Compatible Lie brackets related to elliptic curve

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Abstract

For the direct sum of several copies of sl_n, a family of Lie brackets compatible with the initial one is constructed. The structure constants of these brackets are expressed in terms of θ-functions associated with an elliptic curve. The structure of common Casimir elements for these brackets is investigated. A generalization of this construction to the case of vector-valued θ-functions is presented. The brackets define a multi-hamiltonian structure for the elliptic sl_n-Gaudin model. A different procedure for constructing compatible Lie brackets based on the argument shift method for quadratic Poisson brackets is discussed.

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Introduction

Two Lie brackets $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ defined on the same finite dimensional vector space $\mathbf{V}$ are said to be compatible if

$$[\cdot,\cdot]_u = [\cdot,\cdot]_1 + u[\cdot,\cdot]_2 \tag{0.1}$$

is a Lie bracket for any constant u. As a matter of fact, this notion coincides with the concept of two compatible linear Poisson structures (see [1]). Indeed, the formula

$$\{x_i,\, x_j\} = c^k_{ij} x_k, \qquad i, j = 1, \ldots, N \tag{0.2}$$

defines a Poisson bracket iff c^k_{ij} are structural constants of a Lie algebra and the compatibility of two Poisson brackets of this form is equivalent to the compatibility of the two corresponding Lie structures.

The Casimir functions of the Poisson bracket $\{\cdot,\cdot\}_u$ corresponding to (0.1) are polynomials in u, whose coefficients commute with respect to both Poisson brackets. This way for constructing completely integrable Hamiltonian dynamical systems of compatible Poisson brackets is called the Lenard-Magri scheme [1, 2]. Pairs of compatible Lie brackets have been considered in this context in [3, 4].

However, possible applications of compatible pairs of Lie algebras in the integrability theory are not exhausted by this construction. For example, it was shown in [5] that the system of non-linear hyperbolic equations

$$U_x = [U,\, V]_1, \qquad V_y = [V,\, U]_2, \qquad U, V \in \mathbf{V}$$

is integrable for compatible Lie brackets. If the brackets $[\cdot,\cdot]_{1,2}$ coincide, this system is just the principal chiral model.

Compatible Lie brackets also are closely related to decompositions of infinite-dimensional Lie algebras into a vector direct sum of two subalgebras [6, 7, 8]. Furthermore, it was shown in [9] that any pair of compatible Lie brackets having a common quadratic Casimir function produces a (non-constant) solution of the classical Yang-Baxter equation.

The following classification problem arises: to describe all possible brackets $[\cdot,\cdot]_2$ on a vector space $\mathbf{V}$, compatible with a given semi-simple bracket $[\cdot,\cdot]_1$. Since any semi-simple Lie algebra is rigid, the bracket (0.1) is isomorphic to $[\cdot,\cdot]_1$ for almost all values of the parameter u. It is well known that for the semi-simple case the second bracket $[\cdot,\cdot]_2$ is given by the formula

$$[X,\, Y]_2 = [R(X),\, Y]_1 + [X,\, R(Y)]_1 - R([X,\, Y]_1),$$

where R is a linear operator on $\mathbf{V}$.

Some examples of compatible brackets are known (see [4, 5]). Similar to solutions of classical Yang-Baxter equations ([10]), these examples are (in some sense) rational, trigonometric or elliptic.

In this paper, we construct a class of compatible semi-simple Lie brackets related to elliptic curve. By analogy with other "elliptic" models in integrability theory, one can expect that a very wide class of compatible Lie brackets can be obtained by different degenerations of these basic "elliptic" pairs and by deformations of degenerate pairs, which are usually not so rigid as the elliptic models.

The paper is organized as follows. In Section 1 we present a construction of compatible elliptic $\oplus_{i=1}^{m} sl_n$-Lie brackets. The initial date for our construction is a pair of θ-functions of order m without common zeros on an elliptic curve $\mathcal{E}$. To demonstrate the main idea, let us consider a slightly different situation of two compatible associative structures.

Let $\mathbf{V}$ be the k-dimensional vector space of all polynomials of degree $\le k-1$ in one variable, let μ_1 and μ_2 be given polynomials of degree k without common roots. It is clear that any polynomial Z, where $\deg Z \le 2k-1$, can be uniquely represented in the form $Z = \mu_1 P + \mu_2 Q$, where $P, Q \in \mathbf{V}$. The explicit form of P and Q can be found with the help of the Lagrange interpolation formula. Let us define two multiplications $\circ$ and $\star$ on $\mathbf{V}$ by the formula

$$X\,Y = \mu_1 (X \circ Y) + \mu_2 (X \star Y), \qquad X, Y \in \mathbf{V}.$$

It can be checked that any linear combination of these two products is associative. We consider an analog of this construction for Lie algebras replacing polynomials by θ-functions on $\mathcal{E}$ with values in sl_n.

In Section 2 we investigate properties of the Lie algebra $\mathcal{G}_u$ with the bracket (0.1) constructed in Section 1. Since for generic u the Lie algebra $\mathcal{G}_u$ is isomorphic to $\oplus_{i=1}^{m} sl_n$, we know that the center of the universal enveloping algebra $\mathbf{U}(\mathcal{G}_u)$ is generated by $m(n-1)$ elements of $\mathbf{U}(\mathcal{G}_u)$. More precisely, we have m generators of degree p, where $p = 2, 3, \dots, n$. Usually the elements of the center are called the Casimir elements.

Since (0.1) is linear in u, the Casimir elements can be chosen to be polynomial in u. It turns out that for each p there exists one Casimir element $K_{p,1}$ of degree $p-2$ in u, $m-2$ Casimir elements $K_{p,2}, \dots, K_{p,m-1}$ of degree $p-1$ and one element $K_{p,m}$ of degree p. In particular, there exists one quadratic Casimir element, which does not depend on u. This element can be regarded as an invariant bilinear form, common for both brackets $[\cdot,\cdot]_{1,2}$.

This picture is in accordance with general results and conjectures by Gelfand and Zakharevich [11] about "good" bi-Hamiltonian structures that in our case states, in particular, that $\sum_{i,j}(2\deg_u K_{ij} + 1)$ should be equal to $\dim \mathcal{G}_u = m(n^2-1)$.

In Section 2 we find explicit formulas for of all these Casimir elements K_{ij}.

In Section 3 we generalize the results of Section 1 to the case of vector-valued θ-functions or, which is the same, to the case of l-dimensional indecomposable holomorphic vector bundles over the elliptic curve. As the result, we get an $(l+1)$-dimensional vector space of pairwise compatible Lie brackets.

In Section 4 we discuss a different way for constructing compatible linear Poisson brackets starting with a quadratic Poisson bracket of Sklyanin type. This construction is based on the argument shift method. We conjecture that the family of compatible brackets thus obtained coincides with the brackets constructed in Section 3.

1 Compatible Lie brackets related to scalar θ-functions

Let $\Gamma \subset \mathbb{C}$ be a lattice generated by 1 and τ, where $\operatorname{Im} \tau > 0$. Let $m \in \mathbb{N}$. We denote by $\Theta_m(\tau)$ the vector space of holomorphic functions $\phi : \mathbb{C} \to \mathbb{C}$ such that

$$\phi(z+1) = \phi(z), \qquad \phi(z+\tau) = (-1)^m \exp(-2\pi i m\, z)\ \phi(z).$$

Elements of this vector space are called θ-functions of order m. Properties of θ-functions are described, for example, in the Appendix to the review [12]. In particular, the dimension of $\Theta_m(\tau)$ is equal to m. We fix a generator of $\Theta_1(\tau)$ and denote it by $\theta(z)$. It is known that any element $f \in \Theta_m(\tau)$ has m roots modulo Γ and the sum of these roots is equal to zero modulo Γ. In particular, $\theta(z)$ has only one root modulo Γ at $z = 0$. If $x_1, ..., x_m$ are all roots of $f \in \Theta_m(\tau)$, then $f(z) = c\,\theta(z - x_1) \cdots \theta(z - x_m)$, where c is a constant.

Let us fix relatively prime natural numbers k and n such that $1 \le k < n$. Let $\mathbf{a}$ and $\mathbf{b}$ be $n \times n$-matrices such that

$$\mathbf{a}^n = \mathbf{b}^n = 1, \qquad \mathbf{b}\,\mathbf{a} = \exp\left(\frac{2\pi i k}{n}\right) \mathbf{a}\,\mathbf{b}. \tag{1.1}$$

Note that such a pair of matrices $(\mathbf{a}, \mathbf{b})$ is a necessary ingredient of several "elliptic" constructions related to sl_n (see [10, 13, 14]). It is clear that the matrices $\mathbf{a}^\alpha\, \mathbf{b}^\beta$, where $\alpha, \beta = 0, \dots, n-1,\ (\alpha, \beta) \neq (0,0),$ form a basis in sl_n. The commutator relations between these matrices are given by

$$[\mathbf{a}^{\alpha_1}\, \mathbf{b}^{\beta_1},\, \mathbf{a}^{\alpha_2}\, \mathbf{b}^{\beta_2}] = \left[\exp\left(\frac{2\pi i k \beta_1 \alpha_2}{n}\right) - \exp\left(\frac{2\pi i k \beta_2 \alpha_1}{n}\right)\right] \mathbf{a}^{\alpha_1+\alpha_2} \mathbf{b}^{\beta_1+\beta_2}. \tag{1.2}$$

We denote by $\mathbf{V}_m$ the vector space of holomorphic functions $f : \mathbb{C} \to sl_n$ satisfying the following quasi-periodic conditions:

$$f(z+1) = \mathbf{a} f(z) \mathbf{a}^{-1}, \qquad f(z+\tau) = (-1)^m \exp(-2\pi i m\, z)\ \mathbf{b} f(z) \mathbf{b}^{-1}. \tag{1.3}$$

Note that if $f_1 \in \mathbf{V}_{m_1}$ and $f_2 \in \mathbf{V}_{m_2}$, then $f_1 f_2 \in \mathbf{V}_{m_1+m_2}$. It follows from (1.3) that if

$$f(z) = \sum f_{\alpha,\beta}(z)\, \mathbf{a}^\alpha\, \mathbf{b}^\beta,$$

then

$$\begin{aligned} f_{\alpha,\beta}(z+1) &= \exp\left(-\frac{2\pi i k\, \beta}{n}\right) f_{\alpha,\beta}(z), \\ f_{\alpha,\beta}(z+\tau) &= (-1)^m \exp\left(-2\pi i m z + \frac{2\pi i k\, \alpha}{n}\right) f_{\alpha,\beta}(z). \end{aligned} \tag{1.4}$$

These identities imply that

$$f_{\alpha,\beta}(z) = \exp\left(-\frac{2\pi i k\,\beta}{n}\,z\right)\, g_{\alpha,\beta}\left(z - \frac{k\alpha}{mn} - \frac{k\beta}{mn}\tau\right), \tag{1.5}$$

where $g_{\alpha,\beta}(z)$ belongs to $\Theta_m(\tau)$.

Lemma 1. *Let $\mu_1, \mu_2 \in \Theta_m(\tau)$ be a pair of θ-functions that have no common zeros. Then any element $Z \in \mathbf{V}_{2m}$ can be uniquely represented in the form*

$$Z = \mu_1 P + \mu_2 Q, \qquad P, Q \in \mathbf{V}_m.$$

Proof. Consider a linear mapping $L : \mathbf{V}_m \oplus \mathbf{V}_m \to \mathbf{V}_{2m}$ given by the formula

$$L(P, Q) = \mu_1 P + \mu_2 Q.$$

We should prove that L is an isomorphism. Since $\dim(\mathbf{V}_m \oplus \mathbf{V}_m) = \dim \mathbf{V}_{2m} = 2m(n^2 - 1)$, it suffices to prove that $\mathrm{Ker} L = 0$. Substituting

$$P = \sum P_{\alpha,\beta}(z)\, \mathbf{a}^\alpha\, \mathbf{b}^\beta, \qquad Q = \sum Q_{\alpha,\beta}(z)\, \mathbf{a}^\alpha\, \mathbf{b}^\beta$$

into $L(P, Q) = 0$, we find that $\mu_1(z)P_{\alpha,\beta}(z) + \mu_2(z)Q_{\alpha,\beta}(z) = 0$ for all $(\alpha, \beta) \neq 0$. Since $\mu_1(z)$ and $\mu_2(z)$ have no common roots, we see that any root of $\mu_1(z)$ is a root of $Q_{\alpha,\beta}(z)$. We know that $\mu_1(z) \in \Theta_m(\tau)$ has exactly m zeros modulo Γ and the sum of all these zeros equals 0. It follows from (1.5) that if $Q_{\alpha,\beta}(z) \not\equiv 0$, then $Q_{\alpha,\beta}(z)$ also has exactly m zeros, but their sum is equal to $\frac{k\alpha}{n} + \frac{k\beta}{n}\tau$. Hence $Q_{\alpha,\beta}(z) \equiv 0$, which implies that $Q \equiv P \equiv 0$. ■

Using Lemma 1, for any $f_1, f_2 \in \mathbf{V}_m$ we define $[f_1,\, f_2]_1$ and $[f_1,\, f_2]_2$ by the formula

$$[f_1,\, f_2] = \mu_1 [f_1,\, f_2]_1 + \mu_2 [f_1,\, f_2]_2. \tag{1.6}$$

Proposition 1. *The bilinear operations $[f_1,\, f_2]_1$ and $[f_1,\, f_2]_2$ are compatible Lie brackets on $\mathbf{V}_m$.*

Proof. It is clear that the standard bracket $[f_1,\, f_2] = f_1 f_2 - f_2 f_1$ is a Lie bracket on the vector space $\oplus_{p>0} \mathbf{V}_p$. Lemma 1 shows that $[\cdot, \cdot]_{1,2}$ are well-defined brackets on $\mathbf{V}_m$. Substituting (1.6) into the antisymmetricity condition $[f_1, f_2] + [f_2, f_1] = 0$ for the standard bracket, we get $\mu_1([f_1, f_2]_1 + [f_2, f_1]_1) + \mu_2([f_1, f_2]_2 + [f_2, f_1]_2) = 0$. It follows from Lemma 1 that $[f_1, f_2]_i + [f_2, f_1]_i = 0$ for $i = 1, 2$. Substituting (1.6) into the Jacobi identity for the standard bracket, we obtain an identity of the form $\mu_1^2 P + \mu_1\mu_2 Q + \mu_2^2 R = 0$ for some $P, Q, R \in \mathbf{V}_m$. Using the same argument as in the proof of Lemma 1, one can prove that $P \equiv Q \equiv R \equiv 0$. It is easy to verify that the identities $P = 0$ and $R = 0$ coincide with the Jacobi identity for the brackets $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$, respectively. The identity $Q = 0$ is equivalent to the compatibility of the brackets $[\cdot, \cdot]_{1,2}$. ■

Let $x_i(u)$ be all roots of $\mu_2(z) - u\,\mu_1(z)$:

$$\mu_2(z) - u\,\mu_1(z) = c(u)\theta(z - x_1(u)) \cdots \theta(z - x_m(u)).$$

If $x_1(u), \ldots, x_m(u)$ are distinct modulo Γ, we say that u is regular. For brevity, we use the notation x_j instead of $x_j(u)$.

Now we are going to prove that the linear combination

$$[f_1,\, f_2]_u = [f_1,\, f_2]_1 + u\,[f_1,\, f_2]_2 \tag{1.7}$$

of brackets (1.6) is isomorphic to $\oplus_{i=1}^m sl_n$ for all regular values of u.

Consider the following elements $v_{\alpha,\beta,\gamma}(u,z) \in \mathbf{V}_m$ defined by

$$v_{\alpha,\beta,\gamma}(u,z) = P_{\alpha,\beta,\gamma}(u)\, g_{\alpha,\beta,\gamma}(u,z), \tag{1.8}$$

where

$$P_{\alpha,\beta,\gamma}(u) = \frac{\mu_1(x_\gamma)}{\theta(x_\gamma - x_1)\cdots\hat{\gamma}\cdots\theta(x_\gamma - x_m)\theta(-\frac{k\alpha}{n} - \frac{k\beta}{n}\tau)} \tag{1.9}$$

and

$$\begin{aligned} g_{\alpha,\beta,\gamma}(u,z) = \exp\left(-\frac{2\pi i k\,\beta}{n}\,z\right)\,\theta(z - x_1)\cdots\hat{\gamma}\cdots\theta(z-x_m)\,\times \\ \theta\left(z - x_\gamma - \frac{k\alpha}{n} - \frac{k\beta}{n}\tau\right)\mathbf{a}^\alpha\,\mathbf{b}^\beta \end{aligned} \tag{1.10}$$

Here the symbol $\hat{\gamma}$ means that the factor $\theta(z-x_\gamma)$ is omitted in the product, $\alpha, \beta = 0, \ldots, n-1$, where $(\alpha, \beta) \neq (0,0)$ and $\gamma = 1, \ldots, m$.

Theorem 1. *The elements $v_{\alpha,\beta,\gamma}(u)$ satisfy the following commutator relations*

$$[v_{\alpha_1,\beta_1,\gamma_1},\, v_{\alpha_2,\beta_2,\gamma_2}]_u = 0 \tag{1.11}$$

for $\gamma_1 \neq \gamma_2$ and (cf. (1.2))

$$[v_{\alpha_1,\beta_1,\gamma},\, v_{\alpha_2,\beta_2,\gamma}]_u = \left[\exp\left(\frac{2\pi i k\beta_1\alpha_2}{n}\right) - \exp\left(\frac{2\pi i k \beta_2\alpha_1}{n}\right)\right]\, v_{\alpha_1+\alpha_2,\beta_1+\beta_2,\gamma}. \tag{1.12}$$

The proof of Theorem 1 is based on the following

Lemma 2. *Suppose $x_1, \ldots, x_m \in \mathbb{C}$ are distinct modulo Γ and $x_1 + \cdots + x_m \neq \frac{k\alpha}{n} + \frac{k\beta}{n}\tau$ modulo Γ for $0 \le \alpha, \beta < n$, $(\alpha,\beta) \neq (0,0)$. Then for any $\sigma_1, \ldots, \sigma_m \in sl_n$ there exists a unique element $f \in \mathbf{V}_m$ such that $f(x_\delta) = \sigma_\delta$* for *$\delta = 1, \ldots, m$.*

Proof of Lemma 2. Consider a linear mapping $M : \mathbf{V}_m \to \oplus_{i=1}^m sl_n$ given by the formula $M(f) = (f(x_1), \ldots, f(x_m))$. Since $\dim \mathbf{V}_m = \dim \oplus_{i=1}^m sl_n = m(n^2-1)$, it suffices to prove that $\mathrm{Ker} M = 0$. Suppose $M(f) = 0$, where $f = \sum_{\alpha,\beta} f_{\alpha,\beta}(z)\mathbf{a}^\alpha\mathbf{b}^\beta$. Then $f_{\alpha,\beta}(x_\delta) = 0$ for all α, β, δ. But it follows from (1.5) that if $f_{\alpha,\beta} \not\equiv 0$, then the sum of all zeros for $f_{\alpha,\beta}$ is equal to $\frac{k\alpha}{n} + \frac{k\beta}{n}\tau$. ■

Proof of Theorem 1. The basic idea is to verify that the identities (1.11), (1.12) hold if we substitute $z = x_\delta$, $\delta = 1, \ldots, m$. Then Lemma 2 concludes the proof. It is easy to check that

$$[g_{\alpha_1,\beta_1,\gamma_1},\, g_{\alpha_2,\beta_2,\gamma_2}] = \exp\left(-\frac{2\pi ik(\beta_1+\beta_2)}{n}\, z\right)\, \theta(z-x_1)\cdots\hat{\gamma}_1\cdots\theta(z-x_m)\times$$
$$\theta(z-x_1)\cdots\hat{\gamma}_2\cdots\theta(z-x_m)\times\theta\left(z-x_{\gamma_1}-\frac{k\alpha_1}{n}-\frac{k\beta_1}{n}\tau\right)\times \tag{1.13}$$
$$\theta\left(z-x_{\gamma_2}-\frac{k\alpha_2}{n}-\frac{k\beta_2}{n}\tau\right)\left[\exp\left(\frac{2\pi ik\beta_1\alpha_2}{n}\right)-\exp\left(\frac{2\pi ik\beta_2\alpha_1}{n}\right)\right]\mathbf{a}^{\alpha_1+\alpha_2}\mathbf{b}^{\beta_1+\beta_2}.$$

Using the formula

$$[\cdot,\cdot] = \mu_1[\cdot,\cdot]_u + (\mu_2-u\mu_1)[\cdot,\cdot]_2, \tag{1.14}$$

we find that

$$[g_{\alpha_1,\beta_1,\gamma_1},\, g_{\alpha_2,\beta_2,\gamma_2}]_u(x_\delta) = 0$$

for $\gamma_1 \neq \gamma_2$, which implies (1.11) by Lemma 2. If $\gamma_1 = \gamma_2 = \gamma$, then it follows from (1.13), (1.14) that

$$[g_{\alpha_1,\beta_1,\gamma},\, g_{\alpha_2,\beta_2,\gamma}]_u(x_\delta) = 0$$

for $\delta \neq \gamma$. This proves that (1.12) holds for $z = x_\delta$, $\delta \neq \gamma$. A simple straightforward computation shows that (1.12) also holds for $z = x_\gamma$. ■

Remark 1. It is possible to construct 'trigonometric' and 'rational' degenerations of the elliptic brackets described above. Namely, in the trigonometric case one should replace $\theta(z)$ by $1-\exp(2\pi iz)$, the space $\Theta_m(\tau)$ by the space of functions of the form

$$a_0 + a_1\exp(2\pi iz) + \ldots + a_m\exp(2\pi imz)$$

such that $a_m = (-1)^m a_0$, and the space $\mathbf{V}_m$ by the space of $sl(n)$-valued functions of the form

$$\sum c_{\alpha,\beta}\exp\left(2\pi i\frac{\beta}{n}z\right)\mathbf{a}^\alpha\,\mathbf{b}^\beta, \qquad 0\le\alpha\le n, \quad 0\le\beta\le m\,n,$$

where $c_{\alpha,0} = (-1)^m c_{\alpha,mn}$. In this formula we assume that $\mathbf{a}^n = \mathbf{b}^n = 1$, $\mathbf{b}\,\mathbf{a} = \exp\left(\frac{2\pi i}{n}\right)\mathbf{a}\,\mathbf{b}$.

In the rational case $\theta(z)$ is replaced by z, the space $\Theta_m(\tau)$ by the space of polynomials of the form $\sum_{i=0}^m c_\alpha z^\alpha$, $c_{m-1}=0$, and $\mathbf{V}_m$ by the space of polynomials $\sum_{i=0}^m g_\alpha z^\alpha$, $g_\alpha\in sl_n, g_{m-1}=0$.

2 Structure of Casimir elements

2.1 Casimir elements for sl_n

Let $\mathbf{e}_\alpha$, $\alpha = 1,\ldots,n^2-1$, be a basis in sl_n, and let $\mathbf{e}^\alpha$ be the dual basis with respect to the invariant form $< X, Y >= \mathrm{tr}(XY)$. Then the Casimir elements

$$C_p = \sum_{1\le\alpha_1,\ldots\alpha_p\le n^2-1}\mathrm{tr}(\mathbf{e}^{\alpha_1}\cdots\mathbf{e}^{\alpha_p})\,\mathbf{e}_{\alpha_1}\circ\cdots\circ\mathbf{e}_{\alpha_p}, \qquad p=2,\ldots,n,$$

where $\circ$ denotes the multiplication in the universal enveloping algebra $\mathbf{U}(sl_n)$, generate the center of $\mathbf{U}(sl_n)$.

Let us take $\mathbf{t}_{\alpha,\beta} = \mathbf{a}^\alpha\, \mathbf{b}^\beta$ for a basis in sl_n, where $\mathbf{a}, \mathbf{b}$ are defined by (1.1), $0 \le \alpha, \beta < n$, and $(\alpha, \beta) \neq (0,0)$. Then, up to a common multiplicative constant, the dual basis is given by

$$\mathbf{t}^{\alpha,\beta} = \exp\left(\frac{2\pi i k \alpha\beta}{n}\right) \mathbf{t}_{-\alpha,-\beta}.$$

In this basis the Casimir elements have the form

$$C_p = \sum_{\mathbb{D}_p} \exp\left(\frac{2\pi i k}{n} \sum_{1 \le j_1 \le j_2 \le p} \alpha_{j_1}\beta_{j_2}\right) \mathbf{t}_{\alpha_1,\beta_1} \circ \cdots \circ \mathbf{t}_{\alpha_p,\beta_p},$$

where

$$\mathbb{D}_p = \{\ (\alpha_1, \ldots, \alpha_p,\ \beta_1, \ldots \beta_p) \mid \quad 0 \le \alpha_i, \beta_j < n, \quad (\alpha_j, \beta_j) \neq (0,0),$$
$$\alpha_1 + \cdots + \alpha_p \equiv 0 \bmod n, \quad \beta_1 + \cdots + \beta_p \equiv 0 \bmod n\,\}.$$

In particular, the quadratic Casimir element is given by

$$C_2 = \sum_{\alpha,\beta} \exp\left(\frac{2\pi i k \alpha\beta}{n}\right) \mathbf{t}_{\alpha,\beta} \circ \mathbf{t}_{-\alpha,-\beta}.$$

2.2 Polynomial Casimir elements for $\mathcal{G}_u$

In the previous section, we have equipped the vector space $\mathbf{V}_m$ with the Lie bracket (1.7). For generic u the corresponding Lie algebra $\mathcal{G}_u$ is isomorphic to $\oplus_{i=1}^m sl_n$.

It was shown that the vector space $\mathbf{V}_m$ is isomorphic to $\oplus_{\alpha,\beta}\mathbf{F}_{\alpha,\beta}$, where $\mathbf{F}_{\alpha,\beta}$ is the vector space of holomorphic functions satisfying (1.4). Denote by $\mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ the vector space of holomorphic functions of variables $z_1, \ldots, z_p$ satisfying the conditions

$$\begin{aligned} f(z_1, \ldots, z_j + 1, \ldots, z_p) &= \exp\left(-\frac{2\pi i k\,\beta_j}{n}\right) f(z_1, \ldots, z_p), \\ f(z_1, \ldots, z_j + \tau, \ldots, z_p) &= (-1)^m \exp\left(-2\pi i m z_j + \frac{2\pi i k\,\alpha_j}{n}\right) f(z_1, \ldots, z_p). \end{aligned} \tag{2.1}$$

Calculating the number of free coefficients in the Fourier decomposition of a function satisfying (2.1), we find that $\dim \mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p} = m^p$. In particular, $\dim \mathbf{F}_{\alpha,\beta} = m$. Hence the vector space $\mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ is spanned by the products $f_1(z_1) \cdots f_p(z_p)$, where $f_i(z) \in \mathbf{F}_{\alpha_i,\beta_i}$. In other words, $\mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ is isomorphic to $\mathbf{F}_{\alpha_1,\beta_1} \otimes \cdots \otimes \mathbf{F}_{\alpha_p,\beta_p}$.

We use functions from $\mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ to represent elements of the universal enveloping algebra $\mathbf{U}(\mathcal{G}_u)$. Namely, to the product $f_1(z_1) \cdots f_p(z_p) \in \mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ we assign the element $f_1(z)\mathbf{t}_{\alpha_1,\beta_1} \bullet \cdots \bullet f_p(z)\mathbf{t}_{\alpha_p,\beta_p} \in \mathbf{U}(\mathcal{G}_u)$, where $\bullet$ is the multiplication in $\mathbf{U}(\mathcal{G}_u)$. Denote the corresponding linear mapping from $\oplus_{\mathbb{D}_p}\mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ to $\mathbf{U}(\mathcal{G}_u)$ by σ_p.

Suppose u is regular, that is, all roots $x_1, \ldots, x_m$ of the function $\mu_2(z) - u\mu_1(z)$ are distinct. Let $v_{\alpha,\beta,\gamma}(u, z) = s_{\alpha,\beta,\gamma}(u, z)\, \mathbf{t}_{\alpha,\beta}$ be the basis of $\mathbf{V}_m$ given by (1.8) - (1.10).

By Theorem 1, for each γ the elements of this basis satisfy the same commutator relations as $\mathbf{t}_{\alpha,\beta}$. Hence the center of $\mathbf{U}(\mathcal{G}_u)$ is generated by

$$C_{\gamma,p} = \sum_{\mathbb{D}_p} \exp\left(\frac{2\pi ik}{n} \sum_{1\leq j_1\leq j_2\leq p} \alpha_{j_1}\beta_{j_2}\right) \sigma_p\left(f_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p,\gamma}\right), \tag{2.2}$$

where $f_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p,\gamma}(z_1, \ldots, z_p) = s_{\alpha_1,\beta_1,\gamma}(u, z_1) \cdots s_{\alpha_p,\beta_p,\gamma}(u, z_p)$.

Now our goal is to find linear combinations of generators (2.2) that are polynomial in u. Let $W_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p} \subset \mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ be the vector subspace spanned by $f_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p,\gamma}$, where $\gamma = 1, \ldots, m$. The following statement gives us an inner description of this subspace.

Lemma 3. *The vector space $W_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ consists of holomorphic functions $f(z_1, \ldots, z_p)$ satisfying* (2.1) *and vanishing on the surfaces $\{z_{j_1} = x_{\delta_1}\ \&\ z_{j_2} = x_{\delta_2}\}$ for all $j_1 \neq j_2$ and $\delta_1 \neq \delta_2$.*

Proof. It is clear that the functions $f_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p,\gamma}(z_1, \ldots, z_p)$ enjoy these properties. On the other hand, since $\{s_{\alpha,\beta,\gamma}(z),\ \gamma = 1, \ldots, m\}$ is a basis in $\mathbf{F}_{\alpha,\beta}$, the products $s_{\alpha_1,\beta_1,\gamma_1}(z_1) \cdots s_{\alpha_p,\beta_p,\gamma_p}(z_p)$, where $\gamma_j = 1, \ldots, m,\ j = 1, \ldots, p$, form a basis in $\mathbf{F}_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$. Suppose a function

$$f(z_1, \ldots, z_p) = \sum a_{\gamma_1,\ldots,\gamma_p}\, s_{\alpha_1,\beta_1,\gamma_1}(z_1) \cdots s_{\alpha_p,\beta_p,\gamma_p}(z_p) \tag{2.3}$$

satisfies the conditions of Lemma 3. We have to prove that $a_{\gamma_1,\ldots,\gamma_p} = 0$ if $\gamma_{j_1} \neq \gamma_{j_2}$. To show this, it suffices to substitute $x_{\gamma_{j_1}}$ and $x_{\gamma_{j_2}}$ for z_{j_1} and z_{j_2} in (2.3) and to take into account formulas (1.8) - (1.10). ■

Let $W_p \subset \oplus_{\mathbb{D}_p} W_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$ be the vector space spanned by $\{\oplus_{\mathbb{D}_p} f_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p,\gamma},\ \gamma = 1, \ldots, m\}$. Similarly to the proof of Lemma 3, one can prove the following

Lemma 4. *The vector space W_p consists of elements of the form $\oplus_{\mathbb{D}_p} g_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}$, where $g_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}(z_1, \ldots, z_p)$ satisfies the conditions of Lemma 3 and, in addition,*

$$\exp\left(\frac{2\pi ik}{n} x_\gamma(\beta_1 + \cdots + \beta_p)\right) g_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}(x_\gamma, \ldots, x_\gamma) =$$
$$\exp\left(\frac{2\pi ik}{n} x_\gamma(\beta_1' + \cdots + \beta_p')\right) g_{\alpha_1',\beta_1',\ldots,\alpha_p',\beta_p'}(x_\gamma, \ldots, x_\gamma)$$

for any $\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p,\ \alpha_1', \beta_1', \ldots, \alpha_p', \beta_p'$ and $\gamma = 1, \ldots, m$.

Using Lemmas 3,4, we construct polynomials in u that span W_p for generic u.

Theorem 2. *For arbitrary $g(z) \in \Theta_m(\tau)$, put*

$$\begin{aligned} f_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}(z_1, \ldots, z_p) = \exp\left[-\frac{2\pi ik}{n}(\beta_1 z_1 + \cdots + \beta_p z_p)\right] \sum_{1\leq t\leq p} g(z_t)\, \theta\left(\frac{k\alpha_t}{n} + \frac{k\beta_t}{n}\tau\right) \\ \times \prod_{1\leq j\leq p,\ j\neq t} \frac{\theta(z_t - z_j + \frac{k\alpha_j}{n} + \frac{k\beta_j}{n}\tau)}{\theta(z_t - z_j)} \prod_{1\leq j\leq p,\ j\neq t} (\mu_2(z_j) - u\,\mu_1(z_j)). \end{aligned} \tag{2.4}$$

Then $\oplus_{\mathbb{D}_p} f_{\alpha_1,\beta_1,\dots,\alpha_p,\beta_p}$ *belongs to* W_p *and, therefore, the formula*

$$\sum_{\mathbb{D}_p} \exp\left(\frac{2\pi i k}{n} \sum_{1\le j_1\le j_2\le p} \alpha_{j_1}\beta_{j_2}\right) \sigma_p\left(f_{\alpha_1,\beta_1,\dots,\alpha_p,\beta_p}\right)$$

defines a Casimir element in $\mathbf{U}(\mathcal{G}_u)$.

Proof. We must prove that $f_{\alpha_1,\beta_1,\dots,\alpha_p,\beta_p}$ satisfies the assumptions of Lemmas 3,4. Using the quasi-periodic properties of the functions $\mu_1(z), \mu_2(z) \in \Theta_m(\tau)$ and $\theta(z) \in \Theta_1(\tau)$, one can verify condition (2.1) by a simple computation. To prove that $f_{\alpha_1,\beta_1,\dots,\alpha_p,\beta_p}$ is holomorphic, one can check that the only possible pole at $z_t = z_j$ is canceled after summation. It is clear that if we put $z_{j_1} = x_{\delta_1}$ and $z_{j_2} = x_{\delta_2}$, where $\delta_1 \neq \delta_2$, then all summands in (2.4) vanish. Thus the assumptions of Lemma 3 hold. The assumption of Lemma 4 can be checked by a straightforward computation. ■

Remark 2. If $g(z) \in \Theta_m(\tau)$ does not depend on u, then the function (2.4) is polynomial in u, of degree $p-1$.

Remark 3. Since the Casimir function given by (2.4) is linear in $g(z)$, we have constructed a linear map $T : \Theta_m(\tau) \to$ center of $\mathbf{U}(\mathcal{G}_u)$.

Lemma 5. *The kernel of* T *is generated by the element* $g(z) = \mu_2(z) - u\,\mu_1(z)$ *and, therefore,* $\dim\ \mathrm{Ker}\, T = 1$.

Proof. It follows from (2.4) that

$$T(\mu_2(z) - u\,\mu_1(z)) = \prod_{1\le j\le p} (\mu_2(z_j) - u\,\mu_1(z_j)) \exp\left[-\frac{2\pi i k}{n}(\beta_1 z_1 + \dots + \beta_p z_p)\right] \times$$
$$\sum_{1\le t\le p} \theta\left(\frac{k\alpha_t}{n} + \frac{k\beta_t}{n}\tau\right) \prod_{1\le j\le p,\ j\neq t} \frac{\theta(z_t - z_j + \frac{k\alpha_j}{n} + \frac{k\beta_j}{n}\tau)}{\theta(z_t - z_j)}.$$

Consider the function

$$\frac{T(\mu_2(z) - u\,\mu_1(z))}{\prod_{1\le j\le p}(\mu_2(z_j) - u\,\mu_1(z_j))}.$$

It can be checked that this function is holomorphic and satisfies (2.1) with $m = 0$. Analyzing its Fourier decomposition, we see that such a function is identically zero. Hence $T(\mu_2(z) - u\,\mu_1(z)) = 0$. Suppose now that $T(g(z)) = 0$. Substituting a root x of $\mu_2(z) - u\,\mu_1(z)$ for z in (2.4), we see that $g(x) = 0$. Since $g(z)$ has exactly m zeros $\bmod\,\Gamma$, the function $g(z)$ is proportional to $\mu_2(z) - u\,\mu_1(z)$. ■

It follows from Lemma 5 that $T(\mu_2)$ is a polynomial of degree $p-2$ in u. Therefore, formula (2.4) yields an $(m-1)$-dimensional subspace in the m-dimensional vector space W_p such that one generator of this subspace is the polynomial $T(\mu_2)$ of degree $p-2$ and $m-2$ generators are polynomials of degree $p-1$ in u.

In the following statement we construct a remaining generator of W_p.

Theorem 3. *Let*

$$h_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}(z_1,\ldots,z_p)=\sum_{1\le t\le p}A_t\,\times(\mu_2'(z_t)-u\mu_2'(z_t))\prod_{1\le j\le p,\;j\ne t}(\mu_2(z_j)-u\,\mu_1(z_j))-$$
$$B\,\times\prod_{1\le j\le p}(\mu_2(z_j)-u\,\mu_1(z_j)),$$

where A_t is given by

$$A_t=\exp\left[-\frac{2\pi ik}{n}(\beta_1z_1+\cdots+\beta_pz_p)\right]\theta\left(\frac{k\alpha_t}{n}+\frac{k\beta_t}{n}\tau\right)\prod_{1\le j\le p,\;j\ne t}\frac{\theta(z_t-z_j+\frac{k\alpha_j}{n}+\frac{k\beta_j}{n}\tau)}{\theta(z_t-z_j)}$$

and B is defined by the formula

$$B=\frac{m}{n}\exp\left[-\frac{2\pi ik}{n}(\beta_1z_1+\cdots+\beta_pz_p)\right](B_1+B_2)\,,$$

where

$$B_1=\sum_{1\le t\le p,\,1\le j\le p,\,j\ne t}\theta\left(\frac{k\alpha_t}{n}+\frac{k\beta_t}{n}\tau\right)\frac{\theta'(z_t-z_j+\frac{k\alpha_j}{n}+\frac{k\beta_j}{n}\tau)}{\theta(z_t-z_j)}\times$$
$$\prod_{1\le l\le p,\;l\ne j,t}\frac{\theta(z_t-z_l+\frac{k\alpha_l}{n}+\frac{k\beta_l}{n}\tau)}{\theta(z_t-z_l)}$$

$$B_2=\sum_{1\le t\le p}\theta'\left(\frac{k\alpha_t}{n}+\frac{k\beta_t}{n}\tau\right)\prod_{1\le l\le p,\;l\ne t}\frac{\theta(z_t-z_l+\frac{k\alpha_l}{n}+\frac{k\beta_l}{n}\tau)}{\theta(z_t-z_l)}.$$

Then the formula

$$\sum_{\mathbb{D}_p}\exp\left(\frac{2\pi ik}{n}\sum_{1\le j_1\le j_2\le p}\alpha_{j_1}\beta_{j_2}\right)\sigma_p\left(h_{\alpha_1,\beta_1,\ldots,\alpha_p,\beta_p}\right)$$

defines a Casimir element in $\mathbf{U}(\mathcal{G}_u)$.

Proof is similar to the proof of Theorem 2.

Remark 4. It is clear that the Casimir element constructed in Theorem 3 is polynomial in u, of degree p.

3 Families of compatible Lie brackets associated with vector θ-functions

In this section we generalize the construction in Section 1 replacing the usual θ-functions by vector-valued θ-functions. All proofs are similar to those in Section 1.

Let $\Gamma \subset \mathbb{C}$ be a lattice spanned by 1 and τ, where $\operatorname{Im}\tau > 0$. Our general construction will depend on $d, l, m \in \mathbb{N}$ such that $1 \le l < m$ and m, l are relatively prime. Denote by $V\Theta^d_{m/l}$ the vector space consisting of holomorphic functions $f : \mathbb{C}^{l+1} \to \mathbb{C}$ of variables z, $x_0, \ldots, x_{l-1}$, possessing the following properties:

- $f(z, x_0, \ldots, x_{l-1})$ is a homogeneous polynomial of degree d in variables $x_0, \ldots, x_{l-1}$,
- $$f(z+1, x_0, \ldots, x_{l-1}) = f(z, \mathbf{p}(x_0), \ldots, \mathbf{p}(x_{l-1})),$$
- $$f(z+\tau, x_0, \ldots, x_{l-1}) = \exp\left[-2\pi i \left(\frac{m}{l}z + \frac{m-l-1}{2l}\right) d\right] f(z, \mathbf{q}(x_0), \ldots, \mathbf{q}(x_{l-1})),$$

where
$$\mathbf{p}(x_\alpha) = \exp\left(-2\pi i \frac{m}{l}\alpha\right) x_\alpha, \qquad \mathbf{q}(x_\alpha) = x_{\alpha+1}, \qquad \alpha \in \mathbb{Z}/l\mathbb{Z}.$$

Lemma 6.
$$\dim V\Theta^d_{m/l} = m\,\frac{(l+1)\cdots(l+d-1)}{(d-1)!}.$$

It follows from this formula that in the case $d = 1$, which is of most importance for us, $\dim V\Theta^1_{m/l} = m$.

Remark 5. Our space $V\Theta^d_{m/l}$ is a space of holomorphic sections of an indecomposable vector bundle of degree m and rank l on the elliptic curve. The classification of holomorphic vector bundles on elliptic curves was obtained in the paper [15].

Let $1 \le k < n$ and let k, n be relatively prime. Denote by $\mathbf{V}^d_{m,l}$ the vector space of all holomorphic functions $f : \mathbb{C}^{l+1} \to sl_n$ such that

- $f(z, x_0, \ldots, x_{l-1})$ is a homogeneous polynomial of degree d in variables $x_0, \ldots, x_{l-1}$,
- $$f(z+1, x_0, \ldots, x_{l-1}) = \mathbf{a}\, f(z, \mathbf{p}(x_0), \ldots, \mathbf{p}(x_{l-1}))\, \mathbf{a}^{-1}$$
- $$f(z+\tau, x_0, \ldots, x_{l-1}) = \exp\left[-2\pi i \left(\frac{m}{l}z + \frac{m-l-1}{2l}\right) d\right] \mathbf{b}\, f(z, \mathbf{q}(x_0), \ldots, \mathbf{q}(x_{l-1}))\, \mathbf{b}^{-1},$$

where $\mathbf{a}$ and $\mathbf{b}$ satisfy (1.1).

Lemma 7. *Suppose $\mu_1, \ldots, \mu_{l+1} \in V\Theta^1_{m/l}$ have no common zeros for $(x_0, \ldots, x_{l-1}) \ne 0$. Then any element $Z \in \mathbf{V}^2_{m,l}$ can be uniquely represented in the form*
$$Z = \mu_1 P_1 + \cdots + \mu_{l+1} P_{l+1}, \qquad P_i \in \mathbf{V}^1_{m,l}.$$

It is clear that if $f, g \in \mathbf{V}^1_{m,l}$, then $fg - gf$ belongs to $\mathbf{V}^2_{m,l}$. Using Lemma 7, we define $[\cdot,\cdot]_1, \dots, [\cdot,\cdot]_{l+1}$ by the formula

$$f_1 f_2 - f_2 f_1 = \mu_1 [f_1, f_2]_1 + \cdots + \mu_{l+1} [f_1, f_2]_{l+1}, \qquad f_1, f_2 \in \mathbf{V}^1_{m,l}.$$

Proposition 2. *The bilinear operations $[\cdot,\cdot]_1, \dots, [\cdot,\cdot]_{l+1}$ are Lie brackets on the m-dimensional vector space $\mathbf{V}^1_{m,l}$. All these Lie brackets are pairwise compatible.*

It is clear that any linear combination of brackets from Proposition 2 is a Lie bracket. We call a d-dimensional vector space of pairwise compatible Lie brackets a *d-Lie structure*.

Remark 6. Suppose that one of the sections $\mu_1, \dots, \mu_{l+1} \in V\Theta^1_{m/l}$, say, μ_{l+1}, is nonzero for each z. In this case the subbundle generated by μ_{l+1} is trivial. Consider the quotient bundle modulo this subbundle. It has degree m and rank $l-1$. It is clear that the l-Lie structure obtained from this quotient bundle is a substructure of our $(l+1)$-Lie structure. Counting of parameters shows that any generic l-Lie structure is obtained in this way. Therefore, any $(l+1)$-Lie structure constructed in this section is embedded into an m-Lie structure corresponding to $l = m-1$.

4 Argument shift method for quadratic Poisson brackets

The standard argument shift method allows one to get a family of constant Poisson brackets compatible with any linear Poisson bracket (0.2). Namely, if we perform a shift of coordinates $x_i \mapsto x_i + u a_i$, where a_i are arbitrary constants, we will have as the result an inhomogeneous linear bracket of the form $\{\cdot,\cdot\}_u = \{\cdot,\cdot\} + u\{\cdot,\cdot\}_1$, where the operation $\{\cdot,\cdot\}_1$ is a constant Poisson bracket depending on the shift vector $\mathbf{a} = (a_1, \dots, a_N)$. Moreover, since the shift vector $\mathbf{a}$ is arbitrary, we have got an N-dimensional vector space of constant Poisson brackets such that each of these brackets is compatible with (0.2) and any two of them are pairwise compatible.

Consider now the case of a finite-dimensional quadratic Poisson bracket. Suppose we have a Poisson bracket of the form

$$\{x_i,\, x_j\} = \Gamma^{p,q}_{i,j}\, x_p x_q, \qquad i,j = 1, \dots, N \tag{4.1}$$

The shift $x_i \to x_i + u a_i$ yields a Poisson bracket of the form $\{\cdot,\cdot\}_u = \{\cdot,\cdot\} + u\{\cdot,\cdot\}_1 + u^2\{\cdot,\cdot\}_2$. If the coefficient of u^2 is equal to zero, then this formula defines a compatible pair consisting of the quadratic bracket (4.1) and a linear Poisson bracket. This means that the shift vector $\mathbf{a}$ is not arbitrary one but satisfies the following overdetermined system of algebraic equations:

$$\Gamma^{p,q}_{i,j}\, a_p a_q = 0, \qquad i,j = 1, \dots, N. \tag{4.2}$$

Such a vector is said to be *admissible*. It is clear that the set of admissible vectors coincides with the set of zero-dimensional symplectic leaves of the Poisson structure (4.1). Note that if the set of admissible vectors contains a p-dimensional vector space, then shifting by vectors from this space we obtain p compatible linear brackets and each of them is compatible with the quadratic bracket (4.1).

Let us apply this construction to quadratic elliptic Poisson structures (see [12]). For most of these brackets, the system of equations (4.2) has no non-trivial solutions. Nevertheless, for some important brackets of Sklyanin type non-trivial admissible vectors exist.

Example. Consider the following quadratic Poisson brackets between variables $x_0, \dots, x_7$ (subscripts are taken modulo 8):

$$\begin{aligned}
\{x_i,\, x_{i+1}\} &= p_1 x_i x_{i+1} + k_1 x_{i+2} x_{i+7} - 2k_2 x_{i+3} x_{i+6} + p_2 x_{i+4} x_{i+5},\\
\{x_i,\, x_{i+2}\} &= p_3 (x_{i+1}^2 - x_{i+5}^2),\\
\{x_i,\, x_{i+3}\} &= p_1 x_i x_{i+3} + k_1 x_{i+5} x_{i+6} - 2k_2 x_{i+1} x_{i+2} + p_2 x_{i+4} x_{i+7},\\
\{x_i,\, x_{i+4}\} &= p_4 (x_{i+1} x_{i+3} - x_{i+5} x_{i+7}),
\end{aligned} \tag{4.3}$$

where

$$p_1 = -\frac{1}{2} k_1^{1/2} k_2^{-1/2} (4k_2^2 + k_1^2)^{1/2}, \qquad p_2 = k_2^{1/2} k_1^{-1/2} (4k_2^2 + k_1^2)^{1/2},$$

$$p_3 = k_2^{1/4} k_1^{1/4} (4k_2^2 + k_1^2)^{1/4}, \qquad p_4 = k_2^{-1/4} k_1^{-1/4} (4k_2^2 + k_1^2)^{3/4}$$

k_1, k_2 are arbitrary parameters. These brackets depend on the only essential parameter k_1/k_2.

Brackets (4.3) possess the following four Casimir functions

$$C_i = k_2 (x_i^2 + x_{i+4}^2) + p_3 (x_{i+3} x_{i+5} + x_{i+1} x_{i+7}) + k_1 x_{i+2} x_{i+6}, \qquad i = 0, 1, 2, 3.$$

The admissible vectors are given by

$$\mathbf{a}_{\pm} = (t_1, 0, t_2, 0, \pm t_1, 0, \pm t_2, 0), \qquad \mathbf{b}_{\pm} = (0, t_1, 0, t_2, 0, \pm t_1, 0, \pm t_2),$$

where t_1, t_2 are arbitrary parameters. We see that the admissible vectors form four 2-dimensional vector spaces such that $\mathbb{R}^8$ is their direct sum.

Consider the shift of coordinates defined by $\mathbf{a}_+$. As the result, we get a linear bracket $\{\cdot,\cdot\}_a = t_1 \{\cdot,\cdot\}_1 + t_2 \{\cdot,\cdot\}_2$. Hence, we obtain a pair of compatible linear Poisson brackets $\{\cdot,\cdot\}_{1,2}$. For generic t_1, t_2, the bracket $\{\cdot,\cdot\}_a$ is isomorphic to $gl_2 \oplus gl_2$. It is easy to verify that the bracket $\{\cdot,\cdot\}_a$ has two linear Casimir functions $K_1 = x_0 + x_4$ and $K_2 = x_2 + x_6$. After reducting the linear brackets to the surface $K_1 = K_2 = 0$, we get a pair of compatible $sl_2 \oplus sl_2$ brackets. It is important to mention that the initial quadratic bracket (4.3) cannot be restricted to the surface $K_1 = K_2 = 0$ since K_i are not Casimir functions for (4.3). One can check that the Lenard-Magri scheme applied to the reduced brackets $\{\cdot,\cdot\}_{1,2}$ produces the so(4) Schottky-Manakov top.

In the paper [12], the Poisson algebra (4.3) is denoted by $q_{8,3}(\tau)$. It turns out that the situation is the same for a wide class of quadratic elliptic Poisson algebras.

Theorem 4. *For the quadratic Poisson algebras $q_{mn^2,kmn-1}(\tau)$ (in the notations of [12]), the set of admissible vectors is a union of n^2 components which are m-dimensional vector spaces. The space of generators of the algebra is the direct sum of these spaces.*

Theorem 4 can be proved using the so-called functional realization of these Poisson algebras (see [16, 12]). The proof will be given in another publication.

It is clear that for any m-dimensional vector space of admissible vectors of the Poisson algebra $q_{mn^2,kmn-1}(\tau)$ one obtains after shifting by these vectors, m compatible linear Poisson structures.

Conjecture 1. Each of the corresponding Lie algebras is isomorphic to $\oplus_{i=1}^{m} gl_n$. Moreover, all these Lie algebras have a common center. After factorization with respect to the center one obtains m compatible $\oplus_{i=1}^{m} sl_n$ brackets. These m-Lie structures are isomorphic to the one constructed in Section 3, where $l = m - 1$.

Conjecture 2. Each of the $(l+1)$-Lie structures constructed in Section 3 is a substructure of this m-Lie structure.

Remark 7. The Lenard-Magri scheme applied to the pair of compatible linear brackets described in Section 1,2 gives rise to an integrable model with the $\oplus_{i=1}^{m} sl_n$ Poisson brackets. Probably this integrable system is nothing but the elliptic Gaudin model [17, 18]. However, the family of integrals for the $\oplus_{i=1}^{m} sl_2$-Gaudin model considered in [18] contains one parameter related to the elliptic curve plus $m - 1$ additional constant parameters. In our construction, we have $2m-2$ additional parameters. But if Conjectures 1 and 2 are true, then all these additional parameters are inessential in the following sense. The complete set of integrals is given by the Casimir functions of the quadratic brackets. These integrals depend on the elliptic curve only. Furthermore, there exist linear brackets $\{\cdot,\cdot\}_1, \dots, \{\cdot,\cdot\}_m$ that depend on the elliptic curve only such that any linear combination of these brackets is a Poisson bracket as well. The integrals commute with respect to the whole family of these brackets. If we choose two generic brackets

$$\sum_{i=1}^{m} c_i\{\cdot,\cdot\}_i, \qquad \text{and} \qquad \sum_{i=1}^{m} \bar{c}_i\{\cdot,\cdot\}_i$$

and bring the first one to the canonical form $\oplus_{i=1}^{m} sl_n$ by a linear transformation, then the coefficients c_i appear as parameters in the integrals and the second bracket becomes dependent on parameters $c_i, \bar{c}_i$.

Remark 8. One more construction of families of compatible linear Poisson brackets is known [6]. It would be interesting to understand whether these families coincide with those described in our paper or not.

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