K-energy Maps Integrating Futaki Invariants

by

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Dedicated to the memory of the late Professor Takehiko Miyata

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§0. INTRODUCTION:

One of the long-standing questions in complex differential geometry is the following: given a compact complex connected manifold V with $c_1(V) > 0$, can one find a simple criterion of the existence of an Einstein Kähler metric on V ? At present, there are no definitive answers, but the following conjecture of A. Futaki [3] seems to be reasonable.

(I) GENERALIZED CALABI'S CONJECTURE (FUTAKI): Suppose that the identity component $Aut^0(V)$ of the group of holomorphic automorphisms of V is a reductive algebraic group. If futhermore to each holomorphic vector field on V, the corresponding Futaki invariant vanishes, then V admits an Einstein Kähler metric.

On the other hand, as a characterization of Einstein Hermitian vector bundles on compact Kähler manifolds, S. Kobayashi [5] raised the following:

(II) KOBAYASHI'S CONJECTURE: Let E be an indecomposable holomorphic vector bundle on a compact Kähler manifold W with Kähler metric g_0. Then E admits an Einstein Hermitian metric if and only if E is stable (in the sense of Mumford-Takemoto) with respect to g_0.

Recently, S.K. Donaldson [2] solved (II) for the case where W is a projective algebraic surface. One crucial step of his proof is a construction of a non-linear functional λ from the set of all Hermitian metrics on E to the real numbers such that (1) any critical point of λ is exactly an Einstein Hermitian metric on E and that (2) λ is bounded from below if and only if E is semistable with respect to g.

Although (I) and (II) look quite different, there is some link between these conjectures. Actually even for (I), the same procedure as in Donaldson's work can considerably be carried out.

Fix a Kähler form $\omega_0 = \sqrt{-1}\ \Sigma g_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}}$ on V. In this introduction, for simplicity, we assume that ω_0 represents $2\pi c_1(V)_{\mathbf{R}}$. We denote by K the set of all Kähler forms on V cohomologous to ω_0. Let f_0 be a real valued C^{∞} function on V which is uniquely determined, up to constant, by the equation

$$\bar{\partial}\partial \log\det(g_{\alpha\bar{\beta}}) - \Sigma g_{\alpha\bar{\beta}} dz^{\alpha} \wedge dz^{\bar{\beta}} = \partial\bar{\partial} f_0 \quad .$$

The main purpose of this paper is to prove the following theorem announced earlier in [6].

<u>THEOREM</u>: <u>There exists a mapping</u> $\mu : K \longrightarrow \mathbf{R}$ <u>satisfying the following conditions</u>:

(i) <u>An element</u> ω <u>of</u> K <u>is a critical point of</u> μ <u>if and if</u> ω <u>is Einstein Kähler</u>, (cf. §3).

(ii) Let Y be a holomorphic vector field on V, and ω be an element of K. Put $Y_{\mathbf{R}} := Y + \overline{Y}$ and $y_t := \exp tY_{\mathbf{R}}$ for $t \in \mathbf{R}$. Then $\mu(y_t{}^*\omega)$ is a linear function of t. Namely for every t,

$$\frac{d}{dt}\mu(y_t{}^*\omega) = \int_V (Y_{\mathbf{R}} f_0)\,\omega_0{}^n \Big/ \int_V \omega_0{}^n ,$$

where the right-hand side is the Futaki invariant of V corresponding to the holomorphic vector field Y, (cf. §5).

(iii) If ω is a critical point of μ, then the inequality

$$\frac{d^2}{dt^2}\mu(\theta_t)\Big|_{t=0} \geq 0$$

holds for every smooth path $\{\theta_t \mid -\varepsilon \leq t \leq \varepsilon\}$ in K such that $\theta_0 = \omega$, (cf. §6).

This $\mu : K \longrightarrow \mathbf{R}$ is called the K-energy map of the Kähler manifold (V,ω_0). In view of (i) and (ii) above, one can easily see that if $\int_V (Y_{\mathbf{R}} f_0)\,\omega_0{}^n \neq 0$ for some holomorphic vector field Y on V, then μ cannot have a critical point, i.e., X does not admit any Einstein Kähler metric, which gives another proof of a fundamental theorem of Futaki [3]. Furthermore (i) and (iii) above gives us some indication that Conjecture (I) can be weakened in the following more plausible form.

(III) CONJECTURE: Suppose that $\mathrm{Aut}^0(V)$ is a reductive algebraic group. If μ is bounded from below, then V admits an Einstein Kähler metric.

Several supplements to this paper will be found in [7]. In a forthcoming paper (cf. S. Bando and T. Mabuchi [1]), we shall show the following interesting theorem.

THEOREM: Let E be the set of all Einstein Kähler forms in K, and K^+ be the set of all $\omega \in K$ with positive definite Ricci tensor. Assume that $E \neq \emptyset$. Then

(i) the restriction $\mu|_{K^+} : K^+ \longrightarrow \mathbf{R}$ is bounded from below, and $\mu|_{K^+}$ takes its absolute minimum on E.

(ii) For any ω_1 and ω_2 in E, there exists an element g of $\mathrm{Aut}^0(V)$ such that $g^*\omega_2 = \omega_1$.

We shall also give several generalizations of μ in the latter paper.

In conclusion, I wish to thank all those people who encouraged me and gave me suggestions, and in particular Professors S. Kobayashi and H. Ozeki, and Doctors S. Bando I. Enoki and R. Kobayashi, who helped me again and again during the preparation of this paper. My hearty thanks go also to the Max-Planck-Institut für Mathematik for its hospitality and constant assistance all through my stay in Bonn.

§1. NOTATION AND CONVENTION:

Throughout this paper we fix an arbitrary n-dimensional compact Kähler connected manifold X with Kähler form $\omega_0 = \sqrt{-1}\,\Sigma g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$, where ω_0 is written in terms of holomorphic local coordinates $(z^1, z^2, \ldots, z^n)$. Let

$$K := \left\{ \omega \;\middle|\; \begin{array}{l} \text{Kähler form on } X \text{ which is cohomologous} \\ \text{to } \omega_0 \text{ in } H^{1,1}(X,\mathbf{R}) \end{array} \right\} .$$

For each element $\omega = \sqrt{-1}\,\Sigma g(\omega)_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$ of K, we denote by $\Sigma R(\omega)_{\alpha\bar{\beta}} dz^\alpha \otimes dz^{\bar{\beta}}$ the corresponding Ricci tensor. We put $R(\omega) := \sqrt{-1}\,\Sigma R(\omega)_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$. Then $R(\omega)/2\pi$ represents $c_1(X)_{\mathbf{R}}$ and we have $R(\omega) = \sqrt{-1}\,\bar{\partial}\partial \log \det (g_{\alpha\bar{\beta}}(\omega))$. Furthermore, let $\sigma(\omega)$ (resp. $\square_\omega$) be the corresponding scalar curvature (resp. Laplacian on functions):

$$\sigma(\omega) := \Sigma\, g(\omega)^{\bar{\beta}\alpha} R(\omega)_{\alpha\bar{\beta}} ,$$

$$\square_\omega := \Sigma\, g(\omega)^{\bar{\beta}\alpha} \partial^2/\partial z^\alpha \partial z^{\bar{\beta}} ,$$

where $(g(\omega)^{\bar{\beta}\alpha})$ is the inverse matrix of $(g(\omega)_{\alpha\bar{\beta}})$. For each real valued C^∞ function $\varphi \in C^\infty(X)_{\mathbf{R}}$ on X, we put $\omega_0(\varphi) := \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi$, and let

$$H := \{\varphi \in C^\infty(X)_{\mathbf{R}} \mid \omega_0(\varphi) \in K\} .$$

Note that the natural map

$$\begin{aligned} H &\longrightarrow K \\ \varphi &\longmapsto \omega_0(\varphi) \end{aligned}$$

is surjective. For each $\varphi \in H$, the corresponding $\square_{\omega_0(\varphi)}$, $\sigma(\omega_0(\varphi))$, $R(\omega_0(\varphi))$, $R(\omega_0(\varphi))_{\alpha\bar\beta}$, $g(\omega_0(\varphi))_{\alpha\bar\beta}$, $g(\omega_0(\varphi))^{\bar\beta\alpha}$ will be denoted simply by $\square_\varphi$, $\sigma(\varphi)$, $R(\varphi)$, $R(\varphi)_{\alpha\bar\beta}$, $g(\varphi)_{\alpha\bar\beta}$, $g(\varphi)^{\bar\beta\alpha}$ respectively.

<u>DEFINITION (1.1)</u>: A 1-parameter family $\{\varphi_t | a \le t \le b\}$ of functions in $C^\infty(X)_{\mathbf{R}}$ is said to be <u>smooth</u> (or a <u>smooth path</u>) if the mapping

$$\begin{aligned} [a,b] \times X &\longrightarrow \mathbf{R} \\ (t,\ x) &\longmapsto \varphi_t(x) \end{aligned}$$

is C^∞. We then put $\dot\varphi_t := \partial\varphi_t/\partial t$ and $\ddot\varphi_t = \partial^2\varphi_t/\partial t^2$.

<u>DEFINITION (1.2)</u>: We define the real constants λ and ν by

$$\lambda := 2n\pi \int_X c_1(X)\omega_0^{n-1} / \int_X \omega_0^n, \quad \nu := \lambda/n \quad .$$

Furthermore, to each $\varphi \in C^\infty(X)_{\mathbf{R}}$, we associate an (n,n)-form $V_0(\varphi)$ on X as follows:

$$V_0(\varphi) := \omega_0(\varphi)^n / \int_X \omega_0^n \ .$$

This is normalized so that $\int_X V_0(\varphi) = 1$. Moreover, if ω_0 represents $2\pi c_1(X)_{\mathbf{R}}$, then $\lambda = n$.

DEFINITION (1.3): Let $(z^1, z^2, \ldots, z^n)$ be a system of holomorphic local coordinates on X . For every $f \in C^\infty(X)_{\mathbf{R}}$, we use the following notation:

$$f_\alpha := \partial_\alpha f \ , \ f_{\bar\alpha} := \partial_{\bar\alpha} f \ , \ f_{\alpha\beta} := \partial_\alpha \partial_\beta f \ , \ f_{\bar\alpha\bar\beta} := \partial_{\bar\alpha}\partial_{\bar\beta} f \ ,$$
$$f_{\alpha\bar\beta} := \partial_\alpha \partial_{\bar\beta} f \ , \ f_{\alpha\beta\bar\gamma} := \partial_\alpha \partial_\beta \partial_{\bar\gamma} f \ , \ldots,$$

where we denote by ∂_α (resp. $\partial_{\bar\alpha}, \partial_\beta, \partial_{\bar\beta}, \partial_{\bar\gamma}$) the operator $\partial/\partial z^\alpha$ (resp. $\partial/\partial z^{\bar\alpha}$, $\partial/\partial z^\beta$, $\partial/\partial z^{\bar\beta}$, $\partial/\partial z^{\bar\gamma}$) . Our notation is slightly different from the ordinary one, because for instance, $f_{\alpha\beta}$ is not $\nabla_\beta \nabla_\alpha f$.

§2. Basic Constructions:

This section is a crucial step in the construction of the K-energy map μ. We shall introduce the mappings

$$L : C^{\infty}(X)_{\mathbb{R}} \times C^{\infty}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad (\text{cf. } (2.5)),$$
$$M : \quad H \times H \longrightarrow \mathbb{R}, \quad (\text{cf. } (2.4)),$$

where the latter immediately defines μ, (cf. (2.7), (3.1)). Although the functional L is not essential in later sections, it none the less plays an important role in our forthcoming papers (cf. Mabuchi [7], Bando and Mabuchi [1]).

DEFINITION (2.1): Let S be a non-empty set and A be an additive group. Then a mapping $N : S \times S \longrightarrow A$ is said to satisfy the 1-cocycle condition if

$$\text{i)} \quad N(\sigma_1,\sigma_2) + N(\sigma_2,\sigma_1) = 0 \quad \text{and}$$
$$\text{ii)} \quad N(\sigma_1,\sigma_2) + N(\sigma_2,\sigma_3) + N(\sigma_3,\sigma_1) = 0$$

for all $\sigma_1, \sigma_2, \sigma_3 \in S$.

DEFINITION (2.2). For every $(\varphi', \varphi'') \in H \times H$, we define real numbers $L(\varphi',\varphi'')$, $M(\varphi',\varphi'')$ by

$$(2.2.1) \qquad L(\varphi',\varphi'') := \int_a^b \left(\int_X \dot{\varphi}_t \, V_0(\varphi_t) \right) dt$$

$$M(\varphi',\varphi'') := -\int_a^b \left\{ \int_X \dot{\varphi}_t (\sigma(\varphi_t) - \lambda)\, V_0(\varphi_t) \right\} dt \ , \tag{2.2.2}$$

where $\{\varphi_t | a \leq t \leq b\}$ is an arbitrary piecewise smooth path in H such that $\varphi_a = \varphi'$ and $\varphi_b = \varphi''$.

THEOREM (2.3). $L(\varphi',\varphi'')$ above is independent of the choice of the path $\{\varphi_t | a \leq t \leq b\}$ and therefore well-defined. Moreover,

(2.3.1) L satisfies the 1-cocycle condition, and

(2.3.2) $L(\varphi_1, \varphi_2 + C) = L(\varphi_1,\varphi_2) + C$ for all φ_1,φ_2 and all $C \in \mathbb{R}$.

THEOREM (2.4). $M(\varphi',\varphi'')$ above is independent of the choice of the path $\{\varphi_t | a \leq t \leq b\}$ and therefore well-defined. Moreover,

(2.4.1) M satisfies the 1-cocycle condition, and

(2.4.2) $M(\varphi_1 + C_1, \varphi_2 + C_2) = M(\varphi_1,\varphi_2)$ for all φ_1,φ_2 and all $C_1, C_2 \in \mathbb{R}$.

PROOF of (2.3): Let $\psi(s,t) := s\varphi_t$ for $(s,t) \in [0,1]\times[a,b]$. Since $\{\varphi_t | a \leq t \leq b\}$ is piecewise smooth, there exist a partition $a = a_0 < a_1 < a_2 < \cdots < a_r = b$ of the interval $[a,b]$ such that $\{\varphi_t \mid a_{i-1} \leq t \leq a_i\}$ is smooth for each $i \in \{1,2,\ldots,r\}$.

Step 1: We shall first show that

$$(*)_i \quad \int_{a_{i-1}}^{a_i}\left(\int_X \dot{\varphi}_t V_0(\varphi_t)\right)dt = \int_0^1\left(\int_X \frac{\partial\psi}{\partial s} V_0(\psi)\right)ds \Big|_{t=a_{i-1}}^{t=a_i}$$

Let $\Psi(s,t) := \left(\int_X \frac{\partial\psi}{\partial s}\,\omega_0(\psi)^n \; ds\right) + \left(\int_X \frac{\partial\psi}{\partial t}\,\omega_0(\psi)^n\right)dt$. Then in view of Figure 1, we have

$$\int_\Delta d\,\Psi = \int_{\partial\Delta} \Psi = \textstyle\sum_{i=1}^4 \int_{\gamma_i} \Psi$$

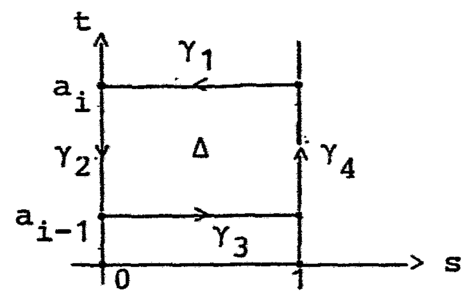

Figure 1.

$$= -\int_0^1\left(\int_X \frac{\partial\psi}{\partial s} V_0(\psi)\right)ds\Big|_{t=a_{i-1}}^{t=a_i} + \int_{a_{i-1}}^{a_i}\left(\int_X \dot{\varphi}_t V_0(\varphi_t)\right)dt \; .$$

Therefore the proof of $(*)_i$ is reduced to showing $d\Psi = 0$. By routine computations, we have

$$d\Psi = dt\wedge ds \int_X \left\{\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial s} V_0(\psi)\right) - \frac{\partial}{\partial s}\left(\frac{\partial\psi}{\partial s} V_0(\psi)\right)\right\}$$

$$= \sqrt{-1}\, dt\wedge ds \int_X \left\{\frac{\partial\psi}{\partial s}\,\partial\bar\partial\left(\frac{\partial\psi}{\partial t}\right) + \frac{\partial\psi}{\partial t}\,\bar\partial\partial\left(\frac{\partial\psi}{\partial s}\right)\right\}\wedge n\omega_0(\psi)^{n-1}\Big/\int_X \omega_0^n$$

$$= \sqrt{-1}\, dt\wedge ds \int_X \left\{-\partial\left(\frac{\partial\psi}{\partial s}\right)\wedge\bar\partial\left(\frac{\partial\psi}{\partial t}\right) - \bar\partial\left(\frac{\partial\psi}{\partial t}\right)\wedge\partial\left(\frac{\partial\psi}{\partial s}\right)\right\}\wedge n\,\omega_0(\psi)^{n-1}\Big/\int_X \omega_0^n$$

$$= 0 \; .$$

Step 2: Adding up the equalities $(*)_i$ $(i=1,2,\ldots,r)$, we obtain

$$\int_a^b \left(\int_X \dot{\varphi}_t V_0(\varphi_t) \right) dt = \int_0^1 \left(\int_X \varphi V_0(s\varphi) \right) ds \Big|_{\varphi=\varphi'}^{\varphi=\varphi''} .$$

This shows that $\int_a^b \left(\int_X \dot{\varphi}_t V_0(\varphi_t) \right) dt$ is independent of the choice of the path $\{\varphi_t \mid a \leq t \leq b\}$. (2.3.1) is also immediate. For (2.3.2), let $\psi_t := \varphi_2 + tC$ $(t \in [0,1])$. Then in view of (2.3.1),

$$L(\varphi_1, \varphi_2 + C) - L(\varphi_1, \varphi_2) = L(\varphi_2, \varphi_2 + C) = \int_0^1 \int_X C V_0(\psi_t)\, dt = C .$$

REMARK (2.5). The above proof is valid even in the case $(\varphi', \varphi'') \in C^\infty(X)_{\mathbb{R}} \times C^\infty(X)_{\mathbb{R}}$. Hence L naturally extends to a functional on $C^\infty(X)_{\mathbb{R}} \times C^\infty(X)_{\mathbb{R}}$. This extended functional (denoted by the same L) can still be defined by (2.2.1) and satisfies (2.3.1) and (2.3.2).

For the proof of (2.4), we need the following Lemma:

LEMMA (2.6). *Suppose that a 2-parameter family* $\{\psi(s,t) \mid (s,t) \in [0,1] \times [a,b]\}$ *of functions in* H *is smooth in the sense that the mapping*

$$[0,1] \times [a,b] \times X \longrightarrow \mathbb{R}$$
$$(s, t, x) \longmapsto (\psi(s,t))(x)$$

is C^∞. *Then there exists a* C^∞ *function* $F = F(s,t,x) \in C^\infty([0,1] \times [a,b] \times X)_{\mathbb{R}}$ *(which is of course unique) such that*

(i) $\partial F/\partial s = -(\square_\psi + \nu)\,(\partial \psi/\partial s)$,

(ii) $\partial F/\partial t = -(\square_\psi + \nu)\ (\partial\psi/\partial t)$,

(iii) $F|_{(s,t) = (0,a)} = 0$ <u>in</u> $C^\infty(X)_{\mathbb{R}}$, and

(iv) $R(\psi) - \nu\omega_0(\psi) = R(\omega_1) - \nu\omega_1 + \sqrt{-1}\ \partial\bar\partial F$,

where we put $\omega_1 := \omega_0(\psi(0,a))$.

<u>PROOF</u>: Using the notation in (1.3), we have

$$(2.6.1)\quad (\partial/\partial t)\ (\square_\psi(\partial\psi/\partial s)) - (\partial/\partial s)\ (\square_\psi(\partial\psi/\partial t))$$

$$= (\partial/\partial t)\ (\Sigma g(\psi)^{\bar\delta\gamma}(\partial\psi/\partial s)_{\gamma\bar\delta}) - (\partial/\partial s)\ (\Sigma g(\psi)^{\bar\alpha\beta}\ (\partial\psi/\partial t)_{\beta\bar\alpha}$$

$$= -\ \Sigma\ \{g(\psi)^{\bar\delta\beta}\ (\partial\psi/\partial t)_{\beta\bar\alpha}\ g(\psi)^{\bar\alpha\gamma}\ (\partial\psi/\partial s)_{\gamma\bar\delta}\}$$

$$+\ \Sigma\ \{g(\psi)^{\bar\alpha\gamma}\ (\partial\psi/\partial s)_{\gamma\bar\delta}\ g(\psi)^{\bar\delta\beta}\ (\partial\psi/\partial t)_{\beta\bar\alpha}\}$$

$$= 0\ .$$

Hence $(\partial/\partial t)\{(\square_\psi + \nu)\ (\partial\psi/\partial s)\} = (\partial/\partial s)\{(\square_\psi + \nu)\ (\partial\psi/\partial t)\}$. Therefore there exists $F(s,t,x) \in C^\infty([0,1] \times [a,b] \times X)$ satisfying (i),(ii) and (iii). For (iv), we first observe that it is true for $(s,t) = (0,a)$. We now have

$$(\partial/\partial s)\ (R(\psi) - \nu\omega_0(\psi)) - (\partial/\partial s)\ (\sqrt{-1}\ \partial\bar\partial F)$$

$$= \sqrt{-1}\ \{\bar\partial\partial(\square_\psi(\partial\psi/\partial s)) - \partial\bar\partial\ (\nu\ \partial\psi/\partial s) - \partial\bar\partial(\partial F/\partial s)\} = 0\ .$$

Similarly $(\partial/\partial t)\ (R(\psi) - \nu\,\omega_0(\psi)) - (\partial/\partial t)\ (\sqrt{-1}\ \partial\bar\partial F) = 0$. Hence we obtain (iv) .

<u>PROOF OF (2.4)</u>: Let $\psi(s,t) := s\,\varphi_t$ for $(s,t) \in [0,1]\times[a,b]$ and $\Psi(s,t)$ be the 1-form

$$\left(\int_X -\frac{\partial\psi}{\partial s}(\sigma(\psi) - \lambda)\, V_0(\psi)\right)ds + \left(\int_X -\frac{\partial\psi}{\partial t}(\sigma(\psi) - \lambda)\, V_0(\psi)\right)dt\,.$$

Then similar to the proof of (2.3), that of (2.4) except for the equality (2.4.2) is reduced to showing $d\Psi = 0$.

Step 1: By Lemma (2.6) applied to our Ψ, there exists a function $F = F(s,t,x) \in C^\infty([0,1] \times [a,b])_{\mathbb{R}}$ satisfying the equalities (i) ~ (iv). First by (iv),

$$\begin{aligned} & n(R(\omega_0) - \nu\omega_0)\ \omega_0(\psi)^{n-1}/\int_X \omega_0{}^n \\ & = n(R(\psi) - \nu\omega_0(\psi) - \sqrt{-1}\partial\bar{\partial} F)\omega_0(\psi)^{n-1}\Big/\int_X \omega_0{}^n \\ & = (\sigma(\psi) - \lambda - \square_\psi F)\, V_0(\psi)\,. \end{aligned}$$

Therefore, introducing the 1-form Φ defined as

$$\begin{aligned} & \Big\{ds \int_X n(R(\omega_0) - \nu\omega_0)\,\frac{\partial\psi}{\partial s}\,\omega_0(\psi)^{n-1} \\ & + dt\int_X n(R(\omega_0) - \nu\omega_0)\,\frac{\partial\psi}{\partial t}\,\omega_0(\psi)^{n-1}\Big\}\Big/\int_X \omega_0{}^n\,, \end{aligned}$$

we obtain

$$\begin{aligned} \Psi &= -\Phi - \left(\int_X \frac{\partial\psi}{\partial s}\,(\square_\psi F)\, V_0(\psi)\right)ds - \left(\int_X \frac{\partial\psi}{\partial t}\,(\square_\psi F)\, V_0(\psi)\right)dt \\ &= -\Phi - \left(\int_X (\square_\psi \frac{\partial\psi}{\partial s})\ F\, V_0(\psi)\right)ds - \left(\int_X (\square_\psi \frac{\partial\psi}{\partial t})\, F\, V_0(\psi)\right) dt\,. \end{aligned}$$

Hence $d\Psi = -\,d\Phi + I\, ds\wedge dt$, where the coefficient I is

$$\int_X \frac{\partial}{\partial t}\left((\square_\psi \frac{\partial\psi}{\partial s})\ F\, V_0(\psi)\right) - \int_X \frac{\partial}{\partial s}\left((\square_\psi \frac{\partial\psi}{\partial t})\ F\, V_0(\psi)\right)\,.$$

In view of the identities $\partial V_0(\psi)/\partial t = \square_\psi(\partial\psi/\partial t)\, V_0(\psi)$ and $\partial V_0(\psi)/\partial s = \square_\psi(\partial\psi/\partial s)\, V_0(\psi)$, (2.6.1) above combined with (i) and (ii) of (2.6) yields

$$I = \int_X (\square_\psi \tfrac{\partial\psi}{\partial s}) \tfrac{\partial}{\partial t}(F\, V_0(\psi)) - \int_X (\square_\psi \tfrac{\partial\psi}{\partial t}) \tfrac{\partial}{\partial s}(F\, V_0(\psi))$$

$$= \int_X \left\{ (\square_\psi \tfrac{\partial\psi}{\partial s}) \left(-\square_\psi \tfrac{\partial\psi}{\partial t} - \nu \tfrac{\partial\psi}{\partial t}\right) + (\square_\psi \tfrac{\partial\psi}{\partial s})\, F\, (\square_\psi \tfrac{\partial\psi}{\partial t}) \right\} V_0(\psi)$$

$$- \int_X \left\{ (\square_\psi \tfrac{\partial\psi}{\partial t}) \left(-\square_\psi \tfrac{\partial\psi}{\partial s} - \nu \tfrac{\partial\psi}{\partial s}\right) + (\square_\psi \tfrac{\partial\psi}{\partial t})\, F\, (\square_\psi \tfrac{\partial\psi}{\partial s}) \right\} V_0(\psi)$$

$$= 0 .$$

Thus, we obtain

$$d\Psi = -d\Phi = ds\wedge dt \int_X n(R(\omega_0) - \nu\omega_0)\left(\tfrac{\partial\psi}{\partial s}\tfrac{\partial}{\partial t} - \tfrac{\partial\psi}{\partial t}\tfrac{\partial}{\partial s}\right)(\omega_0(\psi)^{n-1}) \Big/ \int_X \omega_0{}^n$$

$$= \sqrt{-1}\, ds\wedge dt \int_X n(n-1)\, (R(\omega_0) - \nu\,\omega_0)\left(-\tfrac{\partial\psi}{\partial s}\,\partial\bar\partial(\tfrac{\partial\psi}{\partial t}) - \tfrac{\partial\psi}{\partial t}\,\partial\bar\partial(\tfrac{\partial\psi}{\partial s})\right) \omega_0(\psi)^{n-2} \Big/ \int_X \omega_0{}^n$$

$$= \sqrt{-1}\, ds\wedge dt \int_X n(n-1)\, (R(\omega_0) - \nu\,\omega_0)\left(\bar\partial(\tfrac{\partial\psi}{\partial s})\wedge\partial(\tfrac{\partial\psi}{\partial t}) + \partial(\tfrac{\partial\psi}{\partial t})\wedge\bar\partial(\tfrac{\partial\psi}{\partial s})\right)\omega_0(\psi)^{n-2} \Big/ \int_X \omega_0{}^n$$

$$= 0 .$$

<u>Step 2</u>: We shall finally show (2.4.2). Since $M(\varphi_1+C_1, \varphi_2+C_2) - M(\varphi_1,\varphi_2) = M(\varphi_2, \varphi_2+C_2) - M(\varphi_1, \varphi_1+C_1)$, it suffices to show $M(\varphi, \varphi+C) = 0$ for all φ and C. Let $\psi_t := \varphi+tC$, $t \in [0,1]$. Then

$$M(\varphi,\varphi+C) = -\int_0^1 \left(\int_X C(\sigma(\psi_t) - \lambda)\, V_0(\psi_t)\right) dt = 0 .$$

The proof of (2.4) is now complete.

(2.7) In view of (2.4.2) above, $M : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ factors through $\mathcal{K} \times \mathcal{K}$. Hence we can define a mapping $M : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{R}$ (denoted by the same M) satisfying the 1-cocycle condition by

$$(2.7.1) \qquad M(\omega',\omega'') := M(\varphi',\varphi'') , \quad \text{for all } \omega',\omega'' \in \mathcal{K} ,$$

where φ',φ'' are functions in $\mathcal{H}$ such that $\omega_0(\varphi') = \omega'$ and $\omega_0(\varphi'') = \omega''$. We now put $\mathcal{H}_0 := \{\varphi \in \mathcal{H} \mid L(0,\varphi) = 0\}$. Then the restriction of the mapping $\varphi \in \mathcal{H} \longmapsto \omega_0(\varphi) \in \mathcal{K}$ to $\mathcal{H}_0$ is an isomorphism:

$$\begin{aligned} \mathcal{H}_0 &\cong \mathcal{K} \\ \varphi &\longmapsto \omega_0(\varphi) \end{aligned}$$

Hence we can regard $\mathcal{K}$ as the subset $\mathcal{H}_0$ of $\mathcal{H}$. By this identification, the mapping $M : \mathcal{K} \times \mathcal{K} \longrightarrow \mathbb{R}$ defined just above coincides with the restriction to $\mathcal{H}_0 \times \mathcal{H}_0$ of the original mapping $M : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$.

A 1-parameter family $\{\omega_t \mid a \leq t \leq b\}$ of Kähler forms in $\mathcal{K}$ is said to be <u>smooth</u> (or a <u>smooth path</u>) if it forms a smooth path in $C^\infty(X)_{\mathbb{R}}$ via the identification $\mathcal{K} = \mathcal{H}_0$.

§ 3. K-energy maps and their critical points

DEFINITION (3.1). Let $\mu: K \longrightarrow \mathbb{R}$ be the mapping which associate, to each $\omega \in K$, the real number $\mu(\omega) := M(\omega_0, \omega)$, (cf. (2.7)). This μ is called the K-energy map of the Kähler manifold (X, ω_0). For every $\varphi \in H$, $\mu(\omega_0(\varphi))$ will be denoted by $\mu(\varphi)$ for simplicity.

We write the above μ sometimes as μ_{ω_0} because it depends on the choice of ω_0. If we replace the original ω_0 by another ω_0' cohomologous to ω_0, then the difference between μ_{ω_0} and $\mu_{\omega_0'}$ is just a constant. In fact, for all $\omega \in K$

$$\mu_{\omega_0}(\omega) - \mu_{\omega_0'}(\omega) = M(\omega_0, \omega_0')$$

which is independent of $\omega \in K$. In particular every critical point of μ_{ω_0} is, at the same time, that of $\mu_{\omega_0'}$ and vice versa. Hence "critical points of μ" have an intrinsic meaning in the sense that it depends only on X and on the cohomology class of ω_0 in $H^{1,1}(X, \mathbb{R})$.

THEOREM (3.2). Let $\mu : K \longrightarrow \mathbb{R}$ be the K-energy map of the Kähler manifold (X, ω_0). Then for an arbitrary element ω of K, the following are equivalent:

i) ω is a critical point of μ,
ii) ω has a constant scalar curvature,
iii) ω has the constant scalar curvature λ.

PROOF: Let $\{\varphi_t | -\varepsilon \leq t \leq \varepsilon\}$ be a smooth path in $\mathcal{H}$ such that $\omega_0(\varphi_0) = \omega$. Then by (2.2.2) and (2.7.1),

$$\frac{d}{dt}\mu(\omega_0(\varphi_t))\Big|_{t=0} = \int_X \dot{\varphi}_t\Big|_{t=0} (\sigma(\omega) - \lambda)\, \omega^n \Big/ \int_X \omega_0^{\,n} ,$$

which shows the equivalence of i) and iii). Thus the proof is reduced to showing that ii) implies iii). Since $\int_X (\sigma(\omega) - \lambda)\omega^n = 0$ for every $\omega \in K$, the required implication is now immediate.

DEFINITION (3.3). A compact complex connected manifold with ample anticanonical bundle (or equivalently with $c_1 > 0$) is said to be a Fano manifold. Differential-geometrically, a Fano manifold is a compact complex connected manifold which admits a Kähler metric with positive definite Ricci tensor, (cf. Yau [8]).

THEOREM (3.4). Suppose that X is a Fano manifold and furthermore that ω_0 represents $2\pi c_1(X)_{\mathbb{R}}$. Consider the K-energy map $\mu : K \longrightarrow \mathbb{R}$ of the Kähler manifold (X, ω_0). Then for an arbitrary element ω of K, the following are equivalent:

i) ω is a critical point of μ,

ii) ω is Einstein Kähler,

iii) ω is Einstein Kähler with the constant scalar curvature n.

<u>PROOF</u>: Note that λ is n, (cf. (1.2)). Since X is a Fano manifold, every Kähler form of constant scalar curvature in the cohomology class $c_1(X)_{\mathbb{R}}$ is Einstein. Then (3.4) is straightforward from Theorem (3.2).

§4. Another interpretation of the K-energy map.

Recall that $\mathcal{K}$ is naturally identified with the subset $\mathcal{H}_0$ of $\mathcal{H}$, (cf. (2.7)). In this section, another interpretation of the K-energy map $\mu : \mathcal{H}_0 (= \mathcal{K}) \longrightarrow \mathbb{R}$ of (X, ω_0), (cf. (3.1)), will be given. We shall actually show the following:

THEOREM (4.1). For each $\varphi \in \mathcal{H}$, there exists a unique function $f_\varphi \in C^\infty(X)_{\mathbb{R}}$ such that

(4.1.1) $\sigma(\varphi) - \lambda = \square_\varphi f_\varphi$,

(4.1.2) $\int_X f_\varphi V_0(\varphi) = 0$ if $\varphi = 0$ in $C^\infty(X)_{\mathbb{R}}$, and

(4.1.3) $\frac{\partial}{\partial t}(f_{\varphi_t} - k_{\varphi_t}) = -(\square_{\varphi_t} + \nu)\dot{\varphi}_t$ for every smooth path $\{\varphi_t | a \le t \le b\}$ in $\mathcal{H}$, where for each $\psi \in \mathcal{H}$, we denote by k_ψ the function in $C^\infty(X)_{\mathbb{R}}$ defined by

$$\square_\psi k_\psi = (R(\omega_0) - \nu\,\omega_0) \wedge n\,\omega_0(\psi)^{n-1} / \omega_0(\psi)^n \quad \text{and}$$

$$\int_X k_\psi \omega_0{}^n = 0 .$$

COROLLARY (4.2). Suppose that X is a Fano manifold and furthermore that ω_0 represents $2\pi c_1(X)_{\mathbb{R}}$. Then to each $\varphi \in \mathcal{H}_0$, we can uniquely associate a function $f_\varphi \in C^\infty(X)_{\mathbb{R}}$ (which is the same one as in (4.1)) such that

(4.2.1) $\sigma(\varphi) - n = \square_\varphi f_\varphi$, i.e., $R(\varphi) - \omega_0(\varphi) = \sqrt{-1}\,\partial\bar{\partial} f_\varphi$,

(4.2.2) $\mu(\varphi) = -\int_X f_\varphi V_0(\varphi)$, and

(4.2.3) $\frac{\partial}{\partial t} f_{\varphi_t} = -(\square_{\varphi_t} + 1)\dot{\varphi}_t$ for every smooth path

$\{\varphi_t | a \le t \le b\}$ <u>in</u> $\mathcal{H}_0$.

In view of (4.2.2), the construction of f_φ is crucial to our approach. The key in the definition of f_φ is the following

<u>DEFINTION (4.3)</u>. For each pair $(\varphi', \varphi'') \in \mathcal{H} \times \mathcal{H}$, we define a function $H(\varphi', \varphi'') \in C^\infty(X)_{\mathbb{R}}$ by

(4.3.1) $$H(\varphi', \varphi'') := - \int_a^b (\square_{\varphi_t} + \nu)\, \dot{\varphi}_t \, dt \ ,$$

where $\{\varphi_t \,| a \le t \le b\}$ is an arbitrary piecewise smooth path in $\mathcal{H}$ such that $\varphi_a = \varphi'$ and $\varphi_b = \varphi''$.

<u>THEOREM (4.4)</u>. $H(\varphi', \varphi'')$ <u>above is independent of the choice of the path</u> $\{\varphi_t | a \le t \le b\}$ <u>and therefore well-defined</u>. <u>More-over,</u>

(4.4.1) $H : \mathcal{H} \times \mathcal{H} \longrightarrow C^\infty(X)_{\mathbb{R}}$ <u>satisfies the</u> 1-<u>cocycle condition, a</u>

(4.4.2) $$\{R(\varphi) - \nu\omega_0(\varphi)\} \Big|_{\varphi = \varphi'}^{\varphi = \varphi''} = \sqrt{-1}\, \partial\bar{\partial}\, H(\varphi', \varphi'') \quad .$$

<u>PROOF</u>: In view of the proof of (2.3), we may assume that $\{\varphi_t | a \le t \le b\}$ is a smooth path. Let $\psi(s,t) := s\varphi_t$ for $(s,t) \in [0,1] \times [a,b]$. Then by Lemma (2.6), we obtain a C^∞ function $F(s,t,x) \in C^\infty([0,1] \times [a,b] \times X)_{\mathbb{R}}$ with the properties (i) ~ (iv) of (2.6). For each $(\sigma, \tau) \in [0,1] \times [a,b]$, we set $F_{\sigma,\tau} := F|_{(s,t)=(\sigma,\tau)}$. Then by (i),

$$F_{1,t} - F_{0,t} = - \int_0^1 \left(\square_{\psi(s,t)} + \nu \right) \varphi_t \, ds \ .$$

On the other hand, by (ii) applied to the cases $s = 0$ and $s = 1$,

$$F_{0,b} = F_{0,a},$$
$$F_{1,b} - F_{1,a} = - \int_a^b \left(\square_{\varphi_t} + \nu \right) \dot{\varphi}_t \, dt \ .$$

Combining the three equalities obtained just above, we have

$$- \int_a^b (\square_{\varphi_t} + \nu) \, \dot{\varphi}_t \, dt = (F_{1,t} - F_{0,t}) \Big|_{t=a}^{t=b} = - \int_0^1 (\square_{\psi(s,t)} + \nu) \, \varphi_t ds \Big|_{t=a}^{t=b} \ .$$

The proof, except for (4.3.2), is then straightforward, For (4.3.2), applying (iv) of (2.6) to the cases $(s,t) = (1,a), (1,b)$, we now conclude that

$$\{R(\varphi) - \nu\omega_0(\varphi)\} \Big|_{\varphi = \varphi'}^{\varphi = \varphi''} = \sqrt{-1} \ \partial\bar{\partial} (F_{1,b} - F_{1,a}) = \sqrt{-1} \ \partial\bar{\partial} \, H(\varphi', \varphi'') .$$

We shall now define f_φ for each $\varphi \in H$ and then proceed to the proof of (4.1) and (4.2).

<u>DEFINTITION (4.5)</u>.

(i) For each $\varphi \in H$, we define $f_\varphi \in C^\infty(X)_{\mathbb{R}}$ by $f_\varphi := k_\varphi + H(0,\varphi)$.

(ii) For each $\omega \in K$, let $f_\omega \in C^\infty(X)_{\mathbb{R}}$ denote the function f_{φ_ω}, where φ_ω is the unique element of H_0 such that $\omega = \omega_0(\varphi_\omega)$.

PROOF OF (4.1). Since the uniqueness is easy, it suffices to show f_φ defined in (4.5) satisfies (4.1.1) ~ (4.1.3). First, (4.1.2) is obvious from our definition of f_φ. We next observe that (4.1.3) is an immediate consequence of (4.3.1). For (4.1.1), we apply (4.4.2):

$$R(\varphi) - \nu\omega_0(\varphi) = R(\omega_0) - \nu\omega_0 + \sqrt{-1}\,\partial\bar{\partial}\,H(0,\varphi) \ .$$

Taking the wedge product with $n\,\omega_0^{n-1}$, and then dividing both sides by $\omega_0(\varphi)^n$, we finally obtain

$$\sigma(\varphi) - \lambda = \square_\varphi(k_\varphi + H(0,\varphi)) = \square_\varphi f_\varphi \ .$$

PROOF OF (4.2): Since $R(\omega_0)$ and ω_0 are cohomologous, we have $k_\varphi = f_0$ for every $\varphi \in H_0$, where $f_0 \in C^\infty(X)_{\mathbb{R}}$ is the function defined by the conditions $\int_X f_0\,\omega_0^n = 0$ and $R(\omega_0) - \omega_0 = \sqrt{-1}\,\partial\bar{\partial} f_0$. Since (4.2.3) is then obvious from (4.1.3), the proof is reduced to showing (4.2.2) for f_φ defined in (4.5). Fix an arbitrary $\varphi \in H_0$, and we put $\psi_t := t\varphi - L(0,t\varphi)$, $t \in [0,1]$. Note that $\{\psi_t \mid 0 \le t \le 1\}$ is a smooth path in H_0 connecting 0 with φ. In view of (4.1.2), the proof is further reduced to showing

$$\frac{d}{dt}\mu(\psi_t) = -\frac{d}{dt}\int_X f_{\psi_t} V_0(\psi_t)$$

for every $t \in [0,1]$. We can now finish the proof by the following computation:

$$- \frac{d}{dt}\int_X f_{\psi_t} V_0(\psi_t) = \int_X (\square_{\psi_t} + \nu)\, \dot{\psi}_t V_0(\psi_t) - \int_X f_{\psi_t} (\square_{\psi_t} \dot{\psi}_t) V_0(\psi_t)$$

$$= - \int_X f_{\psi_t} (\square_{\psi_t} \dot{\psi}_t) V_0(\psi_t) = - \int_X (\sigma(\psi_t) - \lambda)\dot{\psi}_t V_0(\psi_t) = \frac{d}{dt}\mu(\psi_t) .$$

§5. Futaki invariants as derivatives of the K-energy map.

Let Aut(X) be the group of holomorphic automorphisms of X, and let $\mathrm{Aut}^0(X)$ be its identity component. For each holomorphic vector field $Y \in \Gamma(X, \mathcal{O}(T(X)))$ on X, we put

$$Y_{\mathbb{R}} := Y + \bar{Y},$$

and we later consider the corresponding 1-parameter group $y_t := \exp t Y_{\mathbb{R}}$, $(t \in \mathbb{R})$. For each $\omega \in K$, let $f_\omega \in C^\infty(X)_{\mathbb{R}}$ be the function defined in (4.5). Recall that

$$\sigma(\omega) - \lambda = \square_\omega f_\omega, \quad (\text{cf. } (4.1.1)).$$

Then a fundamental theorem of Futaki [4] states as follows:

(5.1) For every $Y \in \Gamma(X, \mathcal{O}(T(X)))$, the number $C_{Y,\omega} := \int_X (Y_{\mathbb{R}} f_\omega)\, \omega^n \Big/ \int_X \omega_0{}^n$ doesn't depend on the choice of ω in but depends possibly on the Kähler class K. (Therefore $C_{Y,\omega}$ will be denoted by $C_{Y,K}$.)

(5.2) If there exists a $\tilde{\omega} \in K$ such that $(X, \tilde{\omega})$ is a Kähler manifold of constant scalar curvature, then $C_{Y,K} = 0$ for all $Y \in \Gamma(X, \mathcal{O}(T(X)))$.

The main purpose of this section is to show that the first derivative of the K-energy map $\mu : K \longrightarrow \mathbb{R}$ along each orbit $\{y_t{}^* \omega \mid t \in \mathbb{R}\}$ of the 1-parameter group $\{y_t\}_{t \in \mathbb{R}}$ is nothing but $C_{Y,K}$. Using this fact, we shall give another proof of (5.2) of Futaki's theorem. In a subsequent paper (cf. Bando and Mabuchi [1]), a very simple proof of (5.1) will also be given in a more general situation.

THEOREM (5.3). Let Y be an arbitrary holomorphic vector field on X. Then for all $t \in \mathbb{R}$ and $\omega \in K$

$$\mu(y_t{}^* \omega) = \mu(\omega) + t\, C_{Y,K} .$$

PROOF: For each $t \in \mathbb{R}$, there uniquely exists a function $\varphi_t \in H_0$ such that $y_t{}^* \omega = \omega_0(\varphi_t)$. For simplicity, we write $y_t{}^* \omega$ and $f_{y_t{}^*\omega}$ as ω_t and f_t respectively. We furthermore put $V := \int_X \omega_0{}^n / n!$. Note that

$$L_{Y_{\mathbb{R}}} \omega_t = \frac{\partial}{\partial t} \omega_t = \sqrt{-1}\ \partial\bar{\partial}\ \dot{\varphi}_t .$$

Since $0 = \int_X L_{Y_{\mathbb{R}}}(f_t\, \omega_t{}^n) = \int_X (Y_{\mathbb{R}} f_t)\, \omega_t{}^n + n\sqrt{-1} \int_X f_t \omega_t{}^{n-1} \wedge \partial\bar{\partial}\, \dot{\varphi}_t$,

we have, for every $t \in \mathbb{R}$,

$$\begin{aligned} C_{Y,K} &= \int_X (Y_{\mathbb{R}} f_t)\, \omega_t{}^n/(n!V) = -\, n\sqrt{-1}\int_X f_t \omega_t{}^{n-1} \wedge \partial\bar{\partial}\dot{\varphi}_t / (n!V) \\ &= \sqrt{-1} \int_X \partial f_t \wedge \bar{\partial}\, \dot{\varphi}_t \wedge \omega_t{}^{n-1}/((n-1)!V) = (1/V)\ (\partial f_t, \partial\dot{\varphi}_t)_{L^2(X,\omega_t)} \\ &= -(1/V)\ (\square_{\omega_t} f_t, \dot{\varphi}_t)_{L^2(X,\omega_t)} = -\int_X \dot{\varphi}_t\, (\sigma(\varphi_t) - \lambda)\, V_0(\varphi_t) \\ &= d\,\mu(\omega_t)/dt , \end{aligned}$$

from which the required equality immediately follows.

<u>PROOF OF (5.2)</u> <u>of Futaki's theorem (by assuming (5.1))</u>: By Theorem (3.2), $\mu : \mathcal{K} \longrightarrow \mathbb{R}$ has a critical point at $\tilde{\omega}$. Hence, for an arbitrary $Y \in H^0(X, \mathcal{O}(T(X)))$,

$$C_{Y,\mathcal{K}} = \frac{d}{dt}\mu(y_t{}^*\tilde{\omega})\,|_{t=0} = 0 .$$

For Fano manifolds, we have the following stronger facts:

<u>THEOREM (5.4)</u>. <u>Assume that</u> X <u>is a Fano manifold (where it is not necessary to assume that</u> ω_0 <u>represents a specific class such as</u> $2\pi c_1(X)_{\mathbb{R}}$). <u>Suppose furthermore that there exists a Kähler form</u> $\tilde{\omega} \in \mathcal{K}$ <u>of constant scalar curvature</u>. <u>Then the K-energy map</u> $\mu : \mathcal{K} \longrightarrow \mathbb{R}$ <u>of the compact Kähler manifold</u> (X, ω_0) <u>satisfies</u>

$$\mu(g^*\omega) = \mu(\omega) \qquad (\omega \in \mathcal{K})$$

<u>for all</u> $g \in \mathrm{Aut}(X)$ <u>with</u> $g^*\mathcal{K} = \mathcal{K}$.

<u>PROOF</u>: Since X is a Fano manifold, there exists an $m \in \mathbb{Z}$ $(m \gg 1)$ such that the line bundle $K_X{}^{-m}$ is very ample. Hence $\mathrm{Aut}(X)$ is regarded as a closed (algebraic) subgroup of $PGL(N;\mathbb{C})$ $(= \mathrm{Aut}(\mathbb{P}(H^0(X, \mathcal{O}(K_X^{-m})))))$ (where $N = h^0(X, \mathcal{O}(K_X{}^{-m}))$). Thus for every $g \in \mathrm{Aut}(X)$, we have $g^\alpha \in \mathrm{Aut}^0(X)$ for some positive integer α. Then there exists a sequence $h_0 = e, h_1, h_2, \ldots, h_{r-1}, h_r = g^\alpha$ of points of $\mathrm{Aut}^0(X)$ such that $h_i = h_{i-1} \cdot \exp Y_{i\mathbb{R}}$ $(i = 1, 2, \ldots, r)$ for some

$Y_i \in H^0(X, \mathcal{O}(T(X)))$. We observe, from the defintion of M in §2, that

$$M((g^{j-1})^*\omega, (g^j)^*\omega) = M((g^{j-1})^*\omega, (g^{j-1})^*(g^*\omega)) = M(\omega, g^*\omega)$$

for each j . Hence

$$\alpha(\mu(g^*\omega) - \mu(\omega)) = \alpha M(\omega, g^*\omega) = \sum_{j=1}^{\alpha} M((g^{j-1})^*\omega, (g^j)^*\omega) = M(\omega, (g^{\alpha})^*\omega)$$

$$= \sum_{i=1}^{r} M((h_{i-1})^*\omega, (h_i)^*\omega) = \sum_{i=1}^{r} M((h_{i-1})^*\omega, (\exp(Y_{i\mathbb{R}}))^*(h_{i-1})^*\omega)$$

$$= \sum_{i=1}^{r} C_{Y_i,K} = 0 , \quad \text{(cf. (5.2))} ,$$

as required.

§6. The second variation formula for K-energy maps.

Throughout this section, for simplicity, we assume that X is a Fano manifold with a Kähler form $\omega_0 = \sqrt{-1}\sum g_{\alpha\bar\beta}\, dz^\alpha \wedge dz^{\bar\beta}$ representing $2\pi c_1(X)_{\mathbb{R}}$, (cf. §1). We furthermore fix a smooth path $\{\varphi_t \mid a \le t \le b\}$ in H.

We denote by ∇^t the covariant derivative on the space of 1-forms of the Kähler manifold $(X, \omega_0(\varphi_t))$, and let Λ_t be the Λ-operator

$$\sum a_{\alpha\bar\beta}\, dz^\alpha \wedge dz^{\bar\beta} \xrightarrow{\Lambda_t} \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g(\varphi_t)^{\bar\beta\alpha}\, a_{\alpha\bar\beta}$$

of $(X, \omega_0(\varphi_t))$. Let $f_{\varphi_t} \in C^\infty(X)_{\mathbb{R}}$ be the function defined in (4.5), and we denote this function simply by f_t. Then

$$\dot f_t = -(\square_{\varphi_t} + 1)\,\dot\varphi_t, \quad \text{and}$$

$$R(\varphi_t) - \omega_0(\varphi_t) = \sqrt{-1}\,\partial\bar\partial f_t, \quad \text{i.e.,} \quad \sigma(\varphi_t) - n = \square_{\varphi_t} f_t,$$

for every $t \in [a,b]$, (cf. (4.2)). We shall first prove:

LEMMA (6.1). <u>Let</u> $Y(=\sum_\alpha y^\alpha\, \partial/\partial z^\alpha)$ <u>be an arbitrary complex valued</u> C^∞ <u>global vector field of type</u> (1,0) <u>on</u> X. <u>Then for every</u> $\psi \in C^\infty(X)_{\mathbb{R}}$,

$$(6.1.1)\quad \sqrt{-1}\,\Lambda_t\,\bar\partial\{-(Yf_t)\,\partial\psi + \nabla^t_Y\partial\psi\} + \sqrt{-1}\,\Lambda_t\{(\bar\partial Y)(f_t)\wedge\partial\psi - \nabla^t_{\bar\partial Y}\partial\psi\}$$

$$= Y(-\square_{\varphi_t}\psi - \psi) + (Yf_t)\,\square_{\varphi_t}\psi,$$

where $(\bar{\partial}Y)(f_t) := \sum_{\alpha,\beta} y^{\alpha}{}_{\bar{\beta}} (f_t)_{\alpha} dz^{\bar{\beta}}$ and $\nabla^t_{\bar{\partial}Y} \partial\psi :=$ $\sum_{\alpha,\beta}(y^{\alpha}{}_{\bar{\beta}} dz^{\bar{\beta}} \wedge \nabla^t_{\partial/\partial z^{\alpha}} \partial\psi)$. (We use such notation $y^{\alpha}{}_{\bar{\beta}} = \partial y^{\alpha}/\partial z^{\bar{\beta}}$, $(f_t)_{\alpha} := \partial f_t/\partial z^{\alpha}$, ... as is explained in (1.3).)

PROOF: Fix an arbitrary pair $(t,p) \in \mathbb{R} \times X$. We then choose a system $z = (z^1, z^2, \ldots, z^n)$ of holomorphic local coordinates of X centered at p such that

$$g(\varphi_t)_{\alpha\bar{\beta}}(p) = \delta_{\alpha\beta} \quad \text{and} \quad d(g(\varphi_t)_{\alpha\bar{\beta}})(p) = 0$$

for all α and β . Since there is no fear of confusion, the following $g(\varphi_t)_{\alpha\bar{\beta}}$, $g(\varphi_t)^{\bar{\beta}\gamma}$, $R(\varphi_t)_{\alpha\bar{\beta}}$, f_t, ∇^t, Λ_t, $\Box_{\varphi_t}$ will be denoted simply by $G_{\alpha\bar{\beta}}$, $G^{\bar{\beta}\gamma}$, $R_{\alpha\bar{\beta}}$, f, ∇, Λ, $\Box$ respectively. Then at the point $(t,p) \in \mathbb{R} \times X$,

$$\begin{aligned}
(6.1.2)\quad & \sqrt{-1}\,\Lambda\,\bar{\partial}\{-(Yf)\partial\psi + \nabla_Y \partial\psi\} \\
&= \sum_{\alpha,\beta}(y^{\alpha}{}_{\bar{\beta}} f_{\alpha} \psi_{\beta} + y^{\alpha} f_{\alpha\bar{\beta}} \psi_{\beta} + y^{\alpha} f_{\alpha} \psi_{\beta\bar{\beta}}) \\
&\quad + \sum_{\alpha,\beta}\{-y^{\alpha}{}_{\bar{\beta}} \psi_{\beta\alpha} - y^{\alpha} \psi_{\beta\alpha\bar{\beta}} + y^{\alpha} \textstyle\sum_{\delta} \psi_{\delta}(\partial^2 G_{\beta\bar{\delta}}/\partial z^{\alpha}\partial z^{\bar{\beta}})\} \\
&= \sum_{\alpha,\beta}\{y^{\alpha}{}_{\bar{\beta}} f_{\alpha} \psi_{\beta} + y^{\alpha}(R_{\alpha\bar{\beta}} - \delta_{\alpha\beta})\, \psi_{\beta}\} + (Yf)\Box\psi \\
&\quad + \sum_{\alpha,\beta}(-y^{\alpha}{}_{\bar{\beta}} \psi_{\beta\alpha} - y^{\alpha}\psi_{\beta\bar{\beta}\alpha}) - \sum_{\alpha,\delta} y^{\alpha} R_{\alpha\bar{\delta}} \psi_{\delta} \\
&= \sum_{\alpha,\beta}(y^{\alpha}{}_{\bar{\beta}} f_{\alpha} \psi_{\beta} - y^{\alpha}{}_{\bar{\beta}} \psi_{\beta\alpha}) + Y(-\psi - \Box\psi) + (Yf)\Box\psi .
\end{aligned}$$

On the other hand, at the same point (t,p) ,

$$(6.1.3)\quad \sqrt{-1}\,\Lambda\,\{(\bar{\partial}Y)(f)\wedge\partial\psi - \nabla_{\bar{\partial}Y}\partial\psi\} = -\sum_{\alpha,\beta}(y^{\alpha}{}_{\bar{\beta}} f_{\alpha} \psi_{\beta} - y^{\alpha}{}_{\bar{\beta}} \psi_{\beta\alpha}) .$$

Adding up (6.1.2) and (6.1.3), we obtain (6.1.1).

THEOREM (6.2). (<u>Second variation formula</u>). <u>For every</u> $t \in [a,b]$, <u>we have</u>

$$(6.2.1)\quad \frac{d^2}{dt^2}\mu(\varphi_t) = \frac{1}{V}\,\|\bar\partial Y_t\|^2_{L^2(X,\omega_0(\varphi_t))} - \int_X \{\ddot\varphi_t - \textstyle\sum_{\alpha,\beta} g(\varphi_t)^{\bar\beta\alpha}(\dot\varphi_t)_\alpha(\dot\varphi_t)_{\bar\beta}\}(\sigma(\varphi_t)-n)\,V_0(\varphi_t)\,,$$

<u>where</u> $V := \int_X \omega_0{}^n/n!$ <u>and</u> $Y_t := \sum_{\alpha,\beta} g(\varphi_t)^{\bar\beta\alpha}(\dot\varphi_t)_{\bar\beta}\,\partial/\partial z^\alpha$.

<u>PROOF</u>: We integrate, on X, the equality (6.1.1) applied to $(\psi,Y) = (\dot\varphi_t, Y_t)$. Then by $\int_X \sqrt{-1}(\Lambda_t\,\bar\partial\{\ldots\})V_0(\varphi_t) = 0$, we obtain

$$(6.2.2)\quad \int_X \sqrt{-1}\,\Lambda_t\{(\bar\partial Y_t)(f_t)\wedge\partial\dot\varphi_t - \nabla^t_{\bar\partial Y_t}\partial\dot\varphi_t\}\,V_0(\varphi_t) = \int_X Y_t(-\square_{\varphi_t}\dot\varphi_t - \dot\varphi_t)V_0(\varphi_t) + \int_X (Y_t f_t)(\square_{\varphi_t}\dot\varphi_t)\,V_0(\varphi_t)\,.$$

On the other hand

$$(6.2.3)\quad \frac{d^2}{dt^2}\mu(\varphi_t) = \frac{d}{dt}\int_X -\dot\varphi_t(\square_{\varphi_t} f_t)\,V_0(\varphi_t) = \frac{d}{dt}\{(1/V)\,(\bar\partial\dot\varphi_t, \bar\partial f_t)_{L^2(X,\omega_0(\varphi_t))}\}$$

$$= \int_X \frac{\partial}{\partial t}\{\textstyle\sum_{\alpha,\beta} g(\varphi_t)^{\bar\beta\alpha}(\dot\varphi_t)_{\bar\beta}(f_t)_\alpha V_0(\varphi_t)\}$$

$$= - \int_X \sum_{\alpha,\beta,\gamma,\delta} g(\varphi_t)^{\overline{\beta}\gamma} (\dot{\varphi}_t)_{\gamma\overline{\delta}}\, g(\varphi_t)^{\overline{\delta}\alpha} (\dot{\varphi}_t)_{\overline{\beta}} (f_t)_\alpha V_0(\varphi_t)$$

$$+ \int_X \sum_{\alpha,\beta} g(\varphi_t)^{\overline{\beta}\alpha} (\ddot{\varphi}_t)_{\overline{\beta}} (f_t)_\alpha V_0(\varphi_t)$$

$$+ \int_X Y_t(\dot{f}_t) V_0(\varphi_t) + \int_X (Y_t f_t)(\square_{\varphi_t} \dot{\varphi}_t) V_0(\varphi_t) .$$

Since $\dot{f}_t = -(\square_{\varphi_t} + 1)\dot{\varphi}_t$, the right-hand side of (6.2.2) coincides with the sum of the last two integrals of the bottom of (6.2.3). Hence

$$(6.2.4) \quad \frac{d^2}{dt^2} \mu(\varphi_t) =$$
$$- \int_X \sum_{\alpha,\beta,\gamma,\delta} g(\varphi_t)^{\overline{\beta}\gamma} (\dot{\varphi}_t)_{\gamma\overline{\delta}}\, g(\varphi_t)^{\overline{\delta}\alpha} (\dot{\varphi}_t)_{\overline{\beta}} (f_t)_\alpha V_0(\varphi_t)$$
$$+ \int_X \sum_{\alpha,\beta} g(\varphi_t)^{\overline{\beta}\alpha} (\ddot{\varphi}_t)_{\overline{\beta}} (f_t)_\alpha V_0(\varphi_t)$$
$$+ \int_X \sqrt{-1}\, \Lambda_t \{ (\overline{\partial} Y_t)(f_t) \wedge \partial\dot{\varphi}_t - \nabla^t_{\overline{\partial} Y_t} \partial\dot{\varphi}_t \} V_0(\varphi_t) .$$

Note the following identities:

$$(6.2.5) \quad 0 = \int_X - \sqrt{-1}\, \Lambda_t\, \overline{\partial} (\sum_{\alpha,\beta} g(\varphi_t)^{\overline{\beta}\alpha} (\dot{\varphi}_t)_{\overline{\beta}} (\dot{\varphi}_t)_\alpha\, \partial f_t) V_0(\varphi_t) ;$$

$$(6.2.6) \quad \int_X \sum_{\alpha,\beta} g(\varphi_t)^{\overline{\beta}\alpha} (\ddot{\varphi}_t)_{\overline{\beta}} (f_t)_\alpha V_0(\varphi_t) = (1/V) (\overline{\partial}\ddot{\varphi}_t, \partial f_t)_{L^2(X,\omega_0(\varphi_t))}$$

$$= (1/V) (\ddot{\varphi}_t, -\square_{\varphi_t} f_t)_{L^2(X,\omega_0(\varphi_t))} = - \int_X \ddot{\varphi}_t\, (\sigma(\varphi_t) - n)\, V_0(\varphi_t) .$$

Adding up (6.2.4), (6.2.5) and (6.2.6), we obtain

$$\frac{d^2}{dt^2}\mu(\varphi_t) = \int_X h\, V_0(\varphi_t) \quad ,$$

where $h = h(t,x) \in C^\infty([a,b]\times X)$ is the function defined by

$$h := -\sum_{\alpha,\beta,\gamma,\delta}\left\{g(\varphi_t)^{\bar\beta\gamma}(\dot\varphi_t)_{\gamma\bar\delta}\; g(\varphi_t)^{\bar\delta\alpha}\;(\dot\varphi_t)_{\bar\beta}\;(f_t)_\alpha\right\} - \ddot\varphi_t(\sigma(\varphi_t) - n)$$

$$+ \sqrt{-1}\,\Lambda_t\left\{(\bar\partial Y_t)(f_t)\wedge\partial\dot\varphi_t - \nabla^t_{\bar\partial Y_t}\partial\dot\varphi_t - \bar\partial\left(\sum_{\alpha,\beta} g(\varphi_t)^{\bar\beta\alpha}(\dot\varphi_t)_{\bar\beta}(\dot\varphi_t)_\alpha\,\partial f_t\right)\right\} .$$

On the other hand, writing Y_t as $\sum_\alpha y^\alpha\,\partial/\partial z^\alpha$ (in which we put $y^\alpha := \sum_\beta g(\varphi_t)^{\bar\beta\alpha}(\dot\varphi_t)_{\bar\beta}$), we have

$$(\text{Right-hand side of } (6.2.1)) = \int_X k\, V_0(\varphi_t) \quad ,$$

where $k = k(t,x) \in C^\infty_{\mathbb{R}}([a,b]\times X)$ is the function defined by

$$k := \left\{\sum g(\varphi_t)_{\alpha\bar\beta}\, g(\varphi_t)^{\bar\gamma\delta}(y^\alpha)_{\bar\gamma}\,(\overline{y^\beta})_\delta\right\} - \left(\ddot\varphi_t - \sum g(\varphi_t)^{\bar\beta\alpha}(\dot\varphi_t)_\alpha(\dot\varphi_t)_{\bar\beta}\right)(\sigma(\varphi_t) - n)$$

We fix an arbitrary pair $(t,p) \in [a,b]\times X$ and choose a system $(z^1,z^2,\dots,z^n)$ of holomorphic local coordinates of X centered at p such that

$$g(\varphi_t)_{\alpha\bar\beta}(p) = \delta_{\alpha\beta} \quad \text{and} \quad d(g(\varphi_t)_{\alpha\bar\beta})(p) = 0$$

for all α and β . Then at the point $(t,p) \in [a,b]\times X$,

$$h = \left\{\sum_{\alpha,\gamma} (\dot{\varphi}_t)_{\overline{\gamma\alpha}} (\dot{\varphi}_t)_{\gamma\alpha}\right\} - \ddot{\varphi}_t (\sigma(\varphi_t) - n) + \sum_{\alpha,\gamma} (\dot{\varphi}_t)_\alpha (\dot{\varphi}_t)_{\bar{\alpha}} f_{\gamma\bar{\gamma}}$$

$$= \left\{\sum_{\alpha,\gamma} (\dot{\varphi}_t)_{\overline{\gamma\alpha}} (\dot{\varphi}_t)_{\gamma\alpha}\right\} - (\ddot{\varphi}_t - \sum_\alpha (\dot{\varphi}_t)_\alpha (\dot{\varphi}_t)_{\bar{\alpha}}) (\sigma(\varphi_t) - n) = k$$

as required.

<u>COROLLARY (6.3)</u>. <u>If</u> ω <u>is a critical point of</u> $\mu : K \longrightarrow \mathbb{R}$, <u>then the inequality</u>

$$\frac{d^2}{dt^2} \mu(\theta_t)\Big|_{t=0} \geq 0$$

<u>holds for every smooth path</u> $\{\theta_t \mid -\varepsilon \leq t \leq \varepsilon\}$ <u>in</u> K <u>such that</u> $\theta_0 = \omega$.

<u>REMARK (6.4)</u>. An interesting interpretation of (6.2.1) will be given in a forthcoming paper[7].

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