On the functional equation related to the quantum three-body problem

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Abstract

In the present paper we give the general nondegenerate solution of the functional equation

$$(f(x)+g(y)+h(z))^2 = F(x)+G(y)+H(z),$$

$$x+y+z=0,$$

which related to the exact factorized ground-state wave function for the quantum one-dimensional problem of three different particles with pair interaction.

The purpose of the present paper is to give the general nondegenerate solution of the functional equation

$$(f(x)+g(y)+h(z))^2 = F(x)+G(y)+H(z), \tag{1}$$

$$x+y+z=0,$$

which related to the exact factorized ground-state wave function for the quantum one-dimensional problem of three different particles with pair interaction. We obtain this result as a consequence of a general analytic solution of the following functional equations:

$$\varphi(x+y) = \eta(x)+\eta(y) - \frac{\gamma(x)-\gamma(y)}{\xi(x)-\xi(y)} \tag{2}$$

$$\varphi(x+y) = \varphi(x)+\varphi(y)+\tau(x)\tau(y)A(x+y). \tag{3}$$

Such equations are also interesting from mathematical point of view as new examples of non-classical addition theorems.

1. Let us remind first that an analogous (but more simple) equation for the special case of three identical particles was considered earlier by B. Sutherland [1] and F.Calogero [2]. Namely, in the paper [1] the one-dimensional many-body problem of n identical particles with pair interaction was considered, whose exact ground-state wave function $\Psi_0(x_1, x_2, \ldots, x_n)$ is factorized

$$\Psi_0(x_1, x_2, \ldots, x_n) = \prod_{j<k} \psi(x_j - x_k).$$

It was shown that the logarithmic derivative of $\psi(x)$

$$f(x) = \psi'(x)/\psi(x)$$

should satisfy the functional equation

$$f(x)f(y)+f(y)f(z)+f(z)f(x) = F(x)+F(y)+F(z), \tag{4}$$

$$x+y+z=0,$$

where

$$f(-x) = -f(x), \quad F(-x) = F(x).$$

In [1], partial solutions of equation (4) was also found.

The general solution of equation (4) was found in [2] (see also [3] for review of this and related problems). This solution has the form

$$f(x) = \alpha\zeta(x; g_2, g_3) + \beta x, \tag{5}$$

$$F(x) = \alpha^2 \wp(x; g_2, g_3), \tag{6}$$

where $\zeta(x)$ and $\wp(x)$ are the Weierstrass functions (see for instance [4]).

In the present paper we consider only the three-body problem, but in the general case when all three particles are different from each other.

In this case the ground-state wave function has the form

$$\Psi_0(x_1, x_2, x_3) = \psi_1(x_2 - x_3)\psi_2(x_3 - x_1)\psi_3(x_1 - x_2) \tag{7}$$

and satisfies the Schrödinger equation

$$-\Delta\Psi_0 + U\Psi_0 = E_0\Psi_0, \tag{8}$$

$$U = u_1(x_2 - x_3) + u_2(x_3 - x_1) + u_3(x_1 - x_2). \tag{9}$$

Substituting Ψ_0 from (7) into (8), we obtain

$$\begin{aligned} \Psi_0^{-1}\Delta\Psi_0 = U - E_0 = 3(f_1^2(x_2 - x_3) + f_2^2(x_3 - x_1) + f_3^2(x_1 - x_2)) \\ -(f_1(x_2 - x_3) + f_2(x_3 - x_1) + f_3(x_1 - x_2))^2 \\ +2(f_1'(x_2 - x_3) + f_2'(x_3 - x_1) + f_3'(x_1 - x_2)); \\ f_j = \psi_j'/\psi_j. \end{aligned} \tag{10}$$

Hence, for the potential energy $U(x_1, x_2, x_3)$ to have the form of pair interactions (9), the three functions

$$f(x) = f_1(x), \quad g(y) = f_2(y), \quad h(z) = f_3(z) \tag{11}$$

must satisfy the functional equation

$$\begin{aligned} (f(x) + g(y) + h(z))^2 = F(x) + G(y) + H(z), \\ x + y + z = 0. \end{aligned} \tag{12}$$

The following expression for the potential energies results from (10)–(12).

$$\begin{aligned} u_1(x) &= 3f^2(x) + 2f'(x) - F(x) + \varepsilon_1, \\ u_2(x) &= 3g^2(x) + 2g'(x) - G(x) + \varepsilon_2, \\ u_3(x) &= 3h^2(x) + 2h'(x) - H(x) + \varepsilon_3. \\ \varepsilon_1 &+ \varepsilon_2 + \varepsilon_3 = E_0 \end{aligned} \tag{13}$$

2. Let us consider the meromorphic solutions of the equation

$$(f(x) + g(y) + h(z))^2 = F(x) + G(y) + H(z) \tag{14}$$

$$x + y + z = 0$$

.

Definition. Let us call the solution of Eq. (14) nondegenerate, if each one of the functions f(x), g(x) and h(x) have the pole in finite domain of complex x-plane.

The main result of this paper is the following

Theorem. *The general nondegenerate solution of the equation (14) in the class of meromorphic functions has the form*

$$f(x) = \alpha\zeta(x - a_1; g_2, g_3) + \beta x + \gamma_1, \tag{15}$$

$$g(x) = \alpha\zeta(x - a_2; g_2, g_3) + \beta x + \gamma_2. \tag{16}$$

$$h(x) = \alpha\zeta(x - a_3; g_2, g_3) + \beta x + \gamma_3. \tag{17}$$

$$F(x) = \alpha^2\wp(x - a_1; g_2, g_3) + 2\gamma\alpha\zeta(x - a_1; g_2, g_3) + \gamma^2. \tag{18}$$

$$G(x) = \alpha^2\wp(x - a_2; g_2, g_3) + 2\gamma\alpha\zeta(x - a_2; g_2, g_3) + \gamma^2. \tag{19}$$

$$H(x) = \alpha^2\wp(x - a_3; g_2, g_3) + 2\gamma\alpha\zeta(x - a_3; g_2, g_3) + \gamma^2. \tag{20}$$

where

$$\begin{aligned} a_1 + a_2 + a_3 &= 0, \\ \gamma_1 + \gamma_2 + \gamma_3 &= \gamma. \end{aligned} \tag{21}$$

Proof . The proof of the theorem is divided on several steps.

Let us begin with

Lemma 1. *The functions $(f(x), g(y), h(z))$ satisfy equation (14) for the corresponding functions $(F(x), G(y), H(z))$ if and only if these functions satisfy also the equation*

$$\begin{vmatrix} f''(x) & g''(y) & h''(z) \\ f'(x) & g'(y) & h'(z) \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{22}$$

under condition: $x + y + z = 0$.

Proof. Let us apply the operator $\partial_- \cdot \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial x}$, where $\partial_- = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ to eq.(14). This gives:

$$\frac{\partial}{\partial x}: 2(f'(x) - h'(z))(f(x) + g(y) + h(z)) = F'(x) - H'(z), \tag{23}$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}: 2h''(z)(f(x) + g(y) + h(z)) + 2(f'(x) - h'(z))(g'(y) - h'(z)) = H''(z), \tag{24}$$

$$\partial_- \frac{\partial}{\partial y}\frac{\partial}{\partial x}: h''(z)(f'(x) - g'(y)) + f''(x)(g'(y) - h'(z)) + g''(y)(h'(z) - f'(x)) = 0. \tag{25}$$

Here we use the fact that ∂_- is the differentiation operator, and that $\partial_- h'(z) = \partial_- h''(z) = 0$.

Hence, if functions $(f(x), g(y), h(z))$ satisfy equation (14), then these functions satisfy also equation (25), that can be obviously rewritten in the form (22).

Conversely, let the functions $(f(x), g(y), h(z))$ satisfy (22) and, consequently, (25). The equation (25) may be rewritten as

$$\partial_-[h''(z)(f(x) + g(y) + h(z)) + (f'(x) - h'(z))(g'(y) - h'(z))] = 0;$$

then there is a function $H_1(z)$ satisfying the following equation:

$$h''(z)(f(x) + g(y) + h(z)) + (f'(x) - h'(z))(g'(y) - h'(z)) = H_1(z) \tag{26}$$

Let us note that eq.(26) is equivalent to the equation

$$\frac{\partial}{\partial y}[(f'(x) - h'(z))(f(x) + g(y) + h(z))] = H_1(z)$$

Therefore, there are functions $F_1(x)$ and $H_2(z)$ such that $H_2'(z) = H_1(z)$, and

$$(f'(x) - h'(z))(f(x) + g(y) + h(z)) = F_1(x) - H_2(z). \tag{27}$$

On the other hand, equation (27) is equivalent to

$$\frac{\partial}{\partial x}(f(x) + g(y) + h(z))^2 = 2(F_1(x) - H_2(z)),$$

i. e. there are functions $F(x), G(y)$ and $H(z)$ such that $F'(x) = 2F_1(x), H'(z) = 2H_2(z)$, and

$$(f(x) + g(y) + h(z))^2 = F(x) + G(y) + H(z).$$

Thus Lemma 1 is proved.

Lemma 2. *Equation (14) is invariant under the following transformations:*

$$f(x) \to f_0 + a_1 x + a_2 f(a_3 x + \alpha_1),$$

$$F(x) \to F_0 + a_4 x + a_2^2 F(a_3 x + \alpha_1) + 2a_2 c f(a_3 x + \alpha_1),$$

$$g(y) \to g_0 + a_1 y + a_2 g(a_3 y + \alpha_2).$$

$$G(y) \to G_0 + a_4 y + a_2^2 G(a_3 y + \alpha_2) + 2a_2 c g(a_3 y + \alpha_2),$$

$$h(z) \to h_0 + a_1 z + a_2 h(a_3 z + \alpha_3),$$

$$H(z) \to H_0 + a_4 z + a_2^2 H(a_3 z + \alpha_3) + 2a_2 c h(a_3 z + \alpha_3),$$

where $a_k (k = 1, \dots, 4)$ and c are free parameters

$$f_0 + g_0 + h_0 = c, \quad F_0 + G_0 + H_0 = c^2, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{28}$$

This Lemma is proved by a direct calculation .

Corollary 3. *Taking corresponding values of the parameters $\alpha_1, \alpha_2, \alpha_3$,one can prove that all the functions $(f(x), g(y), h(z))$, $(F(x), G(y), H(z))$ are regular at $x = 0, y = 0, z = 0$, respectively.*

The proof follows from the fact that the set of poles of a meromorphic function of one complex variable is discrete. Thus in what follows we may suppose that all the functions are regular at $x = 0, y = 0, z = 0$.

Definition 4. *Let us call the solution of equation (14) totally degenerate, if at least one of functions $f(x), g(x)$, and $h(x)$ is linear.*

The next Lemma describes all totally degenerate solutions of equation (14).

Lemma 5. *Let $(f(x), g(y), h(z))$, $(F(x), G(y), H(z))$ be a totally degenerate solution of the equation (14).*

Three cases are possible.

1. All three functions $f(x), g(y), h(z)$ are linear. Then

$$f(x) = f_0 + f_1 x, \quad g(y) = g_0 + g_1 y, \quad h(z) = h_0 + h_1 z,$$

$$F(x) = F_0 + F_1 x + (f_1 - g_1)(f_1 - h_1)x^2, \quad G(y) = G_0 + G_1 y + (g_1 - f_1)(g_1 - h_1)y^2,$$
$$H(z) = H_0 + H_1 z + (h_1 - g_1)(h_1 - f_1)z^2$$

Here $f_0, f_1, g_0, g_1, h_0, h_1$ are free parameters.

Let $f_0 + g_0 + h_0 = c$. Then

$$F_0 + G_0 + H_0 = c^2, \quad F_1 = b + 2cf_1, \quad G_1 = b + 2cg_1, \quad H_1 = b + 2ch_1$$

and b is a free parameter.

2. Two of the functions $f(x), g(y), h(z)$ are linear. For example, it is $g(y) = g_0 + g_1 y, \quad h(z) = h_0 + h_1 z$. Then $f(x)$ is an arbitrary function, $g(y) = g_0 + ay, \quad h(z) = h_0 + az, \quad G(y) = G_0 + by, \quad H(z) = H_0 + bz$ and

$$F(x) = [g_0 + h_0 - ax + f(x)]^2 - (G_0 + H_0 - bx).$$

Here g_0, h_0, a, b, G_0, H_0 are free parameters.

3. Only one of the functions $f(x), g(y), h(z)$ is linear. For example, $h(z) = h_0 + h_1 z$. Then

$$f(x) = f_0 + ax + c_1 \exp(\lambda x), \quad g(y) = g_0 + ay + c_2 \exp(\lambda y), \quad h(z) = h_0 + az,$$

$$F(x) = F_0 + bx + c_1 \exp(\lambda x)(2c + c_1 \exp(\lambda x)),$$
$$G(y) = G_0 + by + c_2 \exp(\lambda y)(2c + c_2 \exp(\lambda y)),$$
$$H(z) = H_0 + bz + 2c_1 c_2 \exp(-\lambda z).$$

Here $a, b, c, c_1, c_2, \lambda$ are free parameters, and

$$f_0 + g_0 + h_0 = c, \quad F_0 + G_0 + H_0 = c^2.$$

Proof.

Case 1. It follows from (22) that $f(x), g(y), h(z)$ are arbitrary linear functions. A form of the functions $F(x), G(y), H(z)$ can be reconstructed directly from (14), taking into account the identity $2xy = z^2 - x^2 - y^2$.

Case 2 . We obtain from (22)

$$f''(x)(g_1 - h_1) = 0$$

If $f''(x) \neq 0$, then $g_1 = h_1$ and $f(x)$ is arbitrary. The form of the functions $F(x), G(y), H(z)$ can be reconstructed immediately.

Case 3 . We get from (24):

$$2(f'(x) - h_1)(g'(y) - h_1) = H''(-x - y).$$

If $f'(x)$ and $g'(y)$ are not constants, then according to the classical Cauchy–Pexider result [6] (see also [7]) we obtain :

$$f'(x) - h_1 = \tilde{c}_1 \exp(\lambda x), \quad g'(y) - h_1 = \tilde{c}_2 \exp(\lambda x).$$

where $\tilde{c}_1, \tilde{c}_2$ and λ are free parameters.

Therefore

$$f(x) = f_0 + h_1 x + c_1 \exp(\lambda x), \quad g(y) = g_0 + h_1 y + c_2 \exp(\lambda y),$$

where $c_k = \tilde{c}_k/\lambda, k = 1, 2$. The form of the functions $F(x), G(y), H(z)$ can be reconstructed easily. The Lemma is proved.

So, further we can consider solutions of (14) which are not totally degenerate.

Lemma 6. *On choosing the appropriate values of the parameters $f_0, g_0, h_0, a_1, F_0, G_0$ (see Lemma 2) we obtain*

$$f(0) = g(0) = h(0) = 0, \quad h'(0) = 0, \quad F(0) = G(0). \tag{29}$$

The proof is easy.

Lemma 7. *An appropriate choice of the parameters α_1 and α_2 leads to the relation*

$$f(x) \neq g(x)$$

.

Proof. Suppose on the contrary that

$$f(x + \alpha_1) - f(\alpha_1) \equiv g(x + \alpha_2) - g(\alpha_2) \tag{30}$$

for all α_1 and α_2 in any neighbourhood of the point $x = 0$. On differentiating (30), we obtain

$$\frac{\partial f(x + \alpha_1)}{\partial x} = \frac{\partial f(x + \alpha_1)}{\partial \alpha_1} = f'(\alpha_1),$$

i. e.

$$f(x + \alpha_1) = f'(\alpha_1)x + f(\alpha_1).$$

in contradiction to the assumption about the nondegeneracy of the solution. The Lemma is proved.

Hence, it is sufficient to find all solutions of eq. (1) which are not totally degenerate and satisfy the following additional conditions:

$$f(x) \neq g(x), \quad f(0) = g(0) = h(0), \quad h'(0) = 0, \quad F(0) = G(0) = 0.$$

Then, using the transformations from Lemma 2 we obtain the general nondegenerate solution. In what follows only the nondegenerate solutions of (14) satisfying the above additional conditions will be considered.

Interchanging x and y in eq. (14), we obtain

$$(f(y)+g(x)+h(z))^2 = F(y)+G(x)+H(z). \tag{31}$$

Subtracting (31) from (14) we see that

$$[(f(x)-g(x))-(f(y)-g(y))][(f(x)+g(x))+(f(y)+g(y))+2h(z)] \\ = (F(x)-G(x))-(F(y)-G(y)) \,.$$

Hence,

$$\varphi(x+y) = \eta(x)+\eta(y) - \frac{\gamma(x)-\gamma(y)}{\xi(x)-\xi(y)}, \tag{32}$$

where

$$\varphi(x) = -2h(-x),$$
$$\eta(x) = f(x)+g(x),$$
$$\xi(x) = f(x)-g(x),$$
$$\gamma(x) = F(x)-G(x),$$

and $\varphi(0)=\varphi'(0)=\eta(0)=\gamma(0)=\xi(0)=0$ and $\varphi''(x)\neq 0$.

Definition 8. *Let us call solution $(\varphi,\eta,\xi,\gamma)$ of the equation (32) normalized, if the following initial conditions are satisfied:*

$$\xi'(0)=1, \quad \eta'(0)=0.$$

Lemma 9. *The map*

$$(\varphi,\eta,\xi,\gamma) \to (\varphi,\eta+b_1\xi, b_2\xi, b_2(\gamma+b_1\xi^2)), \tag{33}$$

where b_1 and b_2 are parameters, $b_2\neq 0$, defines a group action. Each orbit of this group contains one and only one normalized solution.

Proof. The first statement may be checked by a direct computation. To proof the second statement, let us differentiate eq. (32) with respect to y. At the point $y=0$ we have:

$$\varphi'(x) = \eta'(0) + \frac{\gamma'(0)}{\xi(x)} - \xi'(0)\frac{\gamma(x)}{\xi(x)^2},$$

Assuming $\varphi(x)$ is regular at $x = 0$ and $\varphi''(x) \neq 0$, it is easy to check that $\xi'(0) \neq 0$.

Applying the transformation (33) with $b_2 = (\xi'(0))^{-1}, b_1 = -\eta'(0)/\xi'(0)$ to the solution $(\varphi, \eta, \xi, \gamma)$, we obtain a normalized solution, and the Lemma is proved.

In what follows solutions are assumed to be normalized, unless the contrary is asserted. Let us now consider the functional equation

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \tau(x)\tau(y)A(x+y), \tag{34}$$

$$\varphi(0) = \varphi'(0) = \tau(0) = \tau''(0) = 0, \quad \tau'(0) = 1.$$

Lemma 10. *For any solution $(\varphi, \eta, \xi, \gamma)$ of the eq. (32), there is a unique solution (φ, τ, A) of the eq. (34) such that*

$$\xi(x) = \frac{\tau(x)}{\tau'(x) - b_3\tau(x)}, \tag{35}$$

$$\eta(x) = \varphi(x) - \varphi'(x)\xi(x), \tag{36}$$

$$\gamma(x) = -\varphi'(x)\xi(x)^2, \tag{37}$$

where $b_3 = \xi''(0)$ is a free parameter.

Proof. Let (φ, τ, A) is some solution of the eq. (34). Then acting on (34) by the operator $\partial_- = (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})$ we obtain

$$0 = \varphi'(x) - \varphi'(y) + (\tau'(x)\tau(y) - \tau(x)\tau'(y))A(x+y),$$

i. e.

$$A(x+y) = -\frac{\varphi'(x) - \varphi'(y)}{\tau'(x)\tau(y) - \tau(x)\tau'(y)}. \tag{38}$$

Hence, we transform the eq. (34) to the equation

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \tau(x)\tau(y)\frac{\varphi'(x) - \varphi'(y)}{\tau(x)\tau'(y) - \tau'(x)\tau(y)}. \tag{39}$$

On the other hand,

$$\frac{\tau(x)\tau(y)}{\tau(x)\tau'(y) - \tau'(x)\tau(y)} = \frac{\tau(x)}{\tau'(x)}\frac{\tau(y)}{\tau'(y)}\frac{1}{(\frac{\tau(x)}{\tau'(x)} - b_3) - (\frac{\tau(y)}{\tau'(y)} - b_3)} = \frac{\xi(x)\xi(y)}{\xi(x) - \xi(y)},$$

where the function $\xi(x)$ may be expressed in terms of $\tau(x)$ by the formula (35) with a free parameter b_3.

Therefore,

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \xi(x)\xi(y)\frac{\varphi'(x) - \varphi'(y)}{\xi(x) - \xi(y)}.$$

On substituting the expressions for $\eta(x)$ and $\gamma(x)$ from (36) and (37) we obtain a solution $(\varphi, \eta, \xi, \gamma)$ of the eq. (32).

Let now $(\varphi, \eta, \xi, \gamma)$ be a solution of eq. (32). On substituting $y = 0$ in eq. (32) we obtain

$$\varphi(x) = \eta(x) - \frac{\gamma(x)}{\xi(x)},$$

i.e. $\gamma(x) = \xi(x)\delta(x)$, where $\delta(x) = \eta(x) - \varphi(x)$, and our initial conditions $\varphi'(0) = \eta'(0) = 0 = \varphi(0) = \eta(0)$ are satisfied.

Hence, $\gamma'(0) = 0$, and from the formula for $\varphi'(x)$ obtained in the course of the proof of Lemma 9 we have

$$\gamma(x) = -\varphi'(x)\xi^2(x),$$

$$\eta(x) = \varphi(x) - \varphi'(x)\xi(x),$$

as asserted in (36) and (37). Let us note that formula (35) may be considered as the differential equation for the function $\tau(x)$. Solving this equation at initial conditions $\tau(0) = 0, \tau'(0) = 1$ we obtain the function $\tau(x)$, if, moreover, we take $b_3 = \xi''(0)$ it will satisfy the condition $\tau''(0) = 0$.

Substituting now the expressions for $\xi(x), \eta(x), \gamma(x)$ into eq. (32) we obtain eq. (39).

Let us apply the operator ∂_- to the eq. (39); we obtain

$$\partial_-\left(\frac{\varphi'(x) - \varphi'(y)}{\tau(x)\tau'(y) - \tau'(x)\tau(y)}\right) \equiv 0.$$

Thus it is shown that the functions $\varphi(x)$ and $\tau(x)$ determine the function $A(x)$ given by the expression (38). The Lemma is proved.

So it was shown, how it is possible to construct all the solutions of the eq. (32) using the solutions of eq. (34).

Now we describe the general analytical solution of equation (34).

Lemma 11. *Let (φ, τ, A) be a solution of equation (34). (Let us remind that $\varphi(0) = \varphi'(0) = \tau(0) = \tau''(0) = 0$ and $\tau'(0) = 1$.) Then the function $u(x) = \varphi'(x)$ is a solution of the differential equation*

$$(u')^2 = c_3u^3 + 4c_2u^2 + 2c_1u + c_0^2, \tag{40}$$

$$u(0) = 0, \quad u'(0) = c_0.$$

The functions $\tau(x)$ and $A(x)$ satisfy the following equations:

$$\frac{\tau'(x)}{\tau(x)} = \frac{1}{2} \quad \frac{u'(x) + c_0}{u(x)}, \tag{41}$$

$$\frac{A'(x)}{A(x)} = \frac{1}{2} \quad \frac{u'(x) - c_0}{u(x)}. \tag{42}$$

If $c_0 = 0$ then $u(x) = \frac{c_1}{2}\tau(x)^2$, and $A(x) = \frac{c_1}{2}\tau(x)$, so that if $c_0 = 0$ then $c_1 \neq 0$.

Proof. Let us consider the first three derivatives with respect to y of the eq. (34)

$$\varphi'(x+y) = \varphi'(y) + \tau(x)[\tau'(y)A(x+y) + \tau(y)A'(x+y)],$$

$$\varphi''(x+y) = \varphi''(y) + \tau(x)[\tau''(y)A(x+y) + 2\tau'(y)A'(x+y) + \tau(y)A''(x+y)],$$

$$\varphi'''(x+y) = \varphi'''(y)+\tau(x)[\tau'''(y)A(x+y)+3\tau''(y)A'(x+y)+3\tau'(y)A''(x+y)+\tau(y)A'''(x+y)].$$

Taking $y = 0$ and making use of the initial conditions for $\varphi(x)$ and $\tau(x)$, we obtain

$$\varphi'(x) = \tau(x)A(x), \tag{43}$$

$$\varphi''(x) = \varphi''(0) + 2\tau(x)A'(x), \tag{44}$$

$$\varphi'''(x) = \varphi'''(0) + \tau(x)[\tau'''(0)A(x) + 3A''(x)]. \tag{45}$$

Let $\varphi_k = \varphi^{(k)}(0)$ and $\tau_3 = \tau'''(0)$. From (43) and (44) we obtain

$$\frac{\varphi''(x) - \varphi_2}{\varphi'(x)} = 2 \quad \frac{A'(x)}{A(x)}; \tag{46}$$

from (45) and (43) it follows that

$$\frac{\varphi'''(x) - \varphi_3}{\varphi'(x)} = \frac{\tau_3 A(x) + 3A''(x)}{A(x)}. \tag{47}$$

Making use of the identity

$$\frac{A''}{A} = (\frac{A'}{A})' + (\frac{A'}{A})^2$$

for the quantity $\varphi'(x) = u(x)$, we obtain the following equation (see eq. (46), (47)):

$$\frac{u'' - \varphi_3}{u} = \tau_3 + 3(\frac{1}{2} \quad \frac{u' - \varphi_2}{u})' + \frac{3}{4} \quad (\frac{u' - \varphi_2}{u})^2.$$

This equation may be rewritten as follows:

$$4(u''-\varphi_3)u = 4\tau_3 u^2 + 6[uu'' - u'(u'-\varphi_2)] + 3(u'-\varphi_2)^2,$$

$$2uu'' - 3(u')^2 + 4\tau_3 u^2 + 4\varphi_3 u + 3\varphi_2^2 = 0. \qquad (48)$$

Let

$$\tau_3 = c_2, \quad \varphi_3 = c_1, \quad \varphi_2 = c_0.$$

The equation (48) admit the integrable factor $u^{-4}u'$ and may be reduced to the equation

$$(u^{-3}(u')^2)' = 4c_2(u^{-1})' + 2c_1(u^{-2})' + c_0^2(u^{-3})'. \qquad (49)$$

Integrating (49) and multiplying the result by u^3 we obtain eq. (40), where c_3 is the integration constant. Equation (42) follows from (46). Then from eq. (43) we obtain:

$$u'(x) = \tau'(x)A(x) + \tau(x)A'(x).$$

It follows from (44) that

$$\tau(x)A'(x) = \frac{u'(x) - c_0}{2}$$

Making use of this fact, we obtain

$$\tau'(x)A(x) = \frac{u'(x) + c_0}{2}.$$

Let us divide this eq. to eq.(43) : $\tau(x)A(x) = u(x)$ we come to the equation (41). Note that if $c_0 = 0$ equations (41),(42), and conditions $\tau(0) = 0, \tau'(0) = 1$ imply

$$u(x) = \frac{c_1}{2}\tau(x)^2, \quad A(x) = \frac{c_1}{2}\tau(x),$$

and it follows, in particular, that $c_1 \neq 0$ if $c_0 = 0$. The Lemma is proved.

Consider the Weierstrass function $\wp(x)$ with parameters g_2 and g_3. We have

$$\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3.$$

Lemma 12. *The general solution of the equation (40) may be written in one of the following equivalent forms:*

$$u(x) = \frac{4}{c_3}(\wp(x+\alpha) - \wp(\alpha)), \qquad (50)$$

$$u(x) = c_1\psi(x) + \frac{c_0^2 c_3}{2}\psi(x)^2 + c_0\psi'(x), \qquad (51)$$

where

$$\psi(x) = \frac{1}{2} \quad \frac{1}{\wp(x) - \frac{1}{3}c_2} \tag{52}$$

Here $\wp(x)$ is the Weierstrass function with parameters:

$$g_2 = 3(\frac{2c_2}{3})^2 - \frac{c_1c_3}{2}, \quad g_3 = -(\frac{2c_2}{3})^3 + \frac{c_1c_2c_3}{6} - (\frac{c_0c_3}{4})^2, \tag{53}$$

and

$$\wp(\alpha) = \frac{c_2}{3}, \quad \wp'(\alpha) = \frac{c_0c_3}{4}.$$

Proof. Formula (50) gives:

$$(u'(x))^2 = \frac{16}{c_3^2}[4\wp(x+\alpha)^3 - g_2\wp(x+\alpha) - g_3].$$

On the other hand,

$$(u'(x))^2 = c_3[\frac{4}{c_3}(\wp(x+\alpha)-\wp(\alpha))]^3 + 4c_2[\frac{4}{c_3}(\wp(x+\alpha)-\wp(\alpha)]^2 + 2c_1[\frac{4}{c_3}(\wp(x+\alpha)-\wp(\alpha))] + c_0^2.$$

Hence

$$16[4\wp(x+\alpha)^3 - g_2\wp(x+\alpha) - g_3] =$$
$$4^3[\wp(x+\alpha) - \wp(\alpha)]^3 + 4^3c_2[\wp(x+\alpha) - \wp(\alpha)]^2 + 8c_1c_3[\wp(x+\alpha) - \wp(\alpha)] + c_0^2c_3^2.$$

Let us compare the coefficients of the terms of the same degree in $\wp(x+\alpha)$. This shows that formula (50) with parameters g_2, g_3 follows from (53). To deduce (51) from (50) one makes use of the addition theorem for the $\wp$-function (cf., e.g., [4]).

$$\wp(x+\alpha) - \wp(\alpha) = -(\wp(x) + 2\wp(\alpha)) + \frac{1}{4}(\frac{\wp'(x) - \wp'(\alpha)}{\wp(x) - \wp(\alpha)})^2.$$

therefore

$$(\wp(x+\alpha) - \wp(\alpha))(\wp(x) - \wp(\alpha))^2 = -(\wp(x) + 2\wp(\alpha))(\wp(x)^2 - 2\wp(x)\wp(\alpha) + \wp(\alpha)^2) +$$
$$\frac{1}{4}(4\wp(x)^3 - g_2\wp(x) - g_3 - 2\wp'(x)\wp'(\alpha) + \wp'(\alpha)^2) =$$
$$3\wp(x)\wp(\alpha)^2 - 2\wp(\alpha)^3 - \frac{g_2}{4}\wp(x) - \frac{1}{4}g_3 - \frac{1}{2}\wp'(x)\wp'(\alpha) + (\frac{\wp'(\alpha)}{2})^2 =$$
$$(3\wp(\alpha)^2 - \frac{1}{4}g_2)(\wp(x) - \wp(\alpha)) - \frac{1}{2}\wp'(x)\wp'(\alpha) + \frac{\wp'(\alpha)^2}{2}.$$

Hence,

$$\wp(x+\alpha)-\wp(\alpha)=\frac{1}{2}\frac{\wp'(x)}{(\wp(x)-\wp(\alpha))^2}\wp'(\alpha)+\frac{3\wp(\alpha)^2-\frac{1}{4}g_2}{\wp(x)-\wp(\alpha)}+\frac{1}{2}(\frac{\wp'(\alpha)}{\wp(x)-\wp(\alpha)})^2. \quad (54)$$

This gives:

$$\wp'(\alpha)=\frac{c_0c_3}{4},\quad 3\wp(\alpha)^2-\frac{1}{4}g_2=\frac{c_1c_3}{8}.$$

Formula (51) follows from eq.(54) on dividing by $\frac{1}{4}c_3$. The Lemma is proved.

Let

$$u_*(x)=\lim_{c_3\to 0}u(x),\quad \psi_*(x)=\lim_{c_3\to 0}\psi(x),\quad \wp_*(x)=\lim_{c_3\to 0}\wp(x).$$

Corollary 13. *If $c_3\to 0$, then the general solution of eq. (40) tends to the function*

$$u_*(x)=c_1(\frac{\cosh 2\sqrt{c_2}x-1}{(2\sqrt{c_2})^2})+c_0\frac{\sinh 2\sqrt{c_2}x}{2\sqrt{c_2}}. \quad (55)$$

Proof. By Lemma 12, the function $\wp_*(x)$ satisfies the equation

$$(\wp_*'(x))^2=4\wp_*(x)^3-3\Big(\frac{2c_2}{3}\Big)^2\wp_*(x)+\Big(\frac{2c_2}{3}\Big)^3=4\Big(\wp_*(x)-\frac{c_2}{3}\Big)^2\Big(\wp_*(x)+\frac{2}{3}c_2\Big).$$

Therefore

$$(\psi_*'(x))^2=\frac{1}{4}\quad(\frac{-\wp_*'(x)}{(\wp_*(x)-\frac{1}{3}c_2)^2})^2=\frac{\wp_*(x)+\frac{2}{3}c_2}{(\wp_*(x)-\frac{1}{3}c_2)^2}=2\psi_*(x)+4c_2\psi_*(x)^2. \quad (56)$$

Differentiating (56) with respect to x, one obtains

$$\psi_*''(x)=4c_2\psi_*(x)+1,$$

$$\psi_*(0)=0,$$

$$\psi_*'(0)=0.$$

Therefore

$$\psi_*(x)=\frac{\cosh 2\sqrt{c_2}x-1}{(2\sqrt{c_2})^2}$$

In view of (51), it follows that

$$u_*(x) = c_1\psi_*(x) + c_0\psi_*'(x).$$

The Corollary is proved.

Note that according to Lemma 11, if the functions (φ, τ, A) satisfy equation (34) then the function $\tau(x)$ is determined uniquely by the equation

$$\frac{\tau'(x)}{\tau(x)} = \frac{1}{2} \quad \frac{u'(x)+c_0}{u(x)}, \tag{57}$$

subject to the initial conditions $\tau(0) = 0, \tau'(0) = 1$, and the function $A(x)$ is determined by the equation (43):

$$A(x) = \frac{u(x)}{\tau(x)}. \tag{58}$$

Below the function φ will be (informally) referred to as a solution of the equation (34).

Theorem 14. *The general solution of the equation (34)*

$$\varphi(x+y) = \varphi(x) + \varphi(y) + \tau(x)\tau(y)A(x+y)$$

is given by the function

$$\varphi(x) = \frac{4}{c_3}(\zeta(\alpha) - \zeta(x+\alpha) - \wp(\alpha)x),$$

$$\varphi(0) = \varphi'(0) = 0$$

where $\zeta(x)$ and $\wp(x)$ are the Weierstrass ζ-function and $\wp$-function with the parameters g_2 and g_3 (see Lemma 12).

Proof. According to the Lemmas (11) and (12) it is sufficient to prove that any function $\varphi(x)$ given by the formula (50) is a solution of equation (34). It is convenient to consider two different cases.
Case 1. $c_3 = 0$.

$$\varphi_*(x) = \lim_{c_3\to 0} \varphi(x)$$

In this case $\varphi_*(x) = \int_0^x u_*(x)dx$ and hence, using the Corollary 13,we obtain

$$\varphi_*(x) = c_1\frac{\sinh 2\sqrt{c_2}x - 2\sqrt{c_2}x}{(2\sqrt{c_2})^3} + c_0\frac{\cosh 2\sqrt{c_2}x - 1}{(2\sqrt{c_2})^2}.$$

Using the elementary identity

$$(e^{(x+y)}-1)=(e^x-1)+(e^y-1)+(e^{\frac{x}{2}}-e^{\frac{-x}{2}})(e^{\frac{y}{2}}-e^{\frac{-y}{2}})e^{\frac{x+y}{2}}. \tag{59}$$

we obtain

$$\begin{aligned}&\sinh 2\sqrt{c_2}(x+y)\\ &\quad= \sinh 2\sqrt{c_2}x + \sinh 2\sqrt{c_2}y + 4\sinh\sqrt{c_2}x\sinh\sqrt{c_2}y\sinh\sqrt{c_2}(x+y),\\ &\cosh 2\sqrt{c_2}(x+y)-1\\ &\quad= \big(\cosh 2\sqrt{c_2}x-1\big)+\big(\cosh 2\sqrt{c_2}y-1\big)+4\sinh\sqrt{c_2}x\sinh\sqrt{c_2}y\cosh\sqrt{c_2}(x+y)\end{aligned}$$

Hence

$$\varphi_*(x+y)=\varphi_*(x)+\varphi_*(y)+\tau_*(x)\tau_*(y)A_*(x+y),$$

where

$$\tau_*(x)=\frac{\sinh\sqrt{c_2}x}{\sqrt{c_2}},\quad A_*(x)=\frac{c_1}{2}\ \frac{\sinh\sqrt{c_2}x}{\sqrt{c_2}}+c_0\cosh\sqrt{c_2}x. \tag{60}$$

Case 2. $c_3\neq 0$.
Then without any restriction we may take $c_3=2$. According to Frobenius-Stickelberger formula the following functions

$$f(x)=\zeta(\alpha_1-\frac{\alpha}{2}-x)-\wp(\alpha)x-\zeta(\alpha_1-\frac{\alpha}{2}), \tag{61}$$

$$g(y)=\zeta(-\alpha_1-\frac{\alpha}{2}-y)-\wp(\alpha)y+\zeta(\alpha_1+\frac{\alpha}{2}), \tag{62}$$

$$h(z)=\zeta(\alpha-z)-\wp(\alpha)z-\zeta(\alpha). \tag{63}$$

are a solution of eq. (14).

Using the considered above reduction of the (14) to the eq.(32) we obtain

$$\varphi(x)=-2h(-x)=2(\zeta(\alpha)-\wp(\alpha)x-\zeta(x+\alpha))$$

that gives the solution of eq. (34). The theorem is proved.

Corollary 15. *The general normalized solution of eq. (32) is given by the formulas*

$$\begin{aligned}\varphi(x)&=\frac{4}{c_3}(\zeta(\alpha)-\zeta(x+\alpha)-\wp(\alpha)x),\\ \xi(x)&=\frac{2u(x)}{c_0-2b_3u(x)+u'(x)},\end{aligned} \tag{64}$$

where $u(x)=\varphi'(x)=\frac{4}{c_3}(\wp(x+\alpha)-\wp(\alpha))$ *and* b_3 *is free parameter.*

$$\eta(x)=\varphi(x)-\varphi'(x)\xi(x),$$

$$\gamma(x) = -\varphi'(x)\xi(x)^2.$$

The proof follows from the theorem 14, formula (61) and Lemma 10. Let us remind that at the proof of Lemma 10 we will give the explicit construction of the solution eq. (32) on the solution of eq. (34).

So, it was proved already that if $(f(x), g(y), h(z))$ is the nondegenerate solution of eq. (14) with additional conditions

$$f(x) \neq g(x), \quad f(0) = g(0) = h(0) = h'(0) = 0. \tag{65}$$

than it is necessary that

$$h(x) = \frac{2}{c_3}(\zeta(\alpha - x) - \wp(\alpha)x - \zeta(\alpha)), \tag{66}$$

where c_3, α and the parameters g_2, g_3 of the Weierstrass $\wp$-function should satisfy the condition of the Lemma (12). Moreover, $c_3 \neq 0$, because if $c_3 = 0$ then, according to Corollary 13, the function $h(x)$ has no poles (see eq. (55)). Then the functions

$$f(x) = \frac{2}{c_3}(\zeta(\alpha_1 - \frac{\alpha}{2} - x) - \wp(\alpha)x - \zeta(\alpha_1 - \frac{\alpha}{2})), \tag{67}$$

$$g(x) = \frac{2}{c_3}(\zeta(-\alpha_1 - \frac{\alpha}{2} - x) - \wp(\alpha)x + \zeta(\alpha_1 + \frac{\alpha}{2})), \tag{68}$$

where α_1 is a free parameter, taken together with the function $h(x)$ from formula (66) give a solution of eq. (14).

Lemma 16. *Let the functions $(f_1(x), g_1(x), h_1(x))$ satisfy the equation (14) and the initial conditions under consideration. Then, if $h_1(x) = h(x)$ is the function from formula (66), then*

$$f_1(x) = s_1 f(x) + s_2 g(x), \tag{69}$$

$$g_1(x) = t_1 f(x) + t_2 g(x), \tag{70}$$

where $f(x)$ and $g(x)$ are given by formulas (67) and (68), and $s_1 + s_2 = 1, t_1 + t_2 = 1$.

Proof.
For the functions, given by formulas (67) and (68) we have:

$$\xi(x) = f(x) - g(x) = \frac{2}{c_3}[\zeta(\alpha_1 - \frac{\alpha}{2} - x) + \zeta(\alpha_1 + \frac{\alpha}{2} + x) - \zeta(\alpha_1 - \frac{\alpha}{2}) - \zeta(\alpha_1 + \frac{\alpha}{2})]. \tag{71}$$

Then

$$\xi'(x) = \frac{2}{c_3}[\wp(\alpha_1 - \frac{\alpha}{2} - x) - \wp(\alpha_1 + \frac{\alpha}{2} + x)],$$

$$\xi''(x) = \frac{2}{c_3}[-\wp'(\alpha_1 - \frac{\alpha}{2} - x) - \wp'(\alpha_1 + \frac{\alpha}{2} + x)]$$

We have that if the quantities α and α_1 are sufficiently close to the point $x = 0$, then $\xi'(0) \neq 0$, and the quantity $\xi''(0)$ gives the free parameter b_3 to construct the general normalized solution of eq. (32). Therefore, in this case the general solution of the equation (32) has the form

$$\varphi(x) = -2h(-x), \quad \eta(x) + b_1\xi(x), \quad b_2\xi(x), \quad b_2(\gamma(x) + b_1\xi(x)^2),$$

where $h(x)$ is the function (66), $\xi(x) = f(x) - g(x)$, and $\eta(x) = f(x) + g(x)$ for the functions (67) and (68), and $\gamma(x) = -\varphi'(x)\xi(x)^2$.

Let us introduce now

$$f_1(x) + g_1(x) = \eta(x) + b_1\xi(x),$$
$$f_1(x) - g_1(x) = b_2\xi(x),$$

we have

$$f_1(x) = \frac{1}{2}\eta(x) + \frac{b_1 + b_2}{2}\xi(x) = s_1 f(x) + s_2 g(x),$$
$$g_1(x) = \frac{1}{2}\eta(x) + \frac{b_1 - b_2}{2}\xi(x) = t_1 f(x) + t_2 g(x),$$

where

$$s_1 = \frac{1}{2} + \frac{b_1 + b_2}{2}, \quad s_2 = \frac{1}{2} - \frac{b_1 + b_2}{2}, \quad t_1 = \frac{1}{2} + \frac{b_1 - b_2}{2}, \quad t_2 = \frac{1}{2} - \frac{b_1 - b_2}{2}$$

The Lemma is proved.

Now we need just find the values of parameters s_1 and t_1, for which the set of functions $(f_1(x), g_1(x), h(x))$ from the Lemma 16 gives the solution of eq. (14).

Let us introduce the notation

$$\det(f, g, h) = \begin{vmatrix} f''(x) & g''(y) & h''(z) \\ f'(x) & g'(y) & h'(z) \\ 1 & 1 & 1 \end{vmatrix}$$

and let us use the following formula (see [WW], p.458)

$$\frac{1}{2}\det(\zeta(x), \zeta(y), \zeta(z)) = \frac{\sigma(x+y+z)\sigma(x-y)\sigma(y-z)\sigma(z-x)}{\sigma^3(x)\sigma^3(y)\sigma^3(z)}$$

If the conditions of the Lemma 16 are satisfied, we have

$$\det(s_1 f(x) + s_2 g(x), \quad t_1 f(y) + t_2 g(y), \quad h(z)) =$$

$$s_1t_1\det(f(x),f(y),h(z))+s_2t_2\det(g(x),g(y),h(z)) \tag{72}$$

On the other hand,

$$\frac{c_3^3}{8}\det(f(x),f(y),h(z))=\frac{c_3^3}{8}\det(\zeta(\alpha_1-\frac{\alpha}{2}-x),\quad \zeta(\alpha_1-\frac{\alpha}{2}-y),\quad \zeta(\alpha-z))$$

$$=\frac{c_3^3}{4}\quad\frac{\sigma(2\alpha_1)\sigma(y-x)\sigma(z-y+\alpha_1-\frac{3}{2}\alpha)\sigma(x-z+\frac{3}{2}\alpha-\alpha_1)}{\sigma^3(\alpha_1-\frac{\alpha}{2}-x)\sigma^3(\alpha_1-\frac{\alpha}{2}-y)\sigma^3(\alpha-z)}, \tag{73}$$

$$\frac{c_3^3}{8}\det(g(x),g(y),h(z))=\frac{c_3}{8}\det(\zeta(-\alpha_1-\frac{\alpha}{2}-x),\quad \zeta(-\alpha_1-\frac{\alpha}{2}-y),\quad \zeta(\alpha-z)$$

$$=\frac{c_3^3}{4}\quad\frac{\sigma(2\alpha_1)\sigma(y-x)\sigma(y-z+\alpha_1+\frac{3}{2}\alpha)\sigma(x-z+\alpha_1+\frac{3}{2}\alpha)}{\sigma^3(\alpha_1+\frac{\alpha}{2}+x)\sigma^3(\alpha_1+\frac{\alpha}{2}+y)\sigma^3(\alpha-z)}. \tag{74}$$

From condition $f(x)\neq g(x)$ we obtain that $\sigma(2\alpha_1)\neq 0$. Comparison of expressions (73) and (74) shows that if $\sigma(2\alpha_1)\neq 0$, then the determinant (72) is equal to zero if and only if $s_1t_1=s_2t_2=0$.

So, we have proved our main result.

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REFERENCES

1. Sutherland B.,*Exact ground-state wave function for a one-dimensional plasma,* Phys. Rev. Lett. **34**, 1083-1085 (1975)

2. Calogero F., *One-dimensional many-body problems with pair interactions whose exact ground-state wave function is of product type,* Lett. Nuovo Cimento **13**, 507-511 (1975)

3. Olshanetsky M. A., Perelomov A.M., *Quantum integrable systems related to Lie algebras,* Phys. Reports **94**,313-404 (1983)

4. Whittaker E., Watson G., *Course of Modern Analysis,* Cambridge Univ. Press (1927)

5. Frobenius G., Stickelberger L., *Ueber die Addition und Multiplication der elliptischen Functionen,* Jour. Reine Angew. Math. **88**, 146-184 (1880)

6. Cauchy A.L., *Cours d'Analyse de l'Ecole Poilyt,1. Analyse algebraique,* **103** (1821); Oeuvres completes (2), **3**, 98-105

7 Abel N.H., *Méthode geñérale pour trouver des fonctions d'une seul quantité variable, lorsqu'une propriété de ces fonctions est exprimee par une équation entre deux variables.* Magazin for Naturvidenskaberne, Aargang I, Bind 1, Christiania, 1823 Oeuvres completes, **1**, Christiania, 1881, 1-10