A PROOF OF A CONJECTURE OF DEGTYAREV ON NON–TORUS PLANE SEXTICS

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Abstract. We show that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ of the complement of an irreducible non–torus sextic C with the set of singularities $4\mathbf{A}_4$ or $4\mathbf{A}_4 \oplus \mathbf{A}_1$ is isomorphic to $\mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$, where $\mathbb{D}_{10}$ is the dihedral group of order 10. This positively answers a conjecture by Degtyarev.

1. Introduction

A sextic $F(X, Y, Z) = 0$ in $\mathbb{CP}^2$ is said to be of *torus type* if there is an expression of the form $F(X, Y, Z) = F_2(X, Y, Z)^3 + F_3(X, Y, Z)^2$, where F_2 and F_3 are homogeneous polynomials of degree 2 and 3 respectively. A conjecture by the second author says that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ of the complement of an irreducible sextic C with simple singularities and which is *not* of torus type is abelian. In [4] we checked this for a number of configurations of singularities, but early this year, Degtyarev [1] observed that this conjecture is false is general. Especially, Degtyarev proved that there exit 8 equisingular deformation families of irreducible non–torus sextics C with simple singularities such that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ factors to the dihedral group $\mathbb{D}_{10}$, one family for each of the following sets of singularities: $4\mathbf{A}_4$, $4\mathbf{A}_4 \oplus \mathbf{A}_1$, $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$, $4\mathbf{A}_4 \oplus \mathbf{A}_2$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$ and $2\mathbf{A}_9$.[1] Furthermore, in the special case where the set of singularities is $4\mathbf{A}_4$, he conjectured that $\pi_1(\mathbb{CP}^2 \setminus C) \simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$ (cf. [1, Conjecture 1.2.1]). The aim of this paper is to prove this conjecture.

Hereafter, we use the term $\mathbb{D}_{10}$–*sextic* for an irreducible sextic $C \subset \mathbb{CP}^2$ with simple singularities such that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ factors to $\mathbb{D}_{10}$ (cf. [2]). By [1, 5, 8], a $\mathbb{D}_{10}$–sextic is not of torus type.

Theorem 1.1. *If C is a $\mathbb{D}_{10}$–sextic with the set of singularities $4\mathbf{A}_4$ (respectively $4\mathbf{A}_4 \oplus \mathbf{A}_1$), then the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ is isomorphic to $\mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$.*

According to [1], there is only one equisingular deformation family of $\mathbb{D}_{10}$–sextics with the set of singularities $4\mathbf{A}_4$ (respectively $4\mathbf{A}_4 \oplus \mathbf{A}_1$). Therefore, to prove the theorem, it suffices to construct a $\mathbb{D}_{10}$–sextic C_1 with four $\mathbf{A}_4$–singularities (respectively a $\mathbb{D}_{10}$–sextic C_2 with four $\mathbf{A}_4$–singularities and one $\mathbf{A}_1$–singularity) — notice that in [1] only the existence of $\mathbb{D}_{10}$–sextics is proved — and show the

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[1]We recall that a point P in a curve C is said to be an $\mathbf{A}_n$–singularity ($n \geq 1$) if the germs (C, P) and $(\{x^2 + y^{n+1} = 0\}, O)$ are topologically equivalent as embedded germs.

isomorphism $\pi_1(\mathbb{CP}^2 \setminus C_i) \simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$ for $i = 1$ (respectively $i = 2$). This is done in sections 2 and 3 respectively.

Note that when this paper was being written, Degtyarev independently found the fundamental groups $\pi_1(\mathbb{CP}^2 \setminus C)$ for all $\mathbb{D}_{10}$–sextics C (cf. [2]). Let us also mention that in addition to the statement about $\mathbb{D}_{10}$–sextics with four $\mathbf{A}_4$–singularities, Degtyarev's Conjecture 1.2.1 in [1] also says that $\pi_1(\mathbb{CP}^2 \setminus C) \simeq \mathbb{D}_{14} \times \mathbb{Z}/3\mathbb{Z}$ for any $\mathbb{D}_{14}$–sextic C with three $\mathbf{A}_6$–singularities (a $\mathbb{D}_{14}$–sextic C is just an irreducible sextic with simple singularities such that $\pi_1(\mathbb{CP}^2 \setminus C)$ factors to the dihedral group $\mathbb{D}_{14}$). This second point of the conjecture is proved in [3].

2. An example of a $\mathbb{D}_{10}$–sextic with the set of singularities $4\mathbf{A}_4$ and the fundamental group of its complement

Let $(X : Y : Z)$ be homogeneous coordinates on $\mathbb{CP}^2$ and (x, y) the affine coordinates defined by $x := X/Z$ and $y := Y/Z$ on $\mathbb{CP}^2 \setminus \{Z = 0\}$, as usual. We consider the following one–parameter family of curves $C(u) : f(x, y, u) = 0$, $u \in \mathbb{C}$, where $f(x, y, u)$ is a polynomial given as $f(x, y, u) = g(x, y^2, u)$, with

$$g(x, y, u) := c_3\, y^3 + c_2\, y^2 + c_1\, y + c_0,$$

and the coefficients $c_3, \ldots, c_0$ are defined as follows:

$$\begin{aligned}
c_3 \;&:=\; -64\,u^3 + 96\,u^2 + 16 + 16\,u^4 - 64\,u,\\
c_2 \;&:=\; 196\,u - 4\,x^2u^6 - 36\,xu^4 + 144\,xu^3 - 226\,xu^2 - 164\,x^2u + 12\,x^2u^4 +\\
&\quad 192\,u^3 - 289\,u^2 + 223\,x^2u^2 - 40\,x + 16\,x^2u^5 - 128\,x^2u^3 - 52 + 160\,xu -\\
&\quad 48\,u^4 + 44\,x^2,\\
c_1 \;&:=\; 56 + 88\,x - 200\,u + 8\,x^2u^6 + 72\,xu^4 - 288\,xu^3 + 454\,xu^2 + 208\,x^2u +\\
&\quad 2\,x^2u^4 - 276\,x^2u^2 + 152\,x^2u^3 - 328\,xu - 192\,u^3 + 48\,u^4 + 290\,u^2 - 64\,x^2 -\\
&\quad 72\,x^3 + 40\,x^4 + 264\,x^3u - 32\,x^2u^5 + 16\,x^4u^5 + 4\,x^3u^6 + 166\,x^4u^2 - 16\,x^3u^5 -\\
&\quad 338\,x^3u^2 - 136\,x^4u + 184\,x^3u^3 - 4\,x^4u^6 - 80\,x^4u^3 - 2\,x^4u^4 - 24\,x^3u^4,\\
c_0 \;&:=\; -20 - 48\,x + 68\,u - x^6u^6 + 144\,xu^3 - 36\,x^6u + 3\,x^4u^6 + 56\,x^5u^3 + 52\,x^2u^2 -\\
&\quad 40\,x^2u - 120\,x^5u^2 + 104\,x^5u - 44\,x^4u^2 - 4\,x^3u^6 - 2\,x^6u^4 - 4\,x^2u^6 + 2\,x^5u^6 -\\
&\quad 8\,x^5u^5 + 298\,x^3u^2 - 24\,x^2u^3 - 240\,x^3u + 40\,x^4u + 18\,x^3u^4 + 39\,x^6u^2 +\\
&\quad 16\,x^4u^3 - 2\,x^5u^4 - 14\,x^2u^4 - 12\,x^4u^5 + 4\,x^6u^5 - 32\,x^5 + 72\,x^3 + 12\,x^6 -\\
&\quad 16\,x^4 + 16\,x^2 + 64\,u^3 - 16\,u^4 - 97\,u^2 - 160\,x^3u^3 + 16\,x^2u^5 + 12\,x^4u^4 -\\
&\quad 228\,xu^2 + 16\,x^3u^5 - 16\,x^6u^3 - 36\,xu^4 + 168\,xu.
\end{aligned}$$

All the curves $C(u)$ in that family are symmetric with respect to the x–axis. All of them have four A_4–singularities located at $(0, \pm 1)$ and $(1, \pm 1)$, except the curves $C(\frac{9\pm\sqrt{33}}{6})$ which obtain, in addition, an A_1–singularity at $(-1, 0)$, and the curve $C(1)$ which is a non–reduced cubic (union of a smooth conic and a line). All the curves are irreducible except $C(1)$. All of them are non–torus curves.

As a test curve with four A_4–singularities, we take the curve $C_1 := C(11/5)$ defined by the equation $f_1(x,y) := f(x,y,11/5) = 0$, where

$$\begin{aligned} a_0 \cdot f_1(x,y) \;:=\; & 518400\,y^6 + (808511\,x^2 - 1435150\,x - 1555825)\,y^4 + \\ & (259536\,x^4 - 1580686\,x^3 - 297122\,x^2 + 2871550\,x + 1556450)\,y^2 - \\ & 45216\,x^6 - 313968\,x^5 + 503423\,x^4 + 1177536\,x^3 - 512014\,x^2 - \\ & 1436400\,x - 519025, \end{aligned}$$

with $a_0 := 15625$. In Fig. 1, we show its real plane section. (In the figures we do not respect the numerical scale.)

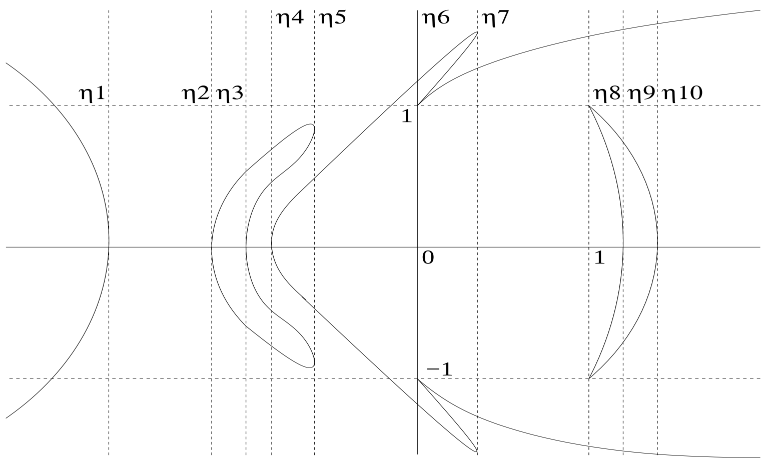

FIGURE 1. Real plane section of C_1

Theorem 2.1. $\pi_1(\mathbb{CP}^2 \setminus C_1) \simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$.

Proof. We use the classical Zariski–van Kampen theorem (cf. [10] and [9]) with the pencil given by the vertical lines $L_\eta\colon x = \eta$, $\eta \in \mathbb{C}$. We always take the point $(0:1:0)$ as the base point for the fundamental groups. This point is nothing but the axis of the pencil, which is also the point at infinity of the lines L_η. Observe that it does not belong to C_1.

The discriminant $\Delta_y(f_1)$ of f_1 as a polynomial in y, which describes the singular lines of the pencil (notice that the line at infinity $Z = 0$ is not singular), is the polynomial in x given by

$$\Delta_y(f_1)(x) = b_0\,(x+1)\,x^{10}\,(408839\,x^2 + 219050\,x - 625)^2\,(x-1)^{10}$$
$$(45216\,x^5 + 268752\,x^4 - 772175\,x^3 - 405361\,x^2 + 917375\,x + 519025),$$

where $b_0 \in \mathbb{Q}\setminus\{0\}$. This polynomial has exactly 10 distinct roots which are all real numbers: $\eta_1 = -7.9192\ldots$, $\eta_2 = -1$, $\eta_3 = -0.7182\ldots$, $\eta_4 = -0.7005\ldots$, $\eta_5 = -0.5386\ldots$, $\eta_6 = 0$, $\eta_7 = 0.0028\ldots$, $\eta_8 = 1$, $\eta_9 = 1.6969\ldots$, and $\eta_{10} = 1.6974\ldots$ The singular lines of the pencil are the lines L_{η_i} $(1 \le i \le 10)$ corresponding to these 10 roots. The lines L_{η_6} and L_{η_8} pass through the singular points of the curve. All the other singular lines are tangent to C_1. See Fig. 1.

We consider the generic line $L_{\eta_6-\varepsilon}$ and choose generators $\xi_1,\dots,\xi_6$ of the fundamental group $\pi_1(L_{\eta_6-\varepsilon}\setminus C_1)$ as in Fig. 2, where $\varepsilon>0$ is small enough. The ξ_j's are (the homotopy classes of) lassos oriented counter–clockwise (see [7] for the definition) around the intersection points of $L_{\eta_6-\varepsilon}$ with C_1. In the figures, a lasso oriented counter–clockwise is always represented by a path ending with a bullet, as in Fig. 3.

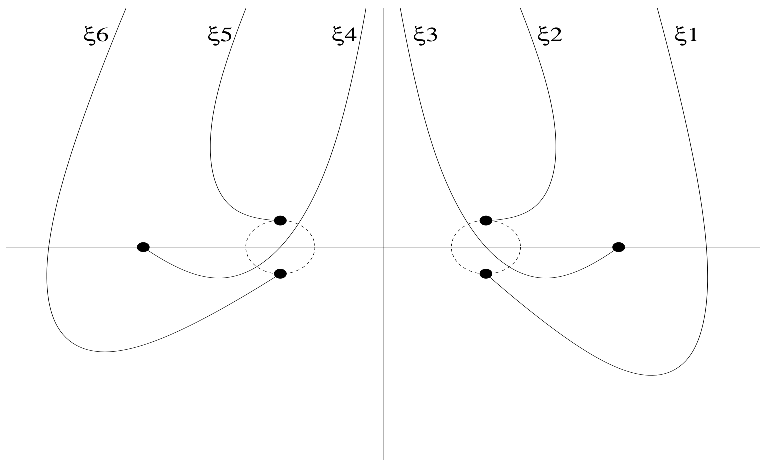

FIGURE 2. Generators at $x=\eta_6-\varepsilon$

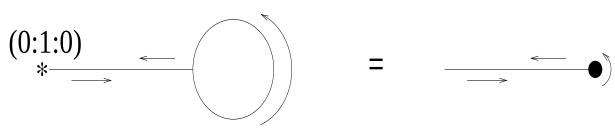

FIGURE 3. Lasso oriented counter–clockwise

The Zariski–van Kampen theorem says $\pi_1(\mathbb{CP}^2\setminus C_1)\simeq\pi_1(L_{\eta_6-\varepsilon}\setminus C_1)/G_1$, where G_1 is the normal subgroup of $\pi_1(L_{\eta_6-\varepsilon}\setminus C_1)$ generated by the monodromy relations associated with the singular lines of the pencil. To determine these relations, we fix a system of generators $\sigma_1,\dots,\sigma_{10}$ for the fundamental group $\pi_1(\mathbb{C}\setminus\{\eta_1,\dots,\eta_{10}\})$ as follows: each σ_i is (the homotopy class of) a lasso oriented counter–clockwise around η_i with base point $\eta_6-\varepsilon$. His tail is a union of real segments and half–circles around the exceptional parameters η_j $(j\neq i)$ located between the base point $\eta_6-\varepsilon$ and η_i. His head is a circle around η_i. For example, for $i=4$, the lasso σ_4 is obtained when the variable x moves on the real axis from $x:=\eta_6-\varepsilon\to\eta_5+\varepsilon$, makes half–turn counter–clockwise on the circle $|x-\eta_5|=\varepsilon$, moves on the real axis from $x:=\eta_5-\varepsilon\to\eta_4+\varepsilon$, runs once counter clockwise on the circle $|x-\eta_4|=\varepsilon$, then comes back on the real axis from $x:=\eta_4+\varepsilon\to\eta_5-\varepsilon$, makes half–turn clockwise on the circle $|x-\eta_5|=\varepsilon$, and moves on the real axis from $x:=\eta_5+\varepsilon\to\eta_6-\varepsilon$ (cf. Fig. 4). For $i=6$, we get σ_6 just by moving x once counter–clockwise on the circle $|x-\eta_6|=\varepsilon$. The monodromy relations around the singular line L_{η_i} are obtained by moving the generic fibre $F\simeq L_{\eta_6-\varepsilon}\setminus C_1$ isotopically 'above' the loop

σ_i so defined, and by identifying the generators ξ_j $(1 \le j \le 6)$ with their own images by the terminal homeomorphism of this isotopy. For details see [10, 9]. Most of the remaining of the proof is to determine these relations.

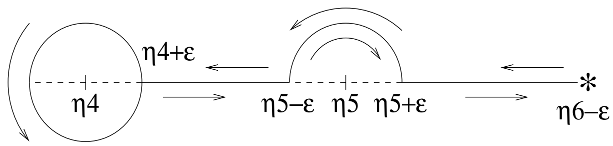

FIGURE 4. Lasso σ_4

Monodromy relations at $x = \eta_5$. In Fig. 5, we show how the generators at $x = \eta_6 - \varepsilon$ (cf. Fig. 2) are deformed when x moves on the real axis from $x := \eta_6 - \varepsilon \to \eta_5 + \varepsilon$. The line L_{η_5} is tangent to the curve at two distinct simple points $P_- = (\eta_5, -0.6132\ldots)$ and $P_+ = (\eta_5, +0.6132\ldots)$, and the intersection multiplicity of this line with the curve at these points is 2. Therefore, by the implicit function theorem, the germ $(C_1, P_\pm)$ is given by

$$x - \eta_5 = \alpha_\pm \cdot (y \mp 0.6132\ldots)^2 + \text{higher terms},$$

where $\alpha_\pm \neq 0$. So, when x runs once counter–clokwise on the circle $|x - \eta_5| = \varepsilon$, the variable y makes half–turn around $\pm 0.6132\ldots$, and therefore the monodromy relations at $x = \eta_5$ are given by

$$\xi_6 = \xi_5 \quad \text{and} \quad \xi_2 = \xi_1. \tag{2.1}$$

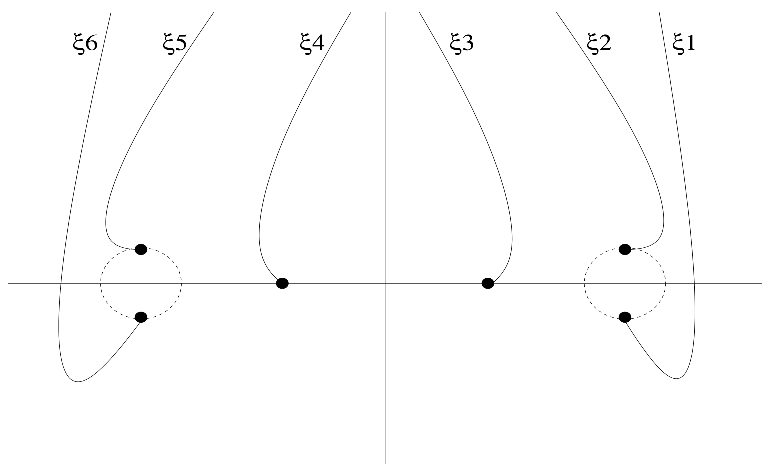

FIGURE 5. Generators at $x = \eta_5 + \varepsilon$

Monodromy relations at $x = \eta_4$. In Fig. 6, we show how the generators at $x = \eta_5 + \varepsilon$ (cf. Fig. 5) are deformed when x makes half–turn counter–clockwise on the circle $|x - \eta_5| = \varepsilon$, then moves on the real axis from $x := \eta_5 - \varepsilon \to \eta_4 + \varepsilon$. The singular line L_{η_4} is tangent to the curve at one simple point P and the intersection multiplicity of this line with the curve at P is 2. Then, as above, the monodromy relation at $x = \eta_4$ is simply given by

(2.2) $$\xi_4 = \xi_3.$$

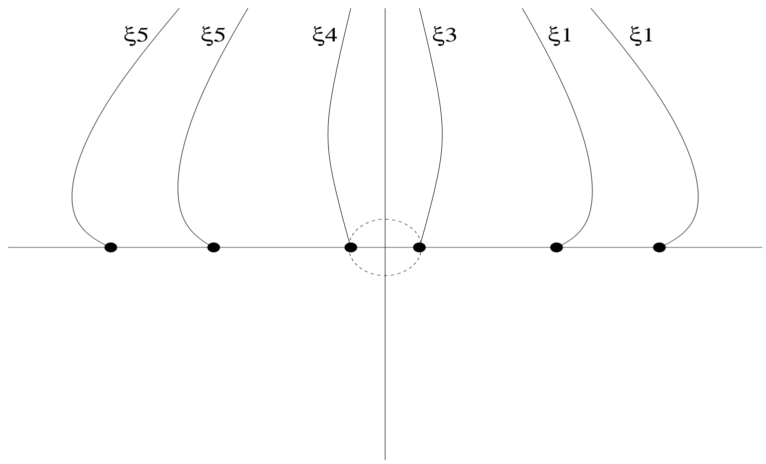

FIGURE 6. Generators at $x = \eta_4 + \varepsilon$

Monodromy relations at $x = \eta_3$. In Fig. 7, we show how the generators at $x = \eta_4 + \varepsilon$ (cf. Fig. 6) are deformed when x makes half–turn counter–clockwise on the circle $|x - \eta_4| = \varepsilon$, then moves on the real axis from $x := \eta_4 - \varepsilon \to \eta_3 + \varepsilon$. The line L_{η_3} is also tangent to the curve at one simple point with intersection multiplicity 2, and the monodromy relation we are looking for is given by

(2.3) $$\xi_5 = \xi_3 \xi_1 \xi_3^{-1}.$$

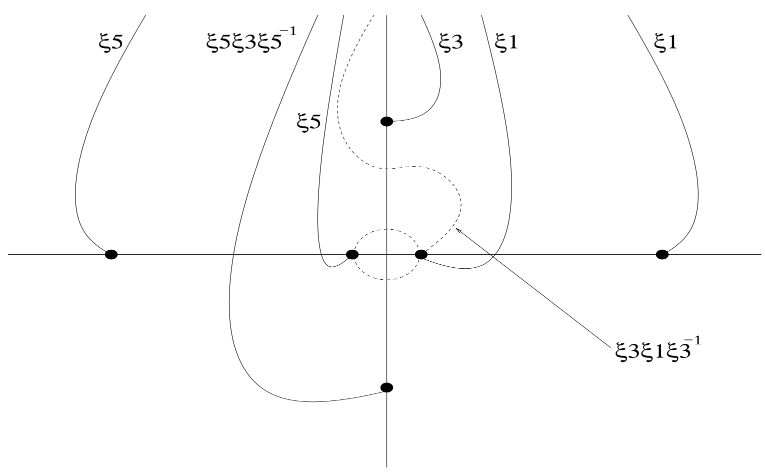

FIGURE 7. Generators at $x = \eta_3 + \varepsilon$

The monodromy relations around the singular lines L_{η_2} and L_{η_1} do not give any new information. The movement of the 6 complex roots of the equation

$f_1(\eta, y) = 0$ for $\eta_1 \le \eta \le \eta_2$ can be chased easily using the real plane section of $g(x, y, 11/5) = 0$ (cf. Fig. 8). For details see [6].

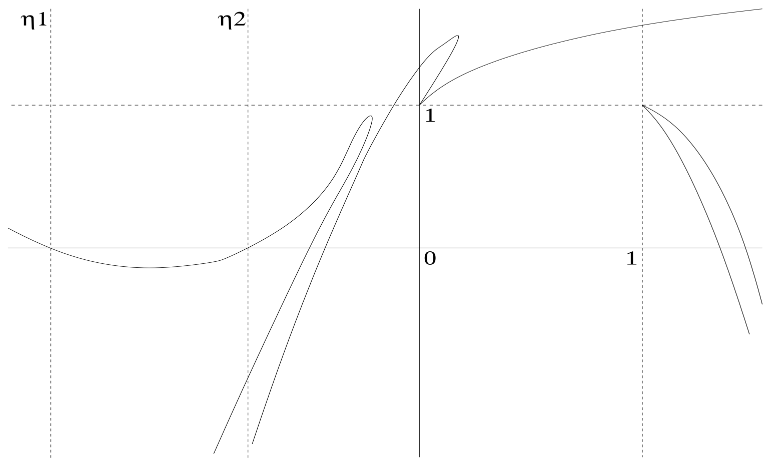

FIGURE 8. Real plane section of $g(x, y, 11/5) = 0$

Monodromy relations at $x = \eta_6$. By (2.1), (2.2) and (2.3), Fig. 2 (which shows the generators at $x = \eta_6 - \varepsilon$) is equivalent to Fig. 9, where

$$\zeta_1 := \xi_3\xi_1 \cdot \xi_3 \cdot (\xi_3\xi_1)^{-1}.$$

The line L_{η_6} passes through the singular points $(0, 1)$ and $(0, -1)$ which are both $\mathbf{A}_4$–singularities. Puiseux parametrizations of the curve at these points are given by

$$x = t^2, \quad y = 1 + \frac{1}{2}t^2 + \frac{359}{200}t^4 + \frac{726}{125}\sqrt{22}\,t^5 + \text{higher terms} \tag{2.4}$$

and

$$x = t^2, \quad y = -1 - \frac{1}{2}t^2 - \frac{359}{200}t^4 - \frac{726}{125}\sqrt{22}\,t^5 + \text{higher terms} \tag{2.5}$$

respectively. Equations (2.4) show that when $x = \varepsilon \exp(i\theta)$ moves once counter–clockwise on the circle $|x - \eta_6| = \varepsilon$, the topological behavior of the two points near 1 in Fig. 7 looks like the movement of two satellites (corresponding to $t = \sqrt{\varepsilon}\exp(i\nu)$, $\nu = \theta/2,\ \theta/2 + \pi$) accompanying a planet. The movement of the planet is described by the term $t^2/2$. It runs once counter–clockwise around 1 (this movement can be ignored in our case). The movement of the satellites around the planet is described by the term $\frac{726}{125}\sqrt{22}\,t^5$. Each of them makes (5/2)–turn counter–clockwise around the planet. Therefore the monodromy relation at $x = \eta_6$ that comes from the singular point $(0, 1)$ is given by

$$\xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 = \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1}. \tag{2.6}$$

Similarly, equations (2.5) show that the monodromy relation at $x = \eta_6$ that comes from the singular point $(0, -1)$ is also given by (2.6).

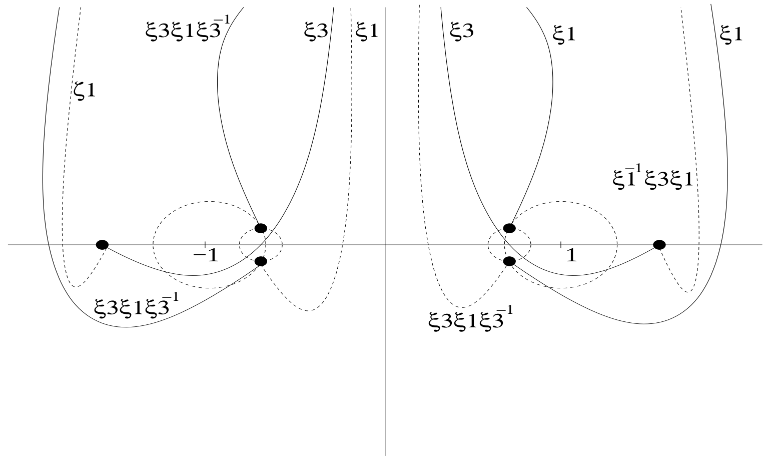

FIGURE 9. Generators at $x = \eta_6 - \varepsilon$

Monodromy relations at $x = \eta_7$. In Fig. 10, we show how the generators at $x = \eta_6 - \varepsilon$ (cf. Fig. 9) are deformed when x makes half–turn counter–clockwise on the circle $|x - \eta_6| = \varepsilon$, then moves on the real axis from $x := \eta_6 + \varepsilon \to \eta_7 - \varepsilon$, where

$$\omega := \xi_3 \xi_1 \xi_3^{-1} \ (= \xi_5 = \xi_6).$$

The line L_{η_7} is tangent to C_1 at two simple points, in both cases with intersection multiplicity 2, and the monodromy relations at $x = \eta_7$ reduce to the following single relation:

$$\xi_1 \xi_3 \xi_1 = \xi_3 \xi_1 \xi_3^{-1} \cdot \xi_1 \cdot \xi_3 \xi_1 \xi_3^{-1}. \tag{2.7}$$

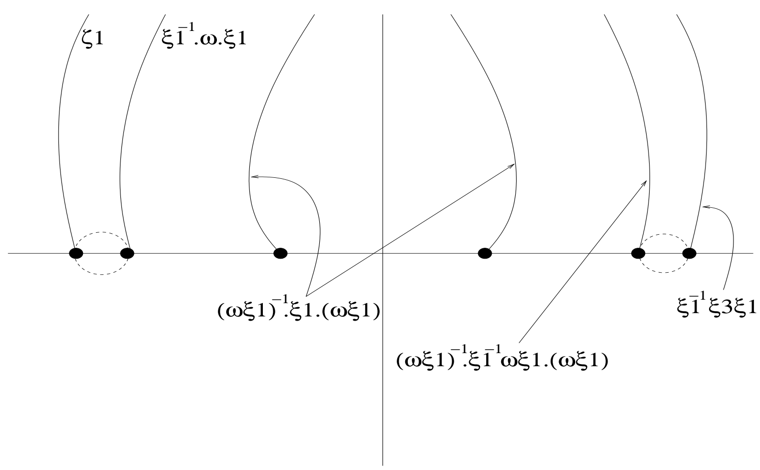

FIGURE 10. Generators at $x = \eta_7 - \varepsilon$

Monodromy relations at $x = \eta_8$. In Fig. 11, we show how the generators at $x = \eta_7 - \varepsilon$ (cf. Fig. 10) are deformed when x makes half–turn counter–clockwise on the circle $|x-\eta_7|=\varepsilon$, then moves on the real axis from $x := \eta_7+\varepsilon \to \eta_8-\varepsilon$, where

$$\begin{aligned}
\zeta_1 &:= (\xi_3\xi_1)\cdot\xi_3\cdot(\xi_3\xi_1)^{-1},\\
\zeta_2 &:= \xi_1^{-1}\cdot\zeta_1\cdot\xi_1,\\
\zeta_3 &:= \xi_1^{-1}\cdot\omega\cdot\xi_1,\\
\zeta_4 &:= (\omega\xi_1)^{-1}\cdot\xi_1\cdot(\omega\xi_1),\\
\zeta_5 &:= (\omega\xi_1)^{-1}\cdot\xi_1^{-1}\omega\xi_1\cdot(\omega\xi_1)=\xi_1^{-1}\xi_3\xi_1 \text{ (by (2.7))},\\
\zeta_6 &:= (\xi_3\xi_1\xi_1)^{-1}\cdot\xi_1\cdot(\xi_3\xi_1\xi_1).
\end{aligned}$$

(To determine dotted lassos, we use the relation (2.7).) The singular line L_{η_8} passes through the singular points $(1,1)$ and $(1,-1)$ which are both $\mathbf{A}_4$–singularities, and Puiseux parametrizations of C_1 at these points are given by

$$x=1+t^2,\quad y=1-\frac{61}{144}\,t^2-\frac{7063}{13824}\,t^4-\frac{125}{684288}\sqrt{22}\,t^5+\text{higher terms}$$

and

$$x=1+t^2,\quad y=-1+\frac{61}{144}\,t^2+\frac{7063}{13824}\,t^4+\frac{125}{684288}\sqrt{22}\,t^5+\text{higher terms}$$

respectively. As above, these equations show that the monodromy relation at $x=\eta_8$ is written as

$$\xi_1\xi_3\xi_1\xi_3\xi_1=\xi_3\xi_1\xi_3\xi_1\xi_3. \tag{2.8}$$

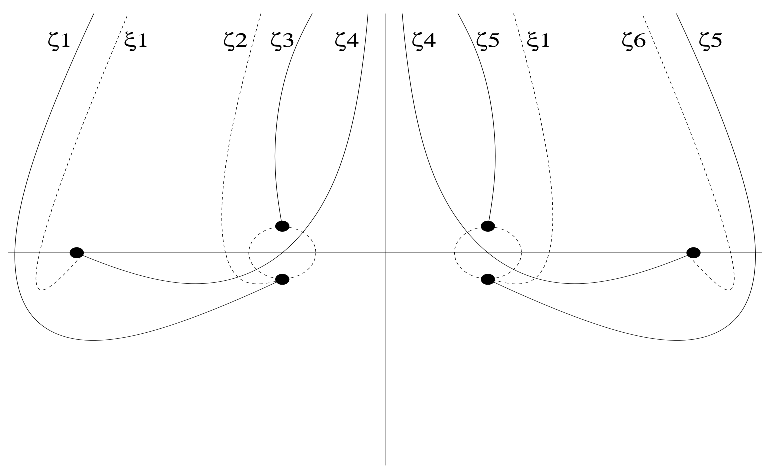

FIGURE 11. Generators at $x=\eta_8-\varepsilon$

The monodromy relations around the singular lines L_{η_9} and $L_{\eta_{10}}$ do not give any new information (details are left to the reader).

Now, by the previous relations, it is easy to check that the vanishing relation at infinity $\xi_6\xi_5\xi_4\xi_3\xi_2\xi_1=e$, where e is the unit element, is written as

$$\xi_3\xi_1\xi_1\cdot\xi_3\xi_1\xi_1=e. \tag{2.9}$$

This relation, combined with (2.7), shows that (2.6) is equivalent to

$$\xi_1\xi_3\xi_1 \cdot \xi_1\xi_3\xi_1 = e. \tag{2.10}$$

Finally, we have proved that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C_1)$ is presented by the generators ξ_1 and ξ_3 and the relations (2.7), (2.8), (2.9) and (2.10).

Simplification of the presentation. By (2.10), the relation (2.8) can be written as

$$\xi_3\xi_1 = \xi_1\xi_3\xi_1 \cdot \xi_3\xi_1\xi_3\xi_1\xi_3,$$

that is,

$$\xi_3\xi_1 = (\xi_1\xi_3)^4. \tag{2.11}$$

In addition, the relation (2.7) can be written as

$$\xi_1\xi_3\xi_1 \cdot \xi_3 \cdot (\xi_1\xi_3\xi_1)^{-1} = \xi_3\xi_1\xi_3^{-1}.$$

Combined with (2.10), this gives

$$\xi_1\xi_3\xi_1 \cdot \xi_3 \cdot (\xi_1\xi_3\xi_1) = \xi_3\xi_1\xi_3^{-1},$$

which is nothing but (2.11). Since the vanishing relation at infinity (2.9) is trivially equivalent to (2.10), it follows that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C_1)$ is presented by the generators ξ_1 and ξ_3 and the relations (2.10) and (2.11). Hence, after the change $a := \xi_1\xi_3\xi_1$ and $b := \xi_1\xi_3$, the presentation is given by

$$\pi_1(\mathbb{CP}^2 \setminus C_1) \simeq \big\langle a,\, b \mid a^2 = e,\, aba = b^4 \big\rangle.$$

Now, we observe that $b^{15} = e$ and b^5 is in the centre of $\pi_1(\mathbb{CP}^2 \setminus C_1)$. Indeed, since $a^2 = e$, the relation $aba = b^4$ gives $b^{16} = ab^4a = b$, that is, $b^{15} = e$ as desired. To show that b^5 is in the centre of $\pi_1(\mathbb{CP}^2 \setminus C_1)$ we write:

$$\begin{aligned} b^5ab^{-5}a^{-1} = b \cdot b^4 \cdot ab^{-5}a^{-1} = b \cdot aba \cdot ab^{-5}a^{-1} = \\ ba \cdot b^{-4} \cdot a^{-1} = ba \cdot a^{-1}b^{-1}a^{-1} \cdot a^{-1} = e. \end{aligned}$$

Hence $\pi_1(\mathbb{CP}^2 \setminus C_1)$ is also presented as:

$$\begin{aligned} \pi_1(\mathbb{CP}^2 \setminus C_1) \;&\simeq\; \big\langle a,\, b \mid a^2 = e,\;\, aba = b^4,\, b^{15} = e,\, b^5a = ab^5 \big\rangle \\ &\simeq\; \big\langle a,\, b,\, c,\, d \mid a^2 = b^{15} = e,\, aba = b^4,\, b^5a = ab^5,\, c = b^6, \\ &\qquad d = b^5,\, da = ad,\, db = bd,\, dc = cd \big\rangle \\ &\simeq\; \big\langle a,\, b,\, c,\, d \mid a^2 = b^{15} = e,\, aba = b^4,\, c = b^6,\, d = b^5, \\ &\qquad b = cd^{-1},\, da = ad,\, db = bd,\, dc = cd \big\rangle \\ &\simeq\; \big\langle a,\, c,\, d \mid a^2 = c^5 = d^3 = e,\, acd^{-1}a = c^4d^{-1},\, da = ad,\, dc = cd \big\rangle \\ &\simeq\; \big\langle a,\, c,\, d \mid a^2 = c^5 = d^3 = e,\, aca = c^4,\, da = ad,\, dc = cd \big\rangle \\ &\simeq\; \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}. \end{aligned}$$

This completes the proof of Theorem 2.1. $\square$

3. An example of a $\mathbb{D}_{10}$–sextic with the set of singularities $4\mathbf{A}_4 \oplus \mathbf{A}_1$ and the fundamental group of its complement

In this section, we consider the curve $C_2 := C(\frac{9+\sqrt{33}}{6})$ defined by the equation $f_2(x,y) := f(x,y,\frac{9+\sqrt{33}}{6}) = 0$, where

$$\begin{aligned} d_0 \cdot f_2(x,y) \quad := \quad & 3867 - 6\,x^3\,y^2\,\sqrt{33} + 6480\,x + 54\,y^2\,x\,\sqrt{33} + 219\,x^2\,y^4\,\sqrt{33} - \\ & 933\,x^4\,\sqrt{33} + 960\,x^3\,\sqrt{33} - 405\,\sqrt{33} - 9270\,y^2 + 2896\,x^5 + \\ & 3723\,x^4 - 8000\,x^3 - 4838\,x^2 - 1376\,x^6 - 432\,x^5\,\sqrt{33} + \\ & 810\,y^2\,\sqrt{33} + 1146\,x^2\,\sqrt{33} - 432\,x\,\sqrt{33} + 288\,x^6\,\sqrt{33} - \\ & 1770\,x^2\,y^2\,\sqrt{33} + 6939\,y^4 - 1536\,y^6 + 10102\,x^2\,y^2 - \\ & 8298\,y^2\,x - 3056\,x^4\,y^2 - 405\,y^4\,\sqrt{33} - 2933\,x^2\,y^4 + \\ & 1818\,y^4\,x + 3482\,x^3\,y^2 + 528\,x^4\,y^2\,\sqrt{33} + 378\,y^4\,x\,\sqrt{33}, \end{aligned}$$

with $d_0 := (3867 - 405\sqrt{33})/(-\frac{677}{18} - \frac{109}{18}\sqrt{33})$ (cf. section 2). We recall that this curve has four $\mathbf{A}_4$–singularities located at $(0,\pm 1)$ and $(1,\pm 1)$, and one $\mathbf{A}_1$–singularity situated at $(-1,0)$. In Fig. 12, we show its real plane section. Near the singular point $(-1,0)$, the equation of C_2 has the following form:

$$\frac{4}{9}\left(4\sqrt{33}+39\right)(x+1)^2 + \left(\frac{8}{3}+\frac{8}{9}\sqrt{33}\right) y^2 + \text{higher terms} = 0.$$

As the leading term $\frac{4}{9}\,(4\sqrt{33}+39)\,(x+1)^2 + (\frac{8}{3}+\frac{8}{9}\sqrt{33})\,y^2$ has no real factorization, the point $(-1,0)$ is an isolated point of the real plane section of the curve.

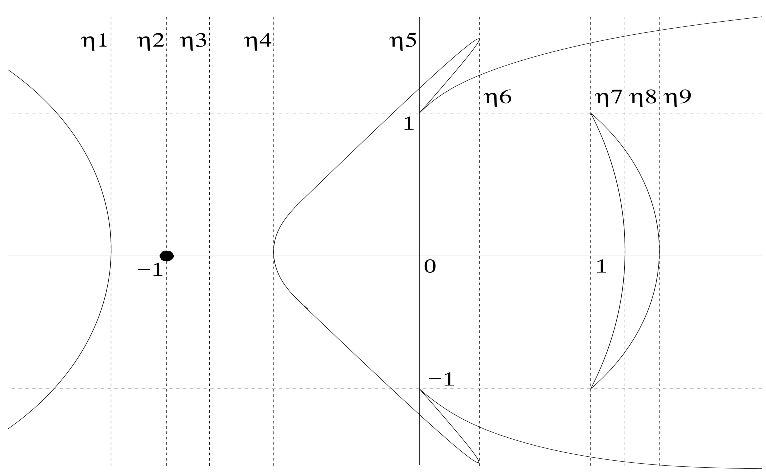

Figure 12. Real plane section of C_2

Theorem 3.1. $\pi_1(\mathbb{CP}^2 \setminus C_2) \simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$.

Proof. We use again the Zariski–van Kampen theorem with the pencil given by the vertical lines $L_\eta \colon x = \eta$, $\eta \in \mathbb{C}$. Observe that the axis of the pencil $(0:1:0)$ does not belong to C_2. The discriminant $\Delta_y(f_2)$ of f_2 as a polynomial in y is the polynomial in x given by

$$\begin{aligned}\Delta_y(f_2)(x) = e_0\,(6592\,x^4 - 14128\,x^3 + 1872\,x^3\sqrt{33} - 7589\,x^2 - 5397\,x^2\sqrt{33}\\ +14586\,x + 1242\,x\sqrt{33} + 11499 + 4347\sqrt{33})\,(x+1)^2\,(x-1)^{10}\,x^{10}\\ (16069\,x^2 + 10680\,x + 774\,x\sqrt{33} - 10917 + 1890\sqrt{33})^2,\end{aligned}$$

where $e_0 \in \mathbb{R}\setminus\{0\}$. This polynomial has exactly 9 roots which are all real numbers: $\eta_1 = -2.2525\ldots$, $\eta_2 = -1$, $\eta_3 = -0.9452\ldots$, $\eta_4 = -0.7814\ldots$, $\eta_5 = 0$, $\eta_6 = 0.0039\ldots$, $\eta_7 = 1$, $\eta_8 = 1.7717\ldots$, and $\eta_9 = 1.7740\ldots$ The singular lines of the pencil are the lines L_{η_i} $(1 \leq i \leq 9)$ corresponding to these 9 roots (notice that the line at infinity is not singular). The lines L_{η_i}, for $i = 2, 5, 7$, pass through the singular points of the curve. All the other singular lines are tangent to C_2. See Fig. 12. The line L_{η_3} intersects the curve at 4 distinct non–real points. It is tangent to C_2 at $(\eta_3, \pm 0.2270\ldots i)$ and the intersection multiplicity of L_{η_3} with C_2 at these two points is 2.

We consider the generic line $L_{\eta_5-\varepsilon}$ and choose generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_5-\varepsilon} \setminus C_2)$ as in Fig. 13. The Zariski–van Kampen theorem says that $\pi_1(\mathbb{CP}^2 \setminus C_2) \simeq \pi_1(L_{\eta_5-\varepsilon} \setminus C_2)/G_2$, where G_2 is the normal subgroup of $\pi_1(L_{\eta_5-\varepsilon} \setminus C_2)$ generated by the monodromy relations around the singular lines L_{η_i} $(1 \leq i \leq 9)$. The latter are given as follows.

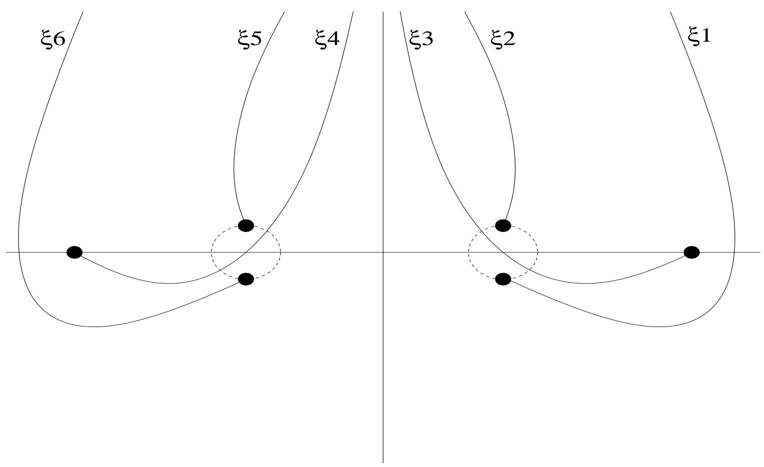

FIGURE 13. Generators at $x = \eta_5 - \varepsilon$

Monodromy relations at $x = \eta_4$. In Fig. 14, we show how the generators at $x = \eta_5 - \varepsilon$ (cf. Fig. 13) are deformed when x moves on the real axis from $x := \eta_5 - \varepsilon \to \eta_4 + \varepsilon$. The line L_{η_4} is tangent to the curve at one simple point with intersection multiplicity 2. Therefore, as above, the monodromy relation around this line is given by

$$\xi_4 = \xi_3. \tag{3.1}$$

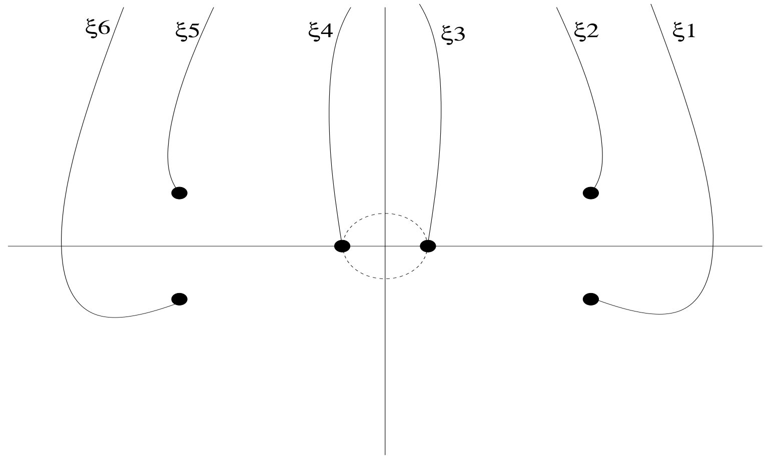

Figure 14. Generators at $x = \eta_4 + \varepsilon$

Monodromy relations at $x = \eta_3$. In Fig. 15, we show how the generators at $x = \eta_4 + \varepsilon$ (cf. Fig. 14) are deformed when x makes half–turn counter–clockwise on the circle $|x - \eta_4| = \varepsilon$, then moves on the real axis from $x := \eta_4 - \varepsilon \to \eta_3 + \varepsilon$. The singular line L_{η_3} is tangent to C_2 at two non–real simple points, in both cases with intersection multiplicity 2, and therefore the monodromy relations we are looking for are given by

$$\xi_5 = \xi_3 \xi_2 \xi_3^{-1} \quad \text{and} \quad \xi_6 = (\xi_5 \xi_3 \xi_2) \cdot \xi_1 \cdot (\xi_5 \xi_3 \xi_2)^{-1}.$$

Equivalently,

$$\xi_5 = \xi_3 \xi_2 \xi_3^{-1} \quad \text{and} \quad \xi_6 = (\xi_3 \xi_2 \xi_2) \cdot \xi_1 \cdot (\xi_3 \xi_2 \xi_2)^{-1}. \tag{3.2}$$

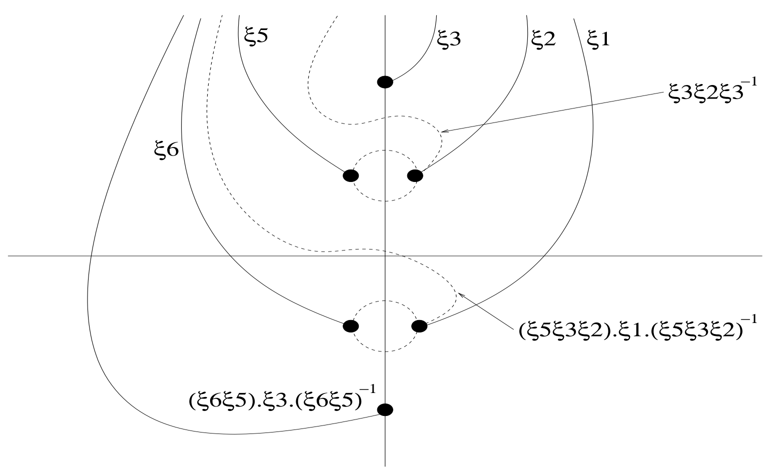

Figure 15. Generators at $x = \eta_3 + \varepsilon$

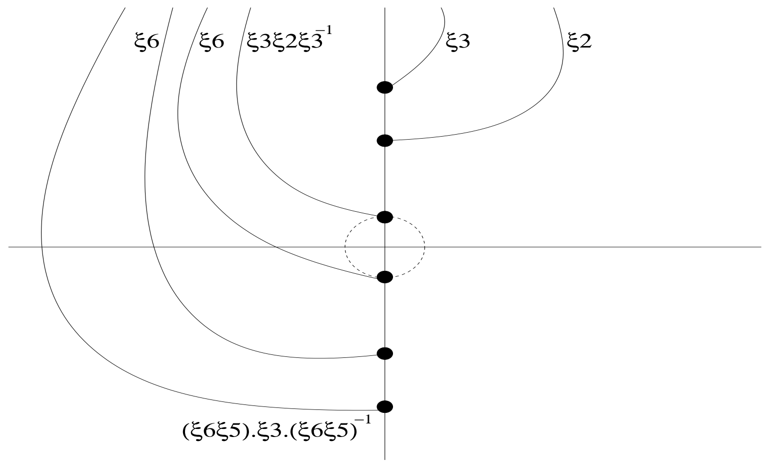

FIGURE 16. Generators at $x = \eta_2 + \varepsilon$

Monodromy relations at $x = \eta_2$. In Fig. 16, we show how the generators at $x = \eta_3 + \varepsilon$ (cf. Fig. 15) are deformed when x makes half–turn counter–clockwise on the circle $|x - \eta_3| = \varepsilon$, then moves on the real axis from $x := \eta_3 - \varepsilon \to \eta_2 + \varepsilon$. The line L_{η_2} passes throuh the singular point $(-1, 0)$ which is an $\mathbf{A}_1$–singularity. At this point, the curve has two branches K_1 and K_2 given by

$$K_1 : \qquad x = -1 + \frac{1}{331}\sqrt{3310 - 5958\sqrt{33}}\, y + \text{higher terms},$$
$$K_2 : \qquad x = -1 - \frac{1}{331}\sqrt{3310 - 5958\sqrt{33}}\, y + \text{higher terms}.$$

These equations show up that when x runs once counter–clockwise on the circle $|x - \eta_2| = \varepsilon$, the points near the origin in Fig. 16 runs once counter–clockwise around it. So the monodromy relation at $x = \eta_2$ is given by

$$\xi_3\xi_2\xi_3^{-1} = \xi_6 \cdot \xi_3\xi_2\xi_3^{-1} \cdot \xi_6^{-1},$$

which can also be written, by (3.2), as

$$\xi_2\xi_1 = \xi_1\xi_2.$$

Monodromy relations at $x = \eta_1$. In Fig. 17, we show how the generators at $x = \eta_2 + \varepsilon$ (cf. Fig. 16) are deformed when x makes half–turn counter–clockwise on the circle $|x - \eta_2| = \varepsilon$, then moves on the real axis from $x := \eta_2 - \varepsilon \to \eta_1 + \varepsilon$. The line L_{η_1} is tangent to C_2 at one simple point, with intersection multiplicity 2, and the monodromy relation at $x = \eta_1$ is given by

$$(\xi_3\xi_2) \cdot \xi_1 \cdot (\xi_3\xi_2)^{-1} = \xi_3\xi_2\xi_3^{-1},$$

that is,

$$\xi_1 = \xi_2. \tag{3.3}$$

In particular, by (3.2), it implies

$$\xi_5 = \xi_6. \tag{3.4}$$

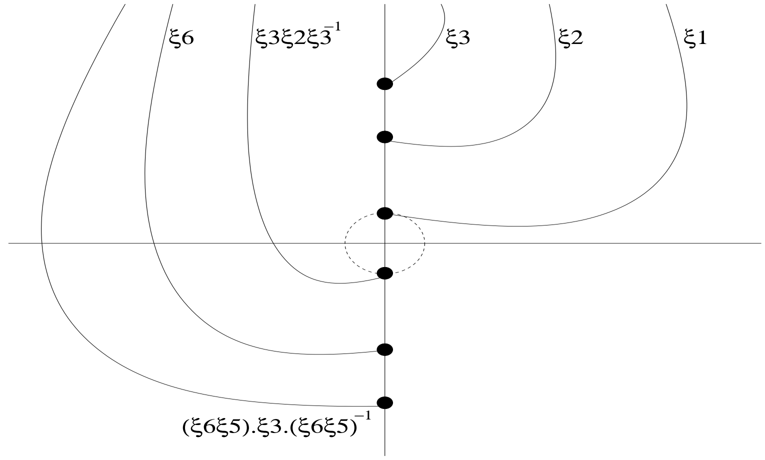

FIGURE 17. Generators at $x = \eta_1 + \varepsilon$

Monodromy relations at $x = \eta_5$**.** By (3.1), (3.2), (3.3) and (3.4), Fig. 13 (which gives the generators at $x = \eta_5 - \varepsilon$) is equivalent to Fig. 18, where

$$\omega := \xi_3\xi_1\xi_3^{-1} \ (= \xi_5 = \xi_6).$$

The line L_{η_5} passes through the singular points $(0, 1)$ and $(0, -1)$ which are both $\mathbf{A}_4$–singularities. Puiseux parametrizations of C_2 at these points are given by

$$x = t^2, \quad y = 1 + \frac{1}{2}\, t^2 + \beta_4\, t^4 + \beta_5\, t^5 + \text{higher terms}$$

and

$$x = t^2, \quad y = -1 - \frac{1}{2}\, t^2 - \beta_4\, t^4 - \beta_5\, t^5 + \text{higher terms}$$

respectively, where $\beta_4, \beta_5 \in \mathbb{R} \setminus \{0\}$. We deduce that the monodromy relation at $x = \eta_5$ is given by

$$\xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 = \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1}. \tag{3.5}$$

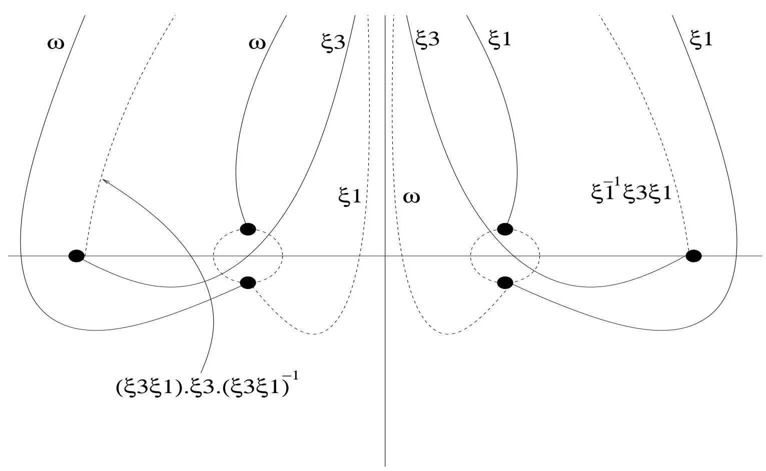

FIGURE 18. Generators at $x = \eta_5 - \varepsilon$

Monodromy relations at $x = \eta_6$. In Fig. 19, we show how the generators at $x = \eta_5 - \varepsilon$ (cf. Fig. 18) are deformed when x makes half–turn counter–clockwise on the circle $|x - \eta_5| = \varepsilon$, then moves on the real axis from $x := \eta_5 + \varepsilon \to \eta_6 - \varepsilon$, where

$$\begin{aligned} \zeta_1 &:= \xi_1^{-1}\omega\xi_1, \\ \zeta_2 &:= (\omega\xi_1)^{-1} \cdot \xi_1 \cdot (\omega\xi_1), \\ \zeta_3 &:= (\omega\xi_1)^{-1} \cdot \xi_1^{-1}\omega\xi_1 \cdot (\omega\xi_1). \end{aligned}$$

The line L_{η_6} is tangent to the curve at two simple points, in both cases with intersection multiplicity 2. So, once more, the monodromy relation around this tangent line is simply given by

$$\xi_1\xi_3\xi_1 = \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1}. \tag{3.6}$$

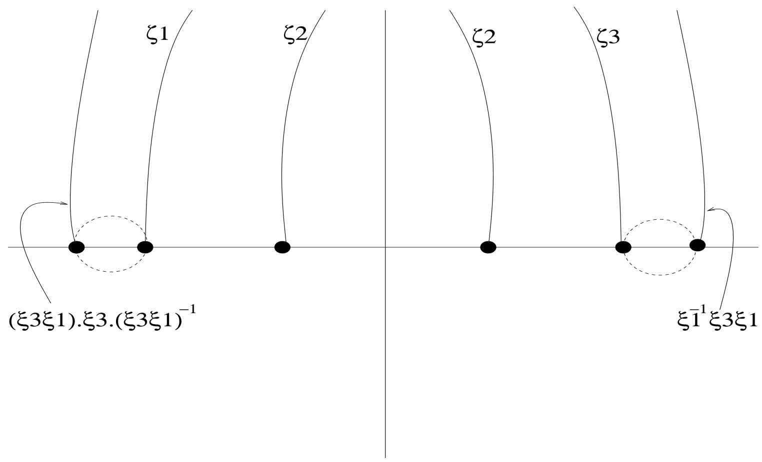

FIGURE 19. Generators at $x = \eta_6 - \varepsilon$

Monodromy relations at $x = \eta_7$. In Fig. 20, we show how the generators at $x = \eta_6 - \varepsilon$ (cf. Fig. 19) are deformed when x makes half–turn counter–clockwise on the circle $|x - \eta_6| = \varepsilon$, then moves on the real axis from $x := \eta_6 + \varepsilon \to \eta_7 - \varepsilon$ (use the relation (3.6) to determine all the lassos). The line L_{η_7} passes through the singular points $(1, 1)$ and $(1, -1)$ which are both $\mathbf{A}_4$–singularities, and Puiseux parametrizations of the curve at these points are given by

$$x = 1 + t^2, \quad y = 1 + \gamma_2\, t^2 + \gamma_4\, t^4 + \gamma_5\, t^5 + \text{higher terms}$$

and

$$x = 1 + t^2, \quad y = -1 - \gamma_2\, t^2 - \gamma_4\, t^4 - \gamma_5\, t^5 + \text{higher terms}$$

respectively, where $\gamma_2, \gamma_4, \gamma_5 \in \mathbb{R} \setminus \{0\}$. Hence the monodromy relation at $x = \eta_7$ is given by

$$\xi_3\xi_1\xi_3\xi_1\xi_3 = \xi_1\xi_3\xi_1\xi_3\xi_1. \tag{3.7}$$

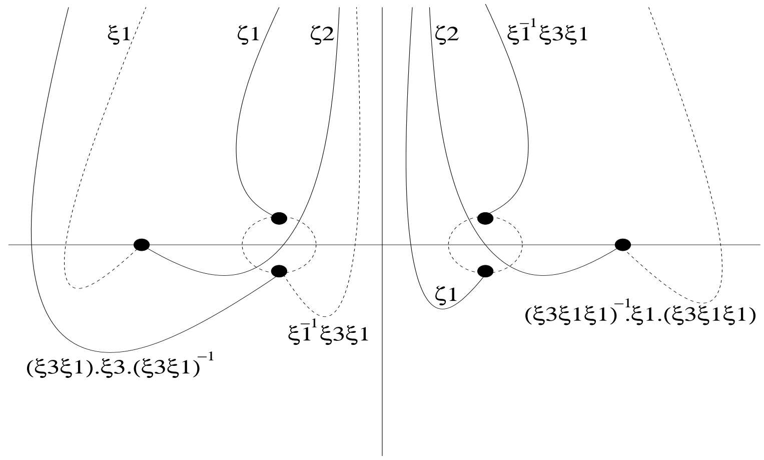

Figure 20. Generators at $x = \eta_7 - \varepsilon$

The monodromy relations around the singular lines L_{η_8} and L_{η_9} do not give any new information (details are left to the reader).

Now, by the previous relations, it is easy to check that the vanishing relation at infinity is written as

$$\xi_3\xi_1\xi_1 \cdot \xi_3\xi_1\xi_1 = e. \tag{3.8}$$

This relation, combined with (3.6), shows that (3.5) is equivalent to

$$\xi_1\xi_3\xi_1 \cdot \xi_1\xi_3\xi_1 = e. \tag{3.9}$$

Finally, we have proved that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C_2)$ is presented by the generators ξ_1 and ξ_3 and the relations (3.6), (3.7), (3.8) and (3.9). We conclude exactly as in section 2. □

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