Additivity Properties of the Width of Knots

Jennifer Schultens

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Abstract

The width of knots satisfies $w(K_1 \# K_2) \geq max(w(K_1), w(K_2))$ for any two knots K_1, K_2. In particular, $w(K_1 \# K_2) \geq \frac{1}{2}(w(K_1) + w(K_2))$.

1 Introduction

The notion of thin position, introduced by D. Gabai [G], has been employed with great success in many geometric constructions. Yet the underlying notion of the width of a knot remains shrouded in mystery. Little is known about the width of specific knots. By stacking a copy of K_1 in thin position on top of a copy of K_2 in thin position, it is easily seen that $w(K_1 \# K_2) \leq w(K_1) + w(K_2) - 2$. The investigation here is the starting point of a more general investigation into the nature of the interaction between incompressible surfaces in a knot complement and the width of a knot. We here show that the width of a composite knot is bounded below by the maximum of the widths of its summands and therefore also by one half the sum of the widths of its summands.

The question as to how the width of a knot behaves under the operation of connected sum of knots is inspired, in part, by a consideration of the fact that the bridge number satisfies the equation $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$, see the paper [S] by H. Schubert, or [Sch2] for a much shorter proof. The shorter proof in [Sch2] crystallized out of an investigation into whether or not thin position arguments clarify the behaviour of bridge number under connected sum. The answer to that question appears to be no. Thin position arguments appear to provide a much more refined analysis than required for the recovery of Schubert's result. On the other hand, thin position can be viewed as a generalization of bridge position, so it is natural to ask to what extent an analogue of Schubert's result holds for the width of a knot.

The proofs in [S], [Sch2] do not carry over to this more refined setting. The argument in [Sch2] fails in settings where the swallow follow torus is too convoluted:

The correspondance between tunnel numbers of knots and bridge numbers of knots, strongly irreducible generalized Heegaard splittings and knots in thin position deserve to be investigated more closely. Suffice it to say that this correspondance played a role in the discovery of the simple argument below. This correspondance also leads one to believe that the degeneration of width under connected sum of knots is possible, i.e., that there might be knots K_1, K_2, such that $w(K_1 \# K_2) <$

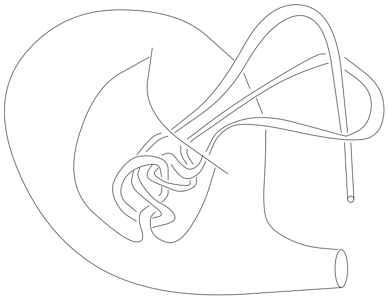

Figure 1: *Convoluted portion of a swallow follow torus*

$w(K_1)+w(K_2)-2$. In the context of this correspondance, it deserves to be pointed out that for small knots (knots whose complements contain no essential closed surfaces), neither width nor tunnel number degenerate under connected sum; i.e., for small knots, width of knots satisfies $w(K_1 \# K_2) = w(K_1) + w(K_2) - 2$ and tunnel number satisfies $t(K_1 \# K_2) = t(K_1) + t(K_2)$. This is proven in [RS] and [Sch1], respectively.

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2 Basic Definitions

For standard definitions and results concerning knots, see [BZ], [L] or [R]. For standard definitions and results pertaining to 3-manifolds, see [H] or [J].

Definition 2.1. *A height function for a 3-manifold M is a Morse function $h : M \to [0,1]$ with exactly two critical points (a minimum and a maximum) in interior(M). Note that the only closed 3-manifold possessing a height function is S^3. Let $h : S^3 \to [0,1]$ be a height function. Then the regular level surfaces of h are spheres. Let $K \subset S^3$ be a knot and let $c_1, \ldots, c_n$ be the critical values of $h|_K$ listed in order so that $h(c_1) < \cdots < h(c_n)$. Choose $r_1, \ldots, r_{n-1}$ so that $c_i < r_i < c_{i+1}$ and set $R_i = h^{-1}(r_i)$. The width of K relative to h, denoted by $w(K,h)$, is $\sum_i |K \cap R_i|$. The width of K, denoted by $w(K)$, is the minimum of this relative width over all height functions. (We say that K is in thin position if it is presented with respect to a height function realizing its width.)*

Here R_i is a thin level if c_i is a maximum value for $h|_K$ and c_{i+1} is a minimum value for $h|_K$; and R_i is a thick level if c_i is a minimum value for $h|_K$ and c_{i+1} is a maximum value for $h|_K$.

Definition 2.2. *Let $K = K_1 \# K_2$ be a composite knot with decomposing sphere S. Then $S^3 - \eta(K \cup S)$ has two components. Each of these components is a torus. Such a torus is called a swallow follow torus. A subannulus of a swallow follow torus T is parallel into $\partial\eta(K_1)$ or $\partial\eta(K_1)$.*

In the first case, we call T the swallow follow torus that swallows K_2 and follows K_1. In the second case, we call T the swallow follow torus that swallows K_1 and *follows K_2.*

We denote the component of $S^3 - T$ containing $K_1\#K_2$ by $int(T)$ and the other by $ext(T)$. Set $C(K_i) = S^3 - \eta(K_i)$. Then for T the swallow follow torus that swallows K_2 and follows K_1, $ext(T) = C(K_1)$. For T the swallow follow torus that swallows K_1 and follows K_2, $ext(T) = C(K_2)$.

Definition 2.3. *Let $K = K_1\#K_2 \subset S^3$ be a composite knot. Suppose that T is a swallow follow torus for $K_1\#K_2$ and $h : S^3 \to [0,1]$ is a height function. We may assume that h_T is a Morse function. Consider the singular foliation, $\mathcal{F}_T$, of T induced by h_T. Let σ be a leaf corresponding to a saddle singularity. Then σ consists of two circles, s_1, s_2, wedged at a point. If either s_1 or s_2 is inessential in T, then we call σ an inessential saddle. Otherwise, σ is an essential saddle.*

A collar of σ has three boundary components $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3$. One of these boundary components, $\tilde{s}_1$, is parallel to s_1, another, $\tilde{s}_2$ is parallel to s_2. We call σ an upper saddle (lower saddle) if the third boundary component, $\tilde{s}_3$, lies above (below) σ.

Let $\sigma, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3$ be as above. We may assume that $\tilde{s}_1$ and $\tilde{s}_2$ are in the same level surface L of h. Then since L is a sphere, $\tilde{s}_1$ and $\tilde{s}_2$ cobound an annulus in L. If a collar of $\tilde{s}_1 \cup \tilde{s}_2$ in this annulus is contained in V, then σ is a nested saddle.

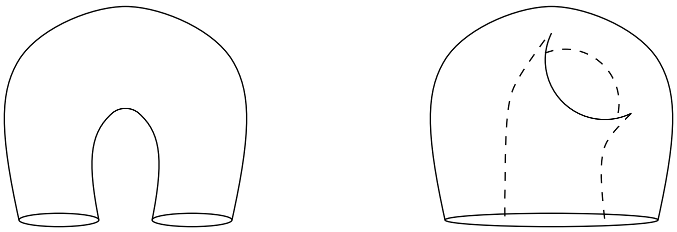

Figure 2: *A nested saddle and an unnested saddle*

As for the case of a knot in thin position, we can define thick and thin levels for swallow follow tori. We may do so without introducing a notion of thin position for a swallow follow torus. (This could easily be done, but is not what is required here.)

Definition 2.4. *Let $c_1, \ldots, c_n$ be the values at which $h|_T$ has saddle singularities and let $r_1, \ldots, r_{n-1}$ be regular values for $h|_T$ with $c_i < r_i < c_{i+1}$. Set $R_i = h^{-1}(r_i)$. Then R_i is a thin level for $h|_T$ if c_i corresponds to an upper saddle and c_{i+1} is corresponds to a lower saddle; and R_i is a thick level for $h|_T$ if c_i corresponds to a lower saddle and c_{i+1} corresponds to an upper saddle. Suppose R is a thin level for $h|_T$ and S is a thick level for $h|_T$, and there are no thick or thin levels between R and S. Then the component W of $S^3 - (R \cup S)$ cobounded by R and S is called a slice. We denote R by $\partial_- W$ and S by $\partial_+ W$. A component cut off by the highest thick level or by the lowest thick level is also called a slice.*

We call W an upper slice if it contains upper saddles and a lower slice if it contains lower saddles.

3 Four lemmata

We here prove a sequence of lemmata providing a solid torus endowed with a height function satisfying the requirements needed in the main construction in the succeeding section.

Lemma 3.1. *Let $h : S^3 \to [0,1]$ be a height function, $K_1, K_2 \subset S^3$ knots and T a swallow follow torus that swallows K_2 and follows K_1. Then for any level surface L, each component of $\partial(L \cap ext(T))$ is either inessential in T or a meridian of $C(K_1)$.*

Proof: Let $\mathcal{F}_T$ be as in Definition 3. Let σ be the highest essential saddle in $\mathcal{F}_T$. For $\tilde{s}_1$, $\tilde{s}_2$, L as in Definition 3, let $\hat{D}_1$, $\hat{D}_2$ be the (disjoint) disks bounded by $\tilde{s}_1$, $\tilde{s}_2$ in a level surface $\tilde{L}$ just above or just below L. As σ is the highest essential saddle in $\mathcal{F}_T$, the portion C of T lying above σ is a planar surface. Furthermore, $\mathcal{F}_T|_C$ contains only inessential saddles. Thus each component of ∂C disjoint from σ bounds a disk in T. It also follows that there is a disk D that can be obtained from C by capping off all components of ∂C disjoint from σ by disks. In particular, σ is an upper saddle.

Now consider $\hat{D}_i$. Any curve in $T \cap interior(\hat{D}_i)$ is contained in D and thus bounds a subdisk of D. A standard innermost disk argument then shows that $\hat{D}_i$ is isotopic rel $\partial\hat{D}_i$ to a disk whose interior is disjoint from T, i.e., lies either entirely in $int(T)$ or entirely in $ext(T)$. Since $\tilde{s}_i$ is parallel to s_i, $\tilde{s}_i$ is essential in T. Furthermore, since $int(T)$ is knotted, T is incompressible, whence $\tilde{s}_i$ is essential in the closure of $ext(T)$. This implies that $\hat{D}_i$ must be isotopic rel $\partial\hat{D}_i$ to a disk whose interior lies entirely in $int(T)$, in particular, $\tilde{s}_i$ is a meridian of $int(T)$. Thus $\tilde{s}_i$ is a meridian of $C(K_i)$.

Now any leaf of $\mathcal{F}$ distinct from $\tilde{s}_i$ is disjoint from $\tilde{s}_i$ and hence must either be inessential, or also a meridian of $C(K_i)$. □

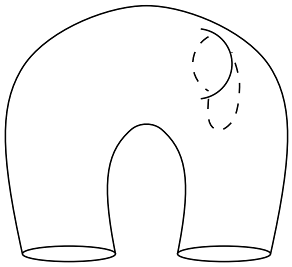

Figure 3: *The highest essential saddle*

The following definition helps us understand the connectivity of $ext(T)$.

Definition 3.2. *Let $c_1, \dots, c_m$ be the critical values of $h|_T$ corresponding to nested saddles and let $C_i = (h|_{ext(T)})^{-1}(c_i)$. (If c_i corresponds to an upper saddle, then a level surface just below C_i has one more component than a level surface just above C_i. The situation is reversed if c_i corresponds to a lower saddle.)*

Consider the component of C_i consisting of two surfaces wedged at a point and let $\tilde{C}_i$ be one of these two surfaces. It is homeomorphic to a component of a level surface just below C_i in the case that c_i corresponds to an upper saddle and homeomorphic to a component of a level surface just above C_i is c_i corresponds to a lower saddle.

Define a graph Γ by assigning a vertex to each component of $ext(T) - (\tilde{C}_1 \cup \dots \cup \tilde{C}_m)$ and connecting two vertices v_1, v_2 by an edge if and only if the components V_1, V_2 of $ext(T) - (\tilde{C}_1 \cup \dots \cup \tilde{C}_m)$ corresponding to v_1, v_2 meet along $\tilde{C}_i$ for some $i \in \{1, \dots, m\}$.

Lemma 3.3. Γ *is a tree.*

Proof: Suppose that γ is a cycle in Γ. We may assume that the vertices, $v_1, \ldots, v_k$ in γ are distinct. We construct a circle α in $ext(T)$ using γ as a model. Since components of $ext(T) - (\tilde{C}_1 \cup \cdots \cup \tilde{C}_m)$ are path components, each vertex gives rise to an arc α_i connecting the components of $\tilde{C}_i$ corresponding to the edges incident to v_i. As $\tilde{C}_i$ is path connected, the endpoints of the α_i can be connected up to form a circle α in $ext(T)$. After a small isotopy, α intersects each component of $\tilde{C}_i - T$ exactly once. Choose one such component, say $\tilde{C}_1$.

By Lemma 3.1, each component of $\partial\tilde{C}_i$ is either inessential in T or a meridian. Consider the surface C obtained by capping off each component of the former type by a pushoff of the disk bounded in T. Then C is a planar surface in $ext(T)$ all of whose boundary components are merdians. Now consider the surface $\hat{C}$ obtained from C by capping off each component of ∂C with a meridian of $int(T)$. Then $\hat{C}$ is a 2-sphere in S^3 that meets the circle α in S^3 exactly once. But this is impossible. Thus there is no cycle in T. $\square$

Lemma 3.4. *Suppose $M = S^3 - \eta(K)$ for some knot $K \subset S^3$ and $h : M \to [0,1]$ a height function whose level surfaces are connected. Then there is a sequence of knots $K_0 = K, K_1, \ldots, K_n$ such that K_n is the unknot, corresponding to a sequence of manifolds $M_0 = C(K_0), \ldots, M_n = C(K_n)$, such that M_n is a solid torus. These manifolds are such that M_i is obtained from M_{i-1} by cutting and pasting along a regular level surface L_{i-1} of h_{i-1} in such a way that each component of $\partial(M_i \cap L_{i-1})$ is reidentified with itself. In particular, there is a sequence of height functions $h_0 : M_0 \to [0,1] = h : M \to [0,1], h_1 : M_1 \to [0,1], \ldots, h_n : M_n \to [0,1]$.*

Furthermore, if h extends to a height function on S^3, then h_i, for $i = 1, \ldots, n$, extends to a height function on S^3.

Proof: Let l be a curve on ∂M that goes through each essential saddle of $h|_{\partial M}$ and is monotone between essential saddles. If l bounds a disk in M, then the conclusions of the lemma hold.

We prove the lemma by induction on the number of thin levels of $h|_{\partial M}$.

Step 1: Constructing disks for l.

Let W be a slice. We may assume that W is an upper slice. Then $l \cap W$ consists of an even number (what goes up must come down) of monotone arcs, Γ_1^W, and arcs containing exactly one critical point, more specifically, a maximum, Γ_2^W. We consider how $l \cap h^{-1}(r)$ varies as r traces out the image of the level surfaces comprising W beginning at the top. It begins with endpoints of Γ_1^W, remains the same until a maximum appears which then splits into two points that can be considered to lie on two monotone arcs ending at the maximum, remains the same until this happens again, ... and ends after a finite number of iterations.

Since level surfaces are connected, any pair of arcs in Γ_1^W can be connected via an arc in $\partial_- W$ and continuously varying copies of this arc in the level surfaces of h comprising W piece together to a disk. (The appearance of maxima is no obstruction, as this one point in a level surface may easily be avoided by the connecting arc.) Call such a disk $D_{\gamma_1,\gamma_1'}$, if it is cobounded, together with one arc in $\partial_- W$ and one arc in $\partial_+ W$ by γ_1 and γ_1'.

Similarly, given an arc $\gamma \in \Gamma_2^W$, piece together a disk D_γ by beginning with the maximum m of γ and then employing the two monotone subarcs of $\gamma - m$ as above. Here D_γ is cobounded by γ together with an arc in $\partial_+ W$.

These disks allow us to prove the lemma under the hypothesis that there are no thin levels:

Step 2: Suppose $M = L \cup_S U$, for L a lower slice and U an upper slice, then the conclusions of the lemma hold.

Set $\mathcal{D}^L = \{D_\gamma \mid \gamma \subset \Gamma_2^L\}$ and $\mathcal{D}^U = \{D_\gamma \mid \gamma \subset \Gamma_2^U\}$. Note that in this case $\Gamma_1^L = \Gamma_1^U = \emptyset$. Denote the arcs in $\cup_{\gamma \subset \Gamma_2^L} \partial D_\gamma \cap S$ by α and the arcs in $\cup_{\gamma \subset \Gamma_2^U} \partial D_\gamma \cap S$ by β. Denote the components of α by $a_1, \ldots, a_k$ and the components of β by $b_1, \ldots, b_k$.

Construct a third set of arcs γ as follows: Consider $S - \alpha$. This is a connected surface, hence there is an arc $g_1 \subset S - \alpha$ with ends on the same punctures as b_1. Note that if $S - (\alpha \cup g_1)$ is disconnected, then a separating circle is formed by g_1 along with one of the a_j. So b_1 shares both endpoints with one of the a_j. Thus $k = 1$, i.e, K is the unknot. Thus $S - (\alpha \cup g_1)$ is connected.

Having constructed $g_1, \ldots, g_i$, such that $S - (\alpha \cup g_1 \cup \cdots \cup g_i)$ is connected, there is an arc $g_{i+1} \subset S - (\alpha \cup g_1 \cup \cdots \cup g_i)$ with ends on the same puncture as b_{i+1}. Here $S - (\alpha \cup g_1 \cdots \cup g_i \cup g_{i+1})$ is also connected unless a separating circle is formed by $g_1 \cdots \cup g_i \cup g_{i+1}$ along with $i+1$ of the a_j. By the same reasoning as in the preceeding paragraph this implies that $i + 1 = k$. Thus either $S - (\alpha \cup g_1 \cdots \cup g_i \cup g_{i+1})$ is connected or $i + 1 = k$.

Cutting M along S and reidentifying in such a way that b_i is identified with g_i for $i = 1, \ldots, k$ produces a knot complement M_1. Let $K_1 \subset S^3$ be the knot of which M_1 is the complement. (Here K_1 is obtained from K by inserting a pure braid near the thick level.) Let h_1 be the height function obtained on M_1. Here K_1, M_1, h_1 satisfy the hypotheses of the lemma. But now $(\cup_{\gamma \subset \Gamma_2^L} \partial D_\gamma) \cap (\cup_{\gamma \subset \Gamma_2^U} \partial D_\gamma) = \emptyset$ and $S \cap ((\cup_{\gamma \subset \Gamma_2^L} \partial D_\gamma) \cup (\cup_{\gamma \subset \Gamma_2^U} \partial D_\gamma))$ is an imbedded circle $\tilde{c}$ that meets each point of $l \cap S$ exactly once. Since S is a connected planar surface, $\tilde{c}$ cuts S into two components S_1, S_2. Now since each component of $\partial S_i - \tilde{c}$ misses l, each such component is inessential in ∂M and hence bounds a disk in ∂M. Thus $\tilde{c}$ bounds a disk $\tilde{D}$.

Hence l bounds the disk $\tilde{D} \cup ((\cup_{\gamma \subset \Gamma_2^L} D_\gamma) \cup (\cup_{\gamma \subset \Gamma_2^U} D_\gamma))$. Whence the lemma follows if there are no thin levels.

Now suppose that the lemma holds if there are up to $n-1$ thin levels and suppose $M = (L_1 \cup_{S_1} U_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (L_n \cup_{S_n} U_n)$ (the L_i's are meant to be lower slices, the U_i's upper slices). Consider $M^1 = L_1 \cup_{S_1} U_1$, $M^2 = M - M^1$. Then $\partial M^1 = \partial_- U_1 \cup (\partial M \cap M^1)$ and $\partial M^2 = \partial_- L_2 \cup (\partial M \cap M^2)$ where $\partial M \cap M^i$ consists of annuli. Pair off the components of $\partial_- U_1 \cap \partial M$ that cobound an annulus in $\partial M \cap M^2$ and those of $\partial_- L_2 \cap \partial M$ that cobound an annulus in $\partial M \cap M^1$.

Construct $\hat{M}^1$ by adjoining an upper slice along $\partial_- U_1$ in which paired off components of $\partial_- U_1 \cap \partial M$ are connected by an annulus with one non nested essential saddle and one maximum. Similarly, construct $\hat{M}^2$. Construct l_i by adjoining to $l \cap M^i$ arcs that run once through the critical point of the saddle in an added annulus and that are monotone otherwise. Then there are no thin levels in $\hat{M}^1$ and $n - 1$ thin levels in $\hat{M}^2$.

Now the inductive hypothesis applies to $\hat{M}^1$ and $\hat{M}^2$. After cut and past operations, l_i bounds a disk $\hat{D}_i$ in $\hat{M}^i$. We may assume that $\hat{D}_i$ intersects each wedge in subdisks constructed as in Step 1. Then each arc in $\hat{D}_1 \cap \partial\_U_1$ and $\hat{D}_2 \cap \partial\_L_2$ cobounds a subdisk of $\hat{D}_i$ contained in $\hat{M}^i - M^i$ and these subdisks are disjoint. (Each such subdisk is bounded by a component of $\partial D_1 \cap \partial\_U_1$ or $\partial D_2 \cap \partial\_L_2$ together with a component of $l_i - l \cap M^i$.) Set $D_i = \hat{D}_i \cap M^i$. Then D_i is a disk in M^i.

Reglueing M^1 and M^2 along F_1 may introduce points of intersection of $D_1 \cap \partial\_U_1$ and $D_2 \cap \partial\_L_2$. The argument in Step 2 shows how to eliminate such points of intersection and create a planar surface $D_1 \cup D_2$. Since the cut and paste operations preserve the connectedness of l, and since $\partial(D_1 \cup D_2) = l$, $D_1 \cup D_2$ is the disk required.

The final comment follows from the fact that the level surfaces of h are in one-to-one correspondance with the level surfaces of h_i. (In particular, $h_i|_{\partial M_i}$ contains no nested saddles.) Indeed, there is a one-to-one correspondance of the boundary components of a level surface of h with the boundary components of a level surface of h_i. Thus if $h : M \to [0,1]$ extends to a height function $\hat{h} : S^3 \to [0,1]$, then this extension is via a solid torus with an appropriate height function. In this solid torus, regular level surfaces are either meridian disks or boundary parallel disks. Via this same height function on this solid torus, $h_i : M_i \to [0,1]$ also extends to a height function $\hat{h}_i : S^3 \to [0,1]$ defined by $\hat{h}_i|_{M_i} = h_i$ and $\hat{h}_i|_{S^3 - M_i} = \hat{h}|_{S^3 - M}$. $\square$

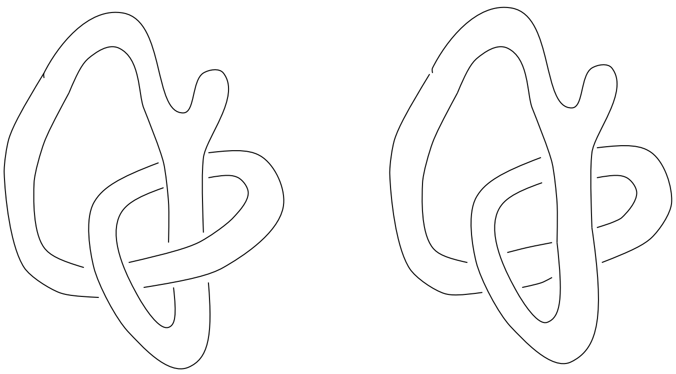

Figure 4: *The transformation of a knot exterior as in Lemma 3.4*

Lemma 3.5. *Let $K, \tilde{K} \subset S^3$ be knots and let T be a swallow follow torus for $K \# \tilde{K}$ that swallows $\tilde{K}$ and follows K. Suppose $M = ext(T)$ and $h : M \to [0,1]$ a height function. Then there is a sequence of knots $K_0 = K, K_1, \ldots, K_n$ such that K_n is the unknot, corresponding to a sequence of manifolds $M_0 = C(K_0), \ldots, M_n = C(K_n)$, such that M_n is a solid torus (i.e., K_n is the unknot). These manifolds are such that M_i is obtained from M_{i-1} by cutting and pasting along a regular level surface L_{i-1} of h_{i-1} in such a way that each component of $\partial(M_i \cap L_{i-1})$ is reidentified with itself. In particular, there is a sequence of height functions $h_0 : M_0 \to [0,1] = h : M \to [0,1], h_1 : M_1 \to [0,1], \ldots, h_n : M_n \to [0,1]$.*

Furthermore, if h extends to a height function on S^3, then h_i, for $i = 1, \ldots, n$, extends to a height function on S^3.

Proof: The proof is by induction on the number of nested saddles in $\mathcal{F}_T$. If there are no nested saddles, then the lemma follows by Lemma 3.4.

Suppose that there are n nested saddles and that the lemma holds whenever there are at most $n-1$ nested saddles. Let v be an endvertex of Γ. Then v corresponds to a component V of $ext(T) - (\tilde{C}_1 \cup \cdots \cup \tilde{C}_m)$ that meets no other nested saddles. Furthermore, V meets only one component of $\tilde{C}_1 \cup \cdots \cup \tilde{C}_m$, say $\tilde{C}_1$. We will assume that c_1 is an upper saddle. The other case follows analogously.

Since $\tilde{C}_1$ is connected, it follows that V is foliated by connected level surfaces. Alter $h|_V$, maintaining the relative positioning of all level surfaces of $h|_V$ to obtain a height function $\bar{h} : ext(T) \to [0,1]$, so that V is contained between C_1 and another level surface L of h, where L is chosen so that $h|_{ext(T)-V}$ has no critical values in $(h(L), c_1)$.

(Note that here if $h|_{ext(T)}$ extends to S^3, $\bar{h}|_{ext(T)}$ may or may not extend to S^3. In S^3, isotoping V upward could force components corresponding to other vertices of Γ to be isotoped upward as well. This would impede progress all together. For this reason we consider the height function $\bar{h}|_{ext(T)}$ and below the height function $\tilde{h}|_{ext(T)}$. It follows from the reasoning below that these temporarily considered height functions help locate the cut and paste operations required.)

We denote the singular point in $\tilde{C}_1$ by c. Let a be a monotone arc from c to a minimum of T in V. Since $\bar{h}|_{ext(T)-V}$ has no critical values in $(\bar{h}(L), c_1)$, there is a monotone arc b from c to a point just below the level surface $\bar{h}^{-1}(h(L))$ in $ext(T)-V$. Consider the height function obtained by modifying $\bar{h}|_{ext(T)}$ along a collar of $a \cup b$ in T in such a way that $a \cup b$ becomes a monotone arc and so that no new critical points are introduced. The effect is that c and the minimum are no longer critical points. (A saddle and a minimum have been canceled.) Denote the new height function by $\tilde{h}|_{ext(T)}$. See fig. 4a.

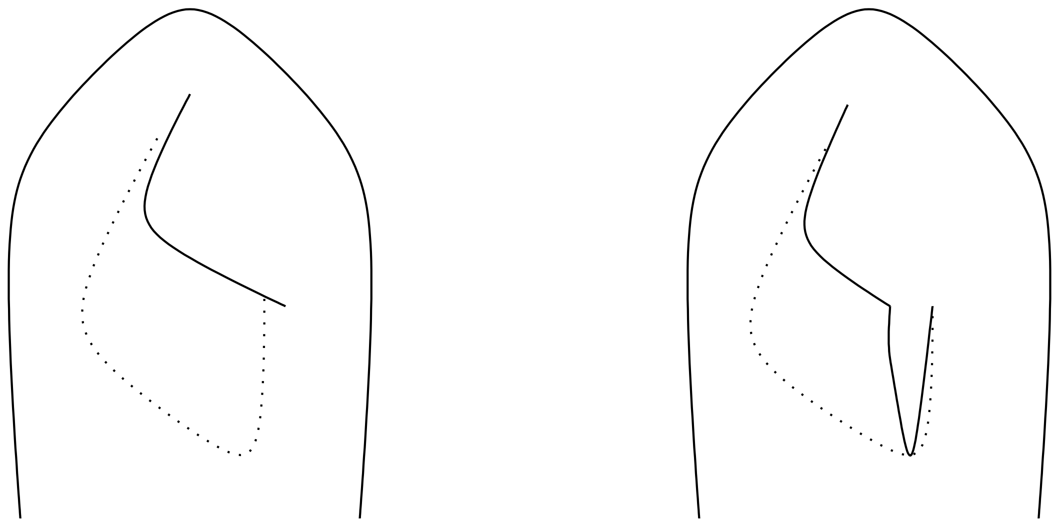

Figure 5: *Transforming h to $\tilde{h}$*

Consider a subarc e of the collar of $a \cup b$ in T that lies in the level surface of $\bar{h}|_{ext(T)}$ containing c. As $\bar{h}|_{ext(T)}$ is modified, e is changed into a new arc d (that has exactly one critical point, a minimum). Simultaneously, a disk, D, cobounded by d and an arc, which we may also call e, connecting the endpoints of e, emerges. Now $ext(T)$ with the new height function $\tilde{h}$ satisfies the inductive hypothesis. Thus the conclusions of the lemma hold and there is a solid torus M_n with height function $\tilde{h}_n$ obtained from $ext(T)$ in the required manner.

Note that the procedure in Step 2 of the proof of Lemma 3.4 may be refined for the setting at hand. Specifically, consider the subarc a_i lying in the level surface $\tilde{S}$ of $\tilde{h}|_{ext(T)}$. Here a_i lies on a component of a level surface $\tilde{S}$ of $\tilde{h}$ but not necessarily of h. See fig. 4b.

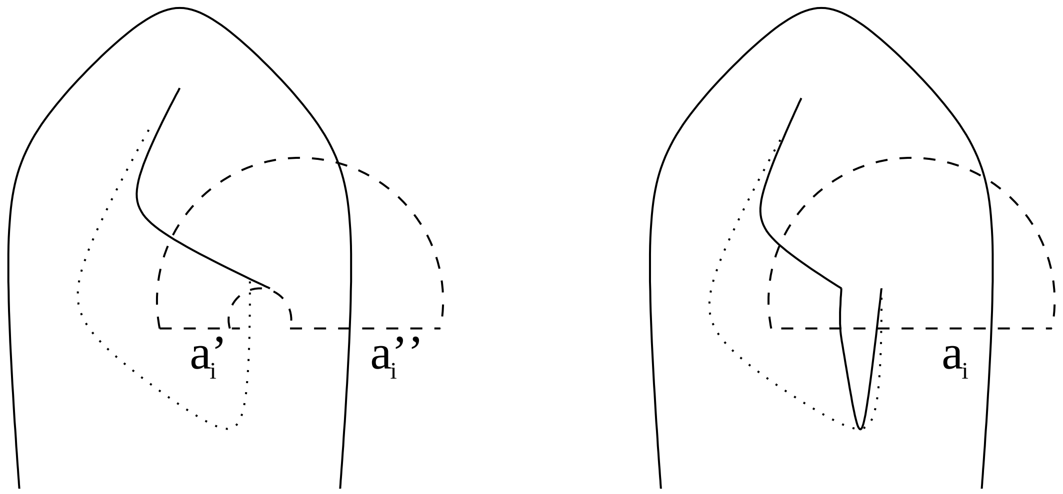

Figure 6: *A disk for a level surface of $\tilde{h}$ yields a disk for h*

Consider the arc $\delta = \tilde{S} \cap D$. Here δ is a separating arc with ends on one component of $\partial\tilde{S}$. It cuts $\tilde{S}$ into two components that can be identified with components of the level surface $\bar{h}^{-1}(\tilde{h}(\tilde{S}))$. It follows from the construction of $\tilde{h}$, that an arc b_i lies entirely on one side of δ. See fig. 4c.

To refine the procedure in Step 2 of the proof of Lemma 3.4 we choose g_i to lie on the same side of δ as b_i. A very similar argument to that in Step 2 of the proof of Lemma 3.4 shows that this can be done. (In the case of a nested saddle that is a lower saddle, we interchange the roles of α and β.)

Thus the cut and paste operation performed on $\tilde{S}$ can be performed so that each point of $D \cap \tilde{S}$ is reidentified with itself. This implies that the cut and paste operation on $\tilde{S}$ can be factored into two cut and paste operations, one on each of the components of $\tilde{S} - D$. Thus the corresponding cut and paste operation can be performed for $\bar{h}|_{ext(T)}$ by factoring it into two cut and paste operations each performed on one of the two components of $\bar{h}^{-1}(\tilde{h}(\tilde{S}))$.

The components of the level surfaces of $\bar{h}|_{ext(T)}$ are in one-to-one correspondance with the components of the level surfaces of $h|_{ext(T)}$. Thus all cut and paste operations on level surfaces of $\bar{h}|_{ext(T)}$ can be realized as successions of cut and paste operations on the level surfaces of $h|_{ext(T)}$.

The final statment follows as before. □

4 the main construction

Theorem 4.1. *$w(K_1 \# K_2) \geq max(w(K_1), w(K_2))$.*

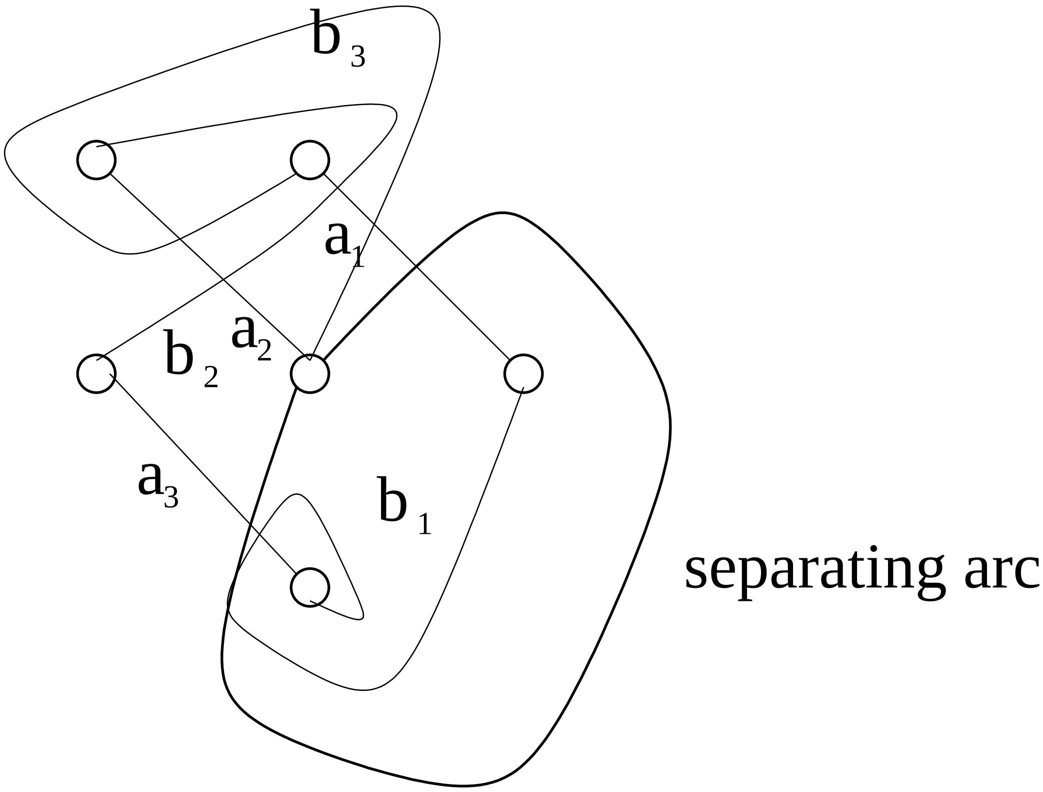

Figure 7: *Intersection pattern of α and β in $\tilde{S}$*

Proof: Let $h : S^3 \to [0,1]$ be a height function that realizes the width of $K = K_1 \# K_2$ and let T be a swallow follow torus that swallows K_1 and follows K_2. We wish to construct a height function $\tilde{h} : S^3 \to [0,1]$ such that $w(K_1, \tilde{h}) = w(K)$.

By construction, $(int(T), K) \cong (solid\, torus, K_1)$. We use this fact as follows: Here $ext(T)$ is homeomorphic to $C(K_2)$. In order to avoid confusion, briefly consider S^3 and remove $ext(T)$, then $int(T)$ is homeomorphic to a closed regular neighborhood of K_2. We do not consider this imbedding of $int(T)$ in S^3. Instead, we consider an abstract copy of $int(T)$. Let D be a meridian disk of $int(T)$, i.e., a properly imbedded disk that meets K once. Then if we attach a solid torus V to $\partial(int(T))$ such that a longitude of ∂V is identified with ∂D, the resulting 3-manifold is S^3, and

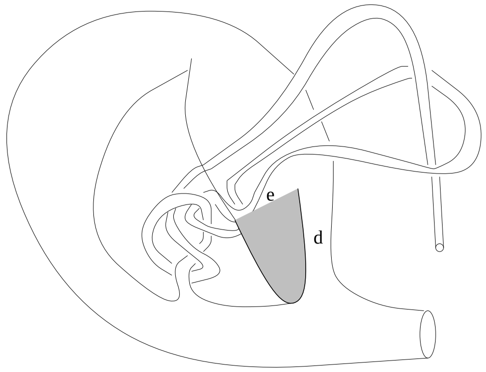

Figure 8: *Modification of the figure in fig. 1*

$(int(T) \cup V, K) \cong (S^3, K_1)$, i.e., K becomes isotopic to K_1 in this S^3.

We now employ h to construct a height function on this latter copy of S^3. To do so, we must extend the Morse function $h|_{int(T)}$ to S^3. Let $V = M_n$ be the solid torus provided by Lemma 3.5. Then V is endowed with a Morse function $h_n : M_n \to [0, 1]$. We denote the Morse function on $int(T) \cup V$ given by $h|_{int(T)}$ on $int(T)$ and h_n on V by $\tilde{h}$. The identification is such that $\partial h_n^{-1}(r)$ matches up with $\partial h|_{int(T)}^{-1}(r)$ for all $r\epsilon[0, 1]$.

(Since $h_n^{-1}(r) = h|_{ext(T)}^{-1}(r)$ for all r, this can be done. In fact, Lemmata 3.4 and 3.5 can be translated into precise instructions on which boundary components to identify. Again, note that this is an abstract identification.)

This provides a Morse function $\tilde{h} : S^3 \to [0, 1]$. Since its critical points are in one-to-one correspondance with the critical points of $h : S^3 \to [0, 1]$, it is a height function. Note that there is now a one-to-one correspondance between the critical points of $h|_{K_1\#K_2}$ and the critical points of $\tilde{h}|_{K_1}$. Thus $w(K_1) \leq w(K_1\#K_2)$.

The proof that $w(K_2) \leq w(K_1\#K_2)$ proceeds analogously. □

5 Corollaries

Corollary 5.1. $w(K_1\#K_2) \geq \frac{1}{2}(w(K_1) + w(K_2))$.

Proof: Suppose $w(K_1) \geq w(K_2)$, then

$$w(K_1\#K_2) \geq w(K_1) = \frac{1}{2}(2w(K_1)) \geq \frac{1}{2}(w(K_1) + w(K_2)).$$

Similarly in the other case. □

Corollary 5.2. $w(K_1\# \dots \#K_n) \geq \frac{1}{n}(w(K_1) + \cdots + w(K_n))$.

Proof: Suppose $w(K_1) \geq max(w(K_2), \dots, w(K_n))$, then

$$w(K_1\# \dots \#K_n) \geq w(K_1) = \frac{1}{n}(n\, w(K_1)) \geq \frac{1}{n}(w(K_1) + \cdots + w(K_n)).$$

Similarly in the other cases. □

Theorem 5.3. *If K is a satellite knot with companion J, then $w(J) \leq w(K)$.*

Proof: The proof is analogous to the one above. □

It remains to find examples of knots with widths exhibiting degenerative behaviour with respect to connected sum.

6 A note on computing the width of a knot

The thick and thin levels of a knot in thin position allow for a computation of the width of a knot.

Lemma 6.1. *Let $S_{i_1}, \ldots, S_{i_k}$ be the thick levels of K and $F_{j_1}, \ldots, F_{j_{k-1}}$ the thin levels. Set $a_{i_l} = |\ K\ \cap S_{i_l}|$ and $b_{j_l} = |\ K\ \cap F_{i_l}|$. Then*

$$w(K) = 2\sum_{l=1}^{k} a_{i_l}^2 - 2\sum_{l=1}^{k-1} b_{j_l}^2.$$

Proof: This can be proven by a direct computation and repeated use of the Gauss Summation Formula. The proof is inspired by a comment by Clint McCrory (his comment concerned the case in which thin position is bridge position, and the illustration is a square of dots.) It is illustrated in fig. 6. Each dot represents two points of intersection with a regular level surface between two critical level surfaces. For instance, the dots in fig. 6 represent the case in which the critical values, listed from the highest to the lowest are a maximum, maximum, maximum, maximum, minimum, minimum, maximum, maximum, minimum, minimum, maximum, maximum, maximum, minimum, minimum, minimum, minimum, minimum.

□

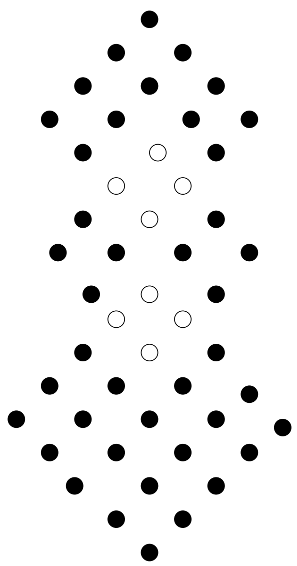

Figure 9: *Dark dots indicate squares that are added, white dots indicate overlap squares that are subtracted*

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Jennifer Schultens
Department of Mathematics and CS
400 Dowman Dr.
Emory University
Atlanta, GA 30322, USA
jcs@mathcs.emory.edu