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Conformal Invariance of Partially Massless Higher Spins

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Abstract

We show that partially massless higher spin theories whose gauge invariance removes their helicity zero modes are conformally invariant in $d = 4$ de Sitter space. This enlarges to all spins the two-entry – Maxwell and partially massless spin 2 – catalog of conformal gauge models.

1 Introduction

In this Letter we present an infinite set of new conformally invariant gauge theories in dimension four constant curvature spaces generalizing the two known, Maxwell and partially massless spin 2, examples. Conformal flatness of these spaces then also implies lightcone propagation [1]. An obvious mechanism to achieve lightlike excitations is gauge invariance, and indeed we recently analyzed such models: partially massless gauge invariant higher spins propagating on the lightcone in de Sitter (dS) backgrounds [4]. While gauge invariance alone is certainly not sufficient to ensure conformal invariance, it turns out that in dimension four, an infinite subset of these gauge invariant higher spin theories is conformal. The first two elements of this conformal subset are the spin 1 Maxwell and spin 2 partially massless theories. [Of course, the non-gauge scalar field can be improved in any dimension.] These theories enjoy "maximal depth" higher derivative (s derivatives at spin s) gauge invariances, with a scalar gauge parameter, just like their Maxwellian cousin. They describe $2s$ lightlike and unitary (for $\Lambda > 0$ – dS space) physical degrees of freedom. Unlike previous attempts [2] to generalize conformal invariance to Maxwell-like theories through higher derivative actions, ours maintain physical, second derivative, order; instead, the number of derivatives appearing in gauge variations increases. As we shall see, the s indices of a spin s field can be balanced by $s-1$ derivatives in gauge transformations.

The results are organized as follows: Section 2 contains a brief review of massive higher spins in constant curvature backgrounds, while Section 3 deals with action principles for these fields. Conformal symmetry in dS backgrounds is discussed in Section 4 and conformally improved scalars are revisited in Section 5. The main result is contained in Section 6 which derives the conformal invariance of maximal depth partially massless fields. Lightlike propagation is displayed in Section 7; Our conclusions comprise the final Section.

2 Higher Spins in Constant Curvature Spaces

Constant curvature,

$$R_{\mu\nu}{}^{\rho\sigma} = -\frac{2\Lambda}{n}\ \delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} \tag{1}$$

spaces in $d \equiv n+1$ dimensions provide consistent backgrounds for higher spin propagation. The cosmological constant Λ is positive for dS space in our conventions. We will often work in the dS steady state patch

$$ds^2 = -dt^2 + e^{2Mt} d\vec{x}^{\,2}\,, \tag{2}$$

where

$$M^2 \equiv \frac{\Lambda}{n}\,. \tag{3}$$

The field equation for physical fields is [3]

$$\Big(D_\mu D^\mu + \Big[(s-1)^2 + (s-2)(n-3) - 3\Big] M^2 - m^2\Big)\, \varphi^{\rm phys}_{\mu_1\ldots\mu_s} = 0\,. \tag{4}$$

Physical fields are classified by the mass parameter m. In steady state coordinates they obey the following criteria for all $m^2 > 0$

$$\varphi_{0\mu_2\ldots\mu_s} = 0 = \varphi^\rho{}_{\rho\mu_2\ldots\mu_s}\,, \tag{5}$$

as well as additional conditions

$$\partial^{\mu_1}\ldots\partial^{\mu_t}\varphi_{\mu_1\ldots\mu_s} = 0\,, \tag{6}$$

when

$$m^2 = (t-1)(2s-t+n-3)M^2\,, \qquad t = 1,\ldots,s\,. \tag{7}$$

The integer t is the "depth" of a partially massless field subject to a mass tuning (7). Depth $t=1$ corresponds to the on-shell condition for a strictly massless spin s field whose physical components are spatially traceless-transverse symmetric tensors. For all values of the depth $1 \le t \le s$, the theory enjoys the higher derivative gauge invariance

$$\delta\varphi_{\mu_1\ldots\mu_s} = D_{(\mu_1}\cdots D_{\mu_t}\xi_{\mu_{t+1}\ldots\mu_s)} + \cdots\,. \tag{8}$$

Non-tuned values of the mass

$$m^2 < (s-1)(s+n-3)M^2\,, \tag{9}$$

correspond to normal massive fields described by spatially traceless tensors. [Unitarity is violated for mass values not obeying (7) or (9) [4].] Strictly massless and massive fields survive in the Minkowski limit $M=0$. Peculiar to dS space are physical fields with values $t \neq 1$ [4]. These partially massless fields of depth t, like their strictly massless counterparts, propagate at the speed of light. Their physical degrees of freedom are spanned by intermediate helicity counts as demanded by (6); they are also unitary in dS backgrounds and obey an energy positivity condition within the intrinsic horizon [5].

3 Actions

Defining

$$\varphi^{\rm phys}_{i_1\dots i_s} \equiv \sqrt{-g}^{\;\frac{s}{n}-\frac{1}{2}} q_\varepsilon\,, \tag{10}$$

where ε stands for a helicity labeling of the spatial indices $i_1 \dots i_s$ subject to (5) and possibly (6), the field equation (4) becomes

$$\Big(-\frac{d^2}{dt^2} + \Delta + \Big[\frac{M(2s+n-4)}{2}\Big]^2 - m^2\Big) q_\varepsilon = 0\,. \tag{11}$$

Here the spatial Laplacian $\Delta \equiv e^{2Mt}\vec{\partial}^2$ carries the only explicit time dependence and we will raise and lower spatial indices with impunity. An action principle follows immediately:

$$S = \int dt \Big(\sum_\varepsilon p_\varepsilon \dot{q}_\varepsilon - H \Big)\,, \tag{12}$$

where the Hamiltonian

$$H \equiv \sum_\varepsilon \int d^n x\, \frac{1}{2}\Big[p_\varepsilon^2 + e^{-2Mt}[\vec{\partial} q_\varepsilon]^2 + \mu^2 q_\varepsilon^2 \Big]\,. \tag{13}$$

Here the "effective mass" is

$$\mu^2 \equiv m^2 - [\frac{M(2s+n-4)}{2}\Big]^2\,. \tag{14}$$

The action (12) may also be obtained by a detailed constraint/Hamiltonian analysis of covariant, higher spin actions [5].

4 Conformal Symmetry

Being conformally flat, dS spacetime enjoys not only an $so(d,1)$ isometry algebra but also an $so(d,2)$ conformal algebra. Perhaps the simplest construction of explicit conformal Killing vectors is in the manifestly conformally flat frame,

$$ds^2 = \frac{-d\tau^2 + d\vec{x}^{\,2}}{M^2\tau^2}\,, \tag{15}$$

where $\tau = -M^{-1} \exp\Big(Mt\Big)$ is the conformal time coordinate. Now we simply write down the flat conformal generators in the coordinates $(\tau, \vec{x})$,

$$
\begin{aligned}
& iP_i = \partial_i \,, \qquad iL = \frac{d}{d\tau} \,, \\
iD = \tau \frac{d}{d\tau} + \vec{x} \cdot \vec{\partial} \,, \qquad & iM_{ij} \;=\; x_i \partial_j - x_j \partial_i \,, \qquad iN_i = x_i \frac{d}{d\tau} + \tau \partial_i \,, \\
iK_i = 2 i x_i D + \Big[-\tau^2 + \vec{x}^{\,2} \Big] \, \partial_i \,, \quad & iJ = -2 i \tau D + \Big[-\tau^2 + \vec{x}^{\,2} \Big] \, \frac{d}{d\tau} \,.
\end{aligned}
\tag{16}
$$

The generators (P_i, D, M_{ij}, K_i) obey the $so(d,1)$ dS isometry algebra while the remaining generators enlarge this to the conformal $so(d,2)$ algebra with corresponding conformal Killing vectors

$$
[iL, ds^2] = -\frac{2}{\tau} \, ds^2 \,, \quad [iN_i, ds^2] = -\frac{2x_i}{\tau} \, ds^2 \,, \quad [iJ, ds^2] = -\frac{2(-\tau^2 + \vec{x}^{\,2})}{\tau} \, ds^2 \,. \tag{17}
$$

In general we will denote the function multiplying $2ds^2$ on the right hand side (namely the divergence of the corresponding conformal Killing vector over d) by α_X:

$$
[iX, ds^2] = 2 \alpha_X \; ds^2 \,. \tag{18}
$$

It is important to note that the conformal $so(d,2)$ algebra is obtained by requiring closure of the generator L and the $so(d-1,1)$ isometry algebra under commutation. This has the pleasant consequence that L and dS invariance are sufficient for a theory to be conformal. In the steady state coordinates (2) we have

$$
iL = e^{Mt} \frac{d}{dt} \,. \tag{19}
$$

5 Scalars

Although our interest is in gauge specimens, improved scalars – conformal in any dimensionality – provide useful calibrations. The improved scalar action

$$
S = -\frac{1}{2} \int dx \sqrt{-g} \left(\partial_\mu \varphi g^{\mu\nu} \partial_\nu \varphi + \frac{1}{6} R \, \varphi^2 \right) \,, \tag{20}
$$

is invariant under conformal transformations

$$\delta\varphi = X\varphi + \left(\frac{d}{2} - 1\right)\ \alpha_X \varphi\,, \tag{21}$$

where $X \equiv \xi^\rho \partial_\rho$, $D_{(\mu}\xi_{\nu)} = \alpha_X\ g_{\mu\nu} = \frac{1}{d} D.\xi\ g_{\mu\nu}$ and $\frac{d}{2} - 1$ is the conformal weight of the field φ.

It is useful for our purposes to spell out this invariance explicitly in a dS background. In the steady state coordinates the action reads

$$S = -\frac{1}{2}\int dt d^n x\ e^{nMt}\left(-\dot{\varphi}^2 + e^{-2Mt}[\vec{\partial}\varphi]^2 + \frac{M^2}{4}\ (n^2-1)\varphi^2\right)\,. \tag{22}$$

As discussed in the previous Section, it suffices to consider the generator $iL = e^{Mt}\frac{d}{dt}$. By computing the right hand side of (21) or by examining the gradient terms in the action, which must be separately conformally invariant, we find

$$\delta_L\varphi = e^{(3-n)Mt/2}\frac{d}{dt}\Big(e^{(n-1)Mt/2}\varphi\Big)\,. \tag{23}$$

It is then easy to verify that the δ_L variation of the Lagrangian in (22) is a total time derivative.

Let us now perform this computation yet again in a first order formulation. Making the field redefinition (10) and a Legendre transformation, the action reads

$$S = \int dt dx^n \left(p\dot{q} - \frac{1}{2}\left[p^2 + e^{-2Mt}[\vec{\partial}q]^2 - \frac{1}{4}M^2 q^2\right]\right)\,. \tag{24}$$

It enjoys the conformal invariance

$$\delta_L q = e^{3Mt/2}\frac{d}{dt}\Big(e^{-Mt/2}q\Big)\,,\qquad \delta_L p = \frac{d}{dt}\Big(e^{3Mt/2}\frac{d}{dt}\Big(e^{-Mt/2}q\Big)\Big)\,. \tag{25}$$

Although the above calculation is a triviality, the resulting conformally invariant action (24) plays a central rôle in what follows.

6 Conformal Partially Massless Higher Spins

Our remaining task is to discover which, if any, of our partially massless theories are also conformal. A Herculean, but perhaps noble feat would be to write down covariant actions at arbitrary values of the spin[1] and determine conformal invariance explicitly. To obtain an answer rapidly, however, we proceed as follows. Firstly, we need only look for invariance with respect to "dS dilations" δ_L. Secondly, a notable feature of free higher spin fields is that despite the complexity of their covariant actions, their first order forms in terms of physical degrees of freedom are extremely simple–see equation (12)!

Now, the crux of our argument: compare actions (12) and (24). Since the dilation δ_L necessarily has no spin dependence, we can ignore the helicity summation $\sum_\varepsilon$. Therefore conformal invariance is guaranteed by choosing the effective mass

$$\mu^2 = -\frac{1}{4}M^2 \,. \tag{26}$$

Setting the mass parameter m^2 to its depth t partially massless value, we therefore require

$$-\frac{1}{4}M^2(2s-2t+n-2)^2 = -\frac{1}{4}M^2 \,, \tag{27}$$

which is solved via

$$t = s + \frac{n \pm 1}{2} - 1 \,. \tag{28}$$

However, the depth t must be both integer and no greater than s, whose only solutions are $n = 3, 1$, *i.e.* spacetime dimensions four and two! In dimension two, there is neither helicity nor dynamics beyond the scalar field: the summation $\sum_\varepsilon$ is empty. This leaves the physical, dimension four, case where we must take $t = s$. The lowest value, $s = t = 0$, represents conformally improved scalars. For $s = t = 1$, we have the single gauge invariance

$$\delta\varphi_\mu = D_\mu \xi \,, \tag{29}$$

which is just Maxwell theory – conformal in four dimensions. For $s = t = 2$, there is a double derivative gauge invariance

$$\delta\varphi_{\mu\nu} = \Big(D_{(\mu}D_{\nu)} + \frac{\Lambda}{3}g_{\mu\nu}\Big)\,\xi \tag{30}$$

and we recover the original spin 2 partially massless theory of [1]. Indeed, these authors obtained it by demanding conformal invariance and found a higher derivative gauge transformation as a consequence; the opposite tack,

[1] Indeed, covariant actions follow from the work of [6].

seeking novel gauge invariances, subsequently led to partially massless, if not necessarily conformal, theories in [3, 4, 5, 8]. We conclude that, in dimension four dS space, partially massless theories with maximal depth gauge invariances

$$\delta\varphi_{\mu_1\ldots\mu_s} = \Big(D_{(\mu_1}\ldots D_{\mu_s)} + \cdots\Big)\,\xi\,, \tag{31}$$

are conformally invariant.

7 Null Propagation

Another, illuminating, way to uncover the conformal invariance of our models is to solve their wave equations. This computation has been carried out in [3]. In steady state coordinates their wave equations are of Bessel type. In general the solutions are, of course, Bessel functions, but Huygens' principle [7] implies that for half integer values of their index ν, these solutions simply reduce to massless plane waves multiplied by slowly varying polynomials. There are cardinally infinitely many such special solutions, but the index $\nu = 1/2$ is special, since it corresponds to a conformally improved scalar. Some details:

Let us work in $d = 4$ where interesting conformal theories live. Fourier transforming $\vec{\partial} \to i\vec{k}$ and rescaling the conformal time coordinate, $\tau \to z = |\vec{k}|\tau$, the field equation for a massive spin s, helicity ε field in de Sitter space reads

$$\frac{d^2q}{dz^2} + \frac{1}{z} + \Big(1 - \frac{\nu^2}{z^2}\Big)q = 0, \tag{32}$$

where $q \equiv |\vec{k}/M|^{3/2-s}q_\varepsilon$ and the index

$$\nu^2 = \frac{1}{4} + s(s-1) - \frac{m^2}{M^2}\,. \tag{33}$$

Setting m^2 to its tuned values (7) yields

$$\nu^2 = \frac{1}{4}(2s - 2t + 1)^2\,. \tag{34}$$

The conformal value $\nu^2 = 1/4$ is obtained only for $s = t$, in agreement with the analysis of the previous Section. In this case, the solution to the wave equation is $q(z) = z^{-1/2}\exp(iz)$, which amounts to massless plane wave propagation since the overall exponential behavior is $\exp(i|\vec{k}|\tau + i\vec{k}\cdot\vec{x})$.

8 Conclusions

We have uncovered an infinite set of conformal, gauge invariant models in four dimensional dS space, generalizing Maxwell's vector theory. These new theories describe spin s, lightlike excitations with helicities $\pm s, \ldots, \pm 1$. The common property that ensures conformal invariance is the elimination of the zero helicity mode through gauge invariance because, only in this way, are the indices of a higher spin field balanced by derivatives appearing in gauge variations. That dimension four is singled out is not surprising given its unique rôle in the Maxwell starting point of our ladder.

The main obstacle to physical viability of fundamental higher spin $s > 2$ models is of course the absence of consistent interactions, at the very least with gravity. As yet no obvious mechanism for partially massless interactions is available; it could conceivably occur within a wider String Theory framework [9]. If so, the subset of models selected by conformal invariance may be particularly tractable.

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