Topological Hochschild cohomology and applications to Morava K-theories

A.Lazarev

1 Introduction

In this paper we prove that Morava K-theories admit a structure of an MU-algebra. By neglect of structure they also admit an A_∞- structure. This problem was first studied in [6]. However, in the cited reference the existence of an A_∞ structure on $K(n)$ was established with a weaker notion of an A_∞ ring spectrum (without the unit condition) and only for odd p.

Using the obtained result we compute the topological Hochschild cohomology of $K(n)$.

This paper is using the language of [2] routinely and relies heavily on the results of the previous paper of the author [3].

2 Morava K-theories as MU-algebras

Theorem 2.1 *Let L_n be the MU-module obtained by killing he regular ideal generated by $(p, x_2, x_4, \ldots, \hat{x}_{2n}, \ldots)$ in the ring $MU_\bullet = Z[x_2, x_4, \ldots]$. Then L_n admits a unique MU-algebra structure (which extends the $MU_\bullet$-algebra structure on $L_{n\bullet} = Z/pZ[x_{2n}]$).*

Proof. We will prove this theorem by induction up the Postnikov tower of L_n. The 1st Postnikov stage of L_n is HZ/p, the Eilenberg-MacLane spectrum mod p which is, of course, an MU-algebra. Suppose that we proved that $L_n^{(i)}$, the $2n(i-1)+1$st Postnikov stage of L_n, is an MU-algebra and show that $L_n^{(i+1)}$ is also such. Notice that the MU-module $L_n^{(i)}$ is gotten from MU by killing the elements x_{2n}^i plus all the remaining polynomial generators in the ring $MU_\bullet$, so that

$$L_n^{(i)} = Z/pZ[x_{2n}]/x_{2n}^i.$$

Lemma 2.2

$$THH_{MU}^\bullet(L_n^{(i)}, HZ/p) = Z/pZ[y_2, y_4, \ldots]$$

where deg $y_{2m} = 2m$ *for* $2m \neq 2n+2$ *and deg* $y_{2n+2} = 2ni+2$

Proof. The first step is to compute $\pi_\bullet(L_n^{(i)} \wedge_{MU} L_n^{(i)})$. Consider the Kunneth spectral sequence

$$Tor_{Z[x_2,x_4,\ldots]}^{st}(Z/pZ[x_{2n}]/x_{2n}^i, Z/pZ[x_{2n}]/x_{2n}^i) \Longrightarrow \pi_{s+t}(L_n^{(i)} \wedge_{MU} L_n^{(i)}).$$

To compute the E_2-term of this spectral sequence we introduce a differential graded algebra

$$A_* = Z[x_2, x_4, \ldots] \otimes \Lambda(z_1, z_3, \ldots),$$

where

$$d(z_1) = p, d(z_3) = x_2, \ldots, d(z_{2n-1}) = x_{2n-2}, d(z_{2n+1}) = x_{2n}^i, d(z_{2n+2}) = x_{2n+2} \ldots$$

Here $degx_{2m} = 2m, degz_{2m+1} = 2m+1$ if $m \neq n$ and $degz_{2n+1} = 2ni+1$. Then A_* is quasiasomorphic to $Z/pZ[x_{2n}]/x_{2n}^i$ and therefore

$$Tor_{Z[x_2,x_4,\ldots]}(Z/pZ[x_{2n}]/x_{2n}^i, Z/pZ[x_{2n}]/x_{2n}^i = A_\star \otimes_{Z[x_2,x_4,\ldots]} Z/pZ[x_{2p^n}]/x_{2p^n}^{i+1}$$

$$= \Lambda_{Z/pZ}(z_1, z_3, \ldots) \otimes Z/pZ[x_{2n}]/x_{2n}^i.$$

Since this is a spectral sequence of Hopf algebras all higher differentials are zero. We do not know apriori whether $L_n^{(i)}$ is a homotopy commutative MU-algebra (if p=2 then it is just plain false). Therefore $\pi_\bullet(L^{(i)} \wedge_{MU} L^{(i)})$ could be noncommutative and there would exist nontrivial multiplicative extensions in our spectral sequence. Let us show that this is not the case. Consider the (skew)-commutators of the elements z_k. On the one hand they should all lie in the bar filtration 2 or higher. On the other hand they should be primitive as commutators of primitive elements. This is possible only if they all equal zero and we see that in fact the algebra $\pi_\bullet(L_n^{(i)} \wedge_{MU} L_n^{(i)})$ is commutative, no mutiplicative extensions are present and

$$\pi_\bullet(L)_n^{(i)} \wedge_{MU} L_n^{(i)}) = \Lambda_{Z/pZ}(z_1, z_3, \ldots) \otimes Z/pZ[x_{2n}]/x_{2n}^i.$$

Next we have the following spectral sequence:

$$E_2^{st} = Ext^{st}_{\Lambda_{Z/pZ}(z_1,z_3,\ldots)\otimes Z/pZ[x_{2n}]/x^i_{2n}}(Z/pZ[x_{2n}]/x^i_{2n}, Z/pZ)$$

$$\Longrightarrow THH^{s+t}_{MU}(L^{(i)}_n, HZ/p).$$

We have

$$E_2^{st} = Ext^{st}_{\Lambda_{Z/pZ}(z_1,z_3,\ldots)}(Z/pZ, Z/pZ) = Z/pZ[y_2, y_4, \ldots],$$

where $deg\ y_{2m} = 2m$ for $m \neq n+1$ and $deg y_{2n+2} = 2ni+2$. This spectral sequence collapses since it is even. Again we do not know if $THH_{MU}(L^{(i)}_n, HZ/p)$ is a homotopy commutative MU-algebra, and therefore there might conceivably be nontrivial multiplicative extensions. To show that this does not happen consider the MU-algebra $THH_{MU}(L^{(i)}_n, L^{(i)}_n)$. The considerations analogous to foregoing show that

$$THH^{\bullet}_{MU}(L^{(i)}_n, L^{(i)}_n) = Z/pZ[y_2, y_4, \ldots] \otimes Z/pZ[x_{2n}]/x^i_{2n}.$$

(no multiplicative extensions are present in this case since the according to the results of J.McClure [4] $THH(A, A)$ is a homotopy commutative spectrum for any A). Next there is a map of algebras

$$THH_{MU}(L^{(i)}_n, L^{(i)}_n) \to THH_{MU}(L^{(i)}_n, HZ/p)$$

induced by the canonical map

$$L^{(i)}_n \to HZ/p$$

and comparing the spectral sequences we conclude that

$$THH_{MU}(L^{(i)}_n, HZ/p) = Z/pZ[y_2, y_4, \ldots].$$

Our lemma is proved.

The next lemma computes the cohomology operations from L_n to HZ/p in the category of MU-modules.

Lemma 2.3

$$[L^{(i)}_n, HZ/p]^{\bullet}_{MU} = \Lambda^*_{Z/pZ}(z_1, z_3, \ldots)$$

where deg $z_{2m+1} = 2m+1$ *for* $m \neq n$ *and deg* $z_{2n+1} = 2ni+1$ *and* $*$ *denotes the dual vector space (over* Z/pZ*)*

Proof. Consider two hypercohomology spectral sequences

$$E_2^{st} = Ext^{st}_{Z/pZ[x_2,x_4,\dots]}(Z[x_{2n}]/x^i_{2n}, Z/pZ) \Longrightarrow [L_n^{(i)}, HZ/p]^{s+t}_{MU}$$

and

$$\overline{E}_2^{st} = Ext^{st}_{Z/pZ[x_2,x_4,\dots]}(Z[x_{2n}]/x^i_{2n}, Z/pZ[x_{2n}/X^i_{2n}]) \Longrightarrow [L_n^{(i)}, L_n^{(i)}]^{s+t}_{MU}$$

and a map between them

$$E_2^{\bullet\bullet} = \Lambda^{\star}_{Z/pZ}(z_1, z_3, \dots) \to \Lambda^{*}_{Z/pZ}(z_1, z_3, \dots) \otimes Z/pZ[x_{2n}]/x^i_{2n} = \overline{E}_2^{\bullet\bullet}$$

induced by the map $L_n^{(i)} \to HZ/p$.

Just as before the spectral sequence $\overline{E}_2^{\bullet\bullet}$ is a spectral sequence of Hopf algebras and collapses for that reason. Therefore $E_2^{\bullet\bullet}$ also collapses and we have

$$E_2^{st} = E_\infty^{st} = \Lambda^*_{Z/pZ}(z_1, z_3, \dots).$$

The next step is to compute the module of derivations from $L_n^{(i)}$ into HZ/p and its image upon the forgetful map into $[L_n, HZ/p]_{MU}$.

Lemma 2.4

$$Der^{\bullet}_{MU}(L_n^{(i)}, HZ/p) = THH^{\bullet-1}_{MU}(L_n^{(i)}, HZ/p)/Z/pZ =$$

$$\Sigma^{-1} Z/pZ[y_2, y_4, \dots]/Z/pZ$$

where as before deg $y_{2m} = 2m$ for $m \neq n$ and deg $y_{2n+2} = 2ni + 2$.

Proof. We have the following homotopy cofibre sequence

$$HZ/p \to THH_{MU}(L_n^{(i)}, HZ/p) \to Der_{MU}(L_n^{(i)}, HZ/p)$$

from which the result follows.

Remark 2.5 *Using this result we can talk unambigously about derivations and singular extensions corresponding to Hochschild cohomology classes of $L_n^{(i)}$ with coefficients in HZ/p having positive degree.*

Consider now the element $\Sigma^{-1}(y_{2n+2}) \in Der^{2ni+1}_{MU}(L_n^{(i)}, HZ/p)$ and its image $l^*(\Sigma^{-1}(y_{2n+2})$ in the group $[L_n, HZ/p]_{MU}$, where l is the canonical forgetful map

$$l : Der_{MU}(L_n^{(i)}, HZ/p) \to [L_n, HZ/p]_{MU}.$$

Lemma 2.6

$$l^*(\Sigma^{-1}(y_{2n+2})) = z^*_{2n+1} \in \Lambda^*_{Z/pZ}(z_1, z_3, \ldots) = [L_n^{(i)}, HZ/p]^{\bullet}_{MU}.$$

Here the element z^*_{2n+1} denotes the primitive element in the coalgebra $[L_n^{(i)}, HZ/p]^{\bullet}_{MU}$ that is dual to z_{2n+1}.

Proof. This is analogous to the statement (ii) of Theorem 8.1 of [3] and is proved similarly. First we find by comparing the spectral sequences for $Der^{\bullet}_{MU}(L_n^{(i)}, HZ/p)$ and $[L_n^{(i)}, HZ/p]_{MU}$ that the image of $\Sigma^{-1}(y_{2n})$ is as stated on the level of the E_∞-terms. Next, this image is contained in the subspace of primitive cohomology operations from $L_n^{(i)}$ to HZ/p, i.e. those operations which are derivations up to homotopy and therefore no higher filtration terms will appear. The lemma is proved.

We now begin to perform the inductive step - proving that $L_n^{(i+1)}$ is an MU-algebra. The cases $i = 1$ and $i > 1$ differ and will be considered separately.

1)When $i = 1$ $L_n^{(i)} = HZ/p$. Take the element

$$y_{2n+2} \in Z/pZ[y_2, y_4, \ldots] = THH^{\bullet}_{MU}(L_n^{(i)}, HZ/p)$$

and consider the associated singular extension of MU-algebras:

$$\Sigma^{2ni} HZ/p \to X \to L_n^{(i)} \tag{1}$$

We want to prove that (1) is the $2n + 1$th Postnikov stage of L_n. For this it suffices to show that the $MU_{\bullet}$-module structure on $X_{\bullet}$ is given by the formula

$$X_{\bullet} = MU_{\bullet}/(p.x_2, x_4, \ldots, x_{2n}^2, x_{2n+2}, \ldots)$$

Notice that (1) gives rise to the following (algebraic) singular extension of $MU_{\bullet}$ - algebras:

$$\Sigma^{2n} Z/pZ \to X_{\bullet} \to Z/pZ \tag{2}$$

If (2) is nontrivial (as an extension of $MU_{\bullet}$-algebras) then we are done - it is elementary to verify that there is only one (up to a scalar) nontrivial singular graded extension of Z/pZ by $\Sigma^{2n} Z/pZ$, namely

$$\Sigma^{2n} Z/pZ \to Z/pZ[x_{2n}]/x_{2n}^2 \to Z/pZ$$

which is what we need. Therefore, we have to eliminate the possibility for (2) to be trivial, i.e. split. Suppose, on the contrary, that this is the case. We will deduce from it that (1) is split in the homotopy category of MU-modules, which would be a contradiction by Lemma (2.6)

Indeed, consider the spectral sequence

$$E_2^{st} = Ext^{st}_{MU_\bullet}(Z/pZ, Z/pZ \oplus \Sigma^{2n} Z/pZ) = Ext^{st}_{MU_\bullet}(L_n^{(i)}, X_\bullet) \Rightarrow [L_n^{(i)}, X]^\bullet_{MU}.$$

Then

$$E_2^{st} = (\Lambda_{Z/pZ}(z_1, z_3, \ldots) \oplus \Sigma^{2n}\Lambda_{Z/pZ}(z_1, z_3, \ldots))^*.$$

Comparing E_2^{st} with the spectral sequence for $[X, X]^\bullet_{MU}$ we see that the former collapses. The element $1 \in E_2^{00}$ corresponds the MU-module map $L_n^{(i)} \to X$ splitting the cofibre sequence (1) and we are done.

2) The case $i > 1$ is a little harder. We want to obtain the singular extension

$$\Sigma^{2ni} HZ/p \to Y \to L_n^{(i)}$$

where $Y_\bullet = MU_\bullet/(p, x_2, \ldots, x_{2n-2}, x_{2n}^{i+1}, x_{2n+2}, \ldots)$.

However the relevant subspace in $THH^\bullet(L_n^{(i)}, HZ/p)$ is 2-dimensional, it is spanned by the elements

$$y_{2n+2}, y_{2ni+2} \in THH^\bullet(L_n^{(i)}, HZ/p) = Z/pZ[y_2, y_4, \ldots]$$

since y_{2n+2} and y_{2ni+2} both have degree $2ni+2$. Take any nonzero element y in this subspace, i.e. a linear combination of y_{2n+2} and y_{2ni+2}. Associated to y is a certain singular extension of $L_n^{(i)}$ by $\Sigma^{2ni} HZ/p$ which gives rise to the following algebraic singular extension of MU-algebras:

$$\Sigma^{2ni} Z/pZ \to Y_\bullet \to L_{n\bullet}^{(i)} \tag{3}$$

Just as before, we show that (3) is nontrivial. But now the space of extensions like (3) is also 2-dimensional. The basis in this space corresponds to two different $MU_\bullet$-algebra structures on $Y_\bullet$. One is given as

$$Y_\bullet = MU_\bullet/(p, x_2, \ldots, x_{2n-2}, x_{2n}^{i+1}, x_{2n+2}, \ldots)$$

and the other as

$$Y_\bullet = MU_\bullet/(p, x_2, \ldots, x_{2n-2}, x_{2n}^i, x_{2n+2}, \ldots, x_{2ni}^2, x_{2ni+2}, \ldots).$$

Let us denote the corresponding algebraic extensions as E_1 and E_2 respectively. We need to choose the element y which gives rise to the extension E_1.

Claim. The element $y_{2ni+2} \in THH^{2ni+2}(L_n^{(i)}, HZ/p)$ gives rise to the extension E_2 (or a multiple of it). Indeed, consider the canonical map $L_n^{(i)} \to L_n^1 = HZ/p$ and the induced map on THH:

$$f : THH(L_n^{(1)}, HZ/p) = Z/pZ[\overline{y}_2, \overline{y}_4, \ldots] \to THH(L_n^{(i)}, HZ/p) =$$

$$Z/pZ[y_2, y_4, \ldots]$$

(here we put bars over the polynomial generators of $THH^\bullet(L_n^{(1)}, HZ/p)$ to distinguish them from the corresponding elements in $THH^\bullet(L_n^{(i)}, HZ/p)$. Then $f(\overline{y}_{2ni+2}) = y_{2ni+2}$. Indeed, the map f is is so on the level of E_2-terms of the corresponding spectral sequences and $f(\overline{y}_{2ni+2})$ should be primitive no higher degree correction terms will appear.

The considerations as in the case $i = 1$ tell us that the element $\overline{y}_{2ni+2}$ gives rise to the algebraic extension (up to a scalar factor)

$$\Sigma^{2ni} Z/pZ \to \overline{Y}_\bullet \to L_{n\bullet}^{(1)} = Z/pZ \tag{4}$$

where

$$\overline{Y}_\bullet = MU_\bullet/(p, x_2, \ldots, x_{2ni}, x_{2ni+2}^2, \ldots).$$

Therefore the extension corresponding to $\overline{y}_{2ni+2}$ is the extension induced from (4) by the augmentation map $Z/pZ[x_{2n}]/x_{2n}^i \to Z/p$ which is nothing but the extension E_2.

Next take an element $y \in THH^\bullet(L_n^{(i)}, HZ/p)$ which is a nonzero linear combination of y_{2n+2} and y_{2ni+2} and not proportional to y_{2ni+2}. The corresponding algebraic extension will have the form $\alpha E_1 + \beta E_2$, where α and β are scalars with $\alpha \neq 0$ and "+" means the Baer sum of extensions. Altering y by adding to it a multiple of y_{2ni+2} and perhaps multiplying it by a scalar we obtain an element $\tilde{y}$ whose corresponding algebraic extension is E_1. (we leave it to the reader to check that addition of elements in $THH^\bullet(L_n^{(i)}, HZ/p)$ corresponds to the Baer sum of associated extensions.)

We proved that the element $\tilde{y}$ descends to the $2ni + 1$st k-invariant of L_n.Inspection shows that there was no freedom in picking the element $\tilde{y}$, i.e. that the MU-algebra structure on $L_n^{(i+1)}$ is unique. With this our theorem is proved.

Corollary 2.7 *The spectrum $k(n)$, the nth connective Morava K-theory admits a structure of an MU-algebra.*

Remark 2.8 *Notice, that we do not claim the uniqueness of this structure. The point is that* $k(n)$ *can be obtained from* MU *by killing the ideal* $(p, x_2, x_4, \ldots, x_2(p^n \hat{-} 1), \ldots)$ *and the preceding theorem applies. However* $k(n)$ *can be obtaibed by killing other ideals as well, which means that even the* $MU_\bullet$*module structure on* $k(n)_\bullet$ *is not unique. Once we agree to call* $k(n)$ *the* MU*-module corresponding to the ideal* $(p, x_2, x_4, \ldots, \hat{x}_{2(p^n-1),\ldots})$ *we have the uniqueness of the* MU*-algebra structure on* $k(n)$.

3 Topological Hochschild cohomology of Morava K-theories

Now we know that Morava K-theory spectra are MU-algebras, and therefore, we can talk about their topological Hochschild cohomology. The computation of topological Hochschild (co)homology of spectra other than Eilenberg-MacLane was initiated by McClure and Staffeldt (cf. [5]). In this section we compute $THH_{MU}(K(n), K(n))$ for all primes p and $THH_S(K(n), K(n))$ for odd primes. The results are similar to the calculation of topological Hochschild cohomology of the Eilenberg-MacLane spectrum HZ/p in[3].Some additional care needs to be taken because $K(n)$ is not a commutative MU-algebra like HZ/p, but this circumstance does not affect the final result. In the future paper we apply these results to realize topologically the Lubin-Tate universal deformation of height n formal groups.

Theorem 3.1
(i)

$$THH^\bullet_{MU}(K(n), K(n)) = K(n)_\bullet[y_2, y_4, \ldots \hat{y}_{2p^n}, \ldots];$$

$$Der^{\bullet-1}_{MU}(K(n), K(n)) = K(n)_\bullet[y_2, y_4, \ldots \hat{y}_{2p^n}, \ldots]/K(n)_\bullet.$$

(ii)

$$[K(n), K(n)]^\bullet_{MU} = \Lambda_{K(n)_\bullet}(z_1, z_3, \ldots, \hat{z}_{2p^n-1}, \ldots).$$

and the map

$$Der^\bullet_{MU}(K(n), K(n)) \to [K(n), K(n)]^\bullet_{MU}$$

sends the elements $y_{2i} \in Der^{2i-1}_{MU}(K(n), K(n))$ *to* $z_{2i-1} \in [K(n), K(n)]^{2i-1}_{MU}$.
(iii) The forgetful map

$$[K(n), K(n)]^\bullet_{MU} \longrightarrow [K(n), K(n)]^\bullet_S$$

maps the elements z_{2p^k-1} *to the operations* $Q_k \in [K(n), K(n)]_S^{2p^k-1}$ *(cf., for instance [1]) and all other z's go to zero.*

Proof. The proof is along the same lines as the proof of Theorem 8.1 of [3], and we will be somewhat sketchy therefore. First compute $\pi_\bullet(K(n) \wedge_{MU} K(n))$. We have the Kunneth spectral sequence

$$Tor_{Z[x_2,x_4,\ldots]}(K(n)_\bullet, K(n)_\bullet) = \Lambda_{K(n)_\bullet}(z_1, z_3, \ldots, z_{2\hat{p^i}-1}, \ldots)$$
$$\Longrightarrow \pi_\bullet(K(n) \wedge_{MU} K(n))$$

which collapses so we have multiplicatively

$$\pi_\bullet K(n) \wedge_{MU} K(n) = \Lambda_{K(n)_\bullet}(z_1, z_3, \ldots, z_{2\hat{p^i}-1}, \ldots).$$

Next consider the spectral sequence

$$E_2^{ij} = Ext^{ij}_{\pi_\bullet K(n)\wedge_{MU}K(n)}(K(n)K(n)) \Longrightarrow THH^{i+j}_{MU}(K(n), K(n)).$$

We have

$$E_2^{ij} = Ext^{ij}_{\Lambda_{K(n)_\bullet}(z_1,z_3,\ldots)}(K(n)K(n)) = K(n)_\bullet[y_2, y_4, \ldots].$$

This spectral sequence collapses, but apriori there could be some nontrivial additive extensions. Indeed, $K(n)$ is not a commutative MU-algebra, and $THH_{MU}(K(n)K(n))$ need not be a $K(n)$-module. In particular we don't know whether $THH^\bullet_{MU}(K(n), K(n))$ is a Z/pZ-algebra. Let us show that this is, in fact, true.

To this end consider the connective Morava K-theory spectrum $k(n)$. This is an MU-algebra and just as above we have a spectral sequence

$$\tilde{E}_2^{ij} = k(n)_\bullet[\tilde{y}_2, \tilde{y}_4, \ldots] \Longrightarrow THH_{MU}(k(n), k(n))$$

(we put tildes over y_i's to distinguish them from the polynomial generators in $THH^\bullet_{MU}(K(n), K(n))$). Notice, that $\tilde{E}_2^{ij}$ is a connected graded algebra whose 0th graded component is Z/pZ. Therefore, in this spectral sequence there is no room for additive extensions and we conclude that

$$THH^\bullet_{MU}(k(n), k(n) = k(n)_\bullet[\tilde{y}_2, \tilde{y}_4, \ldots].$$

Further, consider the following maps of MU-modules:

$$THH_{MU}(K(n), K(n)) \xrightarrow{f} THH_{MU}(k(n), K(n)) \xleftarrow{g} THH_{MU}(k(n), k(n)).$$

Both maps f and g are induced by the localization map $k(n) \to K(n)$. Comparing spectral sequences we see, that f induces an isomorphism

$$THH^{\bullet}_{MU}(K(n), K(n)) \to THH^{\bullet}_{MU}(k(n), K(n))$$

and g induces a monomorphism

$$THH^{\bullet}_{MU}(k(n), k(n)) \to THH^{\bullet}_{MU}(k(n), K(n)).$$

This shows that $THH^{\bullet}_{MU}(k(n), K(n)) = THH^{\bullet}_{MU}(K(n), K(n))$ is a Z/pZ-algebra and part (i) is proved.

Part (ii) is proved similarly to Theorem 8.1 of [3]. An easy calculation shows that

$$[K(n), K(n)]_{MU} = \Lambda_{K(n)_{\bullet}}(z_1, z_3, \ldots, \hat{z}_{2p^i-1}, \ldots).$$

Next we first show that the elements y_{2i} can map to either z_{2i-1} or zero and then considering the induced map on the corresponding spectral sequences we see, that the image of y_{2i-1} is not zero. Part (ii) is proved.

Part (iii) is also proved similar to the corresponding statement in [3]. The only nontrivial thing is to prove that the element $z_{2p^i-1} \in [K(n), K(n)]^{2p^k-1}_{MU}$ considered as a $K(n)$-operation in the category of S-modules is not homotopic to zero. Denote $MU(i,n)$ the MU-module obtained by killing the prime p and all polynomial generators in $MU_{\bullet}$ except for x_{2p^i-2} and x_{2p^i}. Consider the following homotopy cofibre sequence of MU-modules:

$$\Sigma^{2p^i-2} MU(i,n) \xrightarrow{x_{2p^i-2}} MU(i,n) \xrightarrow{p} K(n) \xrightarrow{\partial} \Sigma^{2p^i-1} MU(i,n).$$

Now the map of MU-modules

$$p \circ \partial : K(n) \to \Sigma^{2p^i-1} K(n)$$

is exactly the $K(n)$-operaton Q_i and we are done.

The next theorem describes topological Hochschild cohomology of $K(n)$ as an S-algebra.

Theorem 3.2 *Let p be an odd prime. Then*

$$THH^{\bullet}_{S}(K(n), K(n)) = K(n)_{\bullet}[\alpha_0, \alpha_2, \ldots, \alpha_{n-1}]$$

where $\alpha_k \in THH^{2p^i}(K(n), K(n))$

Proof. We have a spectral sequence

$$E_2^{ij} = Ext_{K(n)_\bullet K(n)}(K(n), K(n)) \Longrightarrow THH_S^\bullet(K(n), K(n))$$

and the calculation of Robinson in [6] shows that

$$E_2^{ij} = K(n)_\bullet[\alpha_0, \alpha_2, \ldots, \alpha_{n-1}].$$

This spectral sequence collapses and to prove the theorem we only need to show that there are no nontrivial additive extensions, that is, that $THH_S^\bullet(K(n), K(n))$ is a Z/pZ-algebra.

There is an algebra map

$$THH_{MU}^\bullet(K(n), K(n)) \to THH_S^\bullet(K(n), K(n)).$$

Since $THH_{MU}^\bullet(K(n), K(n))$ is a Z/pZ-algebra this is only possible if $THH_S^\bullet(K(n), K(n))$ is an Z/pZ-algebra and the theorem is proved.

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