UNSTABLE ATOMICITY AND LOOP SPACES ON LIE GROUPS

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1. A (k-1)-connected space X with $H_k(X,Z_p) \simeq Z_p$ has been defined to be p-atomic if any map $f: X \to X$ which induces an isomorphism $f_*: H_k(X,Z_p) \to H_k(X,Z_p)$ must induce an isomorphism $f_*: H_*(X,Z_p) \to H_*(X,Z_p)$. We describe such X as being C.P. p-atomic (Cohen, Paterson) and consider some variants of the definition. C.P. p-atomicity is an indecomposability condition on a space implying for example that the space has no non trivial mod p homotopy retracts.

We define a space X to be weakly p-atomic if any map $f: X \to X$ is either a mod p homotopy equivalence or for each n, $f_*: \bar{H}_n(X,Z_p) \to \bar{H}_n(X,Z_p)$ is nilpotent under composition. If X is a (k-1)-connected CW complex of finite type, $H_k(X,Z_p) \simeq Z_p$, $k > 1$ and X is either an H-space or a co-H-space (in particular, a suspension), then X is C.P. p-atomic if and only if it is weakly p-atomic (see comment 2 following Proposition 5.1).

The spaces considered satisfy a stronger condition. A space X is defined to be p-atomic if any map $f: X \to X$ is either a mod p homotopy equivalence or $f_*: \bar{H}_*(X,Z_p) \to \bar{H}_*(X,Z_p)$ is nilpotent. Not every weakly p-atomic space is p-atomic, for example, BSU is weakly 2-atomic but is not 2-atomic (see comment 2 following Lemma 3.3).

A p-atomic space X is defined to be p-atomic of degree t if whenever $f_*: \bar{H}_*(X,Z_p) \to \bar{H}_*(X,Z_p)$ is nilpotent, $(f_*)^t = 0$ but there exists some such f with $(f_*)^{t-1} \neq 0$, $t > 0$. The author does not know if every p-atomic space is p-atomic of finite degree. The significance of the degree t will become apparent below.

We consider the 2-atomicity of ΩG where G is a 1-connected simple Lie group.

Theorem 1.1

(a) ΩG is 2-atomic of degree 1 when $G = Sp(m), G_2, F_4, E_7$ or E_8.

(b) ΩE_6 is 2-atomic of degree 3.

(c) $\Omega SU(n)$ is 2-atomic of degree t, where $2^{t-1} < n \le 2^t$.

The spinor groups are not mentioned in Theorem 1.1 because $\Omega Spin(7)$ and $\Omega Spin(8)$ decompose as products at the prime 2 in a non trivial manner; it has not been determined if $\Omega Spin(q)$ is 2-atomic for $q \ge 9$.

For the exceptional Lie groups, we prove a more precise result than Theorem 1.1. Let ΩG be 2-atomic of degree t. We consider the homomorphism of Abelian groups

$$\alpha: [\Omega G, \Omega G] \to \text{Alg.Hom}_{A_2}(H^*(\Omega G, Z_2), H^*(\Omega G, Z_2))$$

where in $[\Omega G, \Omega G]$ loop multiplication induces the group structure and $\text{Alg.Hom}_{A_2}(H^*(\Omega G, Z_2), H^*(\Omega G, Z_2))$ has the corresponding Abelian group addition derived from the Hopf algebra $H^*(\Omega G, Z_2)$. In addition $(f+g).h = f.h + g.h$ in $[\Omega G, \Omega G]$ and $\alpha(f.g) = \alpha(g).\alpha(f)$ where . denotes composition. Let $\underline{n}: \Omega G \to \Omega G$ be defined by $\underline{n}(x) = x^n = (\ldots((x)x)\ldots)$ for $n \in Z$ and let $\underline{Z}$ be the subgroup of $[\Omega G, \Omega G]$ generated by $\underline{1}$. If n is odd, $\underline{n}$ is a mod 2 homotopy equivalence and if n is even, $\alpha(\underline{n})$ is nilpotent (in reduced cohomology). In particular $\alpha(\underline{2}^t) = 0$ and $\alpha(\underline{Z}) \simeq Z_{2^s}$ where $s \le t$. If $f \in [\Omega G, \Omega G]$, then $2^s f = \underline{2}^s.f$ and so 2^s is the exponent of the Abelian group $\alpha(G) = \text{Image } \alpha$. We will show that $s = t$ and $\alpha(G) \simeq Z_{2^t}$ when G is an exceptional Lie group.

We set $\mathcal{I}(G) = \text{R.Alg.Hom}_{A_2}(H^*(\Omega G, Z_2), H^*(\Omega G, Z_2))$, the group of those ring homomorphisms over the Steenrod algebra (acting unstably) which arise as ring homomorphisms of $H^*(\Omega G, Z_{(2)})$ to itself. More explicitly let

$$R.Alg.Hom.(H^*(\Omega G,Z_2),H^*(\Omega G,Z_2)) = \{Alg.Hom.(H^*(\Omega G,Z_{(2)}),H^*(\Omega G,Z_{(2)}))\} \otimes Z_2$$

which is a subgroup of $Alg.Hom.(H^*(\Omega G,Z_2),H^*(\Omega G,Z_2))$ as $H^*(\Omega G,Z_2) = H^*(\Omega G,Z_{(2)}) \otimes Z_2$ and let $\mathcal{I}(G)$ be those homomorphisms defined over the Steenrod algebra. There is an inclusion of groups

$$\alpha(G) \subset \mathcal{I}(G) \subset Alg.Hom_{A_2}(H^*(\Omega G,Z_2),H^*(\Omega G,Z_2)).$$

The theorem which follows determines $\mathcal{I}(G)$ and shows that $\alpha(G) = \mathcal{I}(G)$.

Theorem 1.2

(a) $\mathcal{I}(G) \simeq Z_2$ generated by $\underline{1}^*$ when $G = G_2, F_4, E_7$ or E_8.

(b) $\mathcal{I}(E_6) \simeq Z_8$ generated by $\underline{1}^*$.

It is also true that $\alpha(Sp(m)) = \mathcal{I}(Sp(m)) \simeq Z_2$ generated by $\underline{1}^*$ but this will not be proved in this paper. For $SU(n)$ the situation has not been resolved; $\mathcal{I}(SU(n))$ is a finite Abelian group of exponent 2^t where $2^{t-1} < n \le 2^t$ and the subgroup generated by $\underline{1}^*$ is isomorphic to Z_{2^t}. For small values of n one can check that $\alpha(SU(n)) = \mathcal{I}(SU(n)) \simeq Z_{2^t}$.

A different generalization of atomicity at p has been given in [3]. Let G be a compact, simply connected Lie group and assume that no Spin(q) factor with q > 6 occurs when G is expressed as a product of simple Lie groups.

Theorem 1.3 Let $f: \Omega G \to \Omega G$ induce an isomorphism $f_*: H_2(\Omega G,Z_2) \to H_2(\Omega G,Z_2)$. Then f is a mod 2 homotopy equivalence.

Corollary 1.4 (Proposition 2.1 of [3])

(a) Let $g: G \to G$ induce an isomorphism $g_*: H_3(G,Z_2) \to H_3(G,Z_2)$. Then g is a mod 2 homotopy equivalence.

(b) Let $h: BG \to BG$ induce an isomorphism $h_*: H_4(BG,Z_2) \to H_4(BG,Z_2)$. Then h is a mod 2 homotopy equivalence.

Corollary 1.4 contains little that is new. However it demonstrates that one cannot expect that Theorems 1.1 and 1.3 will extend to odd primes except for a comparatively small number of Lie groups; at odd primes 'most' Lie groups decompose as products. Results on the indecomposability of BG at each prime, where G has the Lie multiplication, can be found in [11].

The more subtle questions associated with ΩG being stably 2-atomic when G is a simple Lie group [6,14,5] will not be considered in this paper although some of the calculations will be used in a subsequent note establishing 2-atomicity in certain cases.

Five sections follow this introduction. The next section is purely algebraic; Corollary 2.5 shows that being 2-atomic or weakly 2-atomic are equivalent conditions on ΩG. In section 3 complex K-theory is used to evaluate certain secondary cohomology operations and to define an unstable operation for complexes without homology 2-torsion derived from the exterior power operator λ^2 in K-theory. Theorems 1.1 and 1.2 are proved in sections 4 and 5 and the proof of Theorem 1.3 is completed in section 6.

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2. The algebra $H^*(\Omega G, Z_{(2)})$

Let M be a finite dimensional strictly positively graded free $Z_{(2)}$-module, where $Z_{(2)}$ is the ring of integers localized at the prime 2, and let $t: M \to M$ be a graded homomorphism inducing an isomorphism $t: M \otimes Z_2 \to M \otimes Z_2$. The following lemma is used repeatedly.

<u>Lemma 2.1</u> For each $v \geq 1$, there exists $n(v)$ such that $r = t^{n(v)}: M \otimes Z_{2^v} \to M \otimes Z_{2^v}$ is the identity.

<u>Proof</u> Let $M = \Sigma M_i$, $1 \leq i \leq \alpha$. The set of isomorphisms $s_i: M_i \otimes Z_2 \to M_i \otimes Z_2$ form a finite group, of exponent $\beta(i)$ say. So $s = t^{\beta(1)\beta(2)\ldots\beta(\alpha)}$ is the identity homomorphism on $M \otimes Z_2$. It follows that $r = s^{2^{v-1}}$ is the identity on $M \otimes Z_{2^v}$.

Let G be a compact 1-connected Lie group. We recall that $H^*(\Omega G, Z)$ is torsion free and is concentrated in even degrees [2]. Also $H^*(\Omega G, Q)$ is a polynomial algebra on $r = \text{rank}(G)$ generators. We define $N(G) = (QH^*(\Omega G, Z_{(2)}))/\text{Torsion}$, the free part of the indecomposable quotient module of $H^*(\Omega G, Z_{(2)})$. Then $N = N(G)$ is a free $Z_{(2)}$-module and a map $f: \Omega G \to \Omega G$ induces a homomorphism $f^*: N \to N$.

Let $\{x_1', x_2', \ldots, x_r'\}$ be a basis for N and choose representative classes $\{x_1, x_2, \ldots, x_r\}$ in $H^*(\Omega G, Z_{(2)})$. Then considering $H^*(\Omega G, Z_{(2)})$ as a subring of $H^*(\Omega G, Q)$ under coefficient inclusion, $\{x_1, x_2, \ldots, x_r\}$ is a set of polynomial generators of $H^*(\Omega G, Q)$.

<u>Proposition 2.2</u>

(a) Let $f^*: N \otimes Z_2 \to N \otimes Z_2$ be an isomorphism. Then

$f^*: H^*(\Omega G, Z_2) \to H^*(\Omega G, Z_2)$ is an isomorphism.

(b) Let $f^*: N \otimes Z_2 \to N \otimes Z_2$ be trivial. Then

$f^*: \bar{H}^*(\Omega G, Z_2) \to \bar{H}^*(\Omega G, Z_2)$ is nilpotent.

<u>Proof</u> Each element of $H^{2m}(\Omega G, Z_{(2)})$ can be expressed uniquely as a polynomial in $\{x_1, x_2, \ldots, x_r\}$ with coefficients in Q. As $H^{2m}(\Omega G, Z_{(2)})$ is finite dimensional, there exists $k = k(m)$ such that for any $w \in H^{2m}(\Omega G, Z_{(2)})$, $2^{k-1}w$ is a polynomial in $\{x_1, x_2, \ldots, x_r\}$ with coefficients in $Z_{(2)}$.

To prove (a), it is sufficient to show that for each n, some iterate of $f^*: H^{2n}(\Omega G, Z_2) \to H^{2n}(\Omega G, Z_2)$ is an isomorphism. As $N_2 \otimes Z_2 = H^2(\Omega G, Z_2)$, f^* is an isomorphism for $n = 1$. We assume that f^* is an isomorphism in dimensions less than 2m. Setting $N = \Sigma H^{2i}(\Omega G, Z_{(2)})$, $1 \le i < m$, in Lemma 2.1, after iteration we can assume that $f^*: \Sigma H^{2i}(\Omega G, Z_{2^k}) \to \Sigma H^{2i}(\Omega G, Z_{2^k})$, $1 \le i < m$, is the identity homomorphism. Let $\bar{w} \in H^{2m}(\Omega G, Z_2)$ have representative $w \in H^{2m}(\Omega G, Z_{(2)})$ which is decomposable as a class in $H^*(\Omega G, Q)$. Let $w = p(x_1, x_2, \ldots, x_r)$ where $2^{k-1}p(x_1, x_2, \ldots, x_r)$ is a polynomial with $Z_{(2)}$ coefficients. If $\dim x_i < 2m$, then $f^*x_i = x_i + 2^k z_i$ and so $f^*w = p(f^*x_1, f^*x_2, \ldots, f^*x_r) = p(x_1, x_2, \ldots, x_r) + 2w'$ where $w' \in H^{2m}(\Omega G, Z_{(2)})$. Thus $f^*\bar{w} = \bar{w}$. So to show that $f^*: H^{2m}(\Omega G, Z_2) \to H^{2m}(\Omega G, Z_2)$ is an isomorphism, we need just consider $f^*\bar{x}_s$ where $x_s \in H^{2m}(\Omega G, Z_{(2)})$. We apply Lemma 2.1 with $M = N$ and $v = 1$. So $f^*x_s = x_s + d_s + 2y_s$ where d_s can be chosen to be decomposable in $H^*(\Omega G, Q)$. Thus $f^*\bar{d}_s = \bar{d}_s$ and $(f^*)^2\bar{x}_s = \bar{x}_s$. This completes the proof of (a).

To prove (b), we first establish a lemma using the techniques of [9]. This relies upon $H^*(\Omega G, Z_{(2)})$ being a free

$Z_{(2)}$-module of finite type and $H^*(\Omega G,Q)$ being a connected, bi-associative, bi-commutative Hopf algebra.

Lemma 2.3 Let $w \in H^{2m}(\Omega G,Z_{(2)})$ with $m > 0$. Then there exists an integer r and classes $w_{2^i} \in D_{2^i}$ such that

$$p = 2^r w + 2^{r-1}w_2 + 2^{r-2}w_4 + \ldots + w_{2^r} \in PH^*(\Omega G,Z_{(2)}),$$

where $D_k = \bar{H}^*(\Omega G,Z_{(2)}).\bar{H}^*(\Omega G,Z_{(2)}). \ \ldots \ .\bar{H}^*(\Omega G,Z_{(2)})$, k factors. If w is decomposable in $H^*(\Omega G,Q)$, then $p = 0$.

Proof In the notations which follow, subscripts denote the filtrations in which elements lie in
$0 = D_s^{2m} \subset D_{s-1}^{2m} \subset \ldots \subset D_1^{2m} = H^{2m}(\Omega G,Z_{(2)})$. Let
$\underline{2}^*: H^*(\Omega G,Z_{(2)}) \to H^*(\Omega G,Z_{(2)})$ be the homomorphism induced from the loop squaring map on ΩG. So $\underline{2}^*(w) = 2w + 2v_2 + v^2$ for some classes v_2 and v.

Suppose in general that $\underline{2}^*(u) = 2u + 2v_s + x$, $s > 1$. Then $\underline{2}^*(u+\lambda v_s) = 2(u+\lambda v_s) + 2v_{s+1} + v_{2s} + x$ for some classes v_{s+1} and v_{2s} where $\lambda = (1-2^{s-1})^{-1}$ and so by repeating this step we can write $\underline{2}^*(u+z_s) = 2(u+z_s) + z_{2s} + x$ for some z_s and z_{2s}. Now we know that $\underline{2}^*(w) = 2w + z_2$, where $z_2 = 2v_2 + v^2$ and so $\underline{2}^*(2w) = 2w + 2z_2$. So there exists w_2 with $\underline{2}^*(2w+w_2) = 2(2w+w_2) + z_4$. The step can be repeated after multiplying by 2. Since $D_s^{2m} = 0$, we obtain $\underline{2}^*(p) = p$ for some p as in the statement of the lemma. Then by Proposition 2.1(c) of [9], p is primitive in $H^*(\Omega G,Z_{(2)})$.

If w is decomposable in $H^*(\Omega G,Q)$, then so is p. But $PH^*(\Omega G,Q) \to QH^*(\Omega G,Q)$ is an isomorphism and so $p = 0$.

We can now complete the proof of Proposition 2.2(b). After finite iteration, it can be assumed that $f^*x_i = 0 \bmod 2$,

$1 \le i \le r$. As inductive hypothesis we assume that $f^*: \Sigma H^{2i}(\Omega G, Z_2) \to \Sigma H^{2i}(\Omega G, Z_2)$, $1 \le i < m$ is trivial. To complete the proof, we show that if $\bar{w} \in H^{2m}(\Omega G, Z_2)$, then $f^*\bar{w} = 0$. It is sufficient to consider $\bar{w}$ with representative $w \in H^{2m}(\Omega G, Z_{(2)})$ which is decomposable in $H^*(\Omega G, Q)$. By Lemma 2.3, there exists an equation

$$2^r w = -2^{r-1}w_2 - 2^{r-2}w_4 - \dots - w_{2^r} \text{ in } H^{2m}(\Omega G, Z_{(2)}).$$

The induction hypothesis implies that $f^*w_{2^i} = 0 \bmod 2^{2^i}$. Therefore $f^*(2^r w) = 0 \bmod 2^{r+1}$ and so $f^*\bar{w} = 0$. This completes the proof of Proposition 2.2.

We record a corollary of the method of proof of Proposition 2.2(b). The set $\{x_1, x_2, \dots, x_r\}$ is as described before the statement of Proposition 2.2 and as usual $\bar{x}_i$ is the mod 2 reduction of x_i.

<u>Corollary 2.4</u> If $f^*\bar{x}_i = 0$, $1 \le i \le r$, then $f^*: \bar{H}^*(\Omega G, Z_2) \to H^*(\Omega G, Z_2)$ is trivial.

<u>Corollary 2.5</u> Let $f: \Omega G \to \Omega G$ be a map. Then $f^*: \bar{H}^*(\Omega G, Z_2) \to \bar{H}^*(\Omega G, Z_2)$ is nilpotent if and only if $f^*: \bar{H}^{2n}(\Omega G, Z_2) \to \bar{H}^{2n}(\Omega G, Z_2)$ is nilpotent for each n.

We require a lemma concerning algebra morphisms of the Hopf algebra $H^*(\Omega SU(n), Z_2)$ for $n \ge 3$. For further information on the Hopf algebra structure, the reader is referred to [10].

<u>Lemma 2.6</u> Let $f: \Omega SU(n) \to \Omega SU(n)$ be a map with $n > 2$. Then $f^*: N(SU(n))_2 \otimes Z_2 \to N(SU(n))_2 \otimes Z_2$ is an isomorphism if and only if $f^*: N(SU(n))_4 \otimes Z_2 \to N(SU(n))_4 \otimes Z_2$ is an isomorphism.

<u>Proof</u> Let $x_2 \in H^2(\Omega SU(n),Z_{(2)})$ be a generator and $x_4 \in H^4(\Omega SU(n),Z_{(2)})$ be chosen so that $\phi(x_4) = x_4\otimes 1 + x_2\otimes x_2 + 1\otimes x_4$. Let s be determined by $2^s \le n - 1 < 2^{s+1}$. Then $x_2^{2^s} \neq 0 \bmod 2$ but $x_2^{2^{s+1}} = 0 \bmod 2$. An elementary coalgebra calculation shows that $x_4^{2^s} \neq 0 \bmod 2$ and $x_2^{2^{s+1}} \neq 0 \bmod 4$. Let $w = 2^{-1}x_2^{2^{s+1}} + x_4^{2^s}$. Then $\bar{w} \in PH^{2^{s+2}}(\Omega SU(n),Z_2) \simeq QH_{2^{s+2}}(\Omega SU(n),Z_2) = 0$. Therefore $2^{-2}x_2^{2^{s+1}} + 2^{-1}x_4^{2^s} \in H^*(\Omega SU(n),Z_{(2)})$.

Now if $f^*x_2 = 0 \bmod 2$, it follows that $f^*x_4 = \varepsilon x_2^2 \bmod 2$ as $2^{-1}x_4^{2^s} \notin H^*(\Omega SU(n),Z_{(2)})$. If $f^*x_4 = \varepsilon x_2^2 \bmod 2$, it follows that $f^*x_2 = 0 \bmod 2$ as $2^{-2}x_2^{2^{s+1}} \notin H^*(\Omega SU(n),Z_{(2)})$. This establishes the lemma.

<u>Comments</u> 1. In section 6 a generalization of Lemma 2.6 will be used. Let $G = SU(n_1) \times SU(n_2) \times \ldots \times SU(n_s)$ and $f: \Omega G \to \Omega G$ induce an isomorphism $f^*: N(G)_2 \otimes Z_2 \to N(G)_2 \otimes Z_2$. Then $f^*: N(G)_4 \otimes Z_2 \to N(G)_4 \otimes Z_2$ is an isomorphism. This can be proved using the same technique as in the proof of Lemma 2.6. It can be assumed that in dimension 2, f^* is the identity. If f^* mod 2 is not an isomorphism in dimension 4, there exists $w_4 = x_4^{(1)} + x_4^{(2)} + \ldots + x_4^{(t)}$ with f^*w_4 mod 2 equal to a polynomial in 2 dimensional elements in $H^4(\Omega G,Z_2)$ and where $x_4^{(i)}$ arises from $SU(m_i)$, $m_1 \le m_2 \le \ldots \le m_t$. Let s be determined by $2^s \le m_t - 1 < 2^{s+1}$. One considers f^*w where $w = 2^{-2}\{(x_2^{(1)})^{2^{s+1}}+\ldots+(x_2^{(t)})^{2^{s+1}}\} + 2^{-1}\{(x_4^{(1)})^{2^s}+\ldots+(x_4^{(t)})^{2^s}\}$.

2. Using Corollary 2.5, Lemma 2.6 and known results on spherical classes in $\Omega SU(n)$ and $\Omega Sp(m)$, it can now be deduced

that these spaces are 2-atomic, c.f. [4]. We will use a different proof to obtain tighter control and because the argument used may have some independent interest.

3. An unstable secondary cohomology operation

All spaces in this section are assumed to have the homotopy type of connected CW-complexes of finite type and to have no 2-torsion in integral homology. The main objective is to define an operator

$$\xi_{4k}\colon H^{4k}(X,Z_2) \to H^{8k}(X,Z_2)/\{\text{Image } Sq^{4k} + \text{Image } Sq^2\},$$

which satisfies two properties: ξ_{4k} is natural and coincides with the secondary operation associated with the Adem relation $Sq^1(Sq^{4k}) + Sq^{4k}(Sq^1) + Sq^2(Sq^{4k-1}) = 0$ on $H^{4k}(\Sigma^2 CP^\infty, Z_2)$.

We use complex K-theory and first recall [7,8] how to define the secondary operation

$\theta_{4k}\colon \text{Ker } Sq^{4k} \cap \text{Ker } Sq^{4k-2} \subset H^{2q}(X,Z_2) \to H^{2q+4k}(X,Z_2)/\text{Image } Sq^{4k}$ by such methods associated with $Sq^1(Sq^{4k}) + Sq^{4k}(Sq^1) + Sq^2Sq^1(Sq^{4k-2}) = 0$.

An element of $H^{2q}(X,Z_2)$ will be denoted by $\bar{x}_{2q}$ and a representative class in $H^{2q}(X,Z_{(2)})$ by x_{2q}. A representative for x_{2q} in $K^0(X,Z_{(2)})_{2q}$ will be u_{2q} where $K^0(X,Z_{(2)})_{2q}/K^0(X,Z_{(2)})_{2q+1} = H^{2q}(X,Z_{(2)})$, as X has no homology 2-torsion. We write $x_{2q} \to \bar{x}_{2q}$, $u_{2q} \to x_{2q}$ and $u_{2q} \to \bar{x}_{2q}$ to describe this situation and for notational convenience assume that $K^0(X)_{2q+4k+1} = 0$. Theorem 6.5 of [1] implies that

$$\left.\begin{array}{l} \psi^2(u_{2q}) = 2^q u_{2q} + 2^{q-1}v_{2q+2} + \dots + 2^{q-2k+1}v_{2q+4k-2} + 2^{q-2k}v_{2q+4k} \\ \text{for classes } v_{2i} \in K^0(X,Z_{(2)})_{2i} \text{ where if } v_{2q+2i} \to \bar{x}_{2q+2i}, \\ Sq^{2i}\bar{x}_{2q} = \bar{x}_{2q+2i}. \end{array}\right\} \quad (3.1)$$

Let $\bar{x}_{2q} \in \text{Ker } Sq^{4q} \cap \text{Ker } Sq^{4k-2}$. Thus $v_{2q+4k-2} = 2u_{2q+4k-2} + v'_{2q+4k}$ and $v_{2q+4k} = 2v''_{2q+4k}$. So (3.1) can be rewritten,

$$\psi^2(u_{2q}) = 2^q u_{2q} + \dots + 2^{q-2k+2}u_{2q+4k-2} + 2^{q-2k+1}u_{2q+4k}.$$

The operator θ_{4k} is defined by $\theta_{4k}(\overline{x}_{2q}) = \overline{x}_{2q+4k}$, where $u_{2q+4k} \to \overline{x}_{2q+4k}$. It is necessary to check that $\theta_{4k}(\overline{x}_{2q})$ is well defined modulo the image of Sq^{4k}; the indeterminacy arises from different choices for x_{2q} representing $\overline{x}_{2q}$. Then θ_{4k} is additive, stable and if $f\colon Y \to X$ is a map of spaces of the type being considered, $f^*[\theta_{4k}(\overline{x}_{2q})] \subset \theta_{4k}(f^*\overline{x}_{2q})$.

The definition of

$$\phi_{4k}\colon \text{Ker } Sq^{4k} \subset H^{2q}(X,Z_2) \to H^{2q+4k}/\{\text{Image } Sq^{4k} + \text{Image } Sq^2\}$$

associated with the Adem relation $Sq^1(Sq^{4k}) + Sq^{4k}(Sq^1) + Sq^2(Sq^{4k-1}) = 0$ is similar but a little more subtle. Let $\overline{x}_{2q} \in \text{Ker } Sq^{4k}$. Again we consider (3.1). If $v_{2q+4k-2} \to \overline{x}_{2q+4k-2}$, $Sq^2\overline{x}_{2q+4k-2} = 0$ as $Sq^2Sq^{4k-2} = 0$. So there exists w_{2q+4k} such that $u_{2q+4k-2} = v_{2q+4k-2} + w_{2q+4k}$ satisfies $\psi^2(u_{2q+4k-2}) = 2^{q+2k-1}u_{2q+4k-2}$. So (3.1) can be rewritten

$$\psi^2(u_{2q}) = 2^q u_{2q} + \ldots + 2^{q-2k+1}u_{2q+4k-2} + 2^{q-2k+1}u_{2q+4k}$$

where $\psi^2(u_{2q+4k-2}) = 2^{q+2k-1}u_{2q+4k-2}$. We define $\phi_{4k}(\overline{x}_{2q}) = \overline{x}_{2q+4k}$ where $u_{2q+4k} \to \overline{x}_{2q+4k}$. One checks that $\phi_{4k}(\overline{x}_{2q})$ is well defined modulo the image of Sq^{4k}, again arising from different choices of x_{2q}, and modulo the image of Sq^2, arising from different choices of $u_{2q+4k-2}$ (if $Sq^{4k-4}\overline{x}_{2q} \neq 0$). Again ϕ_{4k} is additive and stable and if $f\colon Y \to X$ is a map of spaces $f^*[\phi_{4k}\overline{x}_{2q}] \subset \phi_{4k}(f^*\overline{x}_{2q})$.

We can modify the definitions of both θ_{4k} and ϕ_{4k} to obtain an unstable operator by following a similar procedure using the exterior power λ^2 in place of ψ^2. We will just use ξ_{4k} corresponding to ϕ_{4k}. To utilize the notations above, this is most easily defined simply by replacing $\psi^2(u_{4k})$ by

$\psi^2(u_{4k}) - u_{4k}^2 = -2\lambda^2(u_{4k})$ in the definition of ϕ_{4k}. If X is a suspension, ϕ_{4k} and ξ_{4k} are identical, but in general ξ_{4k} is neither additive nor stable; it is natural.

The remainder of this section is concerned with evaluating these operations. First we consider $\phi_{4q}(\bar{x}_2^{2q-1})$ where $\bar{x}_2$ is the generator of $H^2(CP^\infty, Z_2)$. If ξ is the reduced Hopf bundle, $\xi \in K(CP^\infty, Z_{(2)})_2$, $\xi \to \bar{x}_2$ and $\psi^2(\xi) = 2\xi + \xi^2$ and so $\psi^2(\xi^{2q-1}) = \xi^{4q-2} \bmod 2$. But $\psi^2(\xi^{4q-2}) = 2^{4q-2}\xi^{4q-2} + (4q-2)2^{4q-1}\xi^{4q-1} \bmod K(CP^\infty, Z_{(2)})_{8q-1}$, and so $\psi^2(\xi^{4q-2} - (2q-1)\xi^{4q-1}) = 2^{4q-2}(\xi^{4q-2} - (2q-1)\xi^{4q-1})$ $\bmod K(CP^\infty, Z_{(2)})_{8q-1}$. So in the notation used in defining ϕ_{4q}, $\psi^2(\xi^{2q-1}) = 2^{2q-1}\xi^{2q-1} + \ldots + u_{8q-4} + u_{8q-2} \bmod K(CP^\infty, Z_{(2)})_{8q-1}$ where $u_{8q-4} = \xi^{4q-2} - (2q-1)\xi^{4q-1}$, $u_{8q-2} = (2q-1)\xi^{4q-1} \to \bar{x}_2^{4q-1}$. Therefore $\phi_{4q}(\bar{x}_2^{2q-1}) = \bar{x}_2^{4q-1}$ and the indeterminacy is zero.

Let $t = 2^s - 1 + u$ where $0 \leq u < 2^s$. Then $Sq^{2u}\bar{x}_2^{2^s-1} = \bar{x}_2^t$ in $H^*(CP^\infty, Z_2)$ using $Sq^2\bar{x}_2 = \bar{x}_2^2$ and the Cartan formula.

<u>Lemma 3.1</u> (a) In $H^*(CP^\infty, Z_2)$, $\bar{x}_2^t = Sq^{2u}\phi_{2^s}\phi_{2^{s-1}}\cdots\phi_4\bar{x}_2$.

(b) In $H^*(\Sigma^2 CP^\infty, Z_2)$, $(\bar{x}_2^t) = Sq^{2u}\xi_{2^s}\xi_{2^{s-1}}\cdots\xi_4(\bar{x}_2)$.

Lemma 3.1 implies that $\Sigma^r CP^n$ is 2-atomic of degree 1 for all $r \geq 0$ and $1 \leq n \leq \infty$; of course it is only when $n = 2^s - 1$ that this does not follow from the action of the Steenrod algebra.

Recall that $H^*(SU, Z_2) = \Lambda(\bar{x}_3, \bar{x}_5, \ldots, \bar{x}_{2n-1}, \ldots)$ and $H^*(Sp, Z_2) = \Lambda(\bar{x}_3, \bar{x}_7, \ldots, \bar{x}_{4n-1}, \ldots)$, exterior algebras on primitive generators.

<u>Lemma 3.2</u> (a) In $H^*(SU,Z_2)$, $\phi_{2^s}\overline{x}_{2^s-1} = [\overline{x}_{2^{s+1}-1}+\overline{d}]$ where $\overline{d}$ is a decomposable element in the image of Sq^2.

(b) In $H^*(Sp,Z_2)$, $\theta_{2^s}\overline{x}_{2^s-1} = \overline{x}_{2^{s+1}-1}$.

<u>Proof</u> The inclusion $i: \Sigma CP^\infty \to SU$ induces a homomorphism $i^*: H^*(SU,Z_2) \to H^*(\Sigma CP^\infty,Z_2)$ which restricts to an isomorphism $i^*: PH^*(SU,Z_2) \to \overline{H}^*(\Sigma CP^\infty,Z_2)$. This determines the action of the Steenrod algebra and of ϕ_{4k} by Lemma 3.1 and naturality.

The inclusion $j: Sp \to SU$ induces isomorphisms $j^*: PH^{4s-1}(SU,Z_2) \to PH^{4s-1}(Sp,Z_2)$. As Sq^2 is zero on $H^*(Sp,Z_2)$ it follows that $\phi_{2^s}\overline{x}_{2^s-1} = \overline{x}_{2^{s+1}-1}$ in $H^*(Sp,Z_2)$. By considering the definitions of ϕ_{2^s} and θ_{2^s} on $H^*(Sp,Z_2)$, it is clear that θ_{2^s} and ϕ_{2^s} coincide.

<u>Comments</u> 1. One can write $\overline{x}_{2t+1} = Sq^{2u}\phi_{2^s}\phi_{2^{s-1}}\cdots\phi_4\overline{x}_3$ in $H^*(SU,Z_2)$, where for example one chooses the primitive class $\overline{x}_{15}$ to represent $\phi_8\phi_4\overline{x}_3$. In $H^*(Sp,Z_2)$, there is no ambiguity in writing $\overline{x}_{2t+1} = Sq^{2u}\theta_{2^s}\theta_{2^{s-1}}\cdots\theta_4\overline{x}_3$ where $t \equiv 1 \bmod 2$.

2. The operator θ_{2^s} rather than ϕ_{2^s} has been evaluated in $H^*(Sp,Z_2)$ because in the next section we will pass to loop spaces and the indeterminacy of ϕ_{2^s} is there too large.

3. Lemma 3.2 and comment 1 above imply that Sp and Sp(n) are 2-atomic of degree 1. They imply also that SU is weakly 2-atomic.

<u>Lemma 3.3</u> The operator ξ_{2^s} induces an isomorphism

$$\xi_{2^s}: QH^{2^s}(BSU,Z_2) \to QH^{2^{s+1}}(BSU,Z_2),\ s \geq 2.$$

<u>Proof</u> The inclusion $k: \Sigma^2 CP^\infty \to BSU$ giving $k^*: H^*(BSU, Z_2) \to H^*(\Sigma^2 CP^\infty, Z_2)$ induces an isomorphism $k^*: QH^*(BSU, Z_2) \to \overline{H}^*(\Sigma^2 CP^\infty, Z_2)$.

<u>Comments</u> 1. If $H^*(BSU, Z_2) = Z_2[x_4, x_6, x_8, \ldots]$, we may write $\overline{x}_{2t+2} = Sq^{2u}\xi_{2^s}\xi_{2^{s-1}}\ldots\xi_4\overline{x}_4$ provided that this is interpreted as for $H^*(SU, Z_2)$ above.

2. It follows that $BSU(n)$ is 2-atomic and BSU is weakly 2-atomic. By considering the power maps $\underline{2}^s$: $BSU \to BSU$ one sees that BSU is not 2-atomic.

Finally in this section, we give a calculation illustrating difficulties associated with Cartan formulae. Some details about $H^*(\Omega G, Z_2)$ for $G = G_2$ and F_4 can be found in section 5.

<u>Lemma 3.4</u> Let G be the exceptional Lie group G_2 or F_4, and $\overline{x}_2 \in H^2(\Omega G, Z_2)$ the generator. Then

$$\theta_4(\overline{x}_2^2) = \phi_4(\overline{x}_2^2) = \overline{z}_8 \in H^8(\Omega G, Z_2) \simeq Z_2 \text{ which is}$$

indecomposable; $\xi_4(\overline{x}_2^2) = 0$.

<u>Proof</u> We can choose $\xi \in K(\Omega G, Z_{(2)})_2$ with $\xi \to \overline{x}_2$ and $\psi^2(\xi) = 2\xi + \xi^2$. So $\psi^2(\xi^2) = 4\xi^2 + 4\xi^3 + \xi^4$. Now $\xi^4 \to 2z_8$ where $z_8 \to \overline{z}_8$. So $\xi^4 = 2u_8 + u_{10}$. Then $\psi^2(\xi_2^2) = 4\xi_2^2 + 4\xi_2^3 + 2u_8 \bmod K(\Omega G, Z_{(2)})_{10}$. Therefore $\theta_4(\overline{x}_2^2) = \overline{z}_8$. Also $\psi^2(4\xi_2^3 - 6\xi_2^4) = 2^6(4\xi_2^3 - 6\xi_2^4) \bmod K(\Omega G, Z_{(2)})_{10}$ giving $\psi^2(\xi_2^2) = 4\xi_2^2 + 2(2\xi_2^3 - 3\xi_2^4) + 14u_8 \bmod K(\Omega G, Z_{(2)})_{10}$. Thus $\phi_4(\overline{x}_2^2) = \overline{z}_8$ and $\xi_4(\overline{x}_2^2) = 0$. In all cases the indeterminacy is zero.

4. The atomicity of $\Omega SU(n)$ and $\Omega Sp(m)$

In the notation of section 2, $N(SU(n))_{2i} \simeq Z_{(2)}$, $1 \leq i \leq n-1$, generated by x'_{2i} say, and is zero otherwise. Let $x_{2^i} \in H^{2^i}(\Omega SU(n), Z_2)$ represent x'_{2^i}. We choose $x_{2t} \in H^{2t}(\Omega SU(n), Z_2)$ corresponding to x'_{2t} where $2t = 2^i + 2u$, $2u < 2^i$ such that $\overline{x}_{2t} = Sq^{2u}\overline{x}_{2^i}$.

Proposition 4.1 Let $f: \Omega SU(n) \to \Omega SU(n)$ be a map.
(a) If $f^*: N(SU(n))_2 \otimes Z_2 \to N(SU(n))_2 \otimes Z_2$ is an isomorphism, then f is a mod 2 homotopy equivalence.
(b) If $f^*: N(SU(n))_2 \otimes Z_2 \to N(SU(n))_2 \otimes Z_2$ is trivial, then $(f^*)^t: \overline{H}^*(\Omega SU(n), Z_2) \to \overline{H}^*(\Omega SU(n), Z_2)$ is trivial where $2^{t-1} < n \leq 2^t$.

Proof The inclusion $\Omega SU(n) \to \Omega SU = BU$ induces a homotopy equivalence of $(2n-2)$ skeletons and $BU = CP^\infty \times BSU$. Therefore we can use Lemma 3.3 to evaluate cohomology operations in $H^*(\Omega SU(n), Z_2)$ through the appropriate range. By Lemma 2.6, $f^*: N(SU(n))_{2k} \otimes Z_2 \to N(SU(n))_{2k} \otimes Z_2$ is an isomorphism when $k = 1$ if and only if it is an isomorphism when $k = 2$ whenever $n > 2$. Using Lemma 3.3 it follows that $f^*: N(SU(n)) \otimes Z_2 \to N(SU(n)) \otimes Z_2$ is either trivial or an isomorphism for all n. Thus by Proposition 2.2, $f^*: \overline{H}^*(\Omega SU(n), Z_2) \to \overline{H}^*(\Omega SU(n), Z_2)$ is either an isomorphism or is nilpotent. Part (a) now follows from Whitehead's theorem. For part (b), we have $f^*\overline{x}_2 = 0$. Assume as inductive hypotheses that $(f^*)^u\overline{x}_{2s} = 0$ for $2s < 2^{u+1}$ where $1 \leq u < t$. But $f^*\overline{x}_{2^{u+1}} = p(\overline{x}_2, \overline{x}_4, \ldots, \overline{x}_{2^{u+1}-2})$ and so $(f^*)^{(u+1)}\overline{x}_{2^{u+1}} = 0$. The choice of $\overline{x}_{2i}$ ensures that

$(f^*)^{(u+1)}\bar{x}_{2s} = 0$ for $2s < 2^{u+2}$. It follows that $(f^*)^t\colon \bar{H}^*(\Omega SU(n),Z_2) \to \bar{H}^*(\Omega SU(n),Z_2)$ is trivial by Corollary 2.4. This completes the proof of Proposition 4.1.

To complete the proof of Theorem 1.1(c) one considers $\underline{2}^{t-1}\colon \Omega SU(n) \to \Omega SU(n)$. Taking Chern classes $\{c_1,c_2,\ldots,c_{n-1}\}$ in place of $\{x_2,x_4,\ldots,x_{2(n-1)}\}$, $\phi(c_k) = \Sigma c_{k-i} \otimes c_i$ where $c_0 = 1$. So $\underline{2}^*(\bar{c}_{2^{i+1}}) = (\bar{c}_{2^i})^2$ and $\underline{2}^*(\bar{c}_i) = 0$ otherwise. Therefore $(\underline{2}^*)^{t-1}\bar{c}_{2^{t-1}} = (\bar{c}_1)^{2^{t-1}} \neq 0$. Therefore $\Omega SU(n)$ is 2-atomic of degree t.

In the symplectic case $N(Sp(m))_{4i-2} \simeq Z_{(2)}$ for $1 \le i \le m$ and is zero otherwise.

<u>Proposition 4.2</u> $\Omega Sp(m)$ is 2-atomic of degree 1.

<u>Proof</u> The suspension homomorphism $\sigma\colon \bar{H}^*(Sp(m),Z_{(2)}) \to H^{*-1}(\Omega Sp(m),Z_{(2)})$ induces an isomorphism $\sigma\colon QH^*(Sp(m),Z_{(2)}) \to PH^{*-1}(\Omega Sp(m),Z_{(2)}) \simeq N(Sp(m))$. Thus we can choose primitive classes $\{x_2,x_6,\ldots,x_{4m-2}\}$ to represent a basis of $N(Sp(m))$. By Lemma 3.2(b), $\bar{x}_{2t} = Sq^{2u}\theta_{2^s}\theta_{2^{s-1}}\ldots\theta_4\bar{x}_2$ with zero indeterminacy, where $t \equiv 1 \bmod 2$. The result follows again applying Proposition 2.2 and Corollary 2.4.

5. The atomicity of the exceptional Lie groups

The sources of the information listed below are [2,15] and particularly [12] combined with elementary Hopf algebra calculations. Notations are explained at the end of the list.

(a) G_2: $N(G)_{2i} \simeq Z_{(2)}$ for $i = 1$ and 5 and is zero otherwise. In $H^*(\Omega G_2, Z_2)$, $Sq^2\bar{z}_8 = \bar{x}_{10}$.

(b) F_4: $N(G)_{2i} \simeq Z_{(2)}$ for $i = 1,5,7$ and 11 and is zero otherwise. In $H^*(\Omega F_4, Z_2)$, $Sq^2\bar{z}_8 = \bar{x}_{10}$, $Sq^4\bar{x}_{10} = \bar{x}_{14}$ and $Sq^8\bar{x}_{14} = \bar{x}_{22}$.

(c) E_6: $N(E_6)_{2i} \simeq Z_{(2)}$ for $i = 1,4,5,7,8$ and 11 and is zero otherwise. In $H^*(\Omega E_6, Z_2)$, $Sq^2\bar{x}_8 = \bar{x}_{10}$, $Sq^4\bar{x}_{10} = \bar{x}_{14}$, $Sq^8\bar{x}_{14} = \bar{x}_{22}$, $Sq^4\bar{x}_8 = \bar{x}_2^6 \neq 0$, $Sq^2\bar{x}_{16} = \bar{x}_8\bar{x}_{10}$.

(d) E_7: $N(E_7)_{2i} \simeq Z_{(2)}$ for $i = 1,5,7,9,11,13$ and 17 and is zero otherwise. In $H^*(\Omega E_7, Z_2)$, $Sq^4\bar{x}_{10} = \bar{x}_{14}$, $Sq^8\bar{x}_{10} = \bar{x}_{18}$, $Sq^8\bar{x}_{14} = \bar{x}_{22}$, $Sq^4\bar{x}_{22} = \bar{x}_{26}$, $Sq^{16}\bar{x}_{18} = \bar{x}_{34}$ and $Sq^2\bar{x}_{14} = \bar{x}_2^8 \neq 0$.

(e) E_8: $N(E_8)_{2i} \simeq Z_{(2)}$ for $i = 1,7,11,13,17,19,23$ and 29 and is zero otherwise. In $H^*(\Omega E_8, Z_2)$, $Sq^8\bar{x}_{14} = \bar{x}_{22}$, $Sq^4\bar{x}_{22} = \bar{x}_{26}$, $Sq^4\bar{x}_{34} = \bar{x}_{38}$, $Sq^8\bar{x}_{38} = \bar{x}_{46}$, $Sq^2\bar{x}_{14} = \bar{x}_2^8 \neq 0$, $Sq^2\bar{z}_{32} = \bar{x}_{34}$ and $Sq^2\bar{z}_{56} = \bar{x}_{58}$.

We choose classes $x_{2i} \in H^{2i}(\Omega G, Z_{(2)})$ to represent a basis of $N(G)$ as before. With two exceptions, for Hopf algebraic reasons, we can choose x_{2i} to be primitive in $H^*(\Omega G, Z_{(2)})$. The exceptions are x_8 and x_{16} in $H^*(\Omega E_6, Z_{(2)})$; these are chosen so that x_2, x_8 and x_{16} generate a sub-Hopf algebra of $H^*(\Omega E_6, Z_{(2)})$, [9]. Then $\bar{\phi}(\bar{x}_8) = \bar{x}_2^2 \otimes \bar{x}_2^2$ and $\bar{\phi}(\bar{x}_{16}) = \bar{x}_8 \otimes \bar{x}_8 + \bar{x}_2^2 x_8 \otimes \bar{x}_2^2$

$+ \bar{x}_2^2 \otimes \bar{x}_2^2\bar{x}_8 + \bar{x}_2^4 \otimes \bar{x}_8 + \bar{x}_8 \otimes \bar{x}_2^2 + \varepsilon\bar{x}_2^4 \otimes \bar{x}_2^4$; we choose $\varepsilon = 0$.

In $H^*(\Omega G, Z_{(2)})$ with $G = G_2$ or F_4, $x_2^4 = 0 \bmod 2$ and $z_8 = 2^{-1}x_2^4$.

In $H^*(\Omega E_8, Z_{(2)})$, $x_2^{16} = 0 \bmod 2$ and $z_{32} = 2^{-1}x_2^{16}$, $x_{14}^4 = 0 \bmod 2$ and $z_{56} = 2^{-1}x_{14}^4$.

<u>Proposition 5.1</u> Let $G = G_2, F_4, E_7$ or E_8. Then G is 2-atomic of degree 1.

<u>Proof</u> Assume first that $f\colon \Omega G \to \Omega G$ is given inducing the identity homomorphism $f^*\colon H^2(\Omega G, Z_2) \to H^2(\Omega G, Z_2)$. We check that $f^*\bar{x}_{2i} = \bar{x}_{2i}$ for each i. If $G = G_2$ or F_4, it follows that $f^*\bar{z}_8 = \bar{z}_8$ and so in these cases $f^*\bar{x}_{10} = \bar{x}_{10}$ and when $G = F_4$, $f^*\bar{x}_{14} = \bar{x}_{14}$ and $f^*\bar{x}_{22} = \bar{x}_{22}$. Therefore $f\colon \Omega G \to \Omega G$ with $G = G_2$ or F_4 is a homotopy equivalence using Proposition 2.2(a).

If $G = E_7$, it is sufficient to check that $f^*\bar{x}_{10} = \bar{x}_{10}$. Now $f^*\bar{x}_{10} = \varepsilon\bar{x}_{10} + \delta\bar{x}_2^5$. But $Sq^2\bar{x}_{10} = 0$ as $PH^{12}(\Omega E_7, Z_2) = 0$ and $Sq^2\bar{x}_2^5 = \bar{x}_2^6 \neq 0$ and so $\delta = 0$. As $Sq^6\bar{x}_{10} = Sq^2Sq^4\bar{x}_{10} = \bar{x}_2^8 \neq 0$, $f^*\bar{x}_{10} = \bar{x}_{10}$. Hence $f\colon \Omega E_7 \to \Omega E_7$ is a mod 2 homotopy equivalence.

When $G = E_8$, it is sufficient to show that $f^*\bar{x}_{14} = \bar{x}_{14}$. Now $f^*\bar{x}_{14} = \varepsilon\bar{x}_{14} + \delta\bar{x}_2^7$. But $Sq^4\bar{x}_{14} \in PH^{18}(\Omega E_8, Z_2) = 0$ and $Sq^4\bar{x}_2^7 = \bar{x}_2^9 \neq 0$, so $f^*x_{14} = \bar{\varepsilon}x_{14}$. But $Sq^2\bar{x}_{14} = \bar{x}_2^8$ and so $f^*\bar{x}_{14} = \bar{x}_{14}$ which completes the proof for E_8.

Now suppose that $f^*\colon N(G)_2 \otimes Z_2 \to N(G)_2 \otimes Z_2$ is trivial. One can argue as above and establish that $f^*\bar{x}_{2i} = 0$. As a variant, consider $g = 1 + f\colon \Omega G \to \Omega G$. Then $g^*\bar{x}_2 = \bar{x}_2$ and so $g^*\bar{x}_{2i} = \bar{x}_{2i}$ in all cases. But if $\bar{x}_{2t}$ is a lowest dimensional element with $f^*\bar{x}_{2t} = \bar{w}_{2t} \neq 0$, $g^*\bar{x}_{2t} = \bar{x}_{2t} + \bar{w}_{2t}$, which is a contradiction. So $f^*\bar{x}_{2i} = 0$ in all cases and by Corollary 2.4, $f^*: H^*(\Omega G, Z_2) \to H^*(\Omega G, Z_2)$ is trivial.

Comments 1. The proof of Proposition 5.1 establishes Theorem 1.2(a) as well as Theorem 1.1(a). The only properties of $f: \Omega G \to \Omega G$ used in the proof are that $f^* \in \mathcal{I}(G)$.

2. Let X be as in the final sentence of paragraph 2 of the introduction. One can establish the equivalence of definitions mentioned there using an argument similar to that of the final part of the proof of Proposition 5.1. The key fact is that if $f^*: H^{2n}(X, Z_p) \to H^{2n}(X, Z_p)$ is not nilpotent, there exists $\bar{w} \neq 0$ in $H^{2n}(X, Z_p)$ such that $(f^t)^*\bar{w} = \bar{w}$. Alternatively one can use results from [16] for suitable X.

Proposition 5.2 ΩE_6 is 2-atomic of degree 3.

Proof Let $f: \Omega E_6 \to \Omega E_6$ satisfy $f^*\bar{x}_2 = \bar{x}_2$. We show first that $f^*\bar{x}_8 = \bar{x}_8 + \varepsilon\bar{x}_2^4$, which implies by (c) above that $f^*\bar{x}_{2i} = \bar{x}_u$ for $2i = 10, 14$ and 22. For dimensional reasons $f^*\bar{x}_8 = \delta\bar{x}_8 + \varepsilon\bar{x}_2^4$. But $Sq^4\bar{x}_8 = \bar{x}_2^6$ and $Sq^4\bar{x}_2^4 = 0$. So $\delta = 1$.

Let $f^*\bar{x}_{16} = \alpha\bar{x}_{16} + \beta\bar{x}_8^2 + \gamma\bar{x}_2\bar{x}_{14} + \delta\bar{x}_2^3\bar{x}_{10} + \theta\bar{x}_2^4\bar{x}_8 + \zeta\bar{x}_2^8$. Applying Sq^2, we deduce that $f^*(\bar{x}_8\bar{x}_{10}) = \alpha\bar{x}_8\bar{x}_{10} + \gamma\bar{x}_2^2\bar{x}_{14} + \delta\bar{x}_2^4\bar{x}_{10} + \theta\bar{x}_2^4\bar{x}_{10}$. As $f^*(\bar{x}_8\bar{x}_{10}) = (\bar{x}_8 + \varepsilon\bar{x}_2^4)\bar{x}_{10}$, it follows that $\alpha = 1$, $\gamma = 0$ and $\varepsilon = \delta + \theta$. Therefore $f^*\bar{x}_{16} = \bar{x}_{16} + \bar{d}_{16}$ where $\bar{d}_{16}$ is decomposable. So by Proposition 2.2, $f^*: H^*(\Omega E_6, Z_2) \to H^*(\Omega E_6, Z_2)$ is an isomorphism and f is a mod 2 homotopy equivalence.

In a similar manner it follows that if $f^*\bar{x}_2 = 0$, then $f^*\bar{x}_8 = \varepsilon\bar{x}_2^4$, $f^*x_{2i} = 0$ for $2i = 10, 14, 22$ and in the expression for $f^*\bar{x}_{16}$ above, $\alpha = 0$, $\gamma = 0$ and $\varepsilon = \delta + \theta$. Thus $(f^*)^2\bar{x}_{2i} = 0$, $2i \neq 16$ and $(f^*)^2\bar{x}_{16} = \beta\varepsilon\bar{x}_2^8$. So by Corollary

2.4, $(f^*)^3 \colon \bar{H}^*(\Omega E_6, Z_2) \to \bar{H}^*(\Omega E_6, Z_2)$ is zero.

Conversely we consider the loop power maps on ΩE_6. Then $\underline{2}^*(\bar{x}_{2i}) = 0$ for $i = 2, 10, 14$ and 22, $\underline{2}^*(\bar{x}_8) = \bar{x}_2^4$ and $2^*(\bar{x}_{16}) = \bar{x}_8^2$. So $(\underline{2}^*)^2(\bar{x}_{16}) = \bar{x}_2^8 \neq 0$, which establishes Proposition 5.2 and Theorem 1.1(b).

The proof of Theorem 1.2(b) involves further calculations and the fact that $x_{10}^2 = 0 \bmod 2$. Then Hopf algebraic considerations imply that

(c') $Sq^4\bar{x}_{16} = \bar{x}_8\bar{x}_2^6 + \bar{z}_{20}$, where $z_{20} = 2^{-1}x_{10}^2$ and

$Sq^8\bar{x}_{16} = \bar{x}_8^3 + \bar{x}_8^2\bar{x}_2^4 + \bar{x}_8\bar{x}_2^8$.

As in the proof of Proposition 5.2, let $f^*\bar{x}_8 = \bar{x}_8 + \varepsilon\bar{x}_2^4$ and so $f^*\bar{x}_{16} = \bar{x}_{16} + \beta\bar{x}_8^2 + \delta\bar{x}_2^3\bar{x}_{10} + \theta\bar{x}_2^4\bar{x}_8 + \zeta\bar{x}_2^8$, where $\varepsilon = \delta + \theta$. Applying Sq^4 we deduce that $\varepsilon = \theta$ and $\delta = 0$. So $f^*\bar{x}_{16} = \bar{x}_{16} + \beta\bar{x}_8^2 + \varepsilon\bar{x}_2^4\bar{x}_8 + \zeta\bar{x}_2^8$. Applying Sq^8, it follows that $\beta = \varepsilon$. So if $f^*\bar{x}_8 = \bar{x}_8 + \bar{x}_2^4$, then $f^*\bar{x}_{16} = \bar{x}_{16} + \bar{x}_8^2 + \bar{x}_2^4\bar{x}_8 + \zeta\bar{x}_2^8$ where $\zeta = 0$ or 1, and if $f^*\bar{x}_8 = \bar{x}_8$ then $f^*\bar{x}_{16} = \bar{x}_{16} + \zeta\bar{x}_2^8$ where $\zeta = 0$ or 1. In all cases $f^*\bar{x}_{2i} = \bar{x}_{2i}$ otherwise.

Similar computations show that if $f^*\bar{x}_8 = \bar{x}_2^4$, then $f^*\bar{x}_{16} = \bar{x}_8^2 + \zeta\bar{x}_2^8$ and if $f^*\bar{x}_8 = 0$ then $f^*\bar{x}_{16} = \zeta\bar{x}_2^8$ where $\zeta = 0$ or 1; $f^*\bar{x}_{22} = 0$ for the other generators.

Considering the power maps on ΩE_6, one checks that $f^*\bar{x}_{2i} = \underline{r}^*(\bar{x}_{2i})$ for all i where $0 \leq r < 8$ in each case described above and $(\underline{8})^*(\bar{x}_{2i}) = (\underline{0}^*)(\bar{x}_{2i})$. Suppose that $f^*\bar{x}_{2i} = (\underline{s}^*)(\bar{x}_{2i})$ for all i where $0 \leq s < 8$ and let t satisfy $t > 0$, $s + t = 8$. Then $(f+\underline{t})^*(\bar{x}_{2i}) = 0$ for all i and so by Corollary 2.4, $(f+\underline{t})^* \colon \bar{H}^*(\Omega E_6, Z_2) \to \bar{H}^*(\Omega E_6, Z_2)$ is trivial. Therefore on $H^*(\Omega E_6, Z_2)$, $f^* = (f+\underline{8})^* = (f + (\underline{t}+\underline{r}))^*$ $= ((f+\underline{t}) + \underline{r})^* = (\underline{0}+\underline{r})^* = \underline{r}^*$. This completes the proof of Theorem 1.2(b).

6. The proof of Theorem 1.3

Let $G = G(SU) \times G(Sp) \times G(G_2) \times \ldots \times G(E_8)$ where $G(SU)$ is a product of special unitary groups $SU(n_1) \times SU(n_2) \times \ldots \times SU(n_s)$, etc, and the exceptional Lie groups are ordered by increasing rank. Let $f\colon \Omega G \to \Omega G$ induce an isomorphism $f^*\colon N(G)_2 \otimes Z_2 \to N(G)_2 \otimes Z_2$ which as usual we can assume to be the identity. We choose elements in $H^*(\Omega G, Z_{(2)})$ to represent a basis of $N(G) = N(SU(n_1)) \oplus N(SU(n_2)) \oplus \ldots \oplus N(E_8)$ as was done previously. Then $f^*\colon N(G)_{2i} \otimes Z_2 \to N(G)_{2i} \otimes Z_2$ can be represented by a matrix A_{2i} which we must show is non singular, if it is not trivial, and then by iteration we can assume that it is the identity matrix.

The arguments of Proposition 4.2, Proposition 5.1 and Proposition 5.2, and in the unitary case Comment 1 following Lemma 2.6 together with Proposition 4.1 ensure that f_H, the composition

$$\Omega G(H) \xrightarrow{i} \Omega G \xrightarrow{f} \Omega G \xrightarrow{p} \Omega G(H)$$

induces an isomorphism $f_H^*\colon N(G(H))_* \otimes Z_2 \to N(G(H))_* \otimes Z_2$, where H is $SU, Sp, \ldots,$ or E_8, i is the inclusion and p the projection. It follows that, after iteration, each A_{2i} has identity sub-matrices symmetrically positioned about the main diagonal. So to show that each A_{2i} is non singular it is sufficient to show that it is triangular.

Now $N(G)_4 = N(G(SU))_4$ and using Lemma 3.3 and the comments which follow, one deduces that the restriction of f^*,

$$f^*\colon N(G(SU))_* \otimes Z_2 \to N(G)_* \otimes Z_2$$

is the standard inclusion; in dimensions 8 and 16 one uses in addition that 8 and 16 dimensional indecomposable elements of

$H^*(\Omega E_6, Z_2)$ are not in the image of Sq^2. Similarly when H is Sp, G_2 or F_4, the restriction of f*

$$f^*: N(G(H))_* \otimes Z_2 \to N(G)_* \otimes Z_2$$

is the inclusion.

Let $H = E_6$. Then $N(G)_8 = N(G(SU))_8 \oplus N(G(E_6))_8$. Therefore A_8 is a triangular matrix and so non singular. So we can assume that $f^*: N(G(E_6))_{2i} \otimes Z_2 \to N(G)_{2i} \otimes Z_2$ is an inclusion except possibly when 2i = 16.

If $H = E_7$, $N(G)_{10} = N(G(SU))_{10} \oplus N(G(Sp))_{10} \oplus N(G(G_2))_{10}$
$\oplus N(G(F_4))_{10} \oplus N(G(E_6))_{10}$

and so A_{10} is triangular and therefore non singular. Therefore we can assume that the restriction $f^*: N(G(E_7))_* \otimes Z_2 \to N(G)_* \otimes Z_2$ is the inclusion. A similar argument applies for $H = E_8$ considering first $N(G(E_8))_{14}$ and the conclusion is that $f^*: N(G(E_8))_* \otimes Z_2 \to N(G(E_8))_* \otimes Z_2$ is the standard inclusion.

Finally $N(G)_{16} = N(G(SU))_{16} \oplus N(G(E_6))_{16}$ and so A_{16} is triangular and

$$f^*: N(G(E_6))_* \otimes Z_2 \to N(G)_* \otimes Z_2 \text{ is the inclusion.}$$

Thus $f^*: N(G)_* \otimes Z_2 \to N(G_*) \otimes Z_2$ is the identity and the theorem follows from Proposition 2.2(a).

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