Elliptic surfaces with fixed jacobian

Gang Xiao

Gang Xiao
Department of Mathematics
East China Normal University
Shanghai 200062

China

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI/91-30

Elliptic surfaces with fixed jacobian

Gang Xiao*

This is the first part of a study on the geometry of algebraic elliptic surfaces over the complex field. In this paper, we will give the structure of the moduli space of algebraic elliptic surfaces with a fixed jacobian fibration:

Given an elliptic fibration $j : J \to C$ with a unit section such that $q(J) = g(C)$ and k integers $m_1, \dots, m_k$ with $m_i \geq 2$, the coarse moduli space of algebraic elliptic fibrations with j as jacobian fibration and with k multiple fibres of multiplicities $m_1, \dots, m_k$ is a number of disjoint copies of a quasi-projective variety of dimension k, the set of the copies being naturally parametrized by the Shafarevich-Tate group of j. This quasi-projective variety is irreducible when j is stable.

The exact description of this structure is to be found in Theorem 3.9.

Historically, there are two classical approaches to the classification of elliptic surfaces with given jacobian fibration.

The point of view adopted by Kodaira is purely analytic. In [Ko1], Theorem 10.1, Kodaira gives the classification of all analytic elliptic fibrations without multiple fibres associated to a given jacobian fibration. Combined with his logarithmic transforms introduced in [Ko2], one arrives easily at a classification of fibrations with multiple fibres as well.

The main shortcoming of Kodaira's approach is that it does not work well with algebraic fibrations, especially those with multiple fibres. In fact, the operation Kodaira uses to reach fibrations with multiple fibres — logarithmic transforms — almost always transforms an algebraic fibration into a non-algebraic one.

* Also partially supported by CNSF

On the other hand, the étale cohomology theory of Ogg-Shafarevich goes to the other extremity. The set of algebraic fibrations with a given jacobian fibration is described as an étale cohomology group in this theory. This theory is valid for any one-dimensional family of abelian varieties in any characteristics, but at least as long as complex elliptic surfaces are concerned, the heavy cohomological machinary used in the arguments hides away most of the geometric significance of the results. For example, the group structure on the above set does not reveal much of the geometric structure of the moduli space.

Here we propose a new approach for the study of algebraic elliptic surfaces over the complex field, which also brings a link between the above two classical theories. Our method is based on an operation called "fibre twist", which is described in terms of topological surgery, in much the same way as Kodaira's logarithmic transform. When fibre twists of infinite order are allowed, they include logarithmic transforms as a special case; when only finite-order fibre twists are considered, they are just the geometrical interpretation of a torsor. And our basic result in this paper is that every algebraic fibration can be obtained from its jacobian fibration, via a finite step of finite-order fibre twists (Theorem 3.4).

Closely related to the operation of fibre twists is a special kind of divisors on the elliptic surface called n-multisections, which are divisors whose restrictions to general fibres are contained in subgroups generated by elements of order n (Definition 3.1). These divisors behave especially well under the fibre twists, and they characterise in some sense line bundles of degree d on the surface (see the proof of Lemma 3.3).

Using fibre twists and multisections, we are able to get the explicit structure of the moduli of elliptic fibrations with a given jacobian fibration and a given set of multiplicities for multiple fibres (Theorem 3.9). In particular, we find that although in general this moduli has infinitely many components, these components are just copies of one of them, the latter being quasi-projective. So we still have some kind of finite type here.

In the last section, we compare this approach with that of Ogg-Shafarevich, with a new computation of the Shafarevich-Tate group which gives the cotorsion part as well as the formula of Ogg-Shafarevich for the corank. This computation has the advantage of revealing the relation between the Shafarevich-Tate group and the internal geometry of the surface (i.e. the Mordell-Weil group).

Our treatment in this paper is elementary throughout. But as our main goal is the geometry of the moduli space for elliptic surfaces, no effort is made to present the theory in a manner as general as possible.

In a later occasion, we will use the tools developed here to study the moduli spaces of complex algebraic elliptic surfaces. One sees already from the results here a fibre space structure of those moduli spaces: the correspondence between elliptic fibrations and their jacobians gives a projection of such a moduli space onto that of elliptic fibrations with a section; and the fibres of this projection are quotients of the spaces described in Theorem 3.9

by the involution induced by the elliptic involution on the jacobian fibration.

The author is indebted to Igor Dolgachev and Michel Raynaud for their helps concerning Ogg-Shafarevich theory.

§1. Basics of fibre twists

We first give the general definition of a fibre twist, which is *a priori* a topological construction.

Let S be a complete complex analytic surface, with a fibration $f : S \to C$ over a non-singular curve C, whose general fibres are smooth curves of genus $g \geq 1$. By definition, the *sheaf of vertical automorphism groups* $\mathcal{G}$ associated to f is such that for any analytic open set U of C, $\mathcal{G}(U)$ equals the group of automorphisms of $f^{-1}(U)$ fixing each fibre.

Definition 1.1. Let U be an open set of C, s a section of $\mathcal{G}(U)$, p a point in the closure of U. s is *finitely extendable to* p, if for a neiborhood Δ of p, there is a finite cover $\pi : \tilde{\Delta} \to \Delta$ ramified on p, and an open set $\tilde{U}$ in $\tilde{\Delta}$ with $\pi(\tilde{U}) = U$, such that:

Let $f_{\tilde{\Delta}} : S_{\tilde{\Delta}} \to \tilde{\Delta}$ be the relatively minimal smooth model of the pull-back of f by π. Then there is a holomorphic section $\tilde{s}$ in the sheaf of vertical automorphisms of $f_{\tilde{\Delta}}$ such that $\tilde{s}|_{\tilde{U}}$ equals the pull-back of s, and that $\tilde{s}|_{\pi^{-1}(p)}$ is of finite order.

Note that if $\hat{\pi} : \hat{\Delta} \to \tilde{\Delta}$ is a cover ramified on the inverse image of p, and $f_{\hat{\Delta}} : S_{\hat{\Delta}} \to \hat{\Delta}$ is the relatively minimal model of the pull-back of $f_{\tilde{\Delta}}$, then by the uniqueness of $f_{\hat{\Delta}}$, $\tilde{s}$ pulls back to a (unique) section $\hat{s}$ of vertical automorphisms of $S_{\hat{\Delta}}$. This allows us to assume that $f_{\tilde{\Delta}}$ is semi-stable. Then in this case it is easy to see that the order of $\hat{s}|_{(\pi\circ\hat{\pi})^{-1}(p)}$ equals that of $\tilde{s}|_{\pi^{-1}(p)}$, therefore we can further assume that the degree of π is a multiple of the order of $\tilde{s}|_{\pi^{-1}(p)}$.

Let P be a path in the curve C, i.e. a continuous map

$$P : I = [0,1] \longrightarrow C ,$$

such that there exists a finite open cover $U_1, \ldots, U_k$ of I such that P maps each U_i homeomorphically onto its image in C. Let P' be the interior of P (i.e. the image of $I' = (0,1)$). For the sake of simplicity, we will also suppose that the restriction of $\mathcal{G}$ on P' is locally constant, and that the fibres of f over points in P' are smooth.

To define a fibre twist along the path P, we first consider the simple case where P maps I homeomorphically onto its image in C, so that there exists an open neighborhood U of P' with an orientation-preserving homeomorphism $\rho : U \to I' \times (-1,1)$ (for a fixed orientation on $I' \times (-1,1)$), such that $\rho \circ P$ maps I' homeomorphically onto $I' \times \{0\}$.

We may also suppose that the restriction of $\mathcal{G}$ on U is locally constant (hence constant as U is simply connected).

Now let p_0, p_1 be the terminal points of P, $C' = C - \{p_0, p_1\}$ the open curve. Let s be a holomorphic section of $\mathcal{G}$ on U, finitely extendable to p_0 and p_1. Then the *fibre twist* of f along P, with s as *twisting section*, is a fibration $f_1 : S_1 \to C$, constructed as follows:

Let $U^+ = \rho^{-1}((I' \times [0,1))$, $U^- = \rho^{-1}((I' \times (-1,0])$, and let $f_U^+ : S_U^+ \to U^+$ be the restriction of f over U^+. Define similarly S_U^-, S_U, etc. If we glue f_U^+ and f_U^- such that for each point p on P', ${f_U^+}^{-1}(p)$ is mapped to ${f_U^-}^{-1}(p)$ (they are the same fibre of f) via the automorphism $s|_p$, we get a new fibration

$$f_{U,1} : S_{U,1} \longrightarrow U .$$

Then we can glue $f_{U,1}$ back to the old fibration over $C' - U$, to get a new fibration

$$f_1' : S_1' \longrightarrow C' .$$

The condition that s is a holomorphic section guarantees that S_1' is an analytic surface.

Lemma 1.2. *There is a uniquely determined smooth complete analytic surface S_1, with a fibration $f_1 : S_1 \to C$, such that:*

1. The restriction of f_1 over C' equals f_1';

2. there is no (-1)-curve contained in fibres of f_1;

3. when $g = 1$, let t be a continuous section of f over P, and let $t_1'^+, t_1'^-$ be the two imagesections of $t|_{P'}$ in $f_1'|_{P'}$, under the identifications over U^+ and U^-. Then the closures of $t_1'^+$ and $t_1'^-$ in S_1 are two continuous sections of $f_1|_P$.

Proof. The problem is local over the terminals of P. Therefore let p be one of the terminals of P, and Δ a small neighborhood of p isomorphic to the unit disk with p as the center, $\Delta^* = \Delta - \{p\}$. We may also assume

$$P \cap \Delta = \{z \in \Delta | \Re(z) \geq 0, \Im(z) = 0\} \quad .$$

Let $\pi : \tilde{\Delta} \to \Delta$ be a base change as in Definition 1, such that the pull-back fibration $f_{\tilde{\Delta}}$ is semi-stable, and that the degree d of π is a multiple of the order of $\tilde{s}|_{\pi^{-1}(p)}$. Then there is a section $\tilde{s}'$ of finite order of vertical automorphisms of $f_{\tilde{\Delta}}$, such that $\tilde{s}'|_{\pi^{-1}(p)} = \tilde{s}|_{\pi^{-1}(p)}$ (and then we can suppose that the order of $\tilde{s}'$ divides d):

In fact, we can take $\tilde{s}' = \tilde{s}$ when $g > 1$, because an automorphism of a general fibre of $f_{\tilde{\Delta}}$ is always of finite order; when $g = 1$, the central fibre of $f_{\tilde{\Delta}}$ is of type I_m in the notation of Kodaira, and the automorphism group of this fibre is an extension of the automorphism group of the dual graph of this fibre by that of one of the components (the latter being $\mathrm{Aut}(\mathbf{C}^\times) \cong \mathbf{Q}/\mathbf{Z}$ when $m > 0$). Then one verifies directly that any element of finite order in this group can be finite-order extended over $\tilde{\Delta}$. Moreover, in this case $\tilde{t} = \tilde{s} - \tilde{s}'$ is a section composed of translations, hence it can be considered as a section of $f_{\tilde{\Delta}}$, after fixing a unit section. Then as $\tilde{t}$ passes through the neutral component of the central fibre, it can be lifted to a section $\hat{t}$ of the Lie algebra $R^1 f_{\tilde{\Delta}*} \mathcal{O}_{S_{\tilde{\Delta}}}$. Through a trivialisation $R^1 f_{\tilde{\Delta}*} \mathcal{O}_{S_{\tilde{\Delta}}} \cong \tilde{\Delta} \times \mathbf{C}$, $\hat{t}$ becomes a holomorphic function $\tilde{\Delta} \to \mathbf{C}$, which maps the central point $\pi^{-1}(p)$ onto 0.

Let $G = \langle \gamma \rangle \cong \mathbf{Z}_d$ be the Galois group of π, such that the action of γ is a rotation of degree $2\pi/d$. This action of G lifts naturally to an action $\alpha : G \to \mathrm{Aut}(S_{\tilde{\Delta}})$, such that S_Δ is the relatively minimal smooth model of the quotient $S_{\tilde{\Delta}}/\alpha(G)$.

Consider the automorphism $\rho = \alpha(\gamma) \circ \tilde{s}'$ of $S_{\tilde{\Delta}}$. Because $\alpha(\gamma)$ and $\tilde{s}'$ do not necessarily commute, we do not have $\rho^d = 1$ in general. But ρ^d is in any case an automorphism of finite order on $S_{\tilde{\Delta}}$, inducing identity on $\tilde{\Delta}$. If we change π into the base change of degree ed, where e is the order of ρ^d, then $\rho^{ed} = 1$. This allows to suppose $\rho^d = 1$, and get another action $\alpha' : G \to \mathrm{Aut}(S_{\tilde{\Delta}})$, with $\alpha'(\gamma) = \rho$. Let $f_2 : S_{2,\Delta} \to \Delta$ be the relatively minimal smooth model of the quotient $S_{\tilde{\Delta}}/\alpha'(G)$. We now prove that $f_2|_{\Delta^*}$ is isomorphic to $f_1'|_{\Delta^*}$, so that we can glue f_2 to f_1' to get our completion f_1 over p.

Indeed, there is nothing to show when $g > 1$ because $\tilde{s}' = \tilde{s}$. Therefore suppose $g = 1$, and let $\Delta' = \Delta - P \cap \Delta$. We can find an analytic section v of $f_2|_{\Delta'}$, and as f_1' is also a fibre twist of f_2 over Δ (with an image of $\tilde{t}$ as twisting section), we have a canonical isomorphism $\rho : f_2^{-1}(\Delta') \to f_1'^{-1}(\Delta')$. Let $\bar{v}_1, \bar{v}_2$ be the closures of v in $f_1'^{-1}(\Delta^*)$ and $f_2^{-1}(\Delta^*)$. For every point $x \in P \cap \Delta^*$, the restriction of $\bar{v}_i$ to the fibre over x is composed of two points $\bar{v}_{i,o}(x)$ and $\bar{v}_{i,1}(x)$, where $\bar{v}_{i,o}(x)$ is the closure of $v_i|_{U^+}$. We may suppose that $\bar{v}_{1,o}(x) = \bar{v}_{2,0}(x)$ for any $x \in P \cap \Delta^*$, and consider this point as the unit point of the fibre over x. Then $\bar{v}_{2,1} - \bar{v}_{1,1}$ is just the image of $\hat{t}$, via a lift of P onto $\tilde{\Delta}$.

Now let z be a local parameter of Δ, with $z = 0$ at p. Then we have $\hat{t} = \varphi(z)$, where φ is a holomorphic function in $\sqrt[d]{z}$, with $\varphi(0) = 0$. Fix a trivialisation $V = R^1 f_{1*}' \mathcal{O}_{S_1'}|_{\Delta'} \cong \Delta' \times \mathbf{C}$, and let $\theta : V \to f'^{-1}(\Delta')$ be the covering map. Consider the function $\Phi(z) = \frac{1}{2\pi i}\varphi(z) \log z$. Φ defines a holomorphic isomorphism

$$\rho' : f_2^{-1}(\Delta') \longrightarrow f_1'^{-1}(\Delta') \quad ,$$

such that

$$\rho'(x) = \rho(x) - \theta(f_2(x), \Phi(f_2(x))) \quad ,$$

where the addition is made through the group structure of the fibre. One checks immediately that ρ' extends to a holomorphic isomorphism over Δ^*.

Moreover, as Φ has a unique limit 0 at p, condition 3 is satisfied by the fibration f_1 over p.

It remains to show the uniqueness. But this is automatic when $g > 1$, as the central fibre is determined by the moduli map and the local Picard-Lefschetz monodromy. And for $g = 1$, the jacobian fibration of f_1 is determined for the same reason, therefore the only thing to prove is thast the multiplicity of the central fibre over p is determined by condition 3.

To see this, let $f_{2,\Delta} : S_{2,\Delta} \to \Delta$ be another completion of f_1' over p, and let $\pi : \tilde{\Delta} \to \Delta$ be a base change such that the minimal models of the pull-backs of $f_1|_\Delta$ and $f_{2,\Delta}$ are both semi-stable. As these two pull-backs both have a section, they are isomorphic, so we may note them by a same fibration $f_{\tilde{\Delta}} : S_{\tilde{\Delta}} \to \tilde{\Delta}$. Let $G \cong \mathbf{Z}_d$ be the Galois group of π. There are two actions $\alpha_1, \alpha_2 : G \to \mathrm{Aut}(S_{\tilde{\Delta}})$, such that $f_1^{-1}(\Delta)$ and $S_{2,\Delta}$ are respectively the minimal models of the quotients $S_{\tilde{\Delta}}/\alpha_1(G)$ and $S_{\tilde{\Delta}}/\alpha_2(G)$. Now using condition 3, one sees immediately that the restrictions on the central fibre of $f_{\tilde{\Delta}}$ of these two actions of G are identical. QED

Roughly speaking, the operation of fibre twist is to cut the surface S along the inverse image of the path P, then repaste fibre-to-fibre, modulo automorphisms provided by the twisting section s. The fibrations f and f_1 are locally isomorphic except over the terminals of P.

The case of general P is reduced to the above simple case by cutting P into a finite number of simple segments:

Let $0 = p_0 < p_1 < \cdots < p_n = 1$ be a series of points in I such that each I_i is sent homeomorphically to its image by P. Let $P_i : I_i = [p_{i-1}, p_i] \to C$, $i = 1, \ldots, n$, is the i-th segment of P. A section s of $P^{-1}(\mathcal{G})$ is by definition *piecewise holomorphic* if on each I_i, $s|_{P_i}$ is the restriction of a holomorphic section s_i of $\mathcal{G}$ on a tubular neighborhood of P_i, such that $s_i|_{p_i} = s_{i+1}|_{p_i}$ is of finite order for $i = 1, \ldots, n-1$.

Starting from a fibration $f : S \to C$, we can construct a series of fibrations $f_i : S_i \to C$, $i = 1, \ldots, n$, such that f_i is the fibre twist of f_{i-1} along the path P_i, with twisting section s_i. Now the final step $f_n : S_n \to C$ is by definition the *fibre twist* of f along P, with s as *twisting section*.

Just as in the proof of Lemma 1.2, one shows easily that around each point p_i, $i = 1, \ldots, n-1$, the local effects of the i-th and $i+1$-th twists cancel each other, so that f and f_n are locally isomorphic except at p_0 and p_n.

Remark. For a fibre twist Θ transforming $f : S \to C$ into $f_1 : S_1 \to C$, there is an inverse twist Θ^{-1} transforming $f_1 : S_1 \to C$ into $f : S \to C$, which is along the same path of Θ, but uses the inverse of the twisting section of Θ as twisting section.

Example 1.3. Logathrimic transforms are fibre twists.

Let $f : S \to C$ be an elliptic fibration, $p \in C$ a point such that the fibre F over p is of type ${}_mI_n$. Let Δ be a small neighborhood of p isomorphic to the unit disk with p as the center.

The original description of a logarithmic transform centered at p is composed of two successive fibre twists: the first twist is along a ray of Δ, joining p to a point q at the boundary of Δ, with a twisting section of finite order; the second is along the boundary of Δ with q as base point, and with a logarithmic twisting section to kill the multiplicity of the fibre over q introduced by the first fibre twist.

This operation is equivalent to a single fibre twist, along the ray joining p and q, using a twisting section which finite-order extends non-trivially over p, but trivially over q.

Lemma 1.4. *Let $f_1 : S_1 \to C$ be a fibre twist of $f : S \to C$, with a twisting section s of finite order (i.e. for every point p of P, $s|_p$ is of finite order). Then S_1 is algebraic iff S is algebraic.*

In particular, fibre twists on a fibration with $g \geq 2$ always result in algebraic fibrations.

Proof. It suffices to prove one side: S_1 is algebraic when S is.

It results from the condition that s is of finite order that s is contained in a finite sub-group sheaf $\mathcal{H}$ of $\mathcal{G}$.

Let D be a sufficiently ample reduced and irreducible divisor on S. The orbit of D under the action of $\mathcal{H}$ is an effective divisor E on S, and f maps every irreducible component of E surjectively onto C. Therefore it is easy to see that for a sufficiently ample divisor B on C, $E + f^*(B)$ is ample. **QED**

Remark that a twisting section of finite order is always finitely extendable to the terminals, Also by the continuity of the twisting section, if a fibre twist satisfies the condition in Lemma 1.4, the order of $s|_p$ for $p \in P'$ does not depend on p. This order will be called the *sectional order* of the fibre twist.

Through a suitable triangularisation of the base curve C, one shows easily the following result.

Proposition 1.5. *Let $f : S \to C$ and $f_1 : S_1 \to C$ be two fibrations, the general fibres of both are curves of genus g. Suppose that the moduli map $\mu : C \to \bar{\mathcal{M}}_g$ induced by f and f_1 are the same, where $\bar{\mathcal{M}}_g$ is the moduli space of stable curves of genus g. Then f can be transformed into f_1 by a finite number of fibre twists.*

As in [X1], fibre twists of finite sectional order can also be defined in terms of finite base changes, which also works over positive characteristics:

Suppose that a fibre twist of finite sectional order n transforms a fibration $f : S \to C$ into $f_1 : S_1 \to C$, and let $\mathcal{H}$ be the finite sub-group sheaf of $\mathcal{G}$ containing the twisting section. Let $\tilde{\pi} : \tilde{C} \to C$ be the Galois base change such that the pull-back $\tilde{\mathcal{H}}$ of $\mathcal{H}$ is decomposed into sections. Let Σ be the inverse image in $\tilde{C}$ of the terminal points of the twisting path, and let $\tilde{C}' = \tilde{C} - \Sigma$.

The fibre twist on C pulls back to a monodromy homomorphism

$$\mu : \pi_1\left(\tilde{C}'\right) \longrightarrow \mathrm{Aut}\left(\mathcal{H}|_{\tilde{C}'}\right) \quad ,$$

which has finite image due to the generic triviality of $\mathcal{H}$. Let

$$\hat{\pi} : \hat{C} \longrightarrow \tilde{C}$$

be the finite Galois map associated to $\mathrm{Ker}(\mu)$. It follows from the construction that the composite map $\hat{\pi} \circ \tilde{\pi}$ is Galois. Let G be the Galois group.

Now the pull-backs of f and f_1 by $\hat{\pi} \circ \tilde{\pi}$ are isomorphic. Let $\hat{f} : \hat{S} \rightarrow \hat{C}$ be this pull-back. We have two actions of G on $\hat{S}$, ρ and ρ_1, such that S (resp. S_1) is a smooth model of the quotient $\hat{S}/\rho(G)$ (resp. $\hat{S}/\rho_1(G)$). Take a general fibre $\hat{F}$ of $\hat{f}$. The fibre twist is uniquely determined by the action

$$\tau : G \longrightarrow \mathrm{Aut}\left(\hat{F}\right)$$

such that $\tau(\gamma) = \rho_1(\gamma^{-1}) \circ \rho(\gamma)$.

§2. Translating twists

Our main purpose in this paper is to use fibre twists to construct all the elliptic fibrations associated to a given jacobian fibration. This leads to the following definition.

Definition 2.1. A fibre twist between two elliptic fibrations is called *translating*, if the twisting section is composed of translations of finite order.

The following properties are immediate from the definition:

1) Translating twists are algebraic.

2) Elliptic fibrations differing by translating twists have the same jacobian fibration.

3) The only local differences introduced by a translating twist are the multiplicities of the fibres over the terminals of the twisting path.

From now on, we will only consider elliptic fibrations, and all fibre twists will be translating, if not otherwise specified.

Note that in the sheaf $\mathcal{G}$ of vertical automorphism groups associated to an elliptic fibration $f : S \rightarrow C$, the subsheaf consisting of translations is canonically isomorphic to the Néron model of the jacobian fibration. Thus for a fixed jacobian fibration j, translating twists form a group acting on the set Σ_j of (algebraic) elliptic fibrations with j as jacobian, which is easily seen to be abelian. It follows that a twist transforms a fibration into an isomorphic one if and only if it does the same to every fibration in Σ_j. We will call such a fibre twist *cohomologically trivial*.

Furthermore, it is easy to see that a locally trivial translating twist transforms a smooth family of elliptic fibrations with fixed jacobian fibration into a smooth family. Consequently, locally trivial twist is an automorphism of the moduli space (if it exists) of elliptic fibrations with given jacobian.

In order to investigate the effects of a translating twist on the multiplicity of the terminal fibres, we need one more definition.

It is well-known that for a multiple fibre F in an elliptic fibration, the corresponding fibre E in the jacobian fibration is either a smooth elliptic curve, a rational curve with an ordinary double point, or a semi-stable curve composed of a circle of n (-2)-curves. In terms of Kodaira's classification of singular elliptic fibres, E is of type I_n, for $n \geq 0$. Then the type of F is ${}_mI_n$, where m denotes the multiplicity of F.

More precisely, let $f_\Delta : S_\Delta \to \Delta$ (resp. $j : J \to \Delta$) be a local fibration (resp. the jacobian fibration of f_Δ) over the unit disk, with F (resp. E) as central fibre. For any multiple N of m, let $\pi : \Delta \to \Delta$ be the $N : 1$ cyclic cover totally ramified at the center. There is a unique fibration $\tilde{f}_\Delta : \tilde{S}_\Delta \to \Delta$ such that $\tilde{S}_\Delta$ is smooth and does not contain vertical (-1)-curves, and cyclic covers $\Pi : \tilde{S}_\Delta \to S_\Delta$ and $\Pi' : \tilde{S}_\Delta \to J$ of degree N, such that the following diagrams commute:

$$\begin{array}{ccc} \tilde{S}_\Delta & \longrightarrow & S_\Delta \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \Delta \end{array} , \quad \begin{array}{ccc} \tilde{S}_\Delta & \longrightarrow & J \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \Delta \end{array}$$

Let $\tilde{F}$ be the central fibre of $\tilde{f}_\Delta$. $\tilde{F}$ is isomorphic to E when $n = 0$, and is of type ${}_1I_{Nn}$ when $n > 0$. Let $G \cong \mathbf{Z}_N$ be the Galois group of Π. When $n = 0$, consider the induced action of G on $\tilde{F} \cong E$; when $n > 0$, consider the action of G on the dual graph of $\tilde{F}$. The kernel of this action is the subgroup K of order N/m, and the quotient group $H = G/K$ acts freely on $\tilde{F}$ or its dual graph according to the cases. If we fix a generator Γ of G such that the action of Γ on Δ (the base of $\tilde{f}_\Delta$) is a clockwise rotation of degree $2\pi/N$, the image of Γ in H gives a uniquely determined generator γ of order m. We call γ the *direction* of the multiple fibre F (or a direction of order m associated to E).

For a fibre of type I_n, directions of order dividing m form an abelian group A isomorphic to $\mathbf{Z}_m$ (when $n > 0$), or $\mathbf{Z}_m \oplus \mathbf{Z}_m$ (when $n = 0$). And the definition of direction does not depend on the choice of N, in the sense that when $n > 0$ and $m|N_1|N_2$, the base change $\pi_2 : \Delta \to \Delta$ of degree N_2 factorises through $\pi_1 : \Delta \to \Delta$ of degree N_1, such that the direction defined through π_2 is the natural inverse image of that defined through π_1.

Conversely, let $j : J \to \Delta$ be a local fibration with central fibre E of type ${}_1I_n$ and with a unit section, γ a direction of order $m \geq 2$ associated to E, represented by some base change $\pi : \Delta \to \Delta$ or order N divisible by m. We can construct a fibration $f_\Delta : S_\Delta \to \Delta$ with j as jacobian fibration, whose central fibre F is of multiplicity m at direction γ, as follows:

Let $\tilde{f}_\Delta$, $\tilde{F}$ as before, and let D_0 be the inverse image of the unit section on $\tilde{S}_\Delta$, passing through a component Γ_0 of $\tilde{F}$. Let D be the divisor on $\tilde{S}_\Delta$ composed of points of order dividing m in the fibres of $\tilde{f}_\Delta$. Then the monodromy of $\tilde{f}_\Delta$ on D is trivial, in other words D is composed of m^2 sections of $\tilde{f}_\Delta$. The set of these sections form in a natural way a group $\mathcal{G} \cong \mathbf{Z}_m \oplus \mathbf{Z}_m$, with D_0 as the unit element. When $n > 0$, it is easy to see that D passes through m components of $\tilde{F}$, located at regular intervals of Nn/m in $\tilde{F}$. The components of D passing through Γ_0 form a cyclic subgroup $\mathcal{K}$ of order m, such that the components passing through each of the m components of $\tilde{F}$ correspond to a coset of $\mathcal{K}$ (or an element in $\mathcal{H} = \mathcal{G}/\mathcal{K}$). This establishes an isomorphism between $\mathcal{H}$ and A in the case $n > 0$. Of course, we have a trivial isomorphism $\mathcal{K} \cong A$ when $n = 0$.

Now let D_1 be a section in D whose image in A is the direction γ, $\alpha : \tilde{S}_\Delta \to \tilde{S}_\Delta$ the automorphism of $\tilde{S}_\Delta$ sending D_0 to D_1, and inducing trivial automorphism on the base. Let β be the generator of the Galois group of $\Pi' : \tilde{S}_\Delta \to J$ whose action on the base is a clockwise rotation of degree $2\pi/N$. Then the automorphism $\alpha \circ \beta$ generates a group G of order N, and the smooth relatively minimal model S_Δ of the quotient $\tilde{S}_\Delta/G$ has an induced fibration $f_\Delta : S_\Delta \to \Delta$ whose central fibre F is of multiplicity m and direction γ.

Thus we have established the correspondence of the following lemma (from Σ_2 to Σ_1).

Lemma 2.2. *Let $j_\Delta : J_\Delta \to \Delta$ be a local fibration with central fibre E of type ${}_1I_n$, $n \geq 0$. For any integer $N \geq 2$, there is a 1-1 correspondence between the set Σ_1 of directions of order N associated to E, and the set Σ_2 of j-isomorphism classes of local fibrations $f_\Delta : S_\Delta \to \Delta$ having j as jacobian fibration, whose central fibre is of multiplicity N.*

Proof. It remains only to show that the map $\Sigma_2 \to \Sigma_1$ is injective when $n > 0$, in other words with notation as in the discussion preceding the lemma, if α' is another automorphism mapping D_0 to D_1', where D_1 and D_1' intersect the same component of $\tilde{F}$, then the quotient $f_\Delta' : S_\Delta' \to \Delta$ by the group generated by $\alpha \circ \beta$ is j-isomorphic to f_Δ.

In fact, we can write $\mathcal{G} = \mathcal{K} \oplus \mathcal{H}$, with $\mathcal{K} = \langle k \rangle$, $\mathcal{H} = \langle h \rangle$, such that the action of β on $\mathcal{G}$ is an automorphism with $\beta(ak + bh) = (a + b)k + bh$, and that $\alpha(x + y) = x + y + h$ for $x \in \mathcal{K}, y \in \mathcal{H}$. We have a non-trivial homomorphism $\phi : \mathcal{H} \to \mathcal{K}$ such that $\alpha'(x + y) = x + \phi(y) + y + h$. Let $\phi(h) = ak$, for $a \in \mathbf{Z}$. Then if instead of D_0 (i.e. the section 0 in the above representation), we take the section $-ah$ as the unit section of $\tilde{f}_\Delta$, the action of α' becomes the same as that of α with D_0 as unit section. **QED**

We can recover the group sheaf of directions associated to a fibration $f : S \to C$ as follows. Let $j : J \to C$ be the jacobian fibration of f, $m \geq 2$ an integer. Let $\tilde{\mathfrak{D}}$ be the normalisation of the closure in J of points of order dividing m in smooth fibres of j. Let $\mathfrak{D}_m$ be the (singular) curve constructed from $\tilde{\mathfrak{D}}$ by joining into one point every set of points whose image in J are smooth points in a same component of a fibre of type I_n, $n > 0$. Provided with the natural projection to C, $\mathfrak{D}_m$ can be seen as a sheaf of groups on C.

For any fibre E of type I_n in J, when $n = 0$, the group of directions of order dividing m is canonically isomorphic to the restriction of $\mathfrak{D}_m$ on E. if $n > 0$, let $\tilde{F}$ be the central fibre of the pull-back of a local fibration around E via a base change of degree divisible by m. By the fact that the Galois group of the cover of the inverse image over J fixes every component of $\tilde{F}$, we find easily a canonical isomorphism between the directions of E of order dividing m, and the restriction of $\mathfrak{D}_m$ on the image of E. Therefore $\mathfrak{D}_m$ is just the parameter space of directions of order dividing m of f (or of j). And the points of order m in $\mathfrak{D}_m$ form a subvariety D_m, which is the parameter space of directions of order m.

Now we consider the action of fibre twist on the fibre F over a terminal point p of the twisting path P. Let m_1 be the multiplicity of F, m_2 the order of the twisting section s, N a common multiple of m_1 and m_2. Let γ_1 be the direction of F.

First suppose that p is the starting point, but not the ending point, of P. The section s can be considered as a lift of P on $\mathfrak{D}_N$, hence its value over p is a direction γ_2 of order m_2 associated to F. It is clear that the direction of multiplicity of the fibre F_1 over p of the new fibration $f_1 : S_1 \to C$ after fibre twist is $\gamma_1 + \gamma_2$. In particular, the multiplicity of F_1 equals the order of $\gamma_1 + \gamma_2$.

By the same reason, if p is the ending point of P but not the starting point, the direction of F_1 becomes $\gamma_1 - \gamma_2$ with the above notation.

Finally, suppose that P is a loop starting and ending at p. We assume that the fibre F over p is smooth, to simplify notations. In this case s introduces two directions γ_2, γ_3 on F, both of order m_2, one from starting value, the other from ending value. and the direction γ of F_1 is $\gamma_1+\gamma_2-\gamma_3$. On the other hand, we have a monodromy homomorphism

$$\mu_N : \pi_1(C') \to \mathrm{Aut}(\mathfrak{D}_N|_p), \tag{1}$$

where C' is the open subset of C composed of non-critical points of j, such that $\gamma_3 = \mu_N(P)(\gamma_2)$. Therefore $\gamma = \gamma_1 + (1 - \mu_N(P))(\gamma_2)$.

Furthermore, the monodromy μ_N is induced from the Picard-Lefschetz monodromy $M : \pi_1(C') \to \mathrm{Aut}(H^1(F, \mathbf{Z}))$ of j, in the sense that if we note by Γ_i a point in the inverse image of γ_i in the universal cover V of F, there is a point Γ in the inverse image of γ such that $\Gamma = \Gamma_1 + (I_2 - M(P))\Gamma_2$, where $M(P)$ can be considered as a 2×2 integer matrix with a choice of basis. With a slight abuse of notations, we just write

$$\gamma = \gamma_1 + (I_2 - M(P))\gamma_2. \tag{2}$$

Definition 2.3. The *twisting weight* $w = w(f)$ of an elliptic fibration $f : S \to C$ is as follows:

If f has a singular fibre or if the jacobian fibration of f is trivial, $w = 1$.

If f is has smooth non-trivial jacobian fibration, let F be a general fibre of f, and consider the Picard-Lefschetz monodromy homomorphism $\theta : \pi_1(C) \to \operatorname{Aut} H^1(F, \mathbf{Z})$. The image of θ is isomorphic to $\mathbf{Z}_\tau$, where $\tau = 2$, 3, 4 or 6. Define $w = 2$ if $\tau = 2$ or 4, $w = 3$ if $\tau = 3$, $w = 1$ if $\tau = 6$.

We now have the following result.

Theorem 2.4. *Let $j : J \to C$ be an elliptic fibration with a unit section, γ a direction of order m on a point p of C. Suppose that j is non-trivial (or equivalently, $q(J) = g(C)$, or j has non-trivial Picard-Lefschetz monodromy).*

There exist a path P and a translating section s on P, such that translating twists along P using s gives a 1-1 correspondence between the set of elliptic fibrations, with j as jacobian fibration, without multiple fibre over p, and those whose fibre over p is multiple of direction γ. This correspondence preserves algebraicity of the fibrations, and corresponding fibrations are locally isomorphic outside p.

Moreover, the translating twist can be chosen to be of sectional order equal to wm.

Proof. By the existence of inverse twists, we have only to prove one side of the correspondence.

We can suppose that the fibre of j over p is smooth, for the singular case can be easily deduced by taking limits. We have two choices for the possible twisting path P: either a loop with base p, or a path starting at p and ending at a point q over which the fibre of j is singular (note that when the fibre over q is of type I_n, $n > 0$, the fibre twist will not change the multiplicity and direction of this fibre if the twisting section is chosen appropriately).

First consider the case where j has no singular fibres. We take P to be a loop with base p, such that the image of P by the Picard-Lefschetz monodromy $M : \pi_1(C) \to H^1(F, \mathbf{Z})$ is non-trivial. It is well-known that $M(P)$ is a matrix of rotation, so that $I - M(P)$ is a non-singular matrix. We can therefore choose $\gamma_2 = (I - M(P))^{-1}\gamma$, and take the section of finite order on P whose starting value is γ_2, to be the twisting section s. According to (2), we see that for any fibration $f : S \to C$ whose fibre over p is not multiple, the fibre twist of f along P with s as twisting section is a fibration with direction γ over p and locally isomorphic to f elsewhere. The theorem follows therefore in this case.

Note that in this case $M(P)$ is a rotation of order 2,3,4 or 6, and a direct computation gives $Ord(s) = wm$.

In the case where j has a non-semistable fibre over a point $q \in C$, the theorem can be proved in the same manner, by letting P to be a path joining p and q, for such a twist will not introduce multiplicity over q, hence is always locally isomorphic around q. And in this case the twisting section will have the same order as γ.

It remains only the case where j has singular fibres, all of them are of type I_n. We will show that in this case there is a path P from p to a critical value q and a twisting section s on P such that $s|_p = \gamma$, and that $s|_q$ is a point on the neutral component of the fibre of j over q, hence the translating twisting associated to s will only introduce multiplicity to the fibre over p. In particular, we also have the same order for s and γ.

To see this, let $N \geq 3$ be a multiple of the order of γ, $\pi : \tilde{C} \to C$ be the Galois covering such that the inverse image of $\mathfrak{D}_N$ is completely decomposed. Let $\tilde{j} : \tilde{J} \to \tilde{C}$ be the relatively minimal smooth model of the pull-back of j. Then the divisor formed by elements of order dividing N in $\tilde{J}$ is composed of N^2 sections of $\tilde{j}$, which gives a level N structure on the fibres of $\tilde{j}$. This means that the moduli map $\varphi : \tilde{C} \to \mathcal{M}$, where $\mathcal{M}$ is the moduli space of stable curves of genus 1, factors through the principal modular curve C_N of level N, such that $\tilde{j}$ is the model of the pull-back of the elliptic modular fibration $j_N : J_N \to C_N$ of level N as defined in [Shi]. The set of N^2 sections on $\tilde{J}$ is the inverse image of N^2 sections on J_N, which form a group Γ isomorphic to $\mathbf{Z}_N \oplus \mathbf{Z}_N$ ([Shi], Theorem 5.5). The map $\varphi_N : C_N \to \mathcal{M}$ is Galois with group $G \cong SL_2(\mathbf{Z}_N)$. The induced action of G on the set of N-cyclic subgroups of Γ is transitive.

Take any singular fibre F_1 of $\tilde{j}$, which is of type ${}_1I_{N_1}$ with $N|N_1$. The sections in Γ passing through the neutral component of F_1 form an N-cyclic subgroup Γ_1. By the above transitivity, there is a point $\tilde{p}$ in $\tilde{C}$, whose image in C is p, such that the inverse image of γ on $\tilde{p}$ is contained in a section $\tilde{s}$ in Γ_1. Let $\tilde{P}$ be a path in $\tilde{C}$ joining $\tilde{p}$ and $\tilde{j}(F_1)$, P the image of $\tilde{P}$ in C. It is immediate that P and the image s of $\tilde{s}|_{\tilde{P}}$ satisfy our requirements. **QED**

§3. Classification theorems

Definition 3.1. Let $f : S \to C$ be an elliptic fibration, D a reduced divisor on S, not containing curves which are components of a fibre of f. D is called an *n-multisection*, if for a general fibre F of f, $D|_F$ is stable under the translations of order n on F.

We have the following obvious lemma.

Lemma 3.2. *If $f : S \to C$ has an n-multisection, then for any multiple N of n, f has a N-multisection.*

Note that a translating twist of sectional order n transforming $f : S \to C$ into $f_1 : S_1 \to C$ transforms any nN-multisection on S into an nN-multisection on S_1 and vice versa, for any $N \geq 1$.

The following result is communicated to us by M. Raynaud. We give here an elementary proof for the case of characteristic 0, while the original proof of Raynaud is included at the end of this paper.

Lemma 3.3. *An elliptic fibration $f : S \to C$ is algebraic iff it has an n-multisection for some $n \geq 1$.*

Proof. Suppose D is a very ample divisor on S. Let n be the degree of D over C. For any general fibre F of f, there are exactly n^2 points p on F such that np is linearly equivalent to $D|_F$ as divisors on F. The closure in S of the set of all such points form an n-multisection. The converse is an easy consequence of Nakai-Moishezon criterion. **QED**

Remark. The proof of the lemma gives in fact a little bit more: it shows that there is a natural 1-1 correspondence between multisections and divisor classes of positive degree on the generic fibre of f.

Theorem 3.4. *Any algebraic elliptic fibration can be constructed from its jacobian fibration, via a finite number of translating twists.*

Proof. Let $f : S \to C$ be such a fibration. Modulo twists along paths joining two multiple fibres, we can suppose that f has at most 1 multiple fibre, and this fibre, if any, is smooth.

Let $c_1, \ldots, c_l$ be the images of the singular fibres of f, F a general fibre of f, $p = f(F)$. Let q be another point on C, which will be the image of the multiple fibre if any.

Let $\alpha_1, \ldots, \alpha_b, \beta_1, \ldots, \beta_b$, where $b = g(C)$, be a standard set of loops on C, whose conjugate classes generate $\pi_1(C)$. Let γ_i $(i = 1, \ldots, l)$ be small loops around c_i. We can choose base points a_i (resp. b_i, g_i) on α_i (resp. β_i, γ_i), which are all different, and paths P_i (resp. $P'_i, Q_i, Q'_i, G_i, G'_i$) joining p and a_i (resp. p and b_i, q and a_i, q and b_i, p and g_i, q and c_i), such that all these loops and paths are mutually disjoint except at the base points and except that α_i and β_i, as well as G'_i and γ_i, meet transversally at one non-base point.

Let $C' = C - \{c_1, \ldots, c_l, q\}$. The loops α_i, β_i, γ_i together with their joining paths form a free set of generators for $\pi_1(C', p)$. Let D be an n-multisection on S, and consider the monodromy homomorphism $\mu_D : \pi_1(C', p) \to \mathrm{Aut}(D|_F)$ associated to D. Fix any point x in $D|_F$, and identify it with the unique points over a_i, b_i, g_i via paths P_i, P'_i, G_i.

Now modulo a suitable twist along β_i with b_i as base, we may assume that

$$\mu_D(\alpha_i)(x) = x \ .$$

Such a twist might introduce multiple fibre over b_i, but another twist along Q_i' can move this multiplicity over to q. Symmetrically, we get $\mu_D(\beta_i)(x) = x$. And a twist along G_i' will make $\mu_D(\gamma_i)(x) = x$ (note that such a twist will not introduce multiplicity over c_i). But then x is a fixed point under the action of $\pi_1(C')$, hence it belongs to a section of f in D. **QED**

Remark. The proof of the Theorem gives in fact a little bit more: if f has an N-multisection, then the translating twists can be chosen to have sectional orders dividing N.

Definition 3.5. Let $j : J \to C$ be a fixed elliptic fibration with a section. Let $\gamma_1, \dots, \gamma_k$ be a set of directions on j with orders $m_1, \dots, m_k$, no two of them being on a same fibre. Define $\tilde{\Sigma}_{\gamma_1,\dots,\gamma_k}$ (resp. $\Sigma_{\gamma_1,\dots,\gamma_k}$) be the set of elliptic fibrations (resp. set of algebraic elliptic fibrations) with j as jacobian fibration, and with multiple fibres exactly at directions $\gamma_1, \dots, \gamma_k$. Define $\tilde{\Sigma}_\phi$ (resp. Σ_ϕ) be the set of such fibrations (resp. such algebraic fibrations) without multiple fibre.

With this definition, Theorem 2.4 is translated to the following.

Theorem 3.6. *Suppose j is not trivial. Then there is a bijective map*

$$\tilde{\Phi}_{\gamma_1,\dots,\gamma_k} : \tilde{\Sigma}_{\gamma_1,\dots,\gamma_k} \longrightarrow \tilde{\Sigma}_\phi \ ,$$

which induces a bijection

$$\Phi_{\gamma_1,\dots,\gamma_k} : \Sigma_{\gamma_1,\dots,\gamma_k} \longrightarrow \Sigma_\phi \ .$$

Moreover, $\Phi_{\gamma_1,\dots,\gamma_k}$ maps surfaces with an nN-multisection to one with the same property and vice versa for $n \geq 1$, where

$$N = w\text{lcm}(m_1, \dots, m_k).$$

Theorem 3.7. *Suppose j is trivial (i.e. $J \cong E \times C$, where E is an elliptic curve). We consider the directions γ_i as points of finite order in E. Then the following are equivalent:*

a) $\Sigma_{\gamma_1,\dots,\gamma_k}$ is non-empty;

b) there exists a bijective map

$$\tilde{\Phi}_{\gamma_1,\dots,\gamma_k} : \tilde{\Sigma}_{\gamma_1,\dots,\gamma_k} \longrightarrow \tilde{\Sigma}_\phi$$

sending algebraic fibrations onto algebraic ones;

c)

$$\sum_{i=1}^{k} \gamma_i = 0 \ . \tag{3}$$

Proof. In view of Theorem 3.4 and by using induction on k, we have only to notice that translating twists on fibrations with trivial jacobian preserve the condition (3). **QED**

The theory of Ogg-Shafarevich [O] [Sha] shows that Σ_ϕ, hence $\Sigma_{\gamma_1,\ldots,\gamma_k}$, is discrete. See also the next section.

Definition 3.8. Let $j : J \to C$ be an elliptic fibration with a unit section, $m_1, \ldots, m_k$ a set of integers with $k \geq 1$, $m_i > 1$, D_{m_i} the curve of directions of order m_i, as defined in §2, after Lemma 2.2, with projection $\phi_i : D_{m_i} \to C$. Let V' be the dense open subset in $D_{m_1} \times \cdots \times D_{m_k}$:

$$V' = \{(d_1, \ldots, d_k) | \phi_i(d_i) \neq \phi_j(d_j) \text{ if } i \neq j\} \quad .$$

Let G be the symmetric group acting freely on V', exchanging components with equal subscripts. Define

$$V_{m_1,\ldots,m_k} = V'/G \quad .$$

Now we can state our main classification theorem in terms of the variety $V_{m_1,\ldots,m_k}$:

Theorem 3.9. *Let $j : J \to C$ be an elliptic fibration with a unit section. Suppose that j has singular fibres. Let $m_1, \ldots, m_k$ be a set of integers with $k \geq 1$, $m_i > 1$, and let*

$$N = \mathrm{lcm}(m_1, \ldots, m_k) \quad .$$

Let $M_{m_1,\ldots,m_k}$ be the moduli space of elliptic fibrations with jacobian fibration j and k multiple fibres of multiplicities $m_1, \ldots, m_k$. There exists a morphism

$$\Psi : M_{m_1,\ldots,m_k} \longrightarrow \Sigma_\phi \quad ,$$

whose non-empty fibres are isomorphic to each other. Each such fibre M_ϕ has an unramified cover

$$\Lambda : M_\phi \longrightarrow V'_{m_1,\ldots,m_k}$$

of degree d, where $V'_{m_1,\ldots,m_k}$ is a quasi-projective variety with a bijective birational morphism

$$V_{m_1,\ldots,m_k} \longrightarrow V'_{m_1,\ldots,m_k} \quad ,$$

and d is the number of algebraic elliptic fibrations without multiple fibres, with j as jacobian fibration, and with an N-multisection. The value of d will be given by the formula (9) at the end of next section.

Moreover, suppose that the singular fibres of j are all of type I_1. Then M_ϕ is irreducible.

Proof. We first give the map Ψ. Let $f : S \to C$ be a fibration in $M_{m_1,\ldots,m_k}$. Vertical translations of order N generate a subsheaf $\mathcal{H}$ in $\mathcal{G}$, which is almost everywhere locally constant of group $\mathbf{Z}_N \oplus \mathbf{Z}_N$. The quotient of f by the action of $\mathcal{H}$ has a unique relatively minimal model $f^{(N)} : S^{(N)} \to C$. It is easy to see that $f^{(N)}$ has no multiple fibres. Moreover, we have a canonical isomorphism between the jacobians of f and $f^{(N)}$, hence $f^{(N)} \in \Sigma_\phi$. Let

$$\Psi(f) = f^{(N)} \ .$$

The rational map of projection of S to $S^{(N)}$ is called the N*-multiplication map* of S.

Note that any nN-multisection of f is the inverse image of an n-multisection of $f^{(N)}$. In particular, $f^{(N)} \cong j$ iff f has an N-multisection.

Fix a $\varphi \in \Sigma_\phi$ such that the fibre M_φ of Ψ over φ is not empty. As fibre twists of sectional order dividing N do not change the image of the fibration under Ψ, it follows from Theorem 3.6 that M_φ contains fibrations for any choice of directions $\{\gamma_1, \ldots, \gamma_k\}$ such that the order of γ_i equals m_i.

Let $\hat{M}_\varphi$ be the set of fibrations with marked multiple fibres. The group acting on $\hat{M}_\varphi$ exchanging multiple fibres of the same multiplicity induces a finite unramified cover of $\hat{M}_\varphi$ over M_φ, therefore we have only to study the structure of $\hat{M}_\varphi$.

To any element of $\hat{M}_\varphi$ associating the direction of its i-th multiple fibre, we get a map $\mu_i : \hat{M}_\varphi \to D_{m_i}$. The multiple fibre being the image in C of a ramification locus of the N-multiplication map, and D_{m_i} being finite over C, it is not hard to see after a local analysis that there is a bijective morphism

$$\nu_{m_i} : D_{m_i} \longrightarrow D'_{m_i}$$

onto another curve D'_{m_i}, which is locally isomorphic except possibly on points situated in fibres of type I_n, $i \geq 1$, such that for any family $X \to Y$ of fibrations in $\hat{M}_\varphi$, the map $Y \to D'_{m_i}$ induced by $\nu_{m_i} \circ \mu_i$ is a morphism. It follows that the geometric structure on $\hat{M}_\varphi$ is pulled back by the surjective product map

$$\Lambda = \nu_{m_1}\mu_1 \times \cdots \times \nu_{m_k}\mu_k : \hat{M}_\varphi \longrightarrow D'_{m_1} \times \cdots \times D'_{m_k} = \hat{V}' \ ,$$

which is étale of degree d by Theorems 3.6 and 3.4, where d equals the number of fibrations in Σ_ϕ having an N-multisection.

It is easy to construct directly an isomorphism between two fibres M_1, M_2 of Ψ. Take a fibration $f_i : S_i \to C$ in each M_i. We can suppose that f_1 and f_2 are locally isomorphic. There is therefore a locally trivial transformation Θ composed of a finite series of translating twists, such that $\Theta(f_1) = f_2$. As Θ is composed of twists of sectional order dividing N, one sees immadiately that Θ maps M_1 isomorphically onto M_2.

In the rest of this proof, we will assume that all the singular fibres of j are of type I_1. Retake the notations of the proof of Theorem 3.4, and let F_q be the fibre of j over q. For each critical value of c_i, the Picard-Lefschetz monodromy along γ_i may be considered as an element $\mu_i \in \operatorname{Aut} H^1(F_q, \mathbf{Z}) \cong SL_2(\mathbf{Z})$, via the path G'_i.

Lemma 3.10. *After rerouting of the paths $P_i, P'_i, Q_i, Q'_i, G_i$ and G'_i, we can find two critical points, say c_{l-1} and c_l, such that μ_{l-1} and μ_l generate* $\operatorname{Aut} H^1(F_q, \mathbf{Z})$.

Proof. The fixed points of μ_i form a subgroup $Z_i \cong \mathbf{Z}$ in $H^1(F_q, \mathbf{Z})$, with

$$H^1(F_q, \mathbf{Z})/Z_i \cong \mathbf{Z} \ .$$

First, we use our earlier result on the fundamental group of an elliptic surface to show that $H^1(F_q, \mathbf{Z})$ is generated by these Z_i.

Let K be the image of the natural homomorphism

$$\iota : \pi_1(F_q) \longrightarrow \pi_1(J) \ ,$$

which is a normal subgroup of $\pi_1(J)$. We deduce from the fact that $\jmath$ has a section that there is an exact sequence

$$1 \longrightarrow K \longrightarrow \pi_1(J) \longrightarrow \pi_1(C) \longrightarrow 1 \ ,$$

and the loops in the unit section form a subgroup of $\pi_1(J)$, which projects isomorphically onto $\pi_1(C)$. It follows that $\mathrm{Ker}(\iota)$ is generated by the vanishing cycles of singular fibres. With the duality between $\pi_1(F_q) = H_1(F_q, \mathbf{Z})$ and $H^1(F_q, \mathbf{Z})$, the vanishing cycles of the i-th singular fibre are those corresponding to Z_i. Therefore we have an isomorphism

$$K \cong H^1(F_q, \mathbf{Z}) / \langle Z_1, \ldots, Z_l \rangle \ .$$

Now by [X2, Theorem 4], we know that K is trivial.

Next, we show that after reorder of the c_i and rerouting of the paths, we may have $H^1(F_q, \mathbf{Z}) = \langle Z_{l-1}, Z_l \rangle$.

Fix c_l, and write $H = H^1(F_q, \mathbf{Z})/Z_l$. Let c_i, $i = \jmath, \ldots, l-1$ be the points such that the image in $H \cong \mathbf{Z}$ of a generator of Z_i is a non-zero number n_i. We have $(n_\jmath, \ldots, n_{l-1}) = 1$ by the first step.

Let i_1, i_2 be 2 indices between $\jmath$ and $l-1$. If we change the path G'_{i_1} into a non-self-crossing path isotopic in $C' - G'_l$ to G'_{i_1} followed by a loop around c_{i_2}, n_{i_1} will be changed into either $n_{i_1} + n_{i_2}$ or $n_{i_1} - n_{i_2}$, depending on the direction of the loop around c_{i_2} chosen. We can then change accordingly the other paths so that they remain disjoint as before. This allows us to apply Euclidean agorithm on the set $\{n_\jmath, \ldots, n_{l-1}\}$, to arrive at $n_{l-1} = 1$.

Finally, take a generator of Z_{l-1} and one of Z_l as a basis for $H^1(F_q, \mathbf{Z})$. Under this basis, the matrices of μ_{l-1} and μ_l become $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. These 2 matrices generate $SL_2(\mathbf{Z})$. **QED**

Corollary. *D_n is irreducible for any $n \geq 1$. In particular, $\hat{V}'$ is irreducible.*

Proof. In view of the projection of D_n onto C, we have only to show that any two points in $D_n|_q$ can be joined by a path in the smooth part of D_n.

Let Γ_i be the loop on C begining on q, going to γ_i along G'_i and turn around it once, then return to q via G'_i. The lift of Γ_i on D_n starting at a point p ends at $\mu_{D_n}(p) = \bar{\mu}_i(p)$, where $\bar{\mu}_i$ is the class of μ_i in $SL_2(\mathbf{Z}_n) = \operatorname{Aut}\mathfrak{D}_n|_q$.

As $SL_2(\mathbf{Z}_n)$ acts transitively on $D_n|_q$, every point in $D_n|_q$ can be reached from p, by a finite sequence of lifts of loops composed of Γ_{l-1}, Γ_l or their inverses. **QED**

According to the corollary, we have only to show that for any two points f_1, f_2 of $\hat{M}_\varphi$ such that $\Lambda(f_1) = \Lambda(f_2) = p$ is a general point in $\hat{V}'$, there is a path in the smooth part of $\hat{M}_\varphi$ joining f_1 and f_2.

Note that the difference between f_1 and f_2 is a series of fibre twists which transform j into a fibration $f : S \to C$ without multiple fibre, and with an N-multisection.

Lemma 3.11. *f can be reached from j by a finite number of locally trivial twists $\theta_1, \ldots, \theta_n$, such that the twisting section of θ_i is a loop contained in the smooth part of $\mathfrak{D}_N$.*

Proof. Keeping the notations in the proof of Theorem 3.4, our aim is to modify the series of translating twists so that each of them is a locally trivial twist made along a loop not containing critical points of j. The twisting section will then be a loop contained in the smooth part of $\mathfrak{D}_N$.

We first note that the condition that all the singular fibres are of type I_1 allows us to assume that each of the twists is made along a loop with base q. Indeed, we have only to consider the twists along G'_i. Let γ'_i be a small loop around c_i, entirely contained in γ_i, meeting G'_i at a point g'_i. Then in view of (2), the effect on γ_i of the twist along G'_i is equivalent to that of a twist along the loop composed of path $[q, g'_i] \subset G'_i$ followed by γ'_i then by $[g'_i, q] \subset -G'_i$ (with a different twisting section). We will denote this loop simply by γ'_i.

Suppose that c_{l-1} and c_l are as in Lemma 3.10. Let $\theta'_1, \ldots, \theta'_{2b+l}$ be the twists transforming f into j as in the proof of Theorem 3.4 and modified as above to become twists along loops, where $b = g(C)$, and θ_i is along α_i $(i = 1, \ldots, b)$, β_{i-b} $(i = b + 1, \ldots, 2b)$, or γ'_{i-2b} $(i = 2b+1, \ldots, 2b+l)$.

For each i with $1 \le i \le 2b+l-2$, let $\delta_{i,0}$ and $\delta_{i,1}$ be the starting and ending elements of the twisting section of θ'_i, which are elements in $\mathfrak{D}_N|_q$. As the local monodromies along γ'_{l-1}, γ'_l generate $\operatorname{Aut} H^1(F_q, \mathbf{Z})$ by Lemma 3.10, there is a loop ψ_i, composed of γ'_{l-1}, γ'_l and their inverses, such that the monodromy $\mu_{\mathfrak{D}_N}(\psi_i)$ transforms $\delta_{i,1}$ into $\delta_{i,0}$. Let θ_i be the fibre twist along the loop Γ_i consisting of the twisting loop of θ'_i followed by ψ_i, whose twisting section is the unique finite-order extension of the twisting section of θ'_i over Γ_i. Then this twisting section has $\delta_{i,0}$ as starting and ending elements, hence is a loop in (the smooth part of) $\mathfrak{D}_N$.

Let $f' = \theta_1 \circ \cdots \circ \theta_{2b+l-2}(f)$. To prove the lemma, we have only to show that $f' = j$. More precisely, if D' is the image N-multisection of D for f', with the image x' of x as base point, we want to show that $\mu_{D'}(\gamma)(x') = x'$ for any $\gamma \in \pi_1(C', p)$.

Let D_0 be the N-multisection of j containing the unit section. We identify the restrictions of D' and D_0 on the fibres over p, by matching x' with the point on the unit section. Let H be this restriction group. Instead of $\mu_{D'}$, we consider the difference monodromy $\bar{\mu} = \mu_{D'} - \mu_{D_0}$. $\bar{\mu}$ maps any element of $\pi_1(C', p)$ to a translation of H, so we may write

$$\bar{\mu} : \pi_1(C', p) \longrightarrow H \cong \mathbf{Z}_N^2 \ .$$

Now by the above construction, we have $\bar{\mu}(\gamma) = 0$ for $\gamma = \alpha_i$, β_i or $\gamma_1, \ldots, \gamma_{l-2}$, and $\bar{\mu}(\gamma_{l-1}) + \bar{\mu}(\gamma_l) = 0$ for some choice of orientations. To show $\bar{\mu}(\gamma_{l-1}) = \bar{\mu}(\gamma_l) = 0$, we have only to show that the subgroups $\bar{\mu}(\langle\gamma_{l-1}\rangle) \cong \mathbf{Z}_N$ and $\bar{\mu}(\langle\gamma_l\rangle) \cong \mathbf{Z}_N$ have zero intersection.

Indeed, as $\bar{\mu}$ now factors through $\pi_1(C - \{c_{l-1}, c_l\}, p)$, we have

$$\bar{\mu}(\langle\gamma_{l-1}\rangle) = \bar{\mu}\left(\langle\gamma'_{l-1}\rangle\right) \quad , \quad \bar{\mu}(\langle\gamma_l\rangle) = \bar{\mu}(\langle\gamma'_l\rangle)$$

by fixing a path joining p and q. Now the relation

$$\bar{\mu}\left(\langle\gamma'_{l-1}\rangle\right) \cap \bar{\mu}(\langle\gamma'_l\rangle) = 0$$

is a direct consequence of the condition $H^1(F_q, \mathbf{Z}) = \langle Z_{l-1}, Z_l\rangle$, where Z_{l-1} and Z_l are as in the proof of Lemma 3.10. **QED**

Due to Lemma 3.11, we may suppose that the difference between f_1 and f_2 is a twist Θ along a loop s in the smooth part of $\mathfrak{D}_N$.

We can write $s = a_1 s_1 + \cdots + a_k s_k$, where $a_i \in \mathbf{Z}$, and s_i is a loop in D_{m_i}. Then

$$\Theta = a_1\Theta_1 + \cdots + a_k\Theta_k \ ,$$

where Θ_i is the twist along s_i. This allows us to further suppose that s is a loop in D_{m_i}.

Now let $\delta_i \in D_{m_i}$ be the direction of the i-th multiple fibre of f_1. Deforming this multiple fibre along a path P in the smooth part of D_{m_i} joining δ_i and the base $s|_q$ of s, we get a fibration f_1', whose i-th multiple fibre is of direction $s|_q$. Then further deform this multiple fibre along s once, we get an f_2' with $f_2' - f_1' = \Theta$. Finally, deforming the i-th multiple fibre of f_2' back to δ_i via P, we get $f_2 : S_2 \to C$, with $f_2 - f_1 = f_2' - f_1'$ by the commutativity of translating twists. **QED**

§4. Relations with Ogg-Shafarevich theory

We refer to [C-D], Section 5.4 for a recent account of Ogg-Shafarevich theory in the case of elliptic surfaces. We suppose throughout this section that $j : J \to C$ is an elliptic fibration with a unit section, and all elliptic fibrations are algebraic, with j as jacobian fibration. An isomorphism between two such fibrations are called *j-isomorphism*, if it induces the identity map on j.

From cohomological point of view, the set Σ_j of elliptic fibrations with jacobian j equals the set of principal homogenious spaces (torsors) on the generic fibre J_η of j. We have thus a canonical isomorphism $\Sigma_j \cong H^1_{\text{ét}}(\eta, \mathcal{J}_\eta)$, where $\mathcal{J}_\eta$ is the stalk over the generic point η of C of the Néron sheaf of abelian groups associated to j.

Now a translating twist defined via finite base change is just a cocycle on the étale site of η, which defines a torsor. Thus we have a canonical isomorphism between $H^1_{\text{ét}}(\eta, \mathcal{J})$ and the group of translating twists modulo cohomologically trivial twists. With this correspondence in mind, Theorem 3.4 just means that torsors can be represented by a special kind of cocycles: those induced by a cocycle on a finite subgroup scheme.

We have following exact sequence:

$$0 \longrightarrow H^1(C, \mathcal{J}) \longrightarrow \Sigma_j \xrightarrow{\alpha} \sum_{p \in C} \delta(p) \longrightarrow H^2(C, \mathcal{J}) \quad , \tag{4}$$

where $\delta(p)$ is the group of directions of finite order on p, $H^1(C, \mathcal{J})$ is the subgroup of locally trivial torsors (i.e. the Shafarevich-Tate group associated to j).

When j is not trivial, one shows by Ogg-Shafarevich theory that α is surjective, by proving that $H^2(C, \mathcal{J})$ has no torsion [C-D, Theorem 5.4.4]. Our theorem 3.4 gives a direct proof of this surjectivity, and shows moreover that for any element of order n in $\sum \delta(p)$, we can find an inverse image element in Σ_j which is of order at most wn.

Also when j is trivial, the condition (3) in Theorem 3.7 determines the image of α.

Another main result in Ogg-Shafarevich theory is the computation of the Shafarevich-Tate group $H^1(C, \mathcal{J})$, which is shown to be isomorphic to $(\mathbf{Q}/\mathbf{Z})^{b_2 - \rho} \oplus \mathfrak{T}$ in the case of elliptic surfaces, where $\mathfrak{T}$ is a finite group, b_2 and ρ are respectively the second Betti number and the Picard number of J [Ra].

The following is a direct computation of the Shafarevich-Tate group using fibre twists, which also determines the cotorsion $\mathfrak{T}$.

Definition 4.1. Let D_1 and D_2 be two n-multisections on an elliptic fibration $f : S \to C$. D_1 and D_2 are called *congruent* (resp. *isomorphic*) if they are both contained in a same an-multisection for some $a \geq 1$ (resp. if there is a j-automorphism $\alpha : S \to S$, such that $\alpha(D_1) = D_2$).

We denote by M the Mordell-Weil group of sections of j. We may write

$$M \cong \mathbf{Z}^r \oplus T \ , \tag{??}$$

where T is the subgroup of elements of finite order. Note that T is finite when j is not trivial, otherwise $T \cong (\mathbf{Q}/\mathbf{Z})^2$.

Lemma 4.2. *Let D be an n-multisection of f. We have a canonical isomorphism between M and the group of isomorphisms of D with another n-multisection (which may be D itself).*

Under this isomorphism, elements in M of order dividing n correspond to automorphisms of D.

Proof. This follows directly from the fact that M is the group of j-automorphisms of S. **QED**

Lemma 4.3. *Suppose that f has an n-multisection D. Then there is a natural 1-1 correspondence between M/nM and the set Δ_n of isomorphism classes of n-multisections of f.*

Proof. Let $\nu_n : S \to S^{(n)}$ be the n-multiplication map. The image of D being a section in $S^{(n)}$, we can write $S^{(n)} = J$, with $\nu_n(D)$ as the unit section. And ν_n thus gives an isomorphism between M and the set of n-multisections on S. On the other hand, ν_n induces an endomorphism $\epsilon_n : M \to M$, the first M being considered as the group of j-automorphisms on S. ϵ_n is simply the multication by n. Now by Lemma 4.2, an n-multisection D' is isomorphic to D iff its image by ν_n is contained in $\mathrm{Im}(\epsilon_n)$. **QED**

Corollary. *If j is non-trivial, we have*

$$|\Delta_n| = n^r + |T_n| \ ,$$

where $T_n = T/nT$ is isomorphic to the subgroup of T composed of elements of order dividing n; if j is trivial,

$$|\Delta_n| = n^r \ .$$

In particular, there are only a finite number of non-isomorphic n-multisections on an elliptic surface.

Proof. The case of T finite follows directly from the theory of finite abelian groups. Otherwise T is divisible, hence contained in $\mathrm{Im}(\epsilon_n)$ for any n. **QED**

Now for each critical point c_i $(i = 1, \ldots, l)$ of j, let t_i be the number of components in $F_i = j^{-1}(c_i)$,

$$d_i = \begin{cases} 1 & F_i \text{ is semistable} \\ 2 & \text{otherwise,} \end{cases}$$

$$t = \sum_{i=1}^{l} (t_i - 1) \quad , \quad d = \sum_{i=1}^{l} d_i \ .$$

A look into Kodaira's table of singular fibres gives

$$c_2(J) = \sum_{i=1}^{l} (t_i - 1 + d_i) = t + d \ . \tag{6}$$

Now we take the notations in the proof of Theorem 3.4.

Definition 4.4. An *n-marked elliptic fibration* is a triplet (f, D, x), where $f : S \to C$ is an elliptic fibration with jacobian j, D an n-multisection on S, and x a point on $D|_F$, such that f has no multiple fibre outside q, and the multiplicity of F_q is a factor of n. An isomorphism between two n-marked fibrations (f_1, D_1, x_1) and (f_2, D_2, x_2) is a j-isomorphism $\alpha : S_1 \to S_2$ such that $\alpha(x_1) = x_2$. In this case we have automatically $\alpha(D_1) = D_2$.

Let $\Sigma_{n,1}$ be the group of isomorphism classes of n-marked elliptic fibrations, with the group structure induced by that of fibre twists. We first compute $\Sigma_{n,1}$ using arguments in the proof of Theorem 3.4.

Two n-marked fibrations (f_1, D_1, x_1), (f_2, D_2, x_2) are isomorphic if and only if the following diagram commutes:

$$\begin{tikzcd}[column sep=small] & \pi_1(C', p) \arrow[dl, "\mu_{D_1}"'] \arrow[dr, "\mu_{D_2}"] & \\ \mathrm{Aut}(D_1|_F) \arrow[rr, "\alpha"] & & \mathrm{Aut}(D_2|_F) \end{tikzcd}$$

where α is induced by the group isomorphism $\iota : D_1|_F \to D_2|_F$ with $\iota(x_1) = \iota(x_2)$. Identifying $D_1|_F$ with $D_2|_F$ via ι, the above diagram commutes iff

$$\mu_{D_1}(\gamma)(x_1) = \mu_{D_2}(\gamma)(x_1)$$

for every $\gamma = \alpha_i$, β_i or γ_i. Therefore $\Sigma_{n,1}$ is the group of choices for all the $\mu_D(\gamma)(x)$.

For a fixed γ, the choices of $\mu_D(\gamma)(x)$ form a group G_γ isomorphic to $\mathbf{Z}_n^2$ when $\gamma = \alpha_i$ or β_i, and isomorphic to $\mathbf{Z}_n^{d_i}$ when $\gamma = \gamma_i$. The proof of Theorem 3.4 shows that the choices are independent for different γ's, therefore

$$\Sigma_{n,1} = \bigoplus_{\gamma} G_\gamma \cong \mathbf{Z}_n^{4g(C)+d} = \mathbf{Z}_n^{4g(C)+c_2-t}$$

by (6).

The next step is to compute the subgroup $\Sigma_{n,2}$ of $\Sigma_{n,1}$ consisting of n-marked fibrations without multiple fibre. Let $\mathfrak{D}_n|_q \cong \mathbf{Z}_n^2$ be the group of n-directions over q. To each marked fibration f associating the direction of multiplicity of its fibre over q, we get a homomorphism

$$\varphi_q : \Sigma_{n,1} \longrightarrow \mathfrak{D}_n|_q \ ,$$

such that $\Sigma_{n,2} = \operatorname{Ker}\varphi_q$. Noting that φ_q is trivial when j is trivial, we let $\mathfrak{T}_n = \mathfrak{D}_n|_q/\operatorname{Im}\varphi_q$ when j is non-trivial, and $\mathfrak{T}_n = 0$ when j is trivial. Then one sees easily that

$$\Sigma_{n,2} \cong \mathbf{Z}_n^{2g(C)+2g(J)+c_2-t-2} \oplus \mathfrak{T}_n \ . \tag{7}$$

To compute $\mathfrak{T}_n$ for non-trivial j, note that Theorem 2.4 tells that $\mathfrak{T}_n$ is a subgroup of $\mathbf{Z}_w^2$, where w is the twisting weight of j, therefore it is trivial except when j is locally trivial and $w|n$.

Let θ and τ be as in Definition 2.3, and let γ be an element of $\pi_1(C)$ such that $\theta(\gamma)$ generates $\operatorname{Im}\theta$. γ induces an endomorphism $\theta(\gamma)_n : \mathfrak{D}_n|_q \to \mathfrak{D}_n|_q$, and we let K_n be its kernel. Then we have

$$K_n \cong \mathfrak{D}_n|_q/\operatorname{Im}\theta(\gamma)_n = \mathfrak{D}_n|_q/\operatorname{Im}\varphi_q \cong \mathfrak{T}_n$$

by (2). As K_n is simply the group of sections contained in $\mathfrak{D}_n$, we get

$$\mathfrak{T}_n \cong \begin{cases} \mathbf{Z}_2^2 & \tau = 2 \text{ and } \tau|n \\ \mathbf{Z}_3 & \tau = 3 \text{ and } \tau|n \\ \mathbf{Z}_2 & \tau = 4 \text{ and } \tau|n \\ 0 & \text{all other cases.} \end{cases} \tag{8}$$

Now we divide out the congruences. Let H be the subgroup in $\Sigma_{n,2}$ consisting of n-marked fibrations (f, D, x) such that S has an n-multisection D' congruent to D, which contains a section.

Lemma 4.5. *If j is trivial, $H = 0$; otherwise $H = \mathbf{Z}_n^2$.*

Proof. When j is trivial, f contains a section congruent to D iff D is trivial. Therefore $H = 0$ in this case, and we may assume that j is non-trivial.

Due to Lemma 4.3, the n^2-multisection E containing D contains a copy of every isomorphism class of n-multisections congruent to D. Therefore (f, D, x) is in H iff E contains a section.

Moreover, we can write $T_n \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2}$, and consider it as a subgroup of $E|_F$. Then the n-multisection contained in E cuts out a subgroup $I \cong \mathbf{Z}_{n_1 n} \oplus \mathbf{Z}_{n_2 n}$ in F which contains T_n, and E contains a section iff the intersection with F of the sum of sections in E is a coset of T_n in I. This shows that $H \cong I/T_n \cong \mathbf{Z}_n^2$. **QED**

Let $\Sigma_{n,3} = \Sigma_{n,2}/H$. $\Sigma_{n,3}$ is the group of congruence classes of pairs (f, D) where a congruence between (f_1, D_1) and (f_2, D_2) is a j-isomorphism $S_1 \to S_2$ sending D_1 to an n-multisection congruent to D_2. By Lemma 4.3 and the formula (7) for $\Sigma_{n,2}$, we have

$$\Sigma_{n,3} \cong \mathbb{Z}_n^{4q(J)+c_2-t-4} \oplus \mathfrak{T}_n = \mathbb{Z}_n^{b_2-t-2} \oplus \mathfrak{T}_n \ .$$

Now by Lemma 4.3, the group of non-isomorphic congruence classes of n-multisections of a fibration is isomorphic to $\mathbb{Z}_n^r$. Divide out this group from $\Sigma_{n,3}$, we get the group Σ_n of j-isomorphism classes of elliptic fibrations without multiple fibre and with an n-multisection:

$$\Sigma_n \cong \mathbb{Z}_n^{b_2-t-r-2} \oplus \mathfrak{T}_n = \mathbb{Z}_n^{b_2-\rho} \oplus \mathfrak{T}_n \ .$$

Finally, taking limit of n,

Theorem 4.6. (Ogg-Shafarevich) *The Shafarevich-Tate group of j is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \oplus \mathfrak{T}$, where $\mathfrak{T}$ is trivial unless j is locally trivial but not trivial, in the latter case $\mathfrak{T}$ is isomorphic to the torsion part of the Mordell-Weil group of j.*

Remark. Now we get the number d in Theorem 3.9:

$$d = N^{b_2-\rho} \ . \tag{9}$$

(Note that under the assumption of Theorem 3.9, $\mathfrak{T}_n = 0$.)

We close this paper by Raynaud's original proof of Lemma 3.3, which works over any characteristic.

Let $x \in H^1_{\text{ét}}(\eta, \mathcal{J}_\eta)$ be the element corresponding to f. It is known from the theory of torsors that x is of finite order, so let n be this order, and consider the n-multiplication map $S \dashrightarrow S^{(n)}$. Then the torsor element corresponding to the fibration $f^{(n)} : S^{(n)} \to C$ is $nx = 0$, in other words $f^{(n)} \cong j$. Therefore $S^{(n)}$ contains a section, whose inverse image in S is an n-multisection. **QED**

REFERENCES

[BPV] Barth/Peters/Van de Ven: Compact Complex Surfaces, Springer 1984

[C-D] Cossec, F./Dolgachev, I.: Enriques Surfaces I, Progress in Math. 76, Birkhauser 1989

[Do] Dolgachev, I.: Algebraic surfaces with $q = p_g = 0$, in Algebraic Surfaces, Liguori, Napoli (1981)

[Ish] Ishida, Masa-Nori: An elliptic surface covered by Mumford's fake projective plane, Tohoku Math. J. 40, 367-396 (1988)

[Ko1] Kodaira, K.: On compact complex analytic surfaces, I: Annals of Math. 71, 111-152 (1960); II: Annals of Math. 77, 563-626 (1963); III: Annals of Math. 78, 1-40 (1963)

[Ko2] Kodaira, K.: On the structure of compact complex analytic surfaces, I: American J. Math. 86, 751-798 (1964); II: American J. Math. 88, 682-721 (1966); III: American J. Math. 90, 55-83 (1969); IV: American J. Math. 90, 1048-1066 (1969)

[O] Ogg, A.: Cohomology of abelian varieties over function fields, Ann. Math. 76, 185-212 (1962)

[Ra] Raynaud, M.: Charactéristique d'Euler-Poincaré d'un faisceau et cohomologie des varietés abéliennes, in "Dix Exposés sur Cohomologie des Schémas", 12-30, North-Holland 1968

[Sha] Shafarevich, I.: Principal homogeneous spaces over function fields, Proc. Steklov Inst. Math. 64, 316-346 (1961), English translation: AMS Translations 37, 85-114 (1964)

[Shi] Shioda, Tetsuji: On elliptic modular surfaces, J. Math. Soc. Japan 24/1, 20-59 (1972)

[X1] Xiao, Gang: Fibrations of Algebraic Surfaces (in Chinese), to be published by Shanghai Publishing House for Science and Technology

[X2] Xiao, Gang: π_1 of elliptic and hyperelliptic surfaces, to appear