Moduli spaces of vector bundles over ruled surfaces

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Introduction

Let $\pi : X \to C$ be a ruled surface over a smooth algebraic curve C, defined over the complex number field $\mathbb{C}$. Let $c_1 \in Num(X)$ and $c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ be fixed. For any polarization L, denote the moduli space of rank-2 vector bundles stable with respect to L in the sense of Mumford–Takemoto by $\mathcal{M}_L(c_1, c_2)$. Stable 2-vector bundles over a ruled surface have been investigated by many authors; see, for example [T1], [T2], [H-S], [Q1]. Let us mention that Takemoto [T1] showed that there is no rank-2 vector bundle stable with respect to every polarization L in case that $c_1.f$ is odd (f is a fiber of the ruled surface X). In this paper we shall study algebraic 2-vector bundles over ruled surfaces, but we adopt another point of view: we shall study moduli spaces of (algebraic) 2-vector bundles over a ruled surface X, which are defined independent of any ample divisor (line bundle) on X, by taking into account the special geometry of a ruled surface (see [B], [B-St1], [B-St2] and also [Br1], [Br2], [W]).

In section 1 (put for the convenience of the reader) we present (see [B]) two numerical invariants d and r for a 2-vector bundle with fixed Chern classes c_1 and c_2 and we define the set $M(c_1, c_2, d, r)$ of isomorphism classes of bundles with fixed invariants c_1, c_2, d, r. The integer d is given by the splitting of the bundle on the general fibre and the integer r is given by some normalization of the bundle. Recall that the set $M(c_1, c_2, d, r)$ carries a natural structure of an algebraic variety (see [B], [B-St1], [B-St2]). In section

2 we study uniform vector bundles and we prove the existence of algebraic vector bundles given by extensions of line bundles and which are not uniform. In section 3 the main result gives necessary and sufficient conditions for the non-emptiness of the space $M(c_1, c_2, d, r)$ and we apply this result to the moduli space of stable bundles $\mathcal{M}_L(c_1, c_2)$ in the last section.

1 Moduli spaces of rank-2 vector bundles

In this section we shall recall from ([B], [B-St1], [B-St2]) some basic notions and facts.

The notations and the terminology are those of Hartshorne's book [Ha]. Let C be a nonsingular curve of genus g over the complex number field and let $\pi : X \to C$ be a ruled surface over C. We shall write $X \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is normalized. Let us denote by $\mathbf{e}$ the divisor on C corresponding to $\bigwedge^2 \mathcal{E}$ and by $e = -deg(\mathbf{e})$. We fix a point $p_0 \in C$ and a fibre $f_0 = \pi^{-1}(p_0)$ of X. Let C_0 be a section of π such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Any element of $Num(X) \cong H^2(X, \mathbb{Z})$ can be written $aC_0 + bf_0$ with $a,\ b \in \mathbb{Z}$. We shall denote by $\mathcal{O}_C(1)$ the invertible sheaf associated to the divisor p_0 on C. If L is an element of $Pic(C)$ we shall write $L = \mathcal{O}_C(k) \otimes L_0$, where $L_0 \in Pic_0(C)$ and $k = deg(L)$. We also denote by $F(aC_0 + bf_0) = F \otimes \mathcal{O}_X(a) \otimes \pi^* \mathcal{O}_C(b)$ for any sheaf F on X and any $a, b \in \mathbb{Z}$.

Let E be an algebraic rank-2 vector bundle on X with fixed numerical Chern classes $c_1 = (\alpha, \beta) \in H^2(X, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z},\ c_2 = \gamma \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, where $\alpha, \beta, \gamma \in \mathbb{Z}$.

Since the fibres of π are isomorphic to $\mathbb{P}^1$ we can speak about the generic splitting type of E and we have $E|_f \cong \mathcal{O}_f(d) \oplus \mathcal{O}_f(d')$ for a general fibre f, where $d' \leq d,\ d + d' = \alpha$. The integer d is the first numerical invariant of E.

The second numerical invariant is obtained by the following normalization:

$$-r = inf\{l \mid \exists L \in Pic(C), deg(L) = l,\ s.t. H^0(X, E(-dC_0) \otimes \pi^* L) \neq 0\}.$$

We shall denote by $M(\alpha, \beta, \gamma, d, r)$ or $M(c_1, c_2, d, r)$ or M the set of isomorphism classes of algebraic rank-2 vector bundles on X with fixed Chern classes $c_1,\ c_2$ and invariants d and r.

With these notations we have the following result (see [B]):

Theorem 1 *For every vector bundle $E \in M(c_1, c_2, d, r)$ there exist $L_1,\ L_2 \in Pic_0(C)$ and $Y \subset X$ a locally complete intersection of codimension 2 in X, or the empty set, such that E is given by an extension*

$$0 \to \mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2 \to E \to \mathcal{O}_X(d'C_0 + sf_0) \otimes \pi^* L_1 \otimes I_Y \to 0, \qquad (1)$$

where $c_1 = (\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$, $c_2 = \gamma \in \mathbb{Z}$, $d + d' = \alpha$, $d \geq d'$, $r + s = \beta$, $l(c_1, c_2, d, r) := \gamma + \alpha(de - r) - \beta d + 2dr - d^2 e = deg(Y) \geq 0$.

Remark. By applying theorem 1 we can obtain the canonical extensions used in [Br1], [Br2].
Indeed, let us suppose first that $d > d'$. From the exact sequence (1) it follows that

$$\mathcal{O}_C(r) \otimes L_2 \cong \pi_* E(-dC_0)$$

so

$$\mathcal{O}_X(rf_0) \otimes \pi^* L_2 \cong \pi^* \pi_* E(-dC_0)$$

and

$$\mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2 \cong (\pi^* \pi_* E(-dC_0))(dC_0).$$

If $d = d'$ then, by applying π_* to the short exact sequence

$$0 \to \mathcal{O}_X(rf_0) \otimes \pi^* L_2 \to E(-dC_0) \to \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Y \to 0$$

it follows the exact sequence

$$0 \to \mathcal{O}_C(r) \otimes L_2 \to \pi_* E(-dC_0) \to \mathcal{O}_C(s) \otimes L_1 \otimes \mathcal{O}_C(-Z_1) \to 0,$$

where Z_1 is an effective divisor on C with the support $\pi(Y)$. With the notation $Z = \pi^{-1}(Z_1)$, by applying π^* (π is a flat morphism) we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(rf_0) \otimes \pi^* L_2 & \longrightarrow & E(-dC_0) & \longrightarrow & \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Y & \longrightarrow & 0 \\
 & & \wr \uparrow id & & \uparrow \varphi & & \uparrow \psi & & \\
0 & \longrightarrow & \mathcal{O}_X(rf_0) \otimes \pi^* L_2 & \longrightarrow & \pi^* \pi_* E(-dC_0) & \longrightarrow & \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Z & \longrightarrow & 0
\end{array}$$

From the injectivity of ψ we obtain the injectivity of φ. Because of

$$\mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_{Y \subset Z} \cong Coker\ \psi \cong Coker\ \varphi$$

we conclude.

Recall that a set M of vector bundles on a $\mathbb{C}$–scheme X is called *bounded* if there exists an algebraic $\mathbb{C}$-scheme T and a vector bundle V on $T \times X$ such that every $E \in M$ is isomorphic with $V_t = V|_{t \times X}$ for some closed point $t \in T$.

It is well-known (see [K]) that a bounded set of vector bundles has a quotient structure of algebraic variety. For the next result see [B]:

Theorem 2 *The set $M(c_1, c_2, d, r)$ is bounded.*

2 Uniform bundles

In this section we shall use the notations from section 1.

Definition 3 A 2-vector bundle E is called an *uniform bundle* if the splitting type is preserved on all fibres of X.

Theorem 1 allows us to give a criterion for uniformness.

Lemma 4 *Let f be a fibre of X and let us suppose that $I_{Y \cap f \subset f} \cong \mathcal{O}_f(-n)$. Then $E|_f \cong \mathcal{O}_f(d+n) \oplus \mathcal{O}_f(d^{'} - n)$.*

Proof: We suppose that $E|_f \cong \mathcal{O}_f(a) \oplus \mathcal{O}_f(a^{'})$, where $a \geq a^{'}$. Then we have a surjective morphism

$$E|_f \rightarrow \mathcal{O}_f(d^{'}) \otimes I_Y \otimes \mathcal{O}_f$$

in virtue of theorem 1. On the other hand, the restriction of the sequence

$$0 \rightarrow I_Y \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$$

to f gives a surjective morphism

$$I_Y \otimes \mathcal{O}_f \rightarrow I_{Y \cap f \subset f} \cong \mathcal{O}_f(-n).$$

So, we obtain another surjective morphism

$$\mathcal{O}_f(a) \oplus \mathcal{O}_f(a^{'}) \to \mathcal{O}_f(d^{'} - n).$$

By using the inequalities $a \geq a^{'}$, $d \geq d^{'} \geq d^{'} - n$ and the equality $a + a^{'} = d + d^{'} = \alpha$ it follows that $a^{'} = d^{'} - n$ and $a = d + n$.

Corollary 5 *E is an uniform bundle if and only if the set Y (from extension (1)) is empty.*

By means of corollary 5 the uniform bundles are given by extensions of line bundles. It is naturally to ask if the converse is true. Unfortunately, this question has a negative answer, as we shall see.

Proposition 6 *On the rational ruled surface $\mathbb{F}_e$ with $e \geq 1$ there exist non-uniform bundles given by extensions of line bundles.*

For the proof we need some preparations.

Let E be a 2-vector bundle given by an extension

$$0 \to F \to E \to G \to 0, \tag{2}$$

where $F = \mathcal{O}_X(aC_0 + r^{'} f_0) \otimes \pi^* L_2^{'}$, $G = \mathcal{O}_X(a^{'} C_0 + s^{'} f_0) \otimes \pi^* L_1^{'}$ $(L_1^{'}, L_2^{'} \in Pic_0(C))$ are line bundles on X. By means of theorem 1, E sits also in a canonical extension (1). If $a \geq a^{'}$ then E is obviously uniform. So, we shall suppose that $a < a^{'}$.

Lemma 7 *With the above notations we have $d \leq a^{'}$.*

Proof: Indeed, by the restriction of the sequence (2) to a general fibre f we obtain a surjective morphism

$$\mathcal{O}_f(d) \oplus \mathcal{O}_f(d^{'}) \to \mathcal{O}_f(a^{'}).$$

If $d > a^{'}$, then it follows that $d^{'} = a^{'}$ which contradicts the inequalities $a < a^{'}$, $d \geq d^{'}$ $(a + a' = d + d')$.

Lemma 8 *If $d = a^{'}$ then E is uniform.*

Proof: Let f be a fibre of X such that the splitting type of $E|_f$ is different from the generic splitting type of E. According to lemma 4

$$E|_f \cong \mathcal{O}_f(d+n) \oplus \mathcal{O}_f(d' - n),$$

where $n > 0$.

By the restriction of (2) to f we obtain a surjective morphism

$$\mathcal{O}_f(d+n) \oplus \mathcal{O}_f(d' - n) \to \mathcal{O}_f(d).$$

Because of $d + n > d$ it follows $d' - n = d$, contradiction.

Lemma 9 *Let us suppose that $g = 0$ (i.e. X is rational). If $d = a'$ then $E \cong F \oplus G$.*

Proof: Let us observe that we can suppose, without loss of generality, that $a = 0$, $r' = 0$ (by tensoring the sequences (1) and (2) with $\mathcal{O}_X(-aC_0 - r'f_0)$). The sequences (1) and (2) become:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \mathcal{O}_X(\alpha C_0 + \beta f_0) & & & & \\
 & & & \chi \nearrow & \uparrow \varphi & & & & \\
(1') \quad 0 & \longrightarrow & \mathcal{O}_X(dC_0 + rf_0) & \xrightarrow{\psi} & E & \longrightarrow & \mathcal{O}_X(d'C_0 + sf_0) \otimes I_Y & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & \mathcal{O}_X & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

with $\alpha > 0$.

The computation of $c_2(E)$ in $(1')$ implies

$$deg(Y) = dd'e - ds - d'r.$$

Now, $d = \alpha$ and we deduce, by means of lemma 8, that $deg(Y) = 0$ and $d' = 0$, so $s = 0$ (we supposed $\alpha > 0$).

The morphism $\chi = \varphi\psi$ is non-zero, otherwise $\mathcal{O}_X(\alpha C_0 + \beta f_0) \subset \mathcal{O}_X$ (which is a contradiction) so χ is the multiplication by a $\lambda \in \mathbb{C}^*$ and the assertion follows.

In this moment, we are able to give the counter-example announced in Proposition 6.
Proof of Proposition 6 : Let G be $\mathcal{O}_X(2C_0)$ and let F be $\mathcal{O}_X$. Then:

$$dim\ H^1(G^{-1}) = e + 1 \neq 0.$$

For E given by an extension $\xi \in \mathrm{Ext}^1(G, \mathcal{O}_X)$, keeping the notations from section 1, we have $d \leq 2$ (lemma 7) , $d \geq d'$, $d + d' = 2$ and $r + s = 0$.

There are only two possibilities:
(*a*) $d = 2$, $d' = 0$, which implies $E \cong \mathcal{O}_X \oplus \mathcal{O}_X(2C_0)$ (lemma 9).
(*b*) $d = d' = 1$ and, in this case, in the canonical extension (1) of E, we have

$$deg(Y) = dd'e - ds - d'r = e \geq 1.$$

By applying corollary 5, all vector bundles given by non-zero extensions from $\mathrm{Ext}^1(G, \mathcal{O}_X)$ are non-uniform.

3 Non-emptiness of moduli spaces

For a rank-2 vector bundle E, we shall denote by d_E and r_E the invariants of E, when confusions may appear.

Theorem 10 *$M(c_1, c_2, d, r)$ is non-empty if and only if $l := l(c_1, c_2, d, r) \geq 0$ and one of the following conditions holds:*
(I) $2d > \alpha$ *or,*
(II) $2d = \alpha,\ \ \beta - 2r \leq g + l.$

Proof : We observe that if $M \neq \emptyset$ then, by means of the theorem 1, the elements of M sit between 2-vector bundles given by extensions of type (1). So, we conclude that $M \neq \emptyset$ if and only if in the extensions of type (1) there are 2-vector bundles with $d_E = d$ and $r_E = r$.

It is obvious that all the vector bundles given by an extension of type (1) have $d_E = d$ so we shall look for bundles with $r_E = r$.

We shall fix $L_1, L_2 \in Pic_0(C)$ and $Y \subset X$ a locally complete intersection (or the empty set) and we shall denote

$$N_1 = \mathcal{O}_X(d'C_0 + sf_0) \otimes \pi^* L_1$$

$$N_2 = \mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2$$

and $l = deg(Y)$.

Consider the spectral sequence of terms

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(I_Y \otimes N_1, N_2))$$

which converges to

$$\mathrm{Ext}^{p+q}(I_Y \otimes N_1, N_2).$$

We have

$$\mathcal{E}xt^0(I_Y \otimes N_1, N_2) \cong N_2 \otimes N_1^{-1} \text{ and } \mathcal{E}xt^1(I_Y \otimes N_1, N_2) \cong \mathcal{O}_Y.$$

But $H^2(X, N_2 \otimes N_1^{-1}) = 0$ so the exact sequence of lower terms becomes

$$0 \to H^1(X, N_2 \otimes N_1^{-1}) \to \mathrm{Ext}^1(I_Y \otimes N_1, N_2) \to H^0(Y, \mathcal{O}_Y) \to 0.$$

Now, by a result due to Serre (see [O-S-S], Chap.I, 5, [Se]), any element belonging to the group $\mathrm{Ext}^1(I_Y \otimes N_1, N_2)$ which has an invertible image in $H^0(Y, \mathcal{O}_Y)$ defines an extension of the desired form with E a 2-vector bundle.

We put the sequence (1) under the equivalent form

$$0 \to \mathcal{O}_X \to E(-dC_0) \otimes \pi^* L'' \to \mathcal{O}_X((d' - d)C_0 + (s - r)f_0) \otimes \pi^*(\tilde{L}) \otimes I_Y \to 0 \tag{3}$$

where $\tilde{L} = L_1 \otimes L_2^{-1}$, $L'' = \mathcal{O}_C(-r) \otimes L_2^{-1}$ and $deg(L'') = -r$.

From the definition, it follows $r \leq r_E$ for every bundle E given by an extension (1). We distinguish three cases:

(I) $d > d'$. In this case we shall prove that M is non-empty if and only if $l \geq 0$. To do this we prove that *all* vector bundles from extension (1) have $r_E = r$.

We verify that for all $L' \in Pic(C)$ with $deg(L') < 0$ we have

$$H^0(E(-dC_0) \otimes \pi^*(L'' \otimes L')) = 0,$$

which is true because $H^0(L') = 0$ and

$$H^0(\mathcal{O}_X((d' - d)C_0 + (s - r)f_0) \otimes \pi^*(L_1 \otimes L_2^{-1} \otimes L') \otimes I_Y) = 0.$$

(II) a^o. $d = d', r \geq s$. Then M is non-empty if and only if $l \geq 0$. The proof runs like in the first case with the remark $deg(\mathcal{O}_C(s-r) \otimes L_1 \otimes L_2^{-1} \otimes L') < 0$.

(II) b^o. $d = d', r < s$. Then M is non-empty if and only if $l \geq 0$ and $\beta - 2r \leq g + l$.

Let us see first that the natural isomorphism

$$M(2d,\beta,\gamma,d,r)\longrightarrow M(0,\beta,l,0,r)$$

$$E\longrightarrow E(-dC_0)$$

allows us to suppose $d = d^{'} = 0$.

In this case, the sequence (3) becomes

$$0\to\mathcal{O}_X\to E\otimes\mathcal{O}_X(-rf_0)\otimes\pi^*L_2^{-1}\to\mathcal{O}_X((s-r)f_0)\otimes\pi^*(L_1\otimes L_2^{-1})\otimes I_Y\to 0.$$

The definition of the second invariant implies that $r_E = r$ if and only if $E^{'} := \pi_* E\otimes\mathcal{O}_C(-rp_0)\otimes L_2^{-1}$ is normalised. $E^{'}$ belong to an extension

$$0\to\mathcal{O}_C\to E^{'}\to L\to 0 \tag{4}$$

where $L = \mathcal{O}_C((s-r)p_0)\otimes L_1\otimes L_2^{-1}\otimes\mathcal{O}_C(-Z_1)$ with Z_1 an effective divisor on C with support $\pi(Y)$ and $card(Y)\leq deg(Z_1)\leq l = deg(Y)$.

According to a result of Nagata ([N] or [Ha] Ex.V.2.5) , if $E^{'}$ is normalised , then

$$-deg(E^{'}) = r - s + deg(Z_1)\geq -g$$

which proves "$\Rightarrow$".

For "$\Leftarrow$" we choose Y reduced, obtained by intersection between C_0 and l distinct fibres of X. In this case, $Z_1 = \pi(Y)$, $Y\subset Z = \pi^{-1}(Z_1)$.

We have the following short exact sequence

$$0\to I_Z\to I_Y\to I_{Y\subset Z}\to 0 \tag{5}$$

and $Z_1 = p_1 + ... + p_l$, $Z = f_1 + ... + f_l$, where f_i are distinct fibres, $\mathcal{O}_Z = \mathcal{O}_{f_1}\oplus ...\oplus\mathcal{O}_{f_l}$, $I_{Y\subset Z} = \mathcal{O}_{f_1}(-1)\oplus ...\oplus\mathcal{O}_{f_l}(-1)$.

So, the sequence (5) becomes

$$0\to I_Z\to I_Y\to\mathcal{O}_{f_1}(-1)\oplus ...\oplus\mathcal{O}_{f_l}(-1)\to 0.$$

Tensoring by $K_X\otimes N_2^{-1}\otimes N_1$ and taking the long cohomology sequence we obtain an injective map:

$$H^1(K_X\otimes N_2^{-1}\otimes N_1\otimes I_Z)\longrightarrow H^1(K_X\otimes N_2^{-1}\otimes N_1\otimes I_Y).$$

By dualizing, it follows that the natural map

$$\mathrm{Ext}^1(I_Y\otimes N_1,N_2)\xrightarrow{\varphi}\mathrm{Ext}^1(I_Z\otimes N_1,N_2)\cong\mathrm{Ext}^1(L,\mathcal{O}_C)$$

is surjective, which shows that all bundles in (4) are coming from (1) by applying π_*.

According to [Ha] (Ex. V.2.5), there is a *non-empty* open set $V\subset \mathrm{Ext}^1(L,\mathcal{O}_C)$ (don't forget the condition $s-r\le g+l$!) such that all $\xi\in V$ define normalised vector bundles on C.

Now, in $\mathrm{Ext}^1(I_Y\otimes N_1,N_2)$ the set of vector bundles is a non-empty open set U. It is clear that $\varphi^{-1}(V)\cap U\neq\emptyset$ (being open sets in Zariski topology), so we conclude.

4 Moduli of stable bundles

There is an interesting relation between the moduli spaces $M(c_1,c_2,d,r)$ and the Qin's sets $E_\zeta(c_1,c_2)$ (see [Q1], [Q2] for precised definitions).

As in the proof of theorem 10, case (I) we conclude that if ζ is a normalized class reprezenting a non-empty wall of type (c_1,c_2) such that $l_\zeta(c_1,c_2)>0$ then, for $(2d-\alpha,2r-\beta)=\zeta$, $E_\zeta(c_1,c_2)$ and $M(c_1,c_2,d,r)$ are coincident modulo a factor of $Pic_0(C)$ (Qin workes with first Chern class c_1 as an element in $Pic(X)$).

This is a consequence of the following facts:

(a) $l_\zeta(c_1,c_2)=l(c_1,c_2,d,r)$

(b) condition $\zeta^2<0$ implies $2d>\alpha$

(c) in the case $2d>\alpha$ the bundles L_1,L_2 and the set Y from the sequence (1) are uniquely determined by E.

(d) if $l(c_1,c_2,d,r)>0$ then in the sequence (1) the bundles are given only by non-trivial extensions.

In fact it is not hard to see that $M(c_1,c_2,d,r)\neq\emptyset$ iff $E_\zeta(c_1,c_2)\neq\emptyset$ so, by means of theorem 10, $E_\zeta(c_1,c_2)\neq\emptyset$ if $l_\zeta(c_1,c_2)>0$. But we have even more:

Corollary 11 *Let X be a ruled surface different from $\mathbb{P}^1\times\mathbb{P}^1$ and let C be a chamber of type (c_1,c_2) different from C_{f_0}. Then the moduli space $\mathcal{M}_C(c_1,c_2)\neq\emptyset$.*

Proof: From theorem 1.3.3 in [Q2] it follows that

$$\mathcal{M}_C(c_1,c_2)=(\mathcal{M}_{C_1}(c_1,c_2)-\bigsqcup_\zeta E_{(-\zeta)}(c_1,c_2))\bigsqcup_\zeta E_\zeta(c_1,c_2)\ ,$$

where C_1 is the chamber lying above C and sharing with C a non-empty common wall W and ζ runs over all normalised classes representing W. By the above considerations, it follows that $E_\zeta(c_1, c_2) \neq \emptyset$ if $l(c_1, c_2, d, r) > 0$. It remains the case $l(c_1, c_2, d, r) = 0$ and it will be sufficient to prove that

$$h^1(X, N_2 \otimes N_1^{-1}) := dim\ H^1(X, N_2 \otimes N_1^{-1}) > 0$$

(see the proof of Theorem 10).

We have

$$N_2 \otimes N_1^{-1} = \mathcal{O}_X((d-d')C_0 + (r-s)f_0) \otimes \pi^*(L_2 \otimes L_1^{-1})\ ,$$

where $d - d' = 2d - \alpha = u$ and $r - s = 2r - \beta = v$. But $\zeta = uC_0 + vf_0$ is a normalized class and this implies that $u > 0$ and $v < 0$ (see [Q1]).

Because $H^2(X, N_2 \otimes N_1^{-1}) = 0$, the Riemann-Roch Theorem gives the equality:

$$\chi = h^0(X, N_2 \otimes N_1^{-1}) - h^1(X, N_2 \otimes N_1^{-1}) = 1 - g + (1/2)((u+1)(2v - ue) + u(2-2g)).$$

But $\zeta^2 < 0$ gives $u(2v - ue) < 0$; it follows $2v - ue < 0$.

If $g \geq 1$, then obviously $\chi < 0$. If $g = 0$, then $e \geq 0$ and

$$\chi = 1 + v + (u/2)(2(v+1) - e(u+1)).$$

If $e \geq 1$, then $\chi < 0$. For $e = 0$ we get $X = \mathbb{P}^1 \times \mathbb{P}^1$, which we excluded. Thus, in all cases $\chi < 0$; it follows $h^1(X, N_2 \otimes N_1^{-1}) > 0$ and the proof is over.

Remark Let us suppose that $X = \mathbb{P}^1 \times \mathbb{P}^1$ and that C is a chamber of type (c_1, c_2) lying below a non-empty wall defined by a normalized class $\zeta = uC_0 + vf_0$ with $v \leq -2$. Then the same conclusion as in the above corollary holds.

Indeed, in this case we have $\chi = (1+v)(1+u)$. Since $v < -1$, then again $\chi < 0$.

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