Exceptional bundles and moduli spaces of stable vector bundles on Enriques surfaces

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EXCEPTIONAL BUNDLES AND MODULI SPACES OF STABLE VECTOR BUNDLES ON ENRIQUES SURFACES

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§ 0. INTRODUCTION

Moduli spaces of stable vector bundles on algebraic surfaces have been described by several authors.(See § 1 for the definition of stable bundles). Vector bundles on rational surfaces([Ba],[Hu],[D,P]), ruled surfaces([Br],[Q1]), K3 surfaces([Mu1,2],[Ty1,2]), elliptic surfaces([F,M1,2],[F],[O,V]) and some surfaces of general type([Bh],[D,K],[Huy],[Z]) have been studied. In this paper we want to study the moduli spaces of stable bundles on Enriques surfaces. Every Enriques surface has a K3 surface as a universal covering space. Mukai([Mu1]) showed that the moduli space of stable vector bundles on a K3 surfaces has a symplectic structure. We will describe the moduli spaces of stable bundles on Enriques surfaces with relation to those on the corresponding K3 surfaces.

Theorem 1:(1); The image of the moduli space of stable bundles on an Enriques surface by pull back map is a Lagrangian subvariety, exactly the fixed locus of the induced involution, in the moduli space of stable bundles, which is a symplectic manifold, on the covering K3 surface.

(2); The singularities in the moduli space of an Enriques surface are the direct images of finitely many union of the moduli spaces of stable bundles on the K3 surface and the dimension of singular locus is at most (1/2)(dimM + 3)(big codimensional singularty), where M is the moduli space of stable bundles on the Enriques surface.

(3); The pull back map is two to one from the smooth locus of M to the moduli space of stable bundles on the K3 surface, with

no branch points.

We studied also some 2 dimensional components in the moduli spaces of vector bundles of even rank as the first exceptional case to the theorem 1. These are just the anologues of Mukai's result that if the moduli space of stable bundles on a K3 surface is of dimension two and compact, then it is also a K3 surface isogeneous to the original K3 surface(See § 2 for the definition of isogeny). In our Enriques surfaces, we have

Thorem2:If the 2 dimensional component of the moduli space of stable bundles of even rank on an Enriques surface is compact, then it is an Enriques surface isogeneous to the original Enriques surface.

Furthermore we could find a relation involving 4 surfaces(2 K3 surfaces and 2 Enriques surfaces).

Next we considered another exceptional case, exceptional bundles which are simple and rigid(See §3, for definition.) Exceptional bundles have been widely studied by several people. The condition of the existence of exceptional bundles on rational surfaces([D,P],[Ru]), K3 surfaces([Ku1]), Enriques surfaces([D,R]) and surfaces of general type([Ty]) have been determined. Exceptional bundles are very important in particular on K3 surfaces and Enriques surfaces. However, on Enriques surfaces, exceptional bundles of even rank play important roles. On K3 surfaces Kuleshov proved that there exists an exceptional bundle for every possible case(realizing any exceptional vector, see chapter IV) for any K3 surface. However there is some difference between K3 and Enriques surfaces. We showed that there exists an exceptional bundle of rank 2 if and only if X has a smooth rational curve. We also related the exceptional bundles with a nodal cycle(a tree of rational curves)([Ki1]). Here we extend this result to any exceptional bundle of even rank as follows.

Theorem 3:There exists an exceptional bundle E of rank 2k with

E=E(K) if and only if det(E)=N+2L+kK, where N is a nodal cycle and L is any divisor and K is the canonical divisor. In particular, there exists an exceptional bundle E with E=E(K) of even rank if and only if X has a smooth rational curve.

For odd rank case we could generalize the case of rank one case as follows.

Theorem 4:There exists an exceptional bundle of odd rank on any Enriques surface for every possible Chern Classes of exceptional bundle.

For the existence of stable bundles on algebraic surfaces, there are many results, for example, on P^2([D,P]), elliptic surfaces([Fr],[L,O]), K3 surfaces ([Mu2],[Ty1],[Ku2]) and surfaces of general type.([Do],[Gi]) Some of them are on rank two bundles, while others are on bundles of any rank. On rank two bundles on Enriques surfaces, we described not only the structure of the moduli space but also the existence.([Ki2]) On Enriques surfaces there exists a stable bundle of rank 2 for every possible Chern classes of stable bundles. For bundles of arbitrary rank we could find a sufficient condition of the existence of stable bundles, following the methods of Kuleshov([Ku2].)

The contents of this paper will be covered as follows. In chapter one we will cover some preliminaries on Enriques surfaces,stable bundles and some known results and in chapter two we will discuss about the general structure of the moduli space of stable bundles and in the following two chapters we study two exceptional cases, 2-dimensional component of the moduli space of bundles of even rank and exceptional bundles respectively. In the final chapter we will study a sufficient condition of the existence of the stable bundles.

This paper is based on some part of my thesis, where we proved a weaker form of theorem 1 and theorem 3 for rank 2, and has been improved during the stay at Bayreuth University and Max-Planck-Institut in Bonn. I thank to Professor I.Dolgachev for suggesting this problem and guiding and to two institutions for good hospitality with financial support. Takemoto's old result was very important to our work, which I did not notice before. I thank Professor C.Okonek for indicating that paper to me and for some other helpful discussions. I thank Professors, Ono and Borovoi for a lemma in chapter 2 and D. Huybrecht for a good comment in the proof of theorem 1 and constant discussions.

§ I.PRELIMINARIES

1:An Enriques surface X is an algebraic surface whose canonical divisor K_X is not 0, but $2K_X = 0$.

2:Every Enriques surface has an elliptic structure. It has exactly two multiple fibres of multiplicity 2 ,say them F_A, F_B. Then the canonical divisor can be expressed as a difference of two multiple fibres, that is, $K_X = F_A - F_B$.

3. The fundamental group of an Enriques surafce is Z_2, so that the universal covering space is a K3 surface. Let the quotient map be π. That is an etale covering with respect to K_X. So, $\pi_*(O_{\overline{X}}) = O_X \oplus K_X$, $\pi^*(K_X) = O_{\overline{X}}$.

4. An Enriques surface X is called nodal if there exists a smooth rational curve R. (In this case R^2=-2.) Otherwise it is called unnodal. In the 10 dimensional moduli space of Enriques surfaces, a generic one is unnodal, while the nodal ones form a 9 dimensional subspace.

5. A nodal cycle N on an Enriques, or K3 surface is a positive 1-cycle such that $h^1(O_N) = 0$. This is a tree of smooth rational curves.([Ar])

6. We define the slope of E with respect to some ample divisor H, denoted by $\mu_H(E)$, as $(c_1(E) \cdot H)/rank(E)$. A vector bundle E is called H(-semi)-stable if for every subsheaf F, with $0 < rank(F) < rank(E)$, $\mu_H(F) < (\leq)\mu_H(E)$. There exists a moduli space of stable vector bundles which is quasi-projective.

From now on we fix the notations.

X is an Enriques surface and its universal covering space which is a K3 surface is denoted by $\overline{X}$ and the quotient map from $\overline{X}$ to X is π. Let $M_{X,H}(r, c_1, c_2)$ be the moduli space of stable vector bundles with respect to H, where r is the rank of the bundles and c_i is the i-th Chern class. Let also $M_{\overline{X},\pi^*H}(r, c_1, c_2)$ be the moduli space of stable vector bundles with respect to π^*H with rank r, Chern classes c_i. We denote i to be the involution on $\overline{X}$ compatible to π and i^*, the induced involution on $M_{\overline{X}}$.

§ II. GENERAL STRUCTURE THEOREM

Here we interpret the results of Takemoto[Ta] in our Enriques surface X and the covering K3 surface $\overline{X}$.

Theorem[Ta]: (1); if a π^*H-stable bundle F on $\overline{X}$ is not isomorphic to π^*E for any bundle E on X, then $\pi_*(F)$ is H-stable. If F is π^*H-semi-stable, then π_*F is H-semi-stable.

(2);If a simple bundle E on X is isomorphic to E(K), then there exists a simple bundle F on X such that $\pi_*(F) = E$.

Next we introduce the result of Mukai on the moduli spaces of stable bundles on K3 surfaces.

Theorem[Mu1]:The moduli space M of stable bundles on a K3 surface S is smooth and there is a line bundle $L = O_M$ and a skew-symmetric bilinear form $B : TM \times TM \to L$ such that $B \otimes k([F])$ is nondegenerate and canonically isomorphic to the natural pairing $Ext^1(F,F) \times Ext^1(F,F) \to Ext(F,F)$ for any stable bundle F.

Let us begin with a lemma.

Lemma: Let X be an Enriques surface and $\overline{X}$, be the universal covering space of X and F be a simple vector bundle on $\overline{X}$ such that $F = i^*F$. Then there exists a bundle E on X such that $\pi^*E = F$.

Proof: It suffices to prove that there exists a map f: F to F, over involution such that $f^2 = id$. For a given isomorphism $h : F \to i^*F$, let g be a composition of h followed by the natural map $j : i^*F \to F$. Then g is a map from F to F over involution and $g^2 = \lambda Id$ for some $\lambda \neq 0$ in C. Then let f be $(\lambda)^{-1/2}g$. Then f satisfies the required property. □

Remark: This can be generalized to any bundle F whose endomorphism is $M_r(C)$, where r is the rank of E.

Before going into the main theorems, we recall the formular for the dimension of $M = M_X(r, c_1, c_2)$, the moduli space of stable bundles on an Enriques surface X, and the dimension of the tangent space $T_E M$ at $E \in M$.

$$Dim_E M \geq 2rc_2 - (r-1)c_1^2 - r^2 + 1$$

$$Dim T_E M = 2rc_2 - (r-1)c_1^2 - r^2 + 1 + h^2(End E).$$

Here $h^2(End E) = 0$ if $E \neq E(K)$ and 1 if $E = E(K)$. This comes from the fact that for any non-trivial homomorphism between two stable bundles with the same slope is an isomophism. ([O,S,S]) We mean the smallest possible dimension of M by the expected dimension of M.

Then we state our first main result.

Theorem 1: Let $\overline{X}$ be a K3 surface which is a universal covering space of an Enriques surface X and $M_X, (M_{\overline{X}})$ be a moduli space of stable vector bundles on $X(\overline{X})$ (See the fixed notation in § I).

(1);Then M_X is singular at E if and only if $E = E(K)$ except the case that E belongs to a 0-dimensional component (exceptional bundle) or a two dimensional component, where every bundle E satisfies that $E = E(K)$. The singular locus of $M_X = U\pi_*(M^{0,i}_{\overline{X}})$, a union of the direct images of π of finitely many different $M^0_{\overline{X}}$ on $\overline{X}$, where $M^0_{\overline{X}} = (F \in M_{\overline{X}} | F \neq i^*F)$ and its dimension is $\leq (1/2)(dim M_X + 3)$. So, M_X is generically smooth. In particular, if rank is odd, it is everywhere smooth, and if rank=2 then it can have only finitely many isolated singular points.

(2):The pull back map π^* from M^0_X to $M_{\overline{X}}$ is two to one with no branch, where $M^0_X = (E \in M_X | E \neq E(K))$. Explicitly $\pi^*E = \pi^*(E(K))$

(3):The image of M^0_X by π^* is a Lagrangian subvariety in $M_{\overline{X}}$ and is equal to the fixed locus by involution i^*.

Proof of (1): If E is a singular point in M_X then E =E(K), so that E=π_*F for some stable bundle F on X.([Ta],or §1) So, the rank is an even number, 2k. Then $\pi^*E = F \oplus i^*F$. Let the Chern polynomial of F be $1 + c_1(F)t + c_2(F)t^2$. Then, that of i^*F is $1 + (i^*c_1(F))t + c_2(F)t^2$. So,

$$\pi^*c_1(E) = c_1(F) + i^*c_1(F),$$

$$2c_2(E) = \pi^*c_2(E) = c_1(F) \cdot i^*c_1(F) + 2c_2(F).$$

$c_1(F) \cdot \pi^*H = i^*c_1(F) \cdot \pi^*H$ since $\pi^*H = i^*(\pi^*H)$. This implies that $(c_1(F) - i^*c_1(F))^2 \leq 0$ by Hodge Index theorem (equality holds if and only if $c_1(F) = i^*c_1(F)$. Here we can find a relation between the dimensions of $M_X(r = 2k, c_1(E), c_2(E))$, and $M_{\overline{X}}(k, c_1(F), c_2(F))$.

$$dimM_X = 4kc_2(E) - (2k-1)c_1^2(E) - 4k^2 + 1$$

$$= 4k((1/2)(c_1(F) \cdot i^*c_1(F))) - (2k-1)(c_1^2(F) + c_1(F) \cdot i^*c_1(F)) - (4k^2 - 1)$$

$$= 4kc_2(F) - (2k-2)c_1^2(F) - (4k^2 - 1)$$

$$= 2(2kc_2(F) - (k-1)c_1^2(F) - 2k^2 + 2) + (c_1(F) \cdot i^*c_1(F) - c_1^2(F)) - 3.$$

(Here dimM_X is the expected dimension. For the dimension of the component,where E=E(K), we must add by 1.) However we know that

$$(c_1(F) \cdot i^*c_1(F) - c_1^2(F)) \geq 0$$

,where the equality holds if and only if $c_1 = i^*c_1$. So we can conclude that

$$DimM_{\overline{X}}(k, c_1(F), c_2(F)) \leq (1/2)(dimM_X(2k, c_1(E), c_2(E)) + 3),$$

where the equality holds if and only if $c_1 = i^*c_1$. (Or $dimM_{\overline{X}} \leq (1/2)(dimM_X + 2)$, equality holds if and only if $c_1(F) = i^*c_1(F)$ for the component where E=E(K)).

From the above formular, we see that $c_1(F)^2 - c_1(F) \cdot i^*c_1(F) \geq -dimM_X - 3 = B$, so that,

$$2B \leq (c_1(F) - i^*c_1(F))^2 \leq 0.$$

So, there can be only finitely many numbers for $(c_1(F) - i^*c_1(F))^2$ and for a fixed value there can be only finitely many choices for $c_1(F)$ since $(\pi^*H)^{\perp}$ is a negative definite lattice. However π_*F is stable if and only if $F \not\cong i^*F$. So the singular locus of M_X is a finitely many union of direct images of $M_{\overline{X}}^0$. Here the map π_* from these $M_{\overline{X}}$ to the singular locus of M_X is 2 to 1 with no branch. In fact $\pi_*(F) = \pi_*(G)$ implies that $\pi^*(\pi_*(F)) = \pi^*(\pi_*(G))$. This means that $F \oplus i^*F = G \oplus i^*G$. So,

$$Hom(F,G) \oplus Hom(F, i^*G) \oplus Hom(i^*F, G) \oplus Hom(i^*F, i^*G)$$

$$= Hom(F \oplus i^*F, G \oplus i^*G) \neq 0.$$

So this forces that $F = G$, or $F = i^*G$.

So it is 2 to 1 and 1 to 1 if and only if F=i^*F. However in this case π_*F is not stable(splitting). The singular locus is of even dimension and smooth in itself. If the rank is odd, then M is everywhere smooth.

Conversely, if E =E(K), then E=π_*F for some F on X. If E is a smooth point in M, then E=E(K) everywhere in the component M containing E. Then M is a finite union of the direct images of some components of $M_{\overline{X}}$ with possibly different Chern classes. Among them one (and exactly one) has the same dimension as M, call $\overline{M}$, since M is irreducible. From the previous formular, we get

$$dimM = dim\overline{M} \leq (1/2)(dimM + 2).$$

(Note that E=E(K) in M). So, the possible dimension of M is 0 or 2. If dimM=2, then for F in $\overline{M}, c_1(F) = i^*c_1(F)$ and if dimM=0, M has a unique bundle E and $\overline{M}$ has a unique bundle F such that $c_1(F)^2 = c_1(F) \cdot i^*c_1(F) - 2$. If rank is 2, then the singular locus is the direct images of finitely many different line bundles. This completes the proof of (1). □

Remark 1: In fact the singularity of $M_X(2k, c_1, c_2)$ is closely related to the singularity of the curves in the linear system of c_1 (the splitting behaviour of the divisor of $\pi^*(c_1)$ on $\overline{X}$.)

Remark 2: There can be 3 different types of components in M_X, (1) a component M which has the expected dimension and is smooth everywhere, (2) a component M which has the expected dimension, but has some singularity, (3) a component M which has the dimension one bigger than the expected dimension (must be smooth everywhere). The singularity can exist only in the second type. The components with codimension one singularity (of only the second type) can intersect with each other. These things can happen only if dimM=1,3 or 5.

Example 1: The simplest example of M_X with some singularities is $M_X(2, F_A, 1)$, where F_A is a half fibre. Then $M_X(2, F_A, 1) = F_B$, another half fibre. If F_B is singular with an ordinary double point,then the inverse image of F_B is a union of two smooth rational curves $R_1, R_2 = i^*(R_1)(\overline{A_1}$ type). Then the bundle E corresponding to the singularity is just $\pi_* O_{\overline{X}}(R_1)(or R_2)$. Note that $det(\pi_* O_{\overline{X}}(R_1)) = F_B + K = F_A$.

Example 2: We can find many examples of the moduli space M of dimension three whose singularity is an Enriques surface. We can find a moduli space $M_{\overline{X}}(k, c_1, c_2)$ such that $c_1^2 = c_1 \cdot i^* c_1 - 2$(this holds if and only if $c_1 = N + M$, where N is a nodal cycle with $N \cdot i^* N = 0$ and M is a divisor fixed by involution. See §IV Exceptional bundles.) and dim $M_{\overline{X}}$=2. There are many examples with these conditions. Then $M_{\overline{X}}$ is a K3 surface and the dimesion of the corresponding M_X is 3 and the singular locus is just the quotient of that K3 surface with no fixed point, so that it is an Enriques surface. In fact there is no bundle E fixed by involution in $M_{\overline{X}}$ since $c_1(E)$ is not invariant by involution (since N is a nodal cycle and $N \neq i^* N$.)

Example 3: If we choose $M_{\overline{X}}(k, c_1, c_2)$ of dimension 4 with $c_1 = i^* c_1$(there are many choices with odd k), then the image of $M^0_{\overline{X}}$ by π_* is the singular locus of 4 dimension in the 5 dimensional space, M_X.

Proof of (2) First we show that if E is H-stable and is not isomorphic to E(K), then $\pi^* E$ is $\pi^* H$-stable. From the fact that

$$H^0(End \pi^* E) = H^0(End E) \oplus H^0((End E)(K)) = C,$$

$\pi^* E$ is simple. $\pi^* E$ is also a direct sum of stable bundles with the same slope (since the pull back of an Einstein-Hermitian bundle is still Einstein-Hermitian and an Einsten-Hermitian bundle is a direct sum of stable bundles with the same slope.) From these two facts $\pi^* E$ must be $\pi^* H$-stable. From the above equation we conclude also that if E is isomorphic to E(K), then $\pi^* E$ is not simple, just a direct sum of stable bundles. (In fact $\pi^* E = F \oplus i^* F$, for some F such that $\pi_* F = E$.([Ta])) So, π_* is well defined from $M^0_X = (E | E \in M_X, E \neq E(K))$ to $M_{\overline{X}}$. M^0_X is the same as the smooth locus of M_X except two cases as we saw in the proof of (1). Next we show that π^* is 2 to 1 with no branch. If $\pi^*E = \pi^*E'$, then $H^0(\pi^*(E^* \otimes E')) \neq 0$. However,

$$H^0(\pi^*(E^* \otimes E')) = H^0(E^* \otimes E') \oplus H^0(E^* \otimes E'(K)).$$

So, either $H^0(E^* \otimes E') \neq 0$, or $H^0(E^* \otimes E'(K)) \neq 0$. The property of stability implies $E = E'$, or $E = E'(K)$. So, the map π^* is 2 to 1 with no branch. □

Remark: (a) π^* restricted to $M_X^0(2k, D, c_2)$ or $M_X^0(2k, D + K, c_2)$ is still 2 to 1 with no branch. In general, $M_X(2k, D, c_2)$ is not isomorphic to $M_X(2k, D + K, c_2)$. For example, $M_X(2, F_A, 1) = F_B$ is not isomorphic to $M_X(2, F_B, 1) = F_A$. If an exceptional bundle E of even rank exists for det(E)=D, then there is no exceptional bundle for det=D+K (See §IV : Exceptional bundles.) The same is true for two dimensional component of vector bundles of even rank.(See § III: Enriques surfaces as moduli spaces.)

(b)However, π_* restricted to $M_X(2k+1, D, c_2)$ or $M_X(2k+1, D+K, c_2)$ is 1 to 1, so that $M_X(2k+1, D, c_2)(= M_X(2k+1, D+K, c_2))$ is isomorphic to its image.

Proof of (3) First we show that the dimension of $M_X^0(r, c_1, c_2)$ is half of the dimension of $M_{\overline{X}}(r, \pi^*c_1, \pi^*c_2)$.

$$DimM_{\overline{X}} = 2r\pi^*c_2 - (r-1)(\pi^*c_1)^2 - 2r^2 + 2$$

$$= 2(2rc_2 - (r-1)c_1^2 - r^2 + 1) = 2dimM_X^0.$$

Next we show that the pull back of the holomorphic two form ω on $M_{\overline{X}}$ to M_X^0 vanisches. The proof comes easily from the following commuting diagram,

$$\begin{array}{ccccc} Ext^1(\pi^*E, \pi^*E) & \times & Ext^1(\pi^*E, \pi^*E) & \rightarrow & Ext^2(\pi^*E, \pi^*E) \\ \uparrow & & \uparrow & & \uparrow \\ Ext^1(E,E) & \times & Ext^1(E,E) & \rightarrow & Ext^2(E,E), \end{array}$$

and the fact that $Ext^2(E, E) = H^2(EndE) = 0$, for any E $\in M_X^0$. (In the above diagram, $Ext^1(E,E) = T_E M_X$ and $Ext^1(\pi^*E, \pi^*E) = T_{\pi^*E} M_{\overline{X}}$.) So, we can also conclude that the image of M_X^0 is a Lagrangian subvariety in $M_{\overline{X}}$. That the image of M_X^0 is fixed by involution i^* is obvious. Another direction comes from the lemma easily. So the image is exactly the fixed locus by involution. This completes the proof of (3). □

Remark 1: We expect that $M_{\overline{X}}$ is birational to the cotangent bundle of the image of M_X^0 by π^*.

Remark 2: We know that the dimension of $M_X(2k+1, c_1, c_2)$ is even. We expect that $M_X(2k+1, D, c_2)$ is birational to a symmetric power of some Enriques surface. In fact we know many cases that $M_{\overline{X}}$ is birational to a symmetric power of some K3 surface. In this case, the image of M_X is just the fixed locus by involution, so that it is a symmetric power of an Enriques surface, the quotient of that K3 surface. Another example is $M_X(3, c_1, 3)$, where X is a fourfold covering of P^2 and c_1 is a pull back of hyperplane of P^2. Then $c_1^2 = 4$. In this case M is birational to the original Enriques surface X.

Remark 3: We also know that the dimension of $M_X(2k, c_1, c_2)$ is odd except two cases we mentioned. In this case we expect that M_X is birational to the variety with SU_N holonomy. We know one example. Let M denote the moduli space of bundles of rank 2 with $c_1^2 = 6, c_2 = 3$, where c_1 is ample. If $|c_1|$ is not hyperelliptic (there does not exist a divisor $f \neq 0, f^2 = 0$ such that $c_1 \cdot f = 1$.), then M is birational to a double covering of P^3.([Ki2])

§ III. ENRIQUES SURFACES AS MODULI SPACES

Mukai showed that if the moduli space of stable bundles on a K3 surface is of dimension 2, then it is also a K3 surface. More explicitly,

Definition: An algebraic cycle $Z \in H^4(X \times Y, Q)$ on a product of two surfaces X and Y is an isogeny if the homomorphism $f_Z : H^2(X,Q) \rightarrow H^2(Y,Q), t \rightarrow \pi_{*Y}(Z \cdot \pi_X^* t)$ is an isometry, i.e., an isomorphism compatible with cup product pairings. Two surfaces X and Y are isogeneous if there exists an isogeny Z $\in \mathbf{H}^4(X \times Y, Q)$ on $X \times Y$.

Theorem[Mu2]: Let S be an algebraic K3 surface and A moduli space M of stable bundles on S be nonempty and compact. Then M is irreducible and is a K3 surface. Moreover S and M are isogeneous.

Here we study the first exceptional case; a 2-dimensional smooth component, where E=E(K) everywhere.

Theorem 2: If $M_X(2k, c_1, c_2)$ has a component M of dimension two which is compact, then M is an Enriques surface which is isogeneous to the original surface X.

Sketch of Proof: If M is a component of dimention two of the moduli spaces of even rank, then E=E(K) for every E in M and M must be smooth since M is an image of a finite union of even dimensional variety by a degree two map π_* and the dimension of $T_E(M)$ is even. Then by the arguments of Mukai[Mu2] and Huybrecht[Huy], mK_M=0 for some natural number m. To identify one of the four possible surfaces, K3 surface, Abelian surface, hyperelliptic surface and Enriques surface we define a cycle map $f_Z, (f'_Z)$ from $H^*(X)$ to $H^*(M)$(from $H^*(M)$ to $H^*(X)$), where $H^*(W) = \oplus H^i W$, exactly following the methods of Mukai([Mu2]). We define two cycles Z and

Z' on $H^*(X \times M)$.

$$Z = p_X^*(\sqrt{tdX}) \cdot ch(E^*) \cdot p_M^*(\sqrt{tdM})$$

$$Z' = p_X^*(\sqrt{tdX}) \cdot ch(E) \cdot p_M^*(\sqrt{tdM}),$$

where E is a universal bundle on $X \times M$ (we can assume the existence of E without loss of generality and the cycle map defined below is independent of the choice of E), and $\sqrt{tdW}$ is the unique cycle in $H^*_\cdot(W)$ such that the self intersection is td W. Now We define two maps f_Z, f'_Z.

$$f_Z : H^*(X,Q) \rightarrow H^*(M,Q), \alpha \rightarrow p_{M,*}(Z \cdot p_X^* \alpha),$$

$$f'_Z : H^*(M,Q) \rightarrow H^*(X,Q), \beta \rightarrow p_{X,*}(Z' \cdot p_M^* \beta),$$

,where p_X, p_M are projections from $X \times M$ to X and M.

Then $f_Z \circ f'_Z$ is the identity map on $H^*(M)$, so that $H^*(M)$ is a direct summand of $H^*(X)$. Since these are cycle maps, even forms(odd forms) go to even(odd) forms, so M must be an Enriques surface and f_Z is an isomorphism. If we restrict to 4-dimensional cycles of Z and Z', then this induces an isometry from $H^2(X,Q)$ to $H^2(M,Q)$.□

Remark 1: If we use the results of theorem 1, then M is included in a union of the direct images of π of M_i, where exactly one M_i, call $\overline{M}$, is a component of a moduli space $M_{\overline{X}}$ of dimension two and the other M_i s are all exceptional bundles. If $M_{\overline{X}}$ of dimension two is compact, then it is also irreducible[Mu2], so that $M_{\overline{X}} = \overline{M}$. For any F in $\overline{M}$, $c_1(F) = i^* c_1(F)$ as we saw in the proof of theorem 1. If we assume that there is no bundle fixed by involution(for example; $c_2(F)$ is odd for F in $\overline{M}$), then $M_X = \pi_*(\overline{M})$ is compact and irreducible. There are lot's of examples satisfying these conditions.(See [Mu2])

Then in this case we have an interesting diagram which is commuting,

$$\overline{X} \ldots.. \rightarrow M_{\overline{X}}$$

$$\downarrow \pi \downarrow \pi_*$$

$$X \ldots... \rightarrow M_X$$

,where π, π_* are quotient maps of degree two with no ramification and $\ldots . \rightarrow$ indicates isogeny.

In the second cohomology level, we can find another commuting diagram,

$$H^2(\overline{X}, Q) \rightarrow H^2(M_{\overline{X}}, Q)$$

$$\uparrow \pi^* \uparrow \pi_*^*$$

$$H^2(X, Q) \rightarrow H^2(M_X, Q),$$

where vertical maps are one to one and horizontal ones are isomorphisms.

Remark 2: But if $c_2(F)$ in $\overline{M}$, mentioned above, is even and there exist bundles fixed by involution, then M is not compact since M is the direct image of $(\overline{M})^0 = (F | F \neq i^* F, F \in \overline{M})$ and some exceptional bundles. In this case M is birational to P^2.

Remark 3: If M is a component of $M_X(2k+1, c_1, c_2)$ of dimension 2, then in this case also we expect M to be an Enriques surface.

IV.EXCEPTIONAL BUNDLES

First we can associate a vector v(E)=(r,D,s) $\in N \times PicX \times (1/2)Z$ for any vector bundle E on X, where r is the rank of E, D=det(E) and s=$(1/2)c_1(E)^2 - c_2(E) + r/2$. This is a Mukai vector on an Enriques surface.(For an algebraic surface S with $\chi(O_S) = \chi$, s=$(1/2)c_1(E)^2 - c_2(E) + \chi(r/2)$.([Mu2][Ty2])) Then we can define a symmetric bilinear form on that lattice,

$$v_1 \cdot v_2 = (r_1, D_1, s_1) \cdot (r_2, D_2, s_2) = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

Then

$$\chi(E^* \otimes F) = -v(E) \cdot v(F).$$

So, if E is stable, then

$$dimM = v(E) \cdot v(E) + 1 + h^2(EndE),$$

where M is the moduli space containing E. Here we need a definition of exceptional bundles.

Definition: E is exceptional if $h^0(EndE) = 1$ and $h^1(EndE) = 0$.

In this chapter we study the conditions of the existence of exceptional bundles on Enriques surfaces. For an exceptional bundle E of even rank, $v(E) \cdot v(E) = -2$ if and only if $h^2(EndE) = 1$. An exceptional bundle E of even rank such that E=E(K) has the property $h^2(EndE) = 1$. On the otherhand, for an exceptional bundle E of odd rank, $v(E) \cdot v(E) = -1$ if and only if $h^2(EndE) = 0$. There is a general theory on exceptional bundles developed by Russian school([Ty2]), where they emphasized the roles played by exceptional bundles. They play very similar roles played by effective curves of self intersection -2 acting on Picard groups by reflections, in particular on K3 or Enriques surfaces. On K3 surfaces, every exceptional bundle E satisfies $v(E) \cdot v(E) = -2$, while on Enriques surfaces exceptional bundles of even rank can satisfy $v(E) \cdot v(E) = -2$, so that only exceptional bundles of even rank play the role of acting on moduli spaces.

Also there is another difference between K3 surfaces and Enriques surfaces. On K3 surfaces for every divisor D with $D^2 = -2$, D or -D is effective, while on Enriques surfaces it is not always true. It holds only if the Enriques surface is nodal(has a smooth rational curve), since the effective divisor with self intersection -2 must have a smooth rational curve as a component.

There is also an anologue of this fact on exceptional bundles. On K3 surfaces, there always exists an exceptional bundle which realizes the exceptional vector v($v \cdot v = -2$, with the same definition as above, which is a necessary condition for E to be exceptional) More explicitly,

Theorem [Ku] :Suppose that S is a complex algebraic K3 surface. A is an arbitrary ample divisor on S, and v=(r,l,s), $r > 0$, is an exceptional vector belonging to the Mukai lattice on S. Then there exists a simple, μ_A-semi-stable bundle E which realizes the vector v.

However on Enriques surfaces we showed that there exists an exceptional bundle of rank 2 if and only if X is nodal, generalizing the result of Dolgachev and Reider on exceptional bundles of rank 2 with $c_1^2 = 10, c_2 = 3$ ([D,R]). More explicitly,

Theorem [Ki1]: If E is an exceptional bundle of rank 2 such that $v(E) \cdot v(E) = -2$ if and only if E=$E_0(D)$, where D is some divisor and E_0 is a nontrivial extension,

$$0 \to O_X \to E_0 \to O_X(N+K) \to 0,$$

where N is a nodal cycle with N^2=-2 and K is the canonical divisor on X.

We want to generalize this result to any exceptional bundle of even rank. The following is our main theorem.

Theorem 3: There exists an exceptional bundle E such that E=E(K) which realizes a vector v=(2k,D,s) with $v \cdot v = -2$ if

and only if the vector v=v(2k,D,s) with $v \cdot v = -2$ satisfies that D=N+2L+kK, where N is a nodal cycle and L is any divisor. In particular,there exists an exceptional bundle E of even rank such that E=E(K) if and only if X is nodal.

Proof: First we prove that if there exists an exceptional bundle E of even rank which realizes a vector v=(r,D,s) with $v \cdot v = -2$, then D=N+2L+kK, for some nodal cycle N and some divisor L. First we can find a bundle F on $\overline{X}$, such that $\pi_*(F) = E$, so that $\pi^*(E) = F \oplus i^*F$. Then F is simple since

$$h^0(End(\pi^*E)) = h^0(EndE) + h^0((EndE)(K)) = 2.$$

F is also rigid since

$$h^1(End(\pi^*E)) = h^1(EndE) + h^1((EndE)(K)) = 0.$$

So, by the formular in chapter 2, we get

$$c_1(F)^2 = c_1(F) \cdot c_1(i^*F) - 2.$$

However this holds if and only if $det(E) = N + 2L + kK$ for some nodal cycle N and some divisor L on X ([Kil]).

We show the converse that for a vector v=(2k,N+2L+kK,s), for some nodal cycle N and some divisor L on X(X must be a nodal Enriques surface), such that $v \cdot v = -2$, then there exists an exceptional bundle E on X which realizes the vector v. If we consider a vector v=(k,$N_1 + \pi^*L$,s) on $\overline{X}$, where N_1 is a component of $\pi^*N = N_1 + N_2 (N_2 = i^*N_1, N_1 \cdot N_2 = 0)$, then this is an exceptional vector. Then by the theorem of Kuleshov[Ku], there exists an exceptional bundle F which realizes v. Then $\pi_*F = E$ is a bundle which realizes the given vector v. To prove that E is exceptional, we show first that $h^0(EndE) = 1$. To show this we claim that $Hom(F, i^*F) = 0$. If $Hom(F, i^*F) \neq 0$, then $F = i^*F$. This comes from the fact that the composition of h, a nontrivial homomorphism from F to i^*F,with the natural map j from i^*F to F is nontrivial, so that it is an isomorphism, since F is simple. So, F is isomorphic to i^*F, which is a contradiction to the fact that $c_1(F) \neq c_1(i^*F)$. The rigidity of E comes from that $v(E) \cdot v(E) = -2$ and $h^2(EndE) = 1$, since E=E(K). Furthermore we can choose E which is μ_H

-semi-stable, since we can choose F, which is μ_{π^*H} -semi-stable ([Ku]), then by a theorem of Takemoto, π_* preserves the semi-stability. □

Corollary: Every exceptional bundle E of rank 2 with E=E(K)is $\pi_*(O_{\overline{X}}(N + \pi^*L))$ for some nodal cycle N on $\overline{X}$ and some divisor L on X.

Next we consider exceptional bundles of odd rank. These are less important than those of even rank. The simplest case is the case of rank one. We know that every line bundle on Enriques surfaces is exceptional since $h^1(X, O_X) = 0$ (as in K3 surfaces) and these can be identified to the line bundles in K3 surfaces invariant by involution. We want to generalize this fact to exceptional bundles of any odd rank.

Theorem 4: For any vector v=(2k+1,D,s) with $v \cdot v = -1$ on any Enriques surface X, there exists an exceptional bundle E which realizes the vector v.

Proof For given vector v=(2k+1,D,s), we consider a vector v=(2k+1,π^*D,2s). Then this is an exceptional vector on $\overline{X}$. So, by the theorem of Kuleshov[Ku], there exists an exceptional bundle F realizing the vector v. We get $\chi(F^* \otimes i^*F) = 2$, so that there is a nontrivial homomorphism from F to i^*F. The same argument as in the proof of theorem 3 shows that this F is invariant by involution. Then we know that there exists a bundle E on X such that $\pi^*E = F$ by the lemma in chapter 2. This bundle realizes the given vector v. So it suffices to show that E is exceptional. It is easy to see that E is simple and $h^1(EndE) = 0$. Furthermore , if we choose F which is also $\mu_{\pi^*H} - semi - stable$, then E is also μ_H -semi-stable.□

Remark 1: It is interesting to see that exceptional bundles on Enriques surfaces can be used to re-classify the threefolds whose hyperplane sections are Enriques surfaces (See [Co]) just as Mukai re-classified Fano threefolds and Fano variety of co-index 3, using exceptional bundles on K3 surfaces.([Mu3])

Remark 2: It is also interesting to construct exceptional bundles by extensions or divisions by exceptional bundles of even rank ([D,P],[Ru],[Ku1],[Ty2]). For rank 2 case we could construct them explicitly. ([Ki1])

Remark 3: We want to see that the conditions (i) E is exceptional, (ii) E is exceptional and $h^2(EndE) = 1$, (iii) E is exceptional and E=E(K), (iv) E is exceptional and stable, are equivalent for exceptional bundles of even rank. $(iv) \rightarrow (iii) \rightarrow (ii) \rightarrow (i)$ are automatic. We expect that at least (i),(ii) and (iii) are equivalent. We showed that (ii),(iii) and (iv) are equivalent for rank two([Ki1]). In our work, most results are on exceptional bundles with (iii). On exceptional bundles of odd rank, we can also ask the similar question that (i) E is exceptional, (ii) E is exceptional and $h^2(EndE) = 0$, (iii) E is exceptional and stable are equivalent. On K3 surfaces, there are two exceptional bundles with the same numerical invariants, so that they can not be μ_H-stable at the same time for any ample H.([Ku1])

§ V: EXISTENCE OF STABLE BUNDLES

In this chapter we study a sufficient condition of the existence of stable bundles on Enriques surfaces essentially following the methods of Kuleshov([Ku2].) On any Enriques surface X, we can find an ample divisor H and a divisor f such that $H \cdot f = 1$. In fact, we can find an ample divisor H with $H^2 = 2$. Then we can find a divisor f with $H \cdot f = 1$([C,D].)

Here we need another definition of stability. We define the Gieseker slope of E as the polinomial in the integer variable n, $p_H(E(n)) = \chi(E(nH))/rank(E)$. We say E is Gieseker-H- (semi-)stable, if for any subsheaf F($0 < r(F) < r(E)$), one has $p_H(F(n)) < (\leq) p_H(E(n))$, for n sufficiently large. Then we know that if E is μ_H-stable, then E is Gieseker-H-stable and if E is Gieseker-H-semi-stable, then E is μ_H-semi-stable.

Only in this chapter, we call E, H-stable if E is Gieseker-H-stable and μ_H-stable if E is H-stable with the definition in chapter one. (In the previous chapters, we meant μ_H-stable by H-stable.) Here we introduce the results of Kuleshov on K3 surfaces.

Theorem [Ku2] (1): Let S be a smooth K3 surface with Picard group isomorphic to Z. Let v=(r,al,s) be a primitive Mukai vector. If $1/2 \leq a/r \leq 3/2$ and $s \leq 0$, then there is an l-stable torsion free sheaf E on S that realizes the vector. (2): If we replace the condition of primitivity of v with the condition of (r,a)=1, then the vector is v is realized by a μ_l -stable torsion free sheaf. Also, if $r \geq 2$, then E can be chosen to be locally free.

Now we state an anologue of this theorem on Enriques surfaces.

Theorem 5 (1): Let X be a smooth Enriques surface. Let v=(r,D,s) be a Mukai vector such that (r,$D \cdot H$,2s) is a primitive vector and $1/2 \leq (D \cdot H)/r \leq 3/2$ and $s \leq 0$, then there is a H-stable torsion free sheaf E on X that realizes the vector v. (2): Let the

condition be same as (1) except replacing the primitivity of v with (r,$D \cdot H$)=1. Then the vector v is realized by a μ_H-stable torsion free sheaf. Also, if $r \geq 2$, then E can be chosen to be locally free.

Remark: We exactly followed the proof of Kuleshov, except a little modification. This condition depends on H, but we hope that the moduli spaces are independent of the choice of an ample divisor H as was shown to be true in rank 2 case by Qin([Q2].) This is a sufficient condition, but it is not too special since we can normalize E by E' such that $1/2 \leq c_1(E') \cdot H/rank(E') \leq 3/2$ by tensoring by $O_X(f)$. However, this condition is not necessary. Actually, we showed that there exists a μ_H-stable bundle of rank 2 for every possible case($4c_2 - c_1^2 - 3 > 0$).([Ki2])

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