Rational maps, common Julia sets, functional equations

G. Levin* and F. Przytycki**

*
Institute of Mathematics
Hebrew University
91904 Jerusalem
Israel

**
Institute of Mathematics
Polish Academy of Sciences
ul. Sniadeckich 8
00950 Warsaw

Poland

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

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Introduction

We consider the following two problems: to describe all pairs of rational functions f and g such that:

(A) f and g have common measure of maximal entropy,

or

(B) f and g have common Julia set.

In the present paper we solve the problem (A) for arbitrary pair of non-exceptional (see the definition below) rational functions, and we solve the problem (B) in the class Ξ of rational functions with Julia sets not the whole Riemann sphere, a circle (or its arc), without parabolic periodic points and singular domains of the complement of Julia set (see also Remark 1 below). By solution of the problems we mean a functional equation between f and g, which is equivalent to the existence of common measure of maximal entropy (maximal measure), or the existence of common Julia set. A corollary is that in the class Ξ the maximal measure is determined by the common Julia set (rigidity of maximal measure). An application to functional equations is done.

The problems (A) and (B) are closely related to the classical problem of commuting pairs of rational functions. In order to solve the latter problem, Fatou and Julia independently applied what is called now Julia set of a rational function, introduced by them in [F1], [J1]. (Commuting rational functions have a common Julia set J and a common maximal measure.) Discovering fundamental properties of J, Fatou and Julia described [F2], [J2] all commuting rational functions under the restriction that the common Julia set J is not the whole Riemann sphere. Ritt [R1],[R2] gave an algebraic solution of the problem in general: except for explicitly described cases, if f and g commute, then they have a common iteration. These exceptions are exactly the *critically finite rational maps with parabolic orbifolds*, in modern terminology [T]. We call such functions *exceptional.* Recently Eremenko [E] has completed the method by Fatou and Julia studying the common maximal measure of commuting rational functions in the case $J = \overline{\mathbb{C}}$. Note that the problems (A)-(B) are not reduced to the commuting case (see Example below).

The problems (A) and (B) have been studied in [BE], [B1], [B2], [E], [L], [Fe]. In the class of polynomials the solution is known, see for example [BE].

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of the Riemann sphere $\overline{\mathbb{C}}$. Let $J(f)$ denote its Julia set, and $\mu(f)$ its unique probability measure of maximal entropy, [FLM], [Lyu], [M1]. Note that the support of $\mu(f)$ is $J(f)$, and that both the measure and the set are invariant for the iterates of f. In what follows we always assume that all rational functions are *not* *critically finite with a parabolic orbifold.* The critically finite rational maps with parabolic orbifolds are completely classified in [DH]. For such functions the theorems of the paper are not true.

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1. Rational functions with common maximal measure

Theorem A. *Let f, g be two non-exceptional rational functions. The following conditions are equivalent:*

(A1) $\mu(f) = \mu(g)$.

(A2) there exist iterates F of f and G of g, such that, for some natural numbers M and N the following equality holds:

$$(G^{-1} \circ G) \circ G^M = (F^{-1} \circ F) \circ F^N.$$

Here by $G^{-1} \circ G$ (resp. $F^{-1} \circ F$) we mean a single-, or multi-valued function obtained by the analytic continuation of some its branch.

Moreover, $(degG)^M = (degF)^N$.

Example. Let $\deg f = 2$. Consider a branch of two-valued analytic function $h = f^{-1} \circ f$, which is different from identity. Then h is a Mobius transformation (i.e. single-valued). Moreover, the function $g = h \circ f$ is rational and $\mu(f) = \mu(g)$. In this example $(g^{-1} \circ g) \circ g = (f^{-1} \circ f) \circ f$, where $g^{-1} \circ g =$id and $f^{-1} \circ f = h$.

The main ingredients of the proof will be given by Lemma 1 and Lemma 2 below.

Lemma 1. *Let ν be an ergodic f-invariant measure on the Julia set $J(f)$ with positive Lyapunov exponent $\chi = \int \log |f'| d\nu$. Then for every small positive σ and for every λ close to and larger than χ*

(1) there exists a set E of ν-measure $1 - \sigma$,

(2) there exist numbers $r > 0$, $C > 1$, $K > 1$, $\delta > 0$, and $N_0 \in \mathbf{N}$

as follows: for every point $x \in E$, for every $N > N_0$, there exists a set R_N in the interval $(\log(1/\delta), \log(1/\delta) + N\lambda)$, which occupies at least $7/12$ of its length, i.e. the Lebesque measure$(R_N) > \frac{7}{12} N\lambda$ and such that $R_N \subset R_{N+1}$, and for every $t \in R_N$ and for some $n = n(t) \in \mathbf{N}$ the following holds:

(a) the map $f^n : B(x, \exp(-t)) \to \overline{\mathbb{C}}$ is injective and has a distortion bounded by C:

$$1/C < |(f^n)'(x)/(f^n)'(y)| < C,$$

for all $y \in B(x, \exp(-t))$,

(b) $B(f^n(x), r/K) \subset f^n(B(x, \exp(-t))) \subset B(f^n(x), r)$,
(c) $n \to \infty$ *as* $t \to \infty$.

Proof of Lemma 1.

(A) Consider the inverse limit (natural extension in Rohlin terminology [Ro]) $(\tilde{J}, \tilde{f}, \tilde{\nu})$ of (J, f, ν).

Denote by $\pi : \tilde{J} \to J$ the projection on the 0 coordinate and by π_n the projection on n-th coordinate. Then for $\tilde{\nu}$-almost every $\tilde{x} \in \tilde{J}$ there exists $r = r(\tilde{x}) > 0$ such that univalent branches F_n of f^{-n} on $B(\pi(\tilde{x}), r)$ for $n = 1, 2, ...$ for which $F_n(\pi(\tilde{x})) = \pi_{-n}(\tilde{x})$, exist. Moreover for a constant $C = C(\tilde{x}) > 0$

$$\frac{1}{C} < \frac{|F_n'(\pi(\tilde{x}))|}{|F_n'(z)|} < C$$

for every $z \in B(\pi(\tilde{x}), r)$, $n > 0$, (distances and derivatives in the Riemann metric on $\overline{\mathbb{C}}$).

Moreover r and C are measurable functions of $\tilde{x}$.

((A) follows easily from Pesin's theory [Pe]. It is stated explicitely in [PZ, Lemma 1] and a proof of its variant can be found in [Led1] or [P2, Sec.2]. See also [ELyu], [Led2], [M2], [P1, Sec.3].)

(B) Let us fix σ between 0 and 1/4 and find a set $\tilde{E} \subset \tilde{J}$ as follows:

(B1) $\tilde{\nu}(\tilde{E}) > 1 - \sigma$,

(B2) $\nu(E) > 1 - \sigma$, where $E = \pi(\tilde{E})$,

(B3) there exist $r > 0$ and $C > 0$ not depending on $\tilde{x} \in \tilde{E}$ such that univalent branches F_n of f^{-n} on $B(\pi(\tilde{x}), r)$ for $n = 1, 2, ...$ for which $F_n(\pi(\tilde{x})) = \pi_{-n}(\tilde{x})$, exist, and

$$\frac{1}{C} < \frac{|F_n'(\pi(\tilde{x}))|}{|F_n'(z)|} < C$$

for every $z \in B(\pi(\tilde{x}), r)$, $n > 0$,

(B4) $\frac{1}{n} \log |(f^n)'(x)| \to \chi$ as $n \to \infty$ *uniformly* on $x \in E$ (Egoroff's Theorem),

(B5) if $M(\tilde{x}) = \sharp\{n : 1 \le n \le M, \tilde{f}^n(\tilde{x}) \in \tilde{E}\}$, then $M(\tilde{x})/M \to \tilde{\nu}(\tilde{E})$ as $M \to \infty$ *uniformly* on $\tilde{x} \in \tilde{E}$ (Poincaré Recurrence Theorem, Birkhoff Ergodic Theorem, and Egoroff's Theorem).

(C) Let us fix $\lambda > \chi$ so close to χ, that there exists $\varepsilon > 0$ such that $\chi + 2\varepsilon < \lambda$, but

$$\chi - \varepsilon > \frac{2}{3} \frac{\lambda}{1 - 2\sigma}.$$

With this ε, by (B4)-(B5), one can choose n_ε and M_ε such that

$$\chi - \varepsilon < \frac{1}{n} \log |(f^n)'(x)| < \chi + \varepsilon$$

for all $n \geq n_\varepsilon$ and all $x \in E$,

and,

$$\frac{M(\tilde{x})}{M} > 1 - 2\sigma$$

for all $M > M_\varepsilon$ and all $\tilde{x} \in \tilde{E}$.

Then we choose K in such a way that

$$\frac{2}{3} n_\varepsilon \frac{\lambda}{1-2\sigma} < \log(\frac{K}{C^2}) < n_\varepsilon(\chi - \varepsilon).$$

Finally, we set

$$\delta = \frac{r}{C\exp(n_\varepsilon(\chi - \varepsilon))}$$

and $N_0 = n_\varepsilon \frac{\chi}{\varepsilon} + M_\varepsilon + 16\sigma n_\varepsilon$.

Thus we have chosen the set E and the constants r, C, K, δ, and N_0.

(D) Denote by $C(x; r_1, r_2) = B(x, r_1) \setminus \overline{B}(x, r_2) = \{z : r_1 < |z - x| < r_2\}$ the annulus. Let $\tilde{x} \in \tilde{E}$ and $\tilde{f}^n(\tilde{x}) \in \tilde{E}$, and F_n is a branch of f^{-n} on $B(f^n(x), r)$ so that $F_n(f^n(x)) = x$. Then, by (A),

$$F_n(C(f^n(x); \frac{r}{K}, r)) \supset C(x; \frac{r}{K}\frac{C}{a_n}, \frac{r}{Ca_n}), \tag{1}$$

where

$$a_n = |(f^n)'(x)|.$$

(E) Given $x \in E$ and $N > N_0$, let us construct the set R_N. Consider a set $A_N = \{n : n_\varepsilon \leq n \leq N + n_\varepsilon, \tilde{f}^n(\tilde{x}) \in \tilde{E}\}$ and mark all those $n_i \in A_N$, for which $n_{i+1} - n_i \geq n_\varepsilon$. Denote $B_N = \{n_i\}$. If $l = \sharp B_N$, then

$$2n_\varepsilon + (l-1)n_\varepsilon + n_\varepsilon \geq M(\tilde{x}) > M(1 - 2\sigma),$$

where $M = n_\varepsilon + N$, i.e.

$$l \geq \frac{M(1-2\sigma)}{n_\varepsilon} - 1. \tag{2}$$

For every $n_i \in B_N$, let

$$I_i = (\log(\frac{Ca_{n_i}}{r}), \log(\frac{Ka_{n_i}}{rC})).$$

This is an interval since $K > C^2$. Now we can define

$$R_N = \bigcup_{i+1}^{l} I_i.$$

Let us check the following properties of the intervals I_i.

(E1) $I_i \subset (\log(1/\delta, \log(1/\delta) + N\lambda)$.

Indeed,

$$\frac{Ca_{n_1}}{r} : \frac{1}{\delta} = \frac{a_{n_1}}{\exp(n_\varepsilon(\chi - \varepsilon))} > 1.$$

and

$$\frac{Ka_{n_l}}{rC} : \frac{\exp(N\lambda)}{\delta} = \frac{Ka_{n_l}}{C^2 \exp(n_\varepsilon(\chi - \varepsilon))\exp(N\lambda)}$$

$$\leq \frac{K\exp(n_\varepsilon(\chi + \varepsilon))}{\exp(n_\varepsilon(\chi - \varepsilon) + N\lambda)} \leq \frac{K}{C^2} \frac{\exp((N + n_\varepsilon)\chi + \varepsilon(N + n_\varepsilon))}{\exp((N + n_\varepsilon)\chi + (\lambda - \chi)N - \varepsilon n_\varepsilon)}$$

$$= \frac{K}{C^2} \exp(\varepsilon(N + n_\varepsilon) + \varepsilon n_\varepsilon - N(\lambda - \chi)) < \frac{K}{C^2} \exp(\varepsilon(N + n_\varepsilon) + \varepsilon n_\varepsilon - 2N\varepsilon) =$$

$$\frac{K}{C^2} \exp(-\varepsilon(N - n_\varepsilon)) < \exp(n_\varepsilon(\chi - \varepsilon) - \varepsilon(N - n_\varepsilon) < \exp(-\varepsilon(n_\varepsilon \frac{\chi}{\varepsilon} - n_\varepsilon) - n_\varepsilon(\chi - \varepsilon)) = 1.$$

(E2) $I_i \bigcap I_j = \emptyset$, $i \neq j$. Remind that $a_n = |(f^n)'(x)|$, $f^{n_i}(x) \in E$, and $n_{i+1} - n_i \geq n_\varepsilon$. Then

$$\frac{Ka_{n_i}}{rC} : \frac{Ca_{n_{i+1}}}{r} = \frac{K}{C^2} \frac{|(f^{n_i})'(x)|}{|(f^{n_{i+1}})'(x)|} = \frac{K}{C^2} |(f^{n_{i+1} - n_i})'(f^{n_i})(x)|^{-1}$$

$$< \frac{K}{C^2} \exp(-n_\varepsilon(\chi - \varepsilon)) < \exp(n_\varepsilon(\chi - \varepsilon) - n_\varepsilon(\chi - \varepsilon)) = 1.$$

(E3) The measure of R_N equals to

$$\sum_{i=1}^{l} |I_i| \geq l \log \frac{K}{C^2} > [\frac{(N + n_\varepsilon)(1 - 2\sigma)}{n_\varepsilon} - 1] \times \frac{2}{3} \frac{n_\varepsilon \lambda}{1 - 2\sigma} > \frac{2}{3} \frac{7}{8} N\lambda = \frac{7}{12} N\lambda.$$

(F) Let $t \in I_i$. Set $n(t) = n_i$. Then the conclusions (a)-(b) hold because of (D), eq. (1). Clearly, (c) is also true.

Lemma 2. *Let J be the Julia set of a non-exceptional rational function f. Fix a ball $B = B(x, r)$ centered at $x \in J$. Let H_n be a sequence of holomorphic functions in B such that:*

1. The sequence H_n tends to a holomorphic function H in B.

2. For every n and z,

$$z \in B \cap J \Longleftrightarrow H_n(z) \in H_n(B) \cap J.$$

3. If J is the whole Riemann sphere, a circle, or an interval (in some holomorphic coordinates), then additionally, for every n, there is a constant $\alpha > 0$ so that $\mu(H_n(A)) = \alpha\mu(A)$, where $\mu = \mu(f)$ and A is any set such that $H_n : A \to \overline{C}$ is injective.

Then either the limit function H is constant, or $H_n = H$ for all big n.

Remark. A map with the properties 2. and 3. is called in [L] a *local symmetry* on J.

Proof of Lemma 2 : see [L]. For the sake of completeness we reproduce the main steps of the proof here. An idea is to construct many shifts which leave the Julia set invariant. For this we consider a semi-group generated by the *local symmetries* H_n and f^{-n} in neighborhoods of repelling periodic points of f.

I. Let (Φ_n) be any sequence of holomorphic functions univalent on a ball $B(0,\varepsilon)$ such that $q_n = \Phi_n(0) \neq 0$, $n = 1, 2, \ldots$, and $\Phi_n \to \mathrm{id}$ as $n \to \infty$. Given $|\lambda| > 1$, there are $a \neq 0$, $\delta > 0$ and positive integers sequences l_i, n_i such that for every $m \in \mathbf{N}$ and all big i the mappings

$$\Psi_i(z) = \lambda^{l_i - m} \Phi_{n_i}\left(\frac{\Phi_{n_i}^{-1}(z)}{\lambda^{l_i - m}}\right)$$

are defined in $B(0,\delta)$ and

$$\Psi_i(z) \to z + \frac{a}{\lambda^m}$$

as $i \to \infty$.

Indeed, choose l_i and n_i such that $\lambda^{l_i} q_{n_i} \to a \neq 0$. Then we use the expansions

$$\lambda^l \Phi_n\left(\frac{\Phi_n^{-1}(z)}{\lambda^l}\right) = \lambda^l q_n + \alpha_1^{(n)} \Phi_n^{-1}(z) + \sum_{k=2}^{\infty} \alpha_k^{(n)} \left(\frac{\Phi_n^{-1}(z)}{\lambda^{l(1-1/k)}}\right)^k,$$

where $\alpha_k^{(n)}$ are the coefficients of the power series expansion of Φ_n at 0. We have Cauchy's inequalities: $|\alpha_k^{(n)}| < C/(\varepsilon/2)^k$, for some $C > 0$ and all k. With the chosen $l_i \to \infty$ and n_i, it gives us the statement.

II. Let z belong to a half plane $\{\Re z > M_0\}$ and $\phi(z) = z + 1 + o(|z|^{-\gamma})$, $\gamma > 0$, as $z \to \infty$. Given $|\lambda| > 1$ and $c > 0$, there are sequences l_i, n_i and $M > M_0$ such that

$$\lambda^{-n_i} \phi^{l_i}(\lambda^{n_i} z) \to z + c,$$

$i \to \infty$, if $z \in \Pi = \{\Re z > M\}$.

To prove it, we choose a sequence n_i so that $\arg\lambda^{n_i} \to 0$ and then set $l_i = [c|\lambda|^{n_i}]$. Now the asymptotic $\phi^l(z) = z + l + o(l)$ if $z \to \infty$ and $l \to \infty$ leads to the conclusion.

III. There is no open domain U such that $U \bigcap J$ is diffeomorphic to the product of an interval and a Cantor set. A proof (due to A. Eremenko) can be found in [L].

IV. Assume that a limit function H of H_n is not a constant. We can set $H = \mathrm{id}$. We can assume also that H_n are defined and univalent in a ball B centered at a repelling fixed point b of f (passing to an iterate) with multiplier $\lambda = f'(b)$. Let F be a branch of f^{-1} on B contracting to b. We let

$$F_n = H_n^{-1} \circ F \circ H_n.$$

Denote $b_n = H_n^{-1}(b)$. Then $F_n(b_n) = b_n$ and $F_n'(b_n) = 1/\lambda$.

Consider the case $b_n = b$ for some n. Let $R = f \circ F_n$. Then $R(b) = b$ and $R'(b) = 1$. If J coincides with $\overline{C}$, a circle S, or an interval I, then R preserves the measure μ by the assumption 3. Looking at the corresponding Leau flower for R, we see that $R = \mathrm{id}$. Now let J is not $\overline{C}$, S, and I. Assume $R \neq \mathrm{id}$. We make two changes of variable. First, we may assume that locally $f(z) = \lambda z$. Second, after a change $w = Az^{-p}$, with some $A > 0$ and $p \in \mathbf{N}$, the map R turns to a map of the form of p.II, and f turns to $w \mapsto \lambda^{-p} w$. Then applying p.II and returning to the original coordinate z, we see that for each point $x \in J$ close to b, J contains also an analytic arc joining x to b, which corresponds to a horizontal ray in the coordinate w. Then by III J is $\overline{C}$, S, or I. A contradiction. (Another argument is that in the case of a Cantor set of rays in J to b, for a periodic point $b' \neq b$ close to b one has again a Cantor set of arcs to b' which implies $J = \overline{\mathbb{C}}$.

Thus the remaining case is $b_n \neq b$ for all n. We can linearize each F_n by a holomorphic Schroeder map h_n, $h_n(0) = b_n$, and F by h, $h(0) = b$ (so that $h_n = H_n^{-1} \circ h$). Then for passage maps $\Phi_n = h^{-1} \circ h_n$ we apply p.I. If λ is not real we can walk in J in arbitrarily small steps in two different directions which gives $J = \overline{\mathbb{C}}$. If λ is real we walk at least in the direction a. We conclude that J is either $\overline{C}$ or an interval, or J is locally diffeomorphic to the product of a Cantor set and an interval. The latter case is ruled out by p.III. In the first two cases the measure μ is invariant under the shifts (by p.I). It is possible only if f is critically finite with parabolic orbifold (see [E]).

V. Thus $F = F_n$, i.e. F (a branch of f^{-1} in a neighborhood of the repelling periodic point of f) and all H_n commute. So each H_n is linear in some coordinates linearizing F in which b becomes 0. If we apply the result $F = F_n$ to another repelling periodic point of f close to b, we obtain H_n =id. (In [L] the reader can find a different argument.)

Proof of Theorem A.

A1. Let $\mu = \mu(f) = \mu(g)$, $J = J(f) = J(g)$. Since there Lyapunov exponents χ_f and χ_g are positive, we can apply Lemma 1. Take $\sigma = 1/4$, λ_1 close to and bigger than χ_f, and λ_2 close to and bigger than χ_g, and find the set E_1, and numbers $r_1 > 0$, $C_1 > 1$, $K_1 > 1$, $\delta_1 > 0$, and $N_0^1 \in \mathbb{N}$ for f and $r_2 > 0$, $C_2 > 1$, $K_2 > 1$, $\delta_2 > 0$, and $N_0^2 \in \mathbb{N}$ for g. There is a point $x \in E_1 \bigcap E_2$. For this point find the sets R_N^1 for f and R_N^2 for g, for all N big enough. Since these sets occupy more that a half of the intervals $(\log(1/\delta_1, \log(1/\delta_1) + N\lambda_1)$ and $(\log(1/\delta_2), \log(1/\delta_2) + N\lambda_2)$ respectively, one can find a sequence of points $t_i \to \infty$, and two sequences of indexes $n_i^1 \to \infty$, $n_i^2 \to \infty$ such that the maps

$$f^{n_i^1} : B(x, \exp(-t)) \to \overline{C},$$

$$g^{n_i^2} : B(x, \exp(-t)) \to \overline{C}$$

are injective and

$$B(f^{n_i^1}(x), r_1/K_1) \subset f^{n_i^1}(B(x, \exp(-t))) \subset B(f^{n_i^1}(x), r_1),$$

$$B(g^{n_i^2}(x), r_2/K_2) \subset g^{n_i^2}(B(x, \exp(-t))) \subset B(g^{n_i^2}(x), r_2).$$

It is clear now that there exists a ball $B = B(a, r)$, with $a \in J$, and an infinite sequence of maps H_i, which are of the form $g^{l_i} \circ f^{-k_i}$, univalent on B and such that each $H_i(B)$ contains a ball of a fixed positive radius and is contained in other such ball (of a fixed radius). It means, that $\{H_i\}$ is normal in B and the limit functions are not constants.

Now we use Lemma 2. Its assumption 3 holds because Jacobians $\mathrm{Jac}_{\mu(f)} f$ and $\mathrm{Jac}_{\mu(g)} g$ are constant. As we assumed $\mu(f) = \mu(g) = \mu$, $\mathrm{Jac}_\mu H_n$ is constant. (So in Lemma 2 we could state the assumption 3 for every case, not only $J = \overline{\mathbb{C}}$ interval or a circle. This would simplify the proof. However in Section 2, Prop. 1, this is not so.)

Therefore, by Lemma 2, for some natural numbers m, n, k, and l, and for some branches f^{-n} and $f^{-(n+l)}$ defined in B,

$$g^m \circ f^{-n} = g^{m+k} \circ f^{-(n+l)}$$

identically in B. Rewrite it in the form

$$f^{-n} \circ f^{n+l} = g^{-m} \circ g^{m+k}$$

on $f^{-(n+l)}(B)$ and compose nm times.

Then we can set: $G = g^m$, $F = f^n$, $M = nk$, and $N = ml$.

(A2). Let, conversely,

$$(G^{-1} \circ G) \circ G^M = (F^{-1} \circ F) \circ F^N.$$

Observe that this implies

$$(G^{-1} \circ G) \circ G^{iM} = (F^{-1} \circ F) \circ F^{iN}$$

with *the same functions* $G^{-1} \circ G$ *and* $F^{-1} \circ F$ *for all* $i = 1, 2, \ldots$. Because of the uniqness of the measure of maximal entropy, it is enough to show that the measure $\mu = \mu(F)$ is the balanced measure for G^M too. Denote d_F and d_G degrees of F and G. Let us fix any small open domain A. Let $B = G^M(A)$ and A' is a component of $G^{-M}(A)$. Then

$$\mu(A') = d_F^{-2N} \mu((F^{-1} \circ F) \circ F^{2N}(A'))$$

$$= d_F^{-2N} \mu((G^{-1} \circ G) \circ G^{2M}(A')) = d_F^{-2N} \mu((G^{-1} \circ G)(B)). \tag{3}$$

Similarly,

$$\mu(A) = d_F^{-N} \mu((G^{-1} \circ G)(B)). \tag{4}$$

Hence,

$$\mu(A') = d_F^{-N} \mu(A),$$

where $G^M : A' \to A$ is one-to-one (by the choice of A). It follows, $\deg G^M = d_F^N$. The proof is completed.

2. Rational functions with common Julia set

Theorem B. *(On rigidity of maximal measure.) Let f,g be two rational functions without parabolic periodic points and singular domains (Siegel discs, Herman rings), Julia* *sets not $\overline{\mathbb{C}}$, a circle or an interval (in some holomorphic coordinates). Then the following conditions are equivalent:*

(B1) $J(f) = J(g)$.

(B2) $\mu(f) = \mu(g)$.

(B3) there exist iterates F of f and G of g, such that, for some natural numbers M and N the following equality holds:

$$(G^{-1} \circ G) \circ G^M = (F^{-1} \circ F) \circ F^N.$$

Here by $G^{-1} \circ G$ (resp. $F^{-1} \circ F$) we mean a single-, or multi-valued function obtained by the analytic continuation of some its branch.

Moreover, $(degG)^M = (degF)^N$.

In fact, we prove a more general statement:

Proposition 1. *Let f, g be two arbitrary rational functions with the common Julia set $J = J(f) = J(g)$ not being a circle or interval. Suppose there exist periodic sinks p, q for f, g and components U, V of their basins such that $g^m(U) = V$, for some $m \geq 0$. Then the condition (B3) of the Theorem B holds.*

Proof of Proposition 1. Denote by U', V' some periodic components of the basins of p, q and $f^s(U) = U', g^t(V) = V'$ Let ν with index U, U', V, V' be harmonic measure on the appropriate boundary, viewed from p, q in the case of U', V'.

Then $\nu_{U'}$, $\nu_{V'}$ are ergodic invariant measures with positive Lyapunov exponents for f and g respectively. (By passing to iterations one can assume p, q are fixed points.)

Invariance (see for example [P2]): For every continuous $\varphi : U' \to \mathbb{R}$ we have $\int \varphi d\nu = \tilde{\varphi}(p) = \tilde{\varphi}(f(p)) = \widetilde{\varphi \circ f}(p) = \int \varphi \circ f d\nu$, where tilde denotes the harmonic extension to U' (solution of Dirichlet's problem) and $\nu = \nu_{U'}$.

Ergodicity: If $\varphi \circ f = \varphi$ (modν) on $\partial U'$ then $\varphi \circ f^n = \varphi$ for every $n \geq 0$. Hence $\tilde{\varphi} \circ f^n = \widetilde{\varphi \circ f^n} = \tilde{\varphi}$ on U'. Applying this for $n \to \infty$ we obtain $\tilde{\varphi}(p) = \tilde{\varphi}(z)$ for every $z \in U'$. So φ is constant.

Lyapunov exponent: It is not less than half of the entropy h_ν (by [Ru], cf [M], [P2]). Next recall that $h_\nu(f) > 0$ iff f is not an automorphism (in ν), [Pa, Corollary 5.16].

Finally we prove that f is not an automorphism: ν-a.e. $z \in \partial U'$ is accessible along a continuous curve $\gamma \subset U'$ (a Brownian motion path). We can suppose z is not a critical value and for a small $r > 0$ denote by W the component of $U' \cap B(z, r)$ intersecting γ. Let ν_W be a harmonic measure on ∂W (from a point in W). Then $\nu_W(\partial U') > 0$, this follows for example from Dirichlet's regularity of Julia set). There exist at least two branches f^{-1} on W to U', for each we have $\mathrm{Jac}_\nu f^{-1} = \frac{d\nu \circ f^{-1}}{d\nu} > 0$ as f is holomorphic.

In general since f, g are holomorphic, their compositions and inverse branches map sets of positive respective harmonic mesures to the sets of positive harmonic measures. So we can use Lemma 1 to construct an infinite sequence of local symmetries H_i of J in a neighborhood of a point $a \in \partial U'$ of the form $H_i = g^{l_i+t+m} \circ f^{-k_i-s}$ (see Proof of Theorem A).

Remark 1. The Theorem B can be extended to rational functions with parabolic periodic points having simply connected immediate basins.

3. Functional equations

The classical result on commuting rational functions f and g states:

$$f \circ g = g \circ f \Longrightarrow f^m = g^n,$$

for some $m > 0, n > 0$.

Consider another functional equation:

$$f^2 \circ g = f \circ g \circ f,$$

i.e. f commutes with $f \circ g$. It yields

$$f^m = (f \circ g)^n.$$

So the functions are not separated here.

On the other hand, Theorem A gives a way to separate the functions. Indeed, if f commutes with $f \circ g$, then $\mu(f) = \mu(f \circ g) = \mu(g)$, and the conclusion (A2) of Theorem A holds.

One can apply this for any functional equation between f and g whenever one can derive from it the coincidence of the maximal measures of f and g.

References:

[B1] A. F. Beardon. Symmetries of Julia sets. Bull. London Math. Soc. 22 (1990), 576-582.

[B2] A. F. Beardon. Polynomials with identical Julia sets. Complex Variables 17 (1992), 195-200.

[BE] I. N. Baker, A. Eremenko. A problem on Julia sets. Ann. Acad. Sci. Fenn. 12 (1987), 229-236.

[DH] A. Douady, J.H. Hubbard. A proof of Thurston's topological characterization of rational functions. Acta Math. 171,1993.

[E] A. Eremenko. On some functional equations connected with iteration of rational functions. Algebra i Analiz (Leningrad Math. J.), 1.4 (1989), 102-116.

[ELyu] A. Eremenko, M. Lyubich. Dynamics of Analytic transformations. Algebra i Analiz (Leningrad J. Math.) 1.3 (1989), 1-70.

[F1] P. Fatou. Sur les équations fonctionnelles. Bull. Soc. Mat. France 47 (1919), 161-271; 48 (1920), 33-94, 208-314.

[F2] P. Fatou. Sur les fonctions qui admettent plusieurs théorèmes de multiplication. C.R.A.S. 173 (1921), 571-573; Sur l'itération analytique et les substitutions permutables. J. de Math. 2 (1923), 343.

[Fe] J. Fernandez. A note on the Julia set for polynomials. Complex Variables 12 (1989), 83-85.

[FLM] A. Freire, A. Lopes, R. Mañé. An invariant measure for rational maps. Bol. Soc. Bras. Math. 14.1 (1983), 45-62.

[J1] G. Julia. Mémoire sur l'itération des fonctions rationnelles. J. Math. Pure Appl. 8 (1918), 47-245.

[J2] G. Julia. Mémoire sur permutabilité des fractions rationnelles. Ann. Ecole Norm. Sup. 39 (1922), 131-215.

[L] G. M. Levin. On symmetries on a Julia set. Mat. Zametki 48.5 (1990), 72-79 (in Russian); Math.Notes 48.5-6 (1990), 1126-1131 (1991).

[Led1] F. Ledrappier. Some properties of absolutely continuous invariant measures on an interval. Ergod. Th. & Dynam. Sys. 1 (1981), 77-93.

[Led2] F. Ledrappier. Quelques propriétés ergodiques des applications rationnelles. C.R.A.S.P. 299 (1984), 37-40.

[Lyu] M. Lyubich. Entropy properties of rational endomorphisms of the Riemann sphere. Ergod. Th. & Dynam. Sys. 3 (1983), 351-386.

[M1] R. Mañé. On the uniquness of the maximizing measure for rational maps. Bol. Soc. Bras. Math. 14.1 (1983), 27-43.

[M2] R. Mañé. The Hausdorff dimension of invariant probabilities of rational maps. L. N. Math. 1331, Dynamical Systems Valparaiso 1986, Springer.

[Pa] W. Parry. Entropy and Generators in Ergodic Theory. W. A. Benjamin, Inc., New York, 1969.

[Pe] Ja. B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. Russian Math. Surveys 32 (1977), 55-114.

[P1] F. Przytycki. Hausdorff dimension of harmonic measure on the boundary of attractive basin for a holomorphic map. Invent. Math. 80 (1985), 161-179.

[P2] F. Przytycki. Riemann map and holomorphic dynamics. Invent. Math. 85 (1986), 439-455.

[PZ] F. Przytycki, A. Zdunik. Density of periodic sources in the boundary of a basin of attraction for iteration of holomorphic maps, geometric coding trees technique. Fund. Math. (1994).

[R1] J. F. Ritt. On the iteration of rational functions. Trans. Amer. Math. Soc. 21 (1920), 348-356.

[R2] J. F. Ritt. Periodic functions with a multiplication theorem. Trans. Amer. Math. Soc. 23 (1922), 16-25.

[Ro] V. A. Rohlin. Lectures on the entropy theory of transformations with invariant measures. Usp. Mat.Nauk 22.5 (1967), 3-56, (in Russian); Russ. Math.Surv. 22.5 (1967), 1-52.

[Ru] D. Ruelle. An inequality for the entropy of differentiable maps. Bol. Soc. Bras. Mat. 9 (1978), 83-87.

[T] W. Thurston. On combinatorics of iterated rational maps. Preprint 1985.

Genadi Levin: Institute of Mathematics, Hebrew University, 91904 Jerusalem, Israel, and Max-Planck-Institut für Mathematik, 53225 Bonn, Germany.

Feliks Przytycki: Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00950 Warsaw, Poland