It follows from Example 6.8.2 (and the definition of the action of $U^-\#G$ on $\mathcal{R}\#G$) that, for any $s \in G$, the images of elements $x_s \in \mathcal{R}\#G$ and $y_s \in U^-\#G$ in $\mathfrak{End}(\mathcal{R}\#G)$ coincide. ■

C3.6. The Hopf algebra $\mathcal{R}^+ * U_-$. This is an important specialization of the construction of Section 6.9. We assume that the conditions of C3.5 hold; i.e. $\mathcal{R}$ is a Hopf algebra, and the action of U^- on $\mathcal{R}\#G$ is compatible with the comultiplication (cf. Lemma 6.8.1). Let $\mathcal{R}_+$ be the augmentation ideal in $\mathcal{R}\#G$ - the kernel of the coidentity $\epsilon : \mathcal{R}\#G \longrightarrow k$. Denote by L_+ the largest among $U^-\#G$-stable ideals in $\mathcal{R}\#G$ contained in $\mathcal{R}_+$. Set $\mathcal{R}^+ := \mathcal{R}\#G/L_+$. By Lemma 6.9.1, $\mathcal{R}^+$ is a Hopf algebra and the action of $U^-\#G$ on $\mathcal{R}^+$ is compatible with the comultiplication on $\mathcal{R}^+$.

Thus, $\mathcal{R}^+\#(U^-\#G)$ acts on $\mathcal{R}^+$. The quotient, $\mathcal{R}^+ * U_-$, of $\mathcal{R}^+\#(U^-\#G)$ by the annihilator of $\mathcal{R}^+$ is a Hopf algebra. Here U_- denotes the image of $U^-\#G$.

We shall call the kernel, K_-, of the canonical (Hopf algebra) epimorphism from $U^-\#G$ to U_- the *(Hopf) ideal of Serre relations.*

Quantized enveloping algebras are a special case of this construction.

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