ON CUBIC FUNCTORS

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Introduction

Polynomial functors appeared in algebraic topology ((cf. [8])) and proved to be useful for lots of questions in homotopy theory. That is why their study seems to be of interest. Mostly, one deals with "continuous" polynomial functors, which are defined by their values on free groups, and in what follows we consider only such ones. In [7] the author gave a description of all (finitely generated) quadratic functors. This description was done using the technique of the so called *matrix problems*, namely, representations *of bunches of chains*. The reduction to such a matrix problem depends on the presentation of quadratic functors as modules over a special ring, which happens to be an order in a semi-simple algebra. Then we used the procedure of [6]. Now we are going to pass the same way for *cubic functors*. We check that the corresponding ring is again an order in a semi-simple algebra. Unfortunately, in this case, the classification of modules is a *wild problem*, i.e., it includes, in some sense, the classification of all representations of all finitely generated algebras over the residue field $\mathbb{Z}/2$ (cf. Section 2). Thus, there can be no "good" description of all cubic functors.

On the contrary, such a description becomes possible (and rather analogous to that of quadratic functors) if we "make 2 invertible," i.e., consider cubic modules over the ring $\mathbb{Z}[\,1/2\,]$. Such a description is given in Section 3. Just as for quadratic case, this classification problem is *tame*, i.e., indecomposable modules depend on some "discrete" combinatorial data and on at most one "continuous parametre," which is an irreducible polynomial from $\mathbb{Z}/2[\,t\,]$ (or, the same, a closed point of $\mathbb{A}^1_{\mathbb{Z}/2} = \operatorname{Spec}\mathbb{Z}/2[\,t\,]$). We also consider *weakly alternative* cubic functors F, i.e., those with $F(\mathbb{Z}) = 0$. Their classification given in Section 4 is also tame, though this time the corresponding ring has nilpotent ideals. At last, we give a classification of cubic functors with the image being *vector spaces* (Section 5) as well as *torsion free* ones (Section 6). These problems also happens to be tame. We end up with

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one conjecture concerning polynomial functors of higher degrees and some corollaries from this conjecture.

1. Generalities

We suppose all categories to be *pre-additive*, i.e., such that their morphism sets are abelian groups and the composition is bi-additive. On the contrary, we do not suppose the *functors* to be additive, though we always suppose that $F(0)=0$ for a zero morphism. Remind that a *fully additive* category is an additive category such that every idempotent in it splits. If $F:\mathsf{A}\to\mathsf{B}$ is a functor from an additive category A to a fully additive category B and $A=\bigoplus_{k=1}^n A_k$ is an object from A, consider the corresponding embeddings $i_k:A_k\to A$ and projections $p_k:A\to A_k$. Put $e_k=i_kp_k$ and $f(k)=F(e_k)$. Certainly, e_k, hence, $f(k)$, are pairwise orthogonal idempotents and $\operatorname{Im} f(k)\simeq F(A_k)$. Moreover, put $f(kl)=F(e_k+e_l)-f(k)-f(l)$ ($k<l$). Then $f(kl)$ are also idempotents, pairwise orthogonal and orthogonal to all $f(k)$. Hence, $F(A)$ has a direct summand $(\bigoplus_k F(A_k))\oplus(\bigoplus_{k<l}F_2(A_k|A_l))$, where one denotes $F_2(A_k|A_l)=\operatorname{Im} f(kl)$. Define recursively for any $1\le k_1<k_2<\dots<k_r\le n$

$$f(k_1k_2\dots k_m)=F(e_{k_1}+\dots e_{k_m})-\sum_{r<m}\sum_{1\le l_1<\dots<l_r\le m} f(k_{l_1}k_{l_2}\dots k_{l_r})$$

and $F_m(A_1|A_2|\dots|A_m)=\operatorname{Im} f(k_1k_2\dots k_m)$. Then

$$F(A)=\bigoplus_{m\le n}\left(\bigoplus_{k_1k_2\dots k_m} F_m(A_{k_1}|A_{k_2}|\dots|A_{k_m})\right).$$

One easily sees that F_m is indeed a functor $\mathsf{A}^m\to\mathsf{B}$. It is called the m-th *cross-effect* of F. Of course, $F_1=F$. If there is a positive integer n such that $F_m=0$ for $m>n$, one calls F a *polynomial functor*. The smallest possible n with this property is called the *degree* of F. Certainly, the functors of degree 1 are just additive ones. The functors of degree 1 are called *linear*, of degree 2 *quadratic*, of degree 3 *cubic*, etc. For an arbitrary pre-additive category A, one can always consider its *fully additive hull* $\operatorname{add}\mathsf{A}$ and we call polynomial functors $\mathsf{A}\to\mathsf{B}$ those from $\operatorname{add}\mathsf{A}$ to B. Of course, for additive functors we get in this way nothing new as every additive functor $\mathsf{A}\to\mathsf{B}$ can be uniquely (up to isomorphism) prolonged to an additive functor $\operatorname{add}\mathsf{A}\to\mathsf{B}$.

In what follows, we consider the case when $\mathsf{A}=\mathbb{Z}$, the ring of integers, and $\mathsf{B}=\mathbf{R}\text{-}\mathsf{Mod}$, the category of modules over a ring $\mathbf{R}$, mainly even $\mathsf{B}=\mathbf{R}\text{-}\mathsf{mod}$, the category of finitely generated $\mathbf{R}$-modules. Note that $\operatorname{add}\mathbb{Z}=\mathsf{fab}$, the category of finitely generated free abelian groups.

In this case polynomial functors are called *polynomial* $\mathbf{R}$-*modules* and the category of finitely generated polynomial $\mathbf{R}$-modules of degree at most n is denoted by $\mathbf{R}\text{-}\mathsf{mod}^n$. Of course, $\mathbf{R}\text{-}\mathsf{mod}^1$ coincide with the category of finitely generated $\mathbf{R}$-modules. If $\mathbf{R}=\mathbb{Z}$, we are just speaking of "polynomial modules" not precising the target category.

To define a polynomial functor $F:\mathsf{fab}\to\mathsf{B}$, one only has to define the following data:

- objects $F_m=F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$;
- morphisms $h_k^m:F_m\to F_{m+1}$ that are compositions of the morphisms

$$\begin{aligned}F_m(A_1|A_2|\dots|A_m)&\to F(A_1\oplus A_2\oplus\dots\oplus A_m)\\&\to F(A_1\oplus\dots\oplus A_k\oplus A_k\oplus\dots\oplus A_m)\\&\to F_{m+1}(A_1|\dots|A_k|A_k|\dots|A_m),\end{aligned}$$

  where all $A_j=\mathbb{Z}$, the first morphism is the embedding of the direct summand, the last one is the projection onto the direct summand and the middle one is $F(\gamma)$, $\gamma|_{A_j}$ being identity if $j\neq k$ and $\gamma|_{A_k}$ being the diagonal embedding $A_k\to A_k\oplus A_k$;
- morphisms $p_k^m:F_{m+1}\to F_m$ that are compositions of the morphisms

$$\begin{aligned}F_{m+1}(A_1|\dots|A_k|A_k|\dots|A_m)&\to F(A_1\oplus\dots\oplus A_k\oplus A_k\oplus\dots\oplus A_m)\\&\to F(A_1\oplus A_2\oplus\dots\oplus A_m)\\&\to F_m(A_1|A_2|\dots|A_m),\end{aligned}$$

  where again all $A_j=\mathbb{Z}$, the first morphism is the embedding of the direct summand, the last one is the projection onto the direct summand and the middle one is $F(\beta)$, $\beta|_{A_j}$ being identity for $j\neq k$, while $\beta|_{A_k}$ being the codiagonal (summation) mapping $A_k\oplus A_k\to A_k$.

In particular, a cubic $\mathbf{R}$-module F, where $\mathbf{R}$ is a ring, is defined by a diagram of $\mathbf{R}$-modules:

$$(1)\qquad F_1 \underset{p}{\overset{h}{\rightleftarrows}} F_2 \quad \begin{matrix}\xrightarrow{h_1}\\ \xrightarrow{h_2}\\ \xleftarrow{p_1}\\ \xleftarrow{p_2}\end{matrix} \quad F_3$$

One can show that such a diagram corresponds to a cubic module if and only if the following relations hold:

$$
\begin{aligned}
&h_1p_2 = h_2p_1 = 0\,, \quad h_1h = h_2h\,, \quad pp_1 = pp_2\,,\\
&h_ip_ih_i = 2h_i\,, \quad p_ih_ip_i = 2p_i \quad (i = 1, 2)\,,\\
&hph = 2(h + (p_1 + p_2)\overline{h})\,, \quad php = 2(p + \overline{p}(h_1 + h_2))\,,\\
&\overline{h}p + h_1 + h_2 = h_1p_1h_2p_2h_1 + h_2p_2h_1p_1h_2\,,\\
&h\overline{p} + p_1 + p_2 = p_1h_2p_2h_1p_1 + p_2h_1p_1h_2p_2\,,
\end{aligned} \tag{2}
$$

where $\overline{h} = h_1h = h_2h$ and $\overline{p} = pp_1 = pp_2$ [3].

Consider the pre-additive category $\mathbf{A}$ with three objects $1, 2, 3$ and generating morphisms

$$h : 1 \to 2\,,\ p : 2 \to 1\,,\ h_i : 2 \to 3\,,\ p_i : 3 \to 2 \quad (i = 1, 2)$$

subject to the relations (2). Then cubic $\mathbf{R}$-modules are just linear functors $\mathbf{A} \to \mathbf{R}\text{-}\mathsf{mod}$. Sometimes we identify $\mathbf{A}$ with the endomorphism ring $\mathrm{End}(1 \oplus 2 \oplus 3)$. Obviously, the categories of modules over $\mathbf{A}$ and over this ring coincide.

2. Wildness

We are going to show that the description of *all* cubic functors is a *wild problem* in the sense of the representation theory (cf. [5]). Moreover, we show that it is even true for those cubic functors "freely generated by their 1-part," or, the same, for $\mathbf{A}(1,1)$-modules.

Proposition 2.1. *The ring* $\mathbf{A}(1,1)$ *is isomorphic to the subring of* $\mathbb{Z}^3$ *with the basis* $\{\, (1,1,1), (2,0,0), (0,6,0)\,\}$.

Proof. Put $a = ph - \overline{ph}$, $b = ph$. One easily verifies that these two elements generate $\mathbf{A}(1,1)$, $ab = ba = 0$, $a^2 = 2a$ and $b^2 = 6b$. Hence, one gets an isomorphism by mapping $a \mapsto (2,0,0)$, $b \mapsto (0,6,0)$. □

The category of $\mathbf{A}(1,1)$-modules can be embedded into that of $\mathbf{A}$-modules. Namely, every $\mathbf{A}(1,1)$-module M gives rise to a cubic module F, where

$$F_1 = M\,, \quad F_2 = \mathbf{A}(1,2) \otimes_{\mathbf{A}(1,1)} M \quad \text{and} \quad F_3 = \mathbf{A}(1,3) \otimes_{\mathbf{A}(1,1)} M$$

with the obvious action of morphisms. It is known (and easy to check) that this procedure really defines a functor which is a full embedding.

Let Σ_n be the free (non-commutative) algebra $\mathbb{Z}/4\langle\, x_1, x_2, \dots, x_n\,\rangle$ with n generators over the residue ring $\mathbb{Z}/4$.

Proposition 2.2. *Denote by* $\varphi : \mathbf{A}(1,1) \to \Sigma_2$ *the homomorphism mapping* a *to* $2x_1$ *and* b *to* $2x_2$. *For any* Σ_2*-module* L, *let* ${}^{\varphi}L$ *be the* $\mathbf{A}(1,1)$*-module obtained from* L *via the "change of rings"* φ. *Then, for any* Σ_2*-modules* L, L' *which are free as* $\mathbb{Z}/4$*-modules,*

- ${}^{\varphi}L \simeq {}^{\varphi}L'$ *as* $\mathbf{A}(1,1)$*-modules if and only if* $L/2L \simeq L'/2L'$ *as* Σ_2*-modules;*
- $\mathbf{A}(1,1)$*-module* ${}^{\varphi}L$ *is indecomposable if and only if so is* Σ_2*-module* $L/2L$.

The proof is evident. □

Corollary 2.3. *For every* n, *there is a cubic module* $\mathcal{M} \in \mathcal{C}(\Sigma_n)$ *such that, for any* Σ_n*-modules* L, L' *which are free over* $\mathbb{Z}/4$,

- $\mathcal{M} \otimes_{\Sigma_n} L \simeq \mathcal{M} \otimes_{\Sigma_n} L'$ *if and only if* $L/2 \simeq L'/2$;
- $\mathcal{M} \otimes_{\Sigma_n} L$ *is indecomposable if and only if* $L/2$ *is indecomposable.*

Proof. One only has to consider the Σ_2-Σ_n-bimodule $\mathcal{N}$ which is free of rank $n+1$ over Σ_n, while the action of Σ_2 is given by the following rules:

- x_1 is acting as the (upper) Jordan cell;
- x_2 is acting as the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ x_1 & 0 & \dots & 0 & 0 \\ 0 & x_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x_n & 0 \end{pmatrix}$$

Afterwards, one puts $\mathcal{M} = {}^{\varphi}\mathcal{N}$, with φ defined as in Proposition 2.2. All properties are verified immediately (cf., e.g., [5]). □

This Corollary shows that the classification of all cubic modules (even of $\mathbf{A}(1,1)$-modules) is at least so complicated as the classification of n-tuples of matrices over the filed $\mathbb{Z}/2$ up to conjugations, for all n. It is just the situation when one calls such a problem "wild."

3. 2-divisible case

In this section we consider the case of cubic *2-divisible* modules, i.e., cubic functors $F : \mathsf{fab} \to \mathbb{Z}'\text{-}\mathsf{mod}$, where $\mathbb{Z}' = \mathbb{Z}[1/2]$. It happens that here we are able to describe all (finitely generated) cubic modules. Moreover, this description is quite alike that of quadratic $\mathbb{Z}$-modules given in [7]. Certainly, 2-divisible cubic modules correspond to the modules over the category $\mathbf{A}' = \mathbf{A}[1/2]$.

Proposition 3.1. *The category* $\mathbf{A}'$ *is Morita-equivalent to the direct product* $\mathbb{Z}' \times \mathbb{Z}' \times \mathbf{B}$, *where* $\mathbf{B}$ *is the subring in* $\mathbb{Z}' \times \mathrm{Mat}(2,\mathbb{Z}') \times \mathrm{Mat}(2,\mathbb{Z}') \times \mathbb{Z}'$ *consisting of all quadruples of the form:*

$$\left(a, \begin{pmatrix} b_1 & 3b_2 \\ b_3 & b_4 \end{pmatrix}, \begin{pmatrix} c_1 & 3c_2 \\ c_3 & c_4 \end{pmatrix}, d \right)$$

with

$$a \equiv b_1,\ b_4 \equiv c_1,\ c_4 \equiv d \pmod 3)$$

Remark. One can see that a module over the ring $\mathbf{B}$ is given by a diagram of abelian (2-divisible) groups of the following shape:

$$M_1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} M_2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} M_3, \tag{3}$$

satisfying the relations:

$$\begin{aligned} &\alpha_1\beta_1\alpha_1 = 3\alpha_1\,, \quad \beta_1\alpha_1\beta_1 = 3\beta_1\,, \\ &\alpha_2\beta_2\alpha_2 = 3\alpha_2\,, \quad \beta_2\alpha_2\beta_2 = 3\beta_2\,, \\ &\beta_1\beta_2 = 0\,, \quad \alpha_2\alpha_1 = 0\,, \\ &\alpha_1\beta_1 + \beta_2\alpha_2 = 3\,\mathrm{id}_2\,. \end{aligned}$$

We always identify $\mathbf{B}$-modules with such diagrams.

Proof. Put $\mathbf{A}_i = \mathbf{A}'(i,i)$ $(i = 1,2,3)$ and consider the following elements:

in $\mathbf{A}_1$:

$$f_1 = \frac{ph - \overline{p}\overline{h}}{2},\ e_1 = \mathrm{id}_1 - f_1,\ a_1 = \frac{\overline{p}\overline{h}}{2};$$

in $\mathbf{A}_2$:

$$\begin{aligned} &e_2 = \frac{p_1h_1}{2},\ f_2 = \frac{p_2h_2}{2},\ g = \mathrm{id}_2 - e_2 - f_2, \\ &u = hp,\ g_1 = \frac{gu}{2},\ g_2 = g - g_1, \\ &v_i = p_ih_j\ (i,j = 1,2,\ i \neq j),\ v_1' = \frac{5v_1 - v_1v_2v_1}{2}, \\ &a_2 = v_1v_2,\ b_2 = v_2v_1; \end{aligned}$$

in $\mathbf{A}_3$:

$$\begin{aligned} &u_i = h_ip_i\ (i = 1,2),\ e_3 = \frac{u_1}{2},\ f_3 = \mathrm{id}_3 - e_3, \\ &\overline{u} = \overline{h}\overline{p} + e_3,\ a_3 = 2u_2e_3 - \overline{u},\ b_3 = 2e_3u_2 - \overline{u}; \end{aligned}$$

in $\mathbf{A}'(2,1)$:

$$p' = p - \frac{\overline{p}(h_1+h_2)}{2}.$$

On can verify that:

- elements $e_i,\ f_i,\ g_i$ are orthogonal idempotents;
- elements e_1, f_1, a_1 form a $\mathbb{Z}'$-basis of $\mathbf{A}_1$;
- $a_1 = e_1a_1e_1$ and $a_1^2 = 3a_1$;
- elements $e_2, f_2, g_1, g_2, a_2, b_2, v_1, v_2, v_1b_2, a_2v_1$ form a $\mathbb{Z}'$-basis of $\mathbf{A}_2$;
- $g\mathbf{A}_2 = \mathbf{A}_2g = \langle\, g_1, g_2 \,\rangle$;
- $a_2 = e_2a_2e_2\,,\ b_2 = f_2b_2f_2\,,\ a_2^2 = 3a_2\,,\ b_2^2 = 3b_2$;
- $v_1'(v_2/2) = e_2\,,\ (v_2/2)v_1' = f_2$;
- $p'(h/2) = f_1\,,\ (h/2)p' = g_1$;
- elements $e_3, f_3, a_3, b_3, a_3b_3, b_3a_3$ form a $\mathbb{Z}'$-basis of $\mathbf{A}_3$;
- $a_3 = f_3a_3e_3\,,\ b_3 = e_3b_3f_3\,,\ a_3b_3a_3 = 3a_3\,,\ b_3a_3b_3 = 3b_3$;
- $p'(h/2) = f_1$ and $(h/2)p' = g_1$;
- $(p_1/2)h_1 = e_2$ and $h_1(p_1/2) = e_3$;
- $gp_i = h_ig = 0\ (\,i = 1,2\,)$ and $g_2h = pg_2 = 0$;
- $f_2h_i = p_if_2 = 0\ (\,i = 1,2\,)$.

Consider the projective $\mathbf{A}'$-modules:

$$\begin{aligned}
&E_1 = \mathbf{A}'(1,_)e_1,\ F_1 = \mathbf{A}'(1,_)f_1,\\
&E_2 = \mathbf{A}'(2,_)e_2,\ F_2 = \mathbf{A}'(2,_)f_2,\ G_i = \mathbf{A}'(2,_)g_i,\\
&E_3 = \mathbf{A}'(3,_)e_3,\ F_3 = \mathbf{A}'(3,_)f_3,
\end{aligned}$$

They are projective generators of the category of $\mathbf{A}'$-modules. The above equalities imply that:

$F_1 \simeq G_1,\ E_2 \simeq F_2,\ E_2 \simeq E_3;$

$\mathrm{Hom}(G_i, G_j) = 0$ if $i \neq j$;

$\mathrm{Hom}(G_i, E_j) = \mathrm{Hom}(E_j, G_i) = \mathrm{Hom}(G_i, F_3) = \mathrm{Hom}(F_3, G_i) = 0;$

$\mathrm{Hom}(E_1, F_3) = \mathrm{Hom}(F_3, E_1) = 0;$

$\mathrm{End}\, E_1 \simeq e_1\mathbf{A}_1e_1 = \langle\, e_1, a_1 \,\rangle \simeq \mathbf{D};$

$\mathrm{End}\, E_2 \simeq e_2\mathbf{A}_2e_2 = \langle\, e_2, a_2 \,\rangle \simeq \mathbf{D};$

$\mathrm{End}\, F_3 \simeq f_3\mathbf{A}_3f_3 = \langle\, f_3, a_3b_3 \,\rangle \simeq \mathbf{D};$

$\mathrm{End}\, G_i = \mathbb{Z}';$

$\mathrm{Hom}(E_3, F_3) = \langle\, b_3 \,\rangle \simeq \mathbb{Z}';$

$$\mathrm{Hom}(F_3, E_3) = \langle\, a_3 \,\rangle \simeq \mathbb{Z}';$$
$$\mathrm{Hom}(E_1, E_3) = \langle\, \overline{p} \,\rangle \simeq \mathbb{Z}';$$
$$\mathrm{Hom}(E_3, E_1) = \langle\, \overline{h} \,\rangle \simeq \mathbb{Z}'.$$

Here $\mathbf{D}$ denotes the ring $\mathbb{Z}'[t]/(t^2-3)$, which is isomorphic to the subring in $\mathbb{Z}'\times\mathbb{Z}'$ consisting of the pairs $\{\,(a,b)\mid a\equiv b \pmod 3\,\}$. Therefore, the category of $\mathbf{A}'$-modules is equivalent to that of $\mathbf{E}$-modules, where $\mathbf{E}=\mathrm{End}(G_1\oplus G_2\oplus E_1\oplus F_3\oplus E_3)\simeq \mathbb{Z}'\times\mathbb{Z}'\times\mathbf{B}$. □

One can verify that the $\mathbf{A}'$-modules G_i correspond to the functors S^2 and Λ^2, where S and Λ^r denote, as usually, the r-th symmetric and exterior powers. Actually, these functor are quadratic, and it is the main reason that they "stand apart" really cubic ones.

Now the theory of 2-divisible cubic modules becomes quite parallel to that of quadratic modules [7] and one gets the main results just following the same way. As the proofs are also almost the same, we replace them by the exact references to the corresponding items from [7]. We denote by e_i $(\,i=1,2,3\,)$ the natural idempotents in $\mathbf{B}$:

$$e_1 = \left(1, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}, 0\right),$$
$$e_2 = \left(0, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, 0\right),$$
$$e_1 = \left(0, \begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, 1\right).$$

Put $\mathsf{B}_i=\mathbf{B}e_i$; they are just indecomposable projective $\mathbf{B}$-modules. Denote by L_1 the projection of B_1 onto the first component, by L_2 its projection onto the second component; by L_3 the projection of B_2 onto the second component, by L_4 its projection onto the third component; by L_5 the projection of B_3 onto the third and by L_6 its projection onto the forth component. Then L_i are just irreducible torsion free B-modules. Put $\mathbb{Z}_{(p)}=\{\,a/b\mid a,b\in\mathbb{Z},\, p\nmid b\,\}$, $M_{(p)}=\mathbb{Z}_{(p)}\otimes M$.

Theorem 3.2. 1. *2-divisible cubic modules M,N are isomorphic if and only if $M_{(p)}\simeq N_{(p)}$ for every odd prime p.*

2. *Given a set $\{\,N_p\,\}$ $(\,p>2\,)$ where N_p is a cubic $\mathbb{Z}_{(p)}$-module, there is a 2-divisible cubic module M such that $M_{(p)}\simeq N_p$ for all p if and only if almost all (i.e., all but a finite set) of them are torsion free (maybe, zero) and $\mathbb{Q}\otimes N_p\simeq\mathbb{Q}\otimes N_q$ for all p,q. In this case, $\mathbb{Q}\otimes M=\mathbb{Q}\otimes N_p$ and $\mathrm{t}M=\bigoplus_p \mathrm{t}N_p$.*

Cf. [7, Theorem 2.1]. □

Now we define our main personages: string and band modules. We rearrange a bit the description given in [7, Section 4] to make it more comprehensible. We start with some notations and definitions.

Definition 3.3. 1. First define on the set $\{1,2,3,4,5,6\}$ an equivalence relation $-$ with the only non-trivial equivalences $2-3$ and $4-5$ and a symmetric relation $\sim$ (not equivalence!) putting $1\sim 2$, $3\sim 4$, $5\sim 6$.

2. Define also the following mappings acting on every diagram of the form (3):

$$\begin{array}{ll}
\theta(11) = 3\,\mathrm{id}_1 - \beta_1\alpha_1\,, & \theta(22) = \beta_1\alpha_1\,,\\
\theta(23) = \alpha_1\,, & \theta(32) = \beta_1\,,\\
\theta(33) = \beta_2\alpha_2\,, & \theta(44) = \alpha_2\beta_2\\
\theta(45) = \alpha_2\,, & \theta(54) = \beta_2\,,\\
\theta(55) = \alpha_2\beta_2 & \theta(66) = 3\,\mathrm{id}_3 - \alpha_2\beta_2\,.
\end{array} \tag{4}$$

3. Define the function $\nu : \{\{1,2\},\{3,4\},\{5,6\}\} \to \mathbb{N}$ putting $\nu\{1,2\}=1$, $\nu\{3,4\}=2$, $\nu\{5,6\}=3$.

4. A *primary polynomial* over a field $\mathbf{k}$ is, by definition, a power of an irreducible polynomial with the leading coefficient 1.

Now the string and band modules can be defined as follows.

Definition 3.4. 1. A *string diagram* D is a diagram of one of the following types:

$$\text{(i)}\qquad \begin{array}{ccccccccccc}
j_1 & & & & j_2j_3 & & & & j_{2n-2}j_{2n-1} & & \\
& k_1 & & k_2 & & k_3 & & \dots & & k_{2n-1} & \\
& & i_1i_2 & & & & i_3i_4 & & & & i_{2n-1}
\end{array}\,,$$

or

$$\text{(ii)}\qquad \begin{array}{ccccccccc}
& & j_2j_3 & & & & j_{2n-2}j_{2n-1} & & \\
& k_2 & & k_3 & & \dots & & k_{2n-1} & \\
i_2 & & & & i_3i_4 & & & & i_{2n-1}
\end{array}\,,$$

or

$$\text{(iii)}\qquad \begin{array}{ccccccccc}
j_1 & & & & j_2j_3 & & & & j_{2n}\\
& k_1 & & k_2 & & \dots & & k_{2n} & \\
& & i_1i_2 & & & & i_{2n-1}i_{2n} & &
\end{array}\,,$$

where $k\in\mathbb{N}$, $i_m, j_m \in \{1,2,3,4,5,6\}$, satisfying the following conditions:

- $i_{2m-1}\sim i_{2m}$ for every $m=1,\dots,n$.
  (This condition is empty for diagrams (i) and (ii) if $m=n$ and for diagrams (ii) if $m=1$; nevertheless, in these cases we *define* i_1 or i_{2n} so that this condition holds.)

- $j_{2m+1} \sim j_{2m}$ for every $m = 1, \dots, n-1$.
- $i_k - j_k$ for every $m = 1, \dots, 2n$.
  (This condition is empty for diagrams (i) and (ii) if $m = 2n$ and for diagrams (ii) if $m = 1$.)

2. The *string* **B**-*module* corresponding to a string diagram D is the **B**-module $M = M^{\mathrm{D}}$ with the generators

$$\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n, \qquad \mathbf{g}_m \in M_{\nu\{\, i_{2m-1}, i_{2m}\,\}},$$

and the defining relations

$$3^{k_{2m}}\theta(i_{2m}j_{2m})g_m = 3^{k_{2m+1}}\theta(i_{2m+1}j_{2m+1})g_{m+1}$$

for $i = 0, \dots, n$; we put here $g_0 = g_{n+1} = 0$ and omit the relation with $m = n$ for diagrams (i), (ii) and with $m = 1$ for diagrams (ii).

3. A *string cubic module* is one corresponding to a string **B**-module via the Morita-equivalence of Proposition 3.1.

Definition 3.5. 1. A string diagram D of type (iii) is said to be *non-periodic* if it cannot be obtained by a repetition of a smaller diagram D_1 of the same type: $\mathrm{D} \neq \mathrm{D}_1\mathrm{D}_1 \dots \mathrm{D}_1$.

2. A *band data* is a pair $\mathrm{B} = (\mathrm{D}, f(t))$ consisting of a non-periodic string diagram of type (iii) with the additional condition:
   - $j_{2n} \sim j_1$,

   and of a primary polynomial $f(t) = \lambda_1 + \lambda_2 t + \dots + \lambda_d t^{d-1} + t^d$ over the residue field $\mathbb{Z}/3$ with $\lambda_1 \neq 0$.
3. The *band* **B**-*module* corresponding to a band data B is the **B**-module $M = M^{\mathrm{B}}$ with the generators

$$\mathbf{g}_{ml} \ \ (m = 1, \dots, n,\ l = 1, \dots, d), \quad \mathbf{g}_{ml} \in M_{\nu\{\, i_{2m-1}, i_{2m}\,\}},$$

and the defining relations:

$$\begin{aligned} 3^{k_{2m}}\theta(i_{2m}j_{2m})g_{ml} &= 3^{k_{2m+1}}\theta(i_{2m+1}j_{2m+1})g_{m+1,l} \quad \text{if} \quad 1 \leq m < n; \\ 3^{k_{2n}}\theta(i_{2n}j_{2n})g_{nl} &= 3^{k_1}\theta(i_1j_1)g_{1,l+1} \quad \text{if} \quad 1 \leq l < d; \\ 3^{k_{2n}}\theta(i_{2n}j_{2n})g_{nd} &= -3^{k_1}\theta(i_1j_1)\sum_{\nu=1}^{d}\lambda_\nu g_{1\nu}. \end{aligned}$$

4. A *band cubic module* is one corresponding to a band **B**-module via the Morita-equivalence of Proposition 3.1.

Put also $L(i,p,k) = \mathsf{L}_i/p^k\mathsf{L}_i$, where k is a positive integer, $p > 3$ is a prime number, $i \in \{1,2,4,6\}$; $\Lambda^2(p,k) = \Lambda^2/p^k\Lambda^2$ and $\mathrm{S}^2(p,k) = \mathrm{S}^2/p^k\mathrm{S}^2$, where $p \geq 3$ is again a prime number.

Theorem 3.6. 1. *The string and band cubic modules defined above, the modules* $L(i,p,k)$ *and the modules* $\Lambda^2(p,k)$, $\mathrm{S}^2(p,k)$ *are just all finitely generated indecomposable 2-divisible cubic modules.*

2. *The only isomorphisms between these modules are the following:*
   - $M^{\mathrm{D}} \simeq M^{\mathrm{D}^*}$, *where* D^* *denotes the diagram symmetric to* D, *the latter being of type* (ii) *or* (iii)*;*
   - $M^{\mathrm{B}} \simeq M^{\mathrm{B}^{(s)}}$, *where* $\mathrm{B}^{(s)} = (\mathrm{D}^{(s)}, f(t))$ *and* $\mathrm{D}^{(s)}$ *is the* s-th shift *of the diagram* D, *i.e., the diagram*

$$\begin{array}{ccccccccc}
j_{2s+1} & & & & j_{2s+2}j_{2s+3} & & & & j_{2s} \\
& k_{2s+1} & & k_{2s+2} & & \cdots & & k_{2s} & \\
& & i_{2s+1}i_{2s+2} & & & & i_{2s-1}i_{2s} & &
\end{array}$$

   - $M^{\mathrm{B}} \simeq M^{\mathrm{B}^{*(s)}}$, *where* $(\mathrm{D}, f(t))^* = (\mathrm{D}^*, f^*(t))$ *and* $f^*(t) = \lambda_1^{-1} t^d f(1/t)$*;*

Cf. [7, Theorem 5.1] □

Corollary 3.7. 1. *A 2-divisible cubic module* M *is of finite projective dimension if and only if it contains no string summands of types* (i) *and* (iii)*. In this case either* M *is projective (hence, a direct sum of modules isomorphic to* B_i *) or* $\operatorname{pr.dim} M = 1$. *Otherwise,* $\operatorname{pr.dim} M = \infty$.

2. *Every 2-divisible cubic module* M *has a periodic projective resolution*

$$\dots \to P_{n+1} \xrightarrow{\gamma_n} P_n \xrightarrow{\gamma_{n-1}} \dots \xrightarrow{\gamma_1} P_1 \xrightarrow{\gamma_0} P_0 \to M \to 0$$

of period 6, namely, such that $\gamma_{n+6} = \gamma_n$ *for every* $n \geq 2$.

3. *If* M *is an indecomposable non-projective 2-divisible cubic module,* $\mathrm{t}M$ *is its torsion part, then*

$$M/\mathrm{t}M \simeq \begin{cases} \mathsf{L}_{i_{2n}} & \text{if } M \text{ is a string module of type (i)} \\ \mathsf{L}_{i_1} \oplus \mathsf{L}_{i_{2n}} & \text{if } M \text{ is a string module of type (ii)} \\ 0 & \text{otherwise} \end{cases}$$

(Remind that here i_{2n} denotes the unique index such that $i_{2n-1} \sim i_{2n}$ and i_1 denotes the unique index such that $i_2 \sim i_1$.)

Cf. [7, Corollaries 5.3, 5.5]. □

4. Weakly alternative functors

Consider now the case of cubic $\mathbb{Z}$-modules F such that $F(\mathbb{Z}) = 0$. We call them *weakly alternative.* They are actually modules over the category $\mathbf{A}^a = \mathbf{A}/(\mathrm{id}_1)$. The functors Λ^3 and $\Lambda^2 \otimes \mathrm{Id}$ are of this sort. Here again we are able to obtain a complete description.

Denote by $\mathbf{C}_0$ the subring in $\mathbb{Z} \times \mathrm{Mat}(2,\mathbb{Z})$ consisting of the pairs (a,B) with $b_{12} \equiv 0,\ a \equiv b_{11} \pmod 3$.

Proposition 4.1. *The category $\mathbf{A}^a$ is Morita equivalent to the semi-direct product $\mathbf{C} = (\mathbb{Z} \times \mathbf{C}_0) \ltimes T$, where T is the elementary 2-group with three generators ξ, η, θ such that the following relations hold:*

$$\begin{aligned} &\theta = \xi\eta,\ \ \eta\xi = 0, \\ &\eta e = \eta,\ \ e\xi = \xi, \quad \text{where} \quad e = (1,0,0), \\ &\xi\,(0,a,B) = a\xi \quad \text{and} \quad (0,a,B)\,\eta = a\eta \end{aligned}$$

for any pair $(a,B) \in \mathbf{C}_0$.

Proof. The category $\mathbf{A}^a$ has two objects: $2,3$, and is generated by the morphisms $h_i : 2 \to 3,\ p_i : 3 \to 2$ (i=1,2) subject to the following relations, cf. diagram (1):

$$\begin{aligned} &h_ip_ih_i = 2h_i\,, \quad p_ih_ip_i = 2p_i\,, \quad h_ip_j = 0 \ \text{if} \ i \neq j\,, \\ &h_1 + h_2 = h_1p_1h_2p_2h_1 + h_2p_2h_1p_1h_2\,, \\ &p_1 + p_2 = p_1h_2p_2h_1p_1 + p_2h_1p_1h_2p_2\,, \end{aligned}$$

that imply:

$$\begin{aligned} h_ip_i &= h_ip_ih_jp_jh_ip_i\,, \\ 2p_i &= 2p_ih_jp_jh_ip_i\,, \\ 2h_i &= 2h_ip_ih_jp_jh_i \end{aligned}$$

(always $i,j \in \{1,2\},\ i \neq j$). Then $e_i = p_ih_jp_jh_i$ are orthogonal idempotents in $\mathbf{A}^a(2,2)$ which are conjugate as $e_1p_1h_2 = p_1h_2e_2$ and $e_2p_2h_1 = p_2h_1e_1$. Moreover, one can check that $e_3p_1h_2 = p_1h_2e_3 = e_3p_2h_1 = p_2h_1e_3$ where $e_3 = 1 - e_1 - e_2$. Denote this product by θ. Then $2\theta = 0$ and $p_ih_i = \theta + 2e_i$, so $\mathbf{A}^a(2,2)$ is Morita equivalent to $\mathbb{Z} \times \mathbb{Z}[\theta]$.

Just in the same way one can verify that $\mathbf{A}^a(3,3) = \langle f_1, f_2, \alpha_1, \alpha_2, \beta \rangle$, where $f_1 = h_1p_1h_2p_2,\ f_2 = 1 - f_1,\ \alpha_i = h_ip_i,\ \beta = \alpha_2\alpha_1$. Moreover, $f_i\alpha_i = \alpha_if_j = \alpha_i,\ \alpha_1\alpha_2 = 3f_1$ and $\beta^2 = 3\beta$. Therefore, $\mathbf{A}^a(3,3)$ is isomorphic to the subring $\mathbf{C}_0$.

Let $u = p_1h_2p_2,\ v = h_1p_1h_2p_2h_1$. Then $e_1u = uf_1,\ f_1v = ve_1,\ uv = e_1,\ vu = f_1$, so e_1 and f_1 are conjugate in $\mathbf{A}^a$. Put $\xi = p_1 - p_1h_2p_2h_1p_1,\ \eta = h_1 - h_1p_1h_2p_2h_1$. One can also check that

$$\begin{aligned} &2\xi = 0,\ 2\eta = 0,\ \eta\xi = 0,\ \xi\eta = \theta,\ \xi f_3 = \xi,\ f_3\eta = \eta, \\ &e_3\mathbf{A}^a(3,2) = \langle \xi \rangle,\ \mathbf{A}^a(2,3)e_3 = \langle \eta \rangle\,. \end{aligned}$$

Hence, putting $E_3^a = \mathbf{A}^a(2,\_)e_3$, we get that $\mathbf{A}^a$ is Morita equivalent to the endomorphism ring $\mathrm{End}(E_3^a \oplus \mathbf{A}^a(3,\_))$ which is isomorphic to $\mathbf{C}$. $\square$

Describe now (finitely generated) $\mathbf{C}$-modules or, the same, weakly alternative cubic functors. If M is such a module, put $\overline{M} = M/TM$. It is a module over $\mathbb{Z} \times \mathbf{C}$, so $\overline{M} = M_1 \oplus M_2$, where M_1 is an abelian group with $\mathbf{C}_0 M_1 = 0$ and M_2 is a $\mathbf{C}_0$-module. Again, the same observations as in [7, Theorem 2.1] yield the following result.

Proposition 4.2. 1. *Weakly alternative cubic modules M, N are isomorphic if and only if $M_{(p)} \simeq N_{(p)}$ for every prime p.*

2. *Given a set $\{N_p\}$ where N_p is a weakly alternative cubic $\mathbb{Z}_{(p)}$-module, there is a weakly alternative cubic module M such that $M_{(p)} \simeq N_p$ for all p if and only if almost all of them are torsion free (maybe, zero) and $\mathbb{Q} \otimes N_p \simeq \mathbb{Q} \otimes N_q$ for all p, q. In this case $\mathbb{Q} \otimes M = \mathbb{Q} \otimes N_p$ and $\mathrm{t}M = \bigoplus_p \mathrm{t}N_p$.*

Let $\overline{\mathbf{C}} = \mathbf{C}/T = \mathbb{Z} \times \mathbf{C}_0$, $\mathsf{C}_1 = \mathsf{L}_1 = \overline{\mathbf{C}}e$, $\mathsf{C}_2 = \overline{\mathbf{C}}(0,1,e_{11})$ and $\mathsf{C}_3 = \mathsf{L}_4 = \overline{\mathbf{C}}(0,0,e_{22})$. They are all indecomposable projective $\overline{\mathbf{C}}$-modules. Put also $\mathsf{L}_2 = \overline{\mathbf{C}}(0,1,0)$ and $\mathsf{L}_3 = \overline{\mathbf{C}}(0,0,e_{11})$. They are all non-projective indecomposable torsion free $\overline{\mathbf{C}}$-modules. If $p > 2$, the localization $\mathbf{C}_{(p)}$ is torsion free. If, moreover, $p > 3$, this localization is hereditary, hence, any $\mathbf{C}_{(p)}$-module splits into a direct sum of a torsion free and a torsion one, the latter being a direct sum of modules isomorphic to $L(i,k,p) = \mathsf{L}_i/p^k$ for $i \in \{1,2,3\}$, $k \in \mathbb{N}$.

The description of $\mathbf{C}_{(3)}$-modules is similar to that of $\mathbf{B}$-modules in the preceding section. One only has to consider the set $\{1,2,3,4\}$ instead of $\{1,2,3,4,5,6\}$ with the relations $-$ and $\sim$ defined as follows: $3 - 4$ and $2 \sim 3$.

At last, $\mathbf{C}_{(2)} = \mathbf{C}' \times \mathbf{C}''$ where $\mathbf{C}' = (\mathbb{Z}_{(2)} \times \mathbb{Z}_{(2)}) \ltimes T$ and $\mathbf{C}'' \simeq \mathrm{Mat}(2, \mathbb{Z}_{(2)})$. A $\mathbf{C}''$-module is a direct sum of several copies of $(\mathsf{L}_3)_{(2)}$ and of modules isomorphic to $L(3,k,2) = \mathsf{L}_3/2^k$. A $\mathbf{C}'$-module W is given by a diagram of $\mathbb{Z}_{(2)}$-modules of the form:

$$W_1 \underset{\eta}{\overset{\xi}{\rightleftarrows}} W_2$$

such that $2\xi = 0$, $2\eta = 0$, $\eta\xi = 0$. Split both W_1 and W_2 into a direct sum of cyclic modules $C_i = \mathbb{Z}/2^i$ and $C_\infty = \mathbb{Z}_{(2)}$. Then one can check that such a diagram is a direct sum of diagrams $W(\omega)$ and $W(\omega,\pi)$. Here ω is a (finite) word of the form

$$\dots \xi^{i_m} \eta_{j_m} \xi^{i_{m+1}} \eta_{m+1} \xi \dots \quad (i_l, j_l \in \mathbb{N} \cup \{\infty\})$$

containing no subwords of the form ${}_{\infty}\xi,\ {}^{\infty}\eta,\ \eta_1\xi$. In $W(\omega,\pi)$, ω must be of the form

$$ {}_j\xi^{i_1}\eta_{j_1}\xi^{i_2}\eta_{j_2}\xi\dots{}^{i_m}\eta_j $$

with the same restrictions as above and $\pi(t)\neq t^n$ is a primary polynomial over $\mathbb{Z}/2$. Namely, if $W=W(\omega)$, then $W_1=\bigoplus_m C_{i_m}$, $W_2=\bigoplus_m C_{j_m}$, while $\xi(C_{i_m})\subseteq C_{j_{m-1}}$, $\eta(C_{j_m})\subseteq C_{i_m}$ and the induced mappings are the unique non-zero ones of period 2 (we denote them by γ). If $W=W(\omega,\pi)$ and $\deg\pi=n$, then $W_1=\bigoplus_m nC_{i_m}$, $W_2=nC_j\oplus(\bigoplus_m nC_{j_m})$; $\xi(nC_{i_m})\subseteq nC_{j_{m-1}}$, $\eta(nC_{j_m})\subseteq nC_{i_m}$, $\xi(nC_{i_1})\subseteq nC_j$, $\eta(nC_j)\subseteq nC_{i_m}$ and the induced mappings are given by the matrices γI, except the last one, given by $\gamma\Phi$, where Φ is the Frobenius cell corresponding to the polynomial π.

Note that the torsion free part of $W(\omega,\pi)$ is zero, that of $W(\omega)$ consists of at most one cyclic summand, and that of an indecomposable $\mathbf{C}_{(3)}$-module either is trivial, or consists of one or of two cyclic summand (for string modules M^{D} of type, respectively, (i) or (ii), cf. page 9). So, in accordance with Proposition 4.2, the indecomposable $\mathbf{C}$-modules M that are not torsion have the following local components $M_{(p)}$ (we only describe $M_{(2)}$ and $M_{(3)}$ as all other ones are torsion free, hence, uniquely defined):

1. $M_{(2)}=W(\omega)$ where ω contains ξ^∞, $M_{(3)}=M^{\mathrm{D}}$ where D is a string of type (i) or (ii) with $i_{2n-1}=2$ or $i_2=2$ (if both $i_2=i_{2n-1}=2$ in a string of type (ii), there are two possibilities for such M).
2. $M_{(2)}=W(\omega)\oplus W(\omega')$ where both ω and ω' contain ξ^∞, $M_{(3)}=M^{\mathrm{D}}$ where D is a string of type (ii) with $i_2=i_{2n-1}=2$.
3. $M_{(3)}$ is a string module of type (i) or (ii), $M_{(2)}$ is torsion free (thus, uniquely defined).

(Note that the case when $M_{(3)}$ is torsion free is a part of case (1) above.)

5. Cubic vector spaces

Now we consider the *cubic vector spaces*, i.e., cubic functors $F:\mathsf{fab}\to\mathbf{k}\text{-}\mathsf{mod}$ where $\mathbf{k}$ is a field. If $\operatorname{char}\mathbf{k}\neq 2$ they are a partial case of the functors considered in Section 3; hence, we always suppose that $\operatorname{char}\mathbf{k}=2$. In this case a cubic functor F is given by a diagram of

$\mathbf{k}$-vector spaces of the same shape (1) with the relations:

$$
\begin{aligned}
&h_1p_2 = h_2p_1 = 0\,, \quad h_1h = h_2h\,, \quad pp_1 = pp_2\,,\\
&h_ip_ih_i = 0\,, \quad p_ih_ip_i = 0 \quad (i = 1,2)\,,\\
&hph = 0\,, \quad php = 0\,,\\
&\overline{h}p + h_1 + h_2 = h_1p_1h_2p_2h_1 + h_2p_2h_1p_1h_2\,,\\
&h\overline{p} + p_1 + p_2 = p_1h_2p_2h_1p_1 + p_2h_1p_1h_2p_2
\end{aligned} \tag{5}
$$

(just as before, $\overline{h} = h_1h = h_2h$ and $\overline{p} = pp_1 = pp_2$). Denote by C the $\mathbf{k}$-linear category with objects $1, 2, 3$ and generating morphisms $h : 1 \to 2$, $p : 2 \to 1$, $h_i : 2 \to 3$, $p_i : 3 \to 1$ $(i = 1,2)$ satisfying the relations (5). So, a cubic vector space is the same as a C-module (i.e., a linear functor $\mathsf{C} \to \mathbf{k}\text{-}\mathsf{mod}$).

The last two equations imply that

$$\overline{h}\overline{p} + h_ip_i = h_ip_ih_jp_jh_ip_j \quad (i,j = 1,2;\ i \neq j),$$

whence $p_i\overline{h}\overline{p} = \overline{h}\overline{p}h_i = 0$ for $i = 1,2$. Hence, the elements $e_i = h_ip_ih_jp_j$ as well as the elements $f_i = p_ih_jp_jh_i$ $(i,j = 1,2;\ i \neq j)$ are orthogonal idempotents, respectively, in $\mathsf{C}(3,3)$ and in $\mathsf{C}(2,2)$. Thus, in $\operatorname{add}\mathsf{C}$, $3 \simeq 3_0 \oplus 3_1 \oplus 3_2$ and $2 \simeq 2_0 \oplus 2_1 \oplus 2_2$, so that the identity morphisms of 3_i are identified with e_i (with $e_0 = 1 - e_1 - e_2$) and those of 2_i are identified with f_i (with $f_0 = 1 - f_1 - f_2$). In what follows, we write $\mathsf{C}(x,y)$ for the set of morphisms $x \to y$ in $\operatorname{add}\mathsf{C}$. An easy calculation shows that the four objects $2_i, 3_i$ $(i = 1,2)$ are isomorphic in $\operatorname{add}\mathsf{C}$. For instance, as $p_1h_2p_2 = p_1h_2p_2h_1p_1h_2p_2$, this element lies in $\mathsf{C}(3_1, 2_1)$; the element $h_1p_1h_2p_2h_1$ lies in $\mathsf{C}(2_1, 3_1)$ and their products are just f_1 and e_1, whence $2_1 \simeq 3_1$, etc. So we only have to take into account one of these objects, say 3_1. One can also easily check that $\mathsf{C}(3_1, 3_1) = \mathbf{k}e_1$, while $\mathsf{C}(x, 3_1) = \mathsf{C}(3_1, x) = 0$ if $x \in \{1, 2_0, 3_0\}$. Thus, the category C is Morita equivalent to the direct product of the *trivial category* with one object 3_1 and the full subcategory C^* of $\operatorname{add}\mathsf{C}$ with the objects $1, 2_0, 3_0$. One easily check that the cubic module T corresponding to the (unique) indecomposable representation of the trivial part is the following one:

$$
\begin{aligned}
&T_1 = 0\,, \quad T_2 = \langle\, u_1, u_2 \,\rangle\,, \quad T_3 = \langle\, v_1, v_2 \,\rangle\,;\\
&h_1(u_1) = v_1\,,\ h_1(u_2) = 0\,; \quad h_2(u_1) = 0\,,\ h_2(u_2) = v_2\,;\\
&p_1(v_1) = 0\,,\ p_1(v_2) = u_1\,; \quad p_2(v_1) = u_2\,,\ p_2(v_2) = 0\,.
\end{aligned} \tag{6}
$$

(This cubic module corresponds to the functor $\mathbf{k} \otimes \Lambda^2 \otimes \mathrm{Id}$.) So, from now on, we only consider the representations of C^* and for every morphism α from C we denote by the same letter α its restriction onto C^*. As such restrictions of e_i and f_i are zero for $i = 1, 2$, one obtains the relations:

$$h\overline{p} + p_1 + p_2 = 0\,, \qquad \overline{h}p + h_1 + h_2 = 0\,,$$

so we may exclude p_2, h_2 from the generating set. Therefore, C^*-modules are just diagrams of vector spaces

$$F_1 \;\underset{p}{\overset{h}{\rightleftarrows}}\; F_2 \;\underset{p_1}{\overset{h_1}{\rightleftarrows}}\; F_3 \tag{7}$$

with the relations

$$hph = php = h_1p_1h_1 = p_1h_1p_1 = 0\,, \quad h_1p_1 = h_1hpp_1\,. \tag{8}$$

(One easily checks that they imply all relations (5) if we put $h_2 = h_1 + h_1hp$ and $p_2 = p_1 + hpp_1$.)

Consider the subdiagram

$$F_1 \;\underset{p}{\overset{h}{\rightleftarrows}}\; F_2\,.$$

As $hph = php = 0$, it decomposes into a direct sum of the following shape:

$$\begin{array}{ccccccccc}
 & & U_1 & U_2 & U_3 & & U_4 & U_5 & U_6 \\
 & \nearrow & \downarrow & \downarrow & \uparrow & \swarrow & & \uparrow & \\
V_6 & & V_1 & V_2 & V_3 & & V_4 & V_5 &
\end{array}$$

(U_i are the direct summands of F_1, V_i those of F_2, the arrows show the action of h and p when it is non-zero, the corresponding maps being isomorphisms.) It means that h and p are given by the following matrices:

$$h = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9}$$

(I denotes the identity matrices.)

With respect to the decomposition of F_2, h_1 and p_1 are given by the matrices $H = (H_1, H_2, H_3, H_4, H_5, H_6)$ and $P = (P_1, P_2, P_3, P_4, P_5, P_6)^\top$, where $H_i : V_i \to F_3$, $P_i : F_3 \to V_i$. For these matrices the following conditions hold:

- the number of rows of H equals the number of columns of P;
- the number of columns of H_i is the same as the number of rows of P_i for every i;

- the number of columns of H_1 is the same as the number of rows of H_6.

When one applies the isomorphisms of the spaces F_i which do not destroy the form (9) of h and p, they result in elementary transformations of the columns of the matrices H and P such that:

- the transformations of P are *contragredient* to those of H (e.g., when we add the k-th column of H to the l-th one, we have to subtract the l-th column of P from the k-th one, etc.);
- the transformations inside H_1 are the same as those inside H_6;
- one can only add columns of H_i to those of H_j if $i \leq j$ and $(i,j) \neq (3,4)$.

Using such transformations, one can reduce the matrix H to the following form:

$$\left(\begin{array}{cccc|cc|ccc|ccc|cc|cccc}
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

(the vertical lines denote the borders of the matrices H_i). Certainly, we have to subdivide in the same manner the rows and columns of the matrix P. Denote the corresponding blocks by P_{ij} ($1 \leq i \leq 17, 1 \leq j \leq 10$); the horizontal (vertical) stripes of P will be denoted by R_i (respectively, S_j). Then the relations (8) for h_1 and p_1 are equivalent to the following conditions for these blocks:

$$\begin{aligned}
&P_{ij} = 0 \text{ if } i \in \{4, 6, 9, 12, 14, 16, 18\}, \\
&P_{ij} = 0 \text{ if } i \in \{3, 17\} \text{ and } j \neq 10, \\
&P_{8j} = P_{11,j} \text{ for all } j, \\
&P_{3,10} = P_{17,10} \text{ and } P_{i1}P_{3,10} = 0 \text{ for all } i.
\end{aligned}$$

Thus, we only have to consider the matrix $\overline{P}$ obtained from P by crossing out all zero horizontal stripes as well as the stripes R_3 and R_8. A straightforward calculation shows that the automorphisms of the spaces F_3 and V_i which do not destroy the shape of H give rise to the elementary transformations of the matrix $\overline{P}$ satisfying the following conditions:

- the transformations inside R_1 are the same as those inside R_{15} ;
- the transformations inside S_2 are the same as those inside S_9 ;
- the transformations inside R_2 are contragredient to those inside S_8 ;
- the transformations inside R_{11} are contragredient to those inside S_4 ;
- the transformations inside R_{17} are contragredient to those inside S_1 ;
- one can add the columns of S_i to those of S_j if and only if $i \le j$ and $(i,j) \ne (5,6)$;
- one can add the rows of R_i to those of R_j if and only if $i \ge j$ and $(i,j) \ne (10,7)$.

Obviously, one may suppose that the matrix $P_{17,10}$ is of the form $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$. Then the whole stripe S_1 has the form $\begin{pmatrix} 0 & S_1' \end{pmatrix}$, the number of columns in S_1' being the same as that of zero rows in $P_{17,10}$. Moreover, using the elementary transformations described above, one may suppose that the remaining part of the stripe S_{10} is of the form $\begin{pmatrix} S_{10}' & 0 \end{pmatrix}$, the number of columns in S_{10}' being the same as in the zero part of $P_{17,10}$. In what follows, we omit the dash and write S_1 , S_{10} , P_{i1} and $P_{i,10}$ for the remaining (non-zero) parts of the corresponding matrices. Of course, we should no more consider the stripe R_{17} .

Now one immediately sees that we are again in the situation considered in [1]. Namely, we have got the following semi-chains:

$$\mathfrak{E} = \{\, R_i \mid R_1 > R_2 > R_5 > R_7 > R_{11} > R_{13} > R_{15},\ R_5 > R_{10} > R_{11} \,\}\ ,$$
$$\mathfrak{F} = \{\, S_j \mid S_1 < S_2 < S_3 < S_4 < S_5 < S_7 < S_8 < S_9 < S_{10},\ S_4 < S_6 < S_7 \,\}$$

with the involution σ such that

$$\sigma(R_1) = R_{15}\ ,\ \sigma(R_2) = S_8\ ,\ \sigma(R_{11}) = S_4\ ,\ \sigma(S_2) = S_9$$

and $\sigma(x) = x$ if $x \notin \{\, R_1, R_2, R_{11}, R_{15}, S_2, S_4, S_8, S_9 \,\}$. Hence, one can deduce from [1] a list of canonical forms of the matrix P and, therefore, of cubic functors $\mathsf{fab} \to \mathsf{k\text{-}mod}$. It can again be arranged in the form of "strings and bands," though a trifle more sophisticated than in the preceding sections.

Definition 5.1. Put $\mathfrak{X} = (\mathfrak{E} \cup \mathfrak{F}) \setminus \{\, R_{10}, S_6 \,\}$. Write $x - y$ if $x \in \mathfrak{E}$, $y \in \mathfrak{F}$ or vice versa; $x \sim y$ if $\sigma(x) = y \ne x$ or $x = y \in \{\, R_7, S_5 \,\}$. Call R_7, S_5 *special elements.* An $\mathfrak{X}$-*word* (or simply *word*) is a sequence $w = x_1 r_2 x_2 r_3 \dots r_n x_n$, where $x_k \in \mathfrak{X}$, $r_k \in \{\, \sim, - \,\}$, satisfying the following conditions:

- $x_{k-1}r_kx_k$ (in the above defined sense) for all $k=2,\dots,n$;
- $r_k\neq r_{k-1}$ for all $k=3,\dots,n$;

Such a word is called *full* if the following conditions hold:

- either $r_2=\sim$ or $x_1\not\sim y$ for every $y\neq x_1$;
- either $r_n=\sim$ or $x_n\not\sim y$ for every $y\neq x_n$.

A full word is called *special* if x_1 or x_n, but not both of them, is a special element, *bispecial* if both x_1 and x_n are special, and *ordinary* if neither x_1 nor x_n is special. A word w is called *non-symmetric* if $w\neq w^*$, where w^* is the *inverse word*: $w^*=x_nr_nx_{n-1}\dots x_2r_2x_1$.

An $\mathfrak{X}$-word is called *cyclic* if $r_2=r_n=-$ and $x_n\sim x_1$. A cyclic word is called *aperiodic* if it is not of the form $v\sim v\sim\cdots\sim v$ for a shorter word v. The s-th *shift* of a cyclic word w is the word $w^{(s)}=x_{2s+1}r_{2s+2}x_{2s+2}\dots x_n\sim x_1r_2\dots r_{2s}x_{2s}$. A cyclic word w is said to be *shift-symmetric* if $w^{(s)}$ is symmetric for some s. Note that the length n of a cyclic word is always even.

Definition 5.2. A *string datum* D is:

- either an ordinary non-symmetric $\mathfrak{X}$-word w (*ordinary string datum*);
- or a pair (w,δ) consisting of a special word w and $\delta\in\{0,1\}$ (*special string datum*);
- or a quadruple (w,δ_1,δ_2,m) consisting of a bispecial non-symmetric word w, $\delta_1,\delta_2\in\{0,1\}$ and $m\in\mathbb{N}$ (*bispecial string datum*).

Put $\mathrm{D}^*=w^*$ in the first case, $\mathrm{D}^*=(w^*,\delta)$ in the second and $\mathrm{D}^*=(w^*,\delta_2,\delta_1,m)$ in the third one.

A *band datum* B is a pair $(w,\pi(t))$ consisting of an aperiodic cyclic word w and of a *primary polynomial* $\pi(t)\in\mathbf{k}[t]$ (i.e., a power of an irreducible one) such that $\pi(t)\neq t^d$ if w is not shift-symmetric and $\pi(t)\neq(t-1)^d$ if w is shift-symmetric. Put $\mathrm{B}^{(s)}=(w^{(s)},f(t))$ and $\mathrm{B}^*=(w^{*(s)},\lambda^{-1}t^d\pi(1/t))$, where w is not shift-symmetric, $d=\deg\pi$ and $\lambda=\pi(0)$.

Now the results of [1] immediately imply the following

Theorem 5.3. 1. *Every string data* D *defines an indecomposable cubic vector space* V^{D} *called* string cubic space.

2. *Every band data* B *defines an indecomposable cubic vector space* V^{B} *called* band cubic space.
3. *Every indecomposable cubic vector space, except* $\mathbf{k}\otimes\Lambda^2\otimes\mathrm{Id}$, *is isomorphic either to a string or to a band one.*
4. *The only isomorphisms between string and band spaces are the following:*
   - $V^{\mathrm{D}}\simeq V^{\mathrm{D}^*}$ *where* D *is a string data;*

- $V^{\mathrm{B}} \simeq V^{\mathrm{B}(s)}$ *and* $V^{\mathrm{B}} \simeq V^{\mathrm{B}^*(s)}$ *where* B *is a band data and* $s \in \mathbb{N}$.

Moreover, one can deduce from [1] and the preceding considerations an explicit construction of string and band spaces, though it is rather cumbersome and we will not include it here.

6. Torsion free cubic modules

In this section we consider *torsion free* cubic modules, i.e., cubic functors $\mathsf{fab} \to \mathsf{fab}$. Again, we first study them *locally*, i.e., describe cubic functors $\mathsf{fab} \to \mathsf{fab}_{(p)}$, the latter being the category of torsion free finitely generated $\mathbb{Z}_{(p)}$-modules. Denote also by $\mathbb{Z}_p$ the ring of p-adic integers. The latter has the advantage of being complete, that guarantees lifting idempotent endomorphisms modulo p. Note that the calculations of Section 3 easily imply that $\mathbf{A} \otimes \mathbb{Q} \simeq \mathbb{Q}^3 \times \mathrm{Mat}(2,\mathbb{Q}) \times \mathrm{Mat}(4,\mathbb{Q})^2$. In particular, it is a semi-simple *split* $\mathbb{Q}$-algebra, i.e., $\mathrm{End}\, V \simeq \mathbb{Q}$ for every simple module V. Thus, it follows from the standard results of the theory of lattices over orders (cf. [9, Chapter 4, § 1]) that every torsion free $\mathbf{A}_p$-module is actually a completion of a torsion free $\mathbf{A}_{(p)}$-module. Hence, lifting idempotents is possible for $\mathbf{A}_{(p)}$-modules too. Together with the calculations of Section 5 it implies the following result.

Proposition 6.1. *The ring* $\mathbf{A}_{(2)}$ *is isomorphic to the subring of* $\mathbb{Z}_{(2)}^3 \times \mathrm{Mat}(2,\mathbb{Z}_{(2)}) \times \mathrm{Mat}(4,\mathbb{Z}_{(2)})^2$ *consisting of all sextuples* (a_1,a_2,a_3,B,C,D) *satisfying the following congruences* modulo 2:

$$\begin{aligned} &a_1 \equiv b_{11} \equiv c_{11}\,,\ a_2 \equiv b_{22} \equiv c_{22} \equiv c_{33}\,,\ a_3 \equiv c_{44}\,,\\ (*)\qquad &b_{12} \equiv 0 \ \text{ and } \ c_{ij} \equiv 0 \ \text{ if } \ i<j\,. \end{aligned}$$

One can easily check that this ring is an example of the so called *Backström order* [10], i.e., its Jacobson radical coincides with that of an hereditary order $\mathbf{H}$. Thus, torsion free $\mathbf{A}_{(2)}$-modules are in a natural one-to-one correspondence with the representations of a quiver Q over the field $\mathbb{Z}/2$. Namely, the vertices of Q are just simple $\mathbf{A}_{(2)}$-modules $A_1,\dots,A_r$ and simple $\mathbf{H}$-modules $H_1,\dots,H_s$, all arrows are from some A_i to some H_j, and the number of such arrows equals the multiplicity of A_i in H_j. In our example $\mathbf{H}$ consists of the sextuples satisfying the congruences (*) only, $r=4$, $s=10$ and the quiver Q

consists of 4 connected components:

$$
\begin{array}{ccccc}
H_1 & \longleftarrow & A_1 & \longrightarrow & H_4 \\
 & & \downarrow & & \\
 & & H_6 & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & H_5 & & \\
 & & \uparrow & & \\
H_2 & \longleftarrow & A_2 & \longrightarrow & H_7 \\
 & & \downarrow & & \\
 & & H_8 & &
\end{array}
$$

$$
H_3 \longleftarrow A_3 \longrightarrow H_9 \qquad\qquad A_4 \longrightarrow H_{10}
$$

(The numeration is the natural one with respect to the description of $\mathbb{A}_{(2)}$ and $\mathbf{H}$ above.) This quiver is tame and the list of its representations is well known (cf., e.g., [2]). Hence, we can derive the description of torsion free $\mathbf{A}_{(2)}$-modules.

The description of indecomposable torsion free $\mathbf{A}_{(3)}$-modules is given in Section 3. For any other prime p, the localization $\mathbf{A}_{(p)}$ is just $\mathbb{Z}^3_{(p)} \times \mathrm{Mat}(2, \mathbb{Z}_{(p)}) \times \mathrm{Mat}(4, \mathbb{Z}_{(p)})^2$, hence, there are exactly 6 indecomposable (and irreducible) torsion free modules. Therefore, the standard "local–global" procedure [9, 4] implies the following result on torsion free cubic modules.

Proposition 6.2. *Torsion free cubic modules are in one-to-one correspondence with the pairs* (M_2, M_3)*, where* M_p *is a torsion free* $\mathbf{A}_{(p)}$*-module (*$p = 2, 3$*) and* $\mathbb{Q} \otimes M_2 \simeq \mathbb{Q} \otimes M_3$*.*

Proof. It follows from [9, Chapter 4] that such a pair (M_2, M_3) always defines a cubic module M *up to genus.* (Remind that two modules M, N *belong to the same genus* if $M_p \simeq N_p$ for all prime p.) Note that $\Gamma = \mathbb{Z}^3 \times \mathrm{Mat}(2, \mathbb{Z}) \times \mathrm{Mat}(4, \mathbb{Z})^2$ is a maximal order containing $\mathbf{A}$ and two torsion free Γ-modules belonging to the same genus are isomorphic. Applying the results of [4], we see that the isomorphism classes of modules belonging to the same genus as M are in one-to-one correspondence with the double cosets

$$\mathrm{Aut}(\Gamma M) \backslash \mathrm{Aut}(\Gamma M_2) \times \mathrm{Aut}(\Gamma M_3) / \mathrm{Aut}(M_2) \times \mathrm{Aut}(M_3),$$

where ΓM denotes the Γ-submodule in $\mathbb{Q} \otimes M$ generated by M. As $\mathbf{A} \supset 6\Gamma$, we can replace these cosets by

$$\overline{\mathrm{Aut}}(\Gamma M) \backslash \mathrm{Aut}(\Gamma M / 6\Gamma M) / \mathrm{Aut}(M / 6\Gamma M),$$

where $\overline{\mathrm{Aut}}(\Gamma M)$ denotes the image of $\mathrm{Aut}(\Gamma M)$ in $\mathrm{Aut}(\Gamma M / 6\Gamma M)$. But $\mathrm{End}(\Gamma M)$ is just a direct product of matrix algebras $\mathrm{Mat}(n_i, \mathbb{Z})$, and any matrix invertible modulo 6 is an image of an invertible integer matrix. Hence, $\overline{\mathrm{Aut}}(\Gamma M)) = \mathrm{Aut}(\Gamma M/6\Gamma M)$, so every genus only contains one module up to isomorphism. $\square$

Using the arguments analogous to those of [7, Theorem 2.1], one gets the following corollary for *all* cubic functors extending Theorem 3.2.

Corollary 6.3. 1. *Cubic modules M, N are isomorphic if and only if $M_{(p)} \simeq N_{(p)}$ for every prime p.*

2. *Given a set $\{N_p\}$ where N_p is a cubic $\mathbb{Z}_{(p)}$-module, there is a cubic module M such that $M_{(p)} \simeq N_p$ for all p if and only if almost all of them are torsion free (maybe, zero) and $\mathbb{Q} \otimes N_p \simeq \mathbb{Q} \otimes N_q$ for all p, q. In this case $\mathbb{Q} \otimes M = \mathbb{Q} \otimes N_p$ and $\mathrm{t}M = \bigoplus_p \mathrm{t}N_p$.*

It seems plausible that the analogous result is no more true for the polynomial functors of degree 4, but at the moment we do not have a corresponding counterexample.

7. One conjecture

Description of quadratic modules and of cubic 2-divisible modules as well as some other evidence give rise to the following conjecture concerning polynomial modules of higher degrees.

Put $\mathbb{Z}_{<p} = \mathbb{Z}[\,1/(p-1)!\,]$ and denote by $\mathbf{A}^{(p)}$ the subring of the direct product $\mathbb{Z}_{<p} \times \mathbb{Z}_{<p} \times \mathrm{Mat}(2, \mathbb{Z}_{<p})^{p-1}$ consisting of all $(p+1)$-tuples

$$\bigl(a, b, C^1, \dots, C^{p-1}\bigr)$$

such that $a \equiv c^1_{11} \pmod p$, $b \equiv c^{p-1}_{22} \pmod p$ and $c^i_{22} \equiv c^{i+1}_{11} \pmod p$ for every $i = 1, \dots, p-2$.

Conjecture 7.1. *The category $\mathcal{M}(p) = \mathbb{Z}_{<p}\text{-}\mathsf{mod}^p$ is equivalent to the category of modules over $\mathbf{A}^{(p)} \times \mathbb{Z}^r_{<p}$ for appropriate r (depending on p).*

As the ring $\mathbf{A}^{(p)}$ fits the conditions of [6], it would give a complete description of such polynomial functors in strings and bands terms quite similar to that of Section 3. Moreover, easy calculations show that this conjecture would imply the following properties of such polynomial functors, analogous to those of quadratic and 2-divisible cubic functors:

- Two modules from $\mathcal{M}(p)$ are isomorphic if and only if all their localizations are isomorphic.
- Given a set $\{N_q\}$ of polynomial functors ($N_q \in \mathbb{Z}_{(q)}\text{-}\mathsf{mod}^p$, q runs through all primes $\geq p$), there is a functor $M \in \mathcal{M}(p)$ such that $M_{(q)} \simeq N_q$ for all q if and only if almost all N_q are

torsion free and $\mathbb{Q}\otimes N_q \simeq \mathbb{Q}\otimes N_{q'}$ for all q,q'. In this case $\mathrm{t}M=\bigoplus_q \mathrm{t}N_q$ and $\mathbb{Q}\otimes M\simeq \mathbb{Q}\otimes N_q$.

- Any functor from $\mathcal{M}(p)$ is either projective, or of projective dimension 1, or of infinite projective dimension.
- Any functor from $\mathcal{M}(p)$ has a periodic projective resolution of period $2p$.
- If a functor $F\in\mathcal{M}(p)$ is indecomposable and non-projective, its torsion free part is a direct sum of at most two irreducible torsion free modules.

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