# The trace class conjecture in the theory of automorphic forms

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## Contents

## 0. Introduction

Let $G$ be the group of real points of a reductive algebraic group which is defined over $\mathbb{Q}$ and satisfies the same assumptions as in [H,I]. Let $\Gamma$ be an arithmetic subgroup of $G$ and denote by $R_\Gamma$ the right regular representation of $G$ in $L^2(\Gamma\backslash G)$. The theory of Eisenstein series [L1] implies that $L^2(\Gamma\backslash G)$ admits an orthogonal decomposition

$$L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G)$$

where $L^2_d(\Gamma\backslash G)$ is the direct sum of all subspaces of $L^2(\Gamma\backslash G)$ that correspond to irreducible subrepresentations of $R_\Gamma$ and $L^2_c(\Gamma\backslash G)$ is the subspace of $L^2(\Gamma\backslash G)$ where $R_\Gamma$ decomposes continuously. Denote by $R^d_\Gamma$ the restriction of $R_\Gamma$ to $L^2_d(\Gamma\backslash G)$. Let $K$ be a maximal compact subgroup of $G$. Suggested by Selberg's work on the trace formula [S] it is natural to conjecture that for each K-finite $f \in C^\infty_c(G)$, the operator

$$R^d_\Gamma(f) = \int_G f(g) R^d_\Gamma(g)\,dg$$

is of the trace class. This is the so-called trace class conjecture. To establish the trace class property for the operators $R^d_\Gamma(f)$ is, of course, the first step toward a trace formula in the spirit of Selberg. The case $G=SL(2,\mathbb{R})$ was treated by Selberg and the trace class property in this case was first established by him (c.f. [S]). The proof is essentially the same for all real rank one groups. For groups $G$ of $\mathbb{Q}$-rank one the trace class conjecture has been proved by Donnelly [D2] and Langlands. The purpose of this paper is to prove the trace class conjecture in general. In our approach, the trace class conjecture is a consequence of an estimate of the number of eigenvalues of the Casimir operator acting on a fixed K-type.

Before we can state the precise result we have to introduce some notation. First observe that, by passing to a subgroup of finite index, we may assume that $\Gamma$ acts without fixed points on the symmetric space $X = G/K$. Let $\sigma : K \longrightarrow GL(V)$ be an irreducible unitary representation of $K$ and let $E$ be the associated locally homogeneous vector bundle over $\Gamma\backslash X$. The Casimir element of $G$ induces an elliptic second order differential operator $\Delta$ acting in $C_c^\infty(\Gamma\backslash X,E)$. $\Delta$ is essentially selfadjoint in $L^2(\Gamma\backslash X,E)$ and therefore, has a unique selfadjoint extension $\bar{\Delta}$ to an unbounded operator in $L^2(\Gamma\backslash X,E)$. Our main result is the following:

**Theorem 0.1** Let $N(\lambda)$ be the number of linearly independent eigenfunctions of $\bar{\Delta}$ with eigenvalue less than $\lambda$. There exists a constant $C>0$ such that

$$N(\lambda) \leq C(1 + \lambda^{2n})$$

for $\lambda \geq 0$ and $n = \dim X$.

The Paley-Wiener theorem of Clozel and Delorme [C-D] implies then:

**Corollary 0.2** For each K-finite $f \in C_c^\infty(G)$, the operator $R_\Gamma^d(f)$ is of the trace class.

Even more is true. It follows from Theorem 0.1 that $R_\Gamma^d(f)$ is of the trace class for each K-finite $f \in S^1(G)$ where $S^1(G)$ is Harish-Chandra's Schwartz space of integrable rapidly decreasing functions on G. It is also very conceivable that the K-finiteness assumption can be removed by making use of an improved version of Theorem 0.1 which includes the dependence on $\sigma\in\hat{K}$. We think that the following estimation holds

$$N(\lambda) \leq C(1 + (\dim\sigma)^k + \lambda^{2n})$$

with $C>0$ and $k\in\mathbb{N}$ independent of $\sigma\in\hat{K}$. One only has to improve Proposition 3.17. We shall discuss this point elsewhere. Another observation is that Corollary 0.2 implies the corresponding result for the adèlic case (c.f.§8).

We shall now describe the content of this paper and the main steps of the proof of Theorem 0.1. First we observe that the discrete spectrum has a further decomposition

$$L^2_d(\Gamma\backslash G) = L^2_{cus}(\Gamma\backslash G) \oplus L^2_{res}(\Gamma\backslash G)$$

into the direct sum of the space of cusp forms $L^2_{cus}(\Gamma\backslash G)$ and the residual spectrum $L^2_{res}(\Gamma\backslash G)$. For cuspidal eigenfunctions the estimation claimed in Theorem 0.1 is true by Donnelly's results [D1]. Therefore, it remains to investigate the residual spectrum. It follows from Langlands' theory of Eisenstein systems that $L^2_{res}(\Gamma\backslash G)$ is spanned by "iterated residues" of cuspidal Eisenstein series (c.f.[L1,§7]). This statement will be made more precise in §8. Using this description of the residual spectrum, the proof of Theorem 0.1 can be reduced to the following problem: For a given cuspidal Eisenstein series, we have to estimate the number of its singular hyperplanes which are real and intersect a fixed compact set containing the origin. But the singularities of a cuspidal Eisenstein series are essentially the same as the singularities of the corresponding intertwining operator. Using the factorization property of the intertwining operators, one can reduce everything to cuspidal Eisenstein series associated to rank one $\mathbb{Q}$-parabolic subgroups of the Levi components of $\mathbb{Q}$-parabolic subgroups of G. Thus, we only have to consider rank one cuspidal Eisenstein series. This is the first step.

Let $\mathcal{P}$ be a class of associate rank one parabolic subgroups of G which are defined over $\mathbb{Q}$. The theory of Eisenstein series associates to $\mathcal{P}$ a sequence of intertwining operators

$$C(s) : \mathcal{E}_{cus}(\sigma,\mathcal{O}) \longrightarrow \mathcal{E}_{cus}(\sigma,\mathcal{O})$$

where $\mathcal{E}_{cus}(\sigma,\mathcal{O})$ is a finite-dimensional space of automorphic forms and $C(s)$ is a linear operator which is a meromorphic function of $s\in\mathbb{C}$ (c.f. §3). The problem is now to estimate the number of poles, counted to multiplicity, of $\det C(s)$ in a finite interval of the real line. In §3 we consider those poles of $\det C(s)$ which are contained in the half-plane $\mathrm{Re}(s)>0$. Let $t\in\mathbb{R}$. Using the analytic properties of $C(s)$, it follows that the number of poles of $\det C(s)$ in $\mathrm{Re}(s)>0$ is bounded by the dimension of $\mathcal{E}_{cus}(\sigma,\mathcal{O})$ times the number of points $s_o\in\mathbb{R}^+$ such that

$$C(s_o)\Phi = -e^{2s_ot}\Phi \tag{0.3}$$

for some non-zero $\Phi\in\mathcal{E}_{cus}(\sigma,\mathcal{O})$. Since $\mathcal{E}_{cus}(\sigma,\mathcal{O})$ consists of cusp forms on the Levi components of a finite number of parabolic subgroups in $\mathcal{P}$, we can use Donnelly's results [D1] to estimate the dimension of $\mathcal{E}_{cus}(\sigma,\mathcal{O})$. On the other hand, each

$\Phi \in \mathcal{E}_{cus}(\sigma, \mathcal{O})$ gives rise to a rank one cuspidal Eisenstein series $E(\Phi,s)$, $s\in\mathbb{C}$. It is known that the constant term of $E(\Phi,s)$ along any rational parabolic subgroup of $G$, which is not in $\mathcal{P}$, vanishes. Furthermore, the constant term of $E(\Phi,s)$ along any $P\in\mathcal{P}$ is described by $C(s)$. Let $\Lambda^T E(\Phi,s_0)$ be the Eisenstein series $E(\Phi,s_0)$ truncated at level t (c.f. §3). If (0.3) is satisfied then $\Lambda^T E(\Phi,s_0)$ belongs to the Sobolev space $H^1(\Gamma\backslash X,E)$ and all its constant terms vanish in a neighborhood of infinity. On the space of all these sections of $E$ we introduce an auxiliary selfadjoint operator $\Delta_T$ which has pure point spectrum. In the two-dimensional case this operator was first introduced by Lax and Phillips [L-P] and has been employed by Colin de Verdiere in [Co]. Under the assumption that (0.3) is satisfied, the truncated Eisenstein series $\Lambda^T E(\Phi,s_0)$ is an eigenfunction of $\Delta_T$. Then we generalize the method of Donnelly [D1] to get an estimate on the growth of the number of eigenvalues of $\Delta_T$. Combining these results gives the desired estimation for the number of poles of $\det C(s)$ in $\mathrm{Re}(s)>0$.

The next step is to show that $\det C(s)$ can be written as

$$\det C(s) = \frac{F_1(s)}{F_2(s)} \tag{0.4}$$

where $F_1(s)$ and $F_2(s)$ are entire functions of finite order. In the case of $SL(2,\mathbb{R})$ this result is due to Selberg (c.f.[He,Ch.VI, §11] for a complete proof). Our proof of this result is based on §4 where we develop a new method of analytic continuation of rank one cuspidal Eisenstein series. This method is essentially an extension of the method employed by Colin de Verdiere [Co] in the case of $SL(2,\mathbb{R})$. In the higher rank case, the geometry of $\Gamma\backslash X$ is much more complicated so that several technical difficulties arise. In §5 we employ the results of §4 to establish (0.4).

Using (0.4) together with Hadamard's factorization theorem, we obtain in §6 the following product formula

$$(0.5) \qquad \det C(s) = \det C(0)\, q^{s} \prod_{j=1}^{l} \frac{s+\sigma_j}{s-\sigma_j} \prod_{\eta} \frac{s+\bar{\eta}}{s-\eta}$$

Here $\sigma_1,\ldots,\sigma_l \in \mathbb{R}^+$ are the poles of $\det C(s)$ in the half-plane $\mathrm{Re}(s) \geq 0$ and $\eta$ runs over all poles of $\det C(s)$ in $\mathrm{Re}(s) < 0$. $q$ is a certain constant which satisfies $\log(q) \leq C \dim \mathcal{E}_{cus}(\sigma,\mathcal{O})$ and $C$ is a constant which depends only on $\mathcal{P}$. For $SL(2,\mathbb{R})$ this product formula is also due to Selberg (c.f. [He, Ch.VI, §12]).

In §7 we first estimate the integral

$$(0.6) \qquad \int_{-\Lambda}^{\Lambda} \frac{d}{ds} \log \det C(i\lambda)\, d\lambda$$

in terms of $\Lambda$ and the orbit type $\mathcal{O}$. Let $P \in \mathcal{P}$ with Langlands decomposition $P = NAM$. $M$ is the group of real points of a reductive algebraic group defined over $\mathbb{Q}$ and $\Gamma_M = N\Gamma \cap M$ is an arithmetic subgroup of $M$. The orbit type $\mathcal{O}$ determines an eigenvalue $\mu$ of the Casimir operator $\Omega_M$ acting on sections of the locally homogeneous vector bundle $E_M$ over $\Gamma_M\backslash M/K \cap M$ associated to $\sigma|K \cap M$. Using facts established in §3, we show that the integral (0.6) is bounded from above by the number of eigenvalues less than $\Lambda^2 + \mu + |\rho_P|^2$ of the operator $\Delta_T$ times the dimension of $\mathcal{E}_{cus}(\sigma,\mathcal{O})$. This enables us to estimate (0.6) by $C(1 + \Lambda^n + \mu^n)$, $n = \dim X$. Now we can use (0.5) to compute the logarithmic derivative of $\det C(s)$. The formula we obtain shows that $d/ds \log \det C(i\lambda)$ is essentially of the form $\Sigma_1 + \Sigma_2$ where $\Sigma_1$ is the sum over all poles in $Re(s) < 0$ and $\Sigma_2$ the sum over all poles in $Re(s) > 0$. The point is that each term in $\Sigma_1$ (resp. $\Sigma_2$) is negative (resp. positive). By §3 we can estimate the integral of $\Sigma_2$ over $[-1,1]$ and therefore, we can also estimate the integral of $\Sigma_1$ over $[-1,1]$. But this integral is bounded from below by a fixed constant times the number of real poles of $\det C(s)$ in a finite interval $[-c,0]$, $c>0$. This completes the estimation of the number of real poles of $\det C(s)$ in a finite interval.

Finally, in §8 we prove Theorem 0.1. We also indicate briefly how the adèlic version of Corollary 0.2 can be deduced from Corollary 0.2.

Our method to prove the trace class conjecture has also other applications. It yields, for example, estimates for the number of zeros of principal L-functions for $GL(n)$. This will be discussed elsewhere. It is also an interesting question to understand to what extent in the case of the Laplace operator the locally symmetric structure of $\Gamma\backslash X$ is relevant for Theorem 0.1 to hold.

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**1. Preliminaries**

1.1 Let $G$ be the group of real points of a reductive algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ which satisfies the same assumptions as in [H,I,§1]. $\Gamma$ will denote an arithmetic subgroup of G. We fix a maximal compact subgroup $K$ of $G$ and set $X = G/K$. Throughout this paper we shall assume that $\Gamma$ acts without fixed points on X. By $\Theta$ we shall denote the Cartan involution of G with respect to K.

1.2 The Lie algebra of a Lie group $G, H, \ldots$ is denoted by the corresponding l.c. German letter $\mathfrak{g}, \mathfrak{h}, \ldots$. By $U(\mathfrak{g})$ we shall denote the universal envelopind algebra of the complexified Lie algebra $\mathfrak{g}\otimes\mathbb{C}$ and by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. $Z(\mathfrak{g})$ contains the Casimir element $\Omega_G$ (or simply $\Omega$ ) of $G$ with respect to an admissible real valued bilinear form $F$ on $\mathfrak{g}$ (c.f. [B-G]).

1.3. Let $\mathbf{P}$ be a parabolic subgroup of $G$ defined over $\mathbb{Q}$. The group $P$ of real points of $\mathbf{P}$ is called a $\mathbb{Q}$-parabolic subgroup of $G$. We may decompose $P$ as

$$P = N_P A_P M_P \qquad (1.1)$$

(or just $NAM$, if there is no danger of confusion) where $N_P$ is the unipotent radical of P, $A_P M_P$ is the unique Levi subgroup of $P$ stable under $\Theta$ and $A_P$ is the identity component of the group of real points of the maximal $\Theta$-stable torus of the $\mathbb{Q}$-split radical of $\mathbf{P}$. The decomposition (1.1) is called Langlands decomposition of $P$. $A_P$ is called special split component of P. The rank of $P$ is defined to be the dimension of $A_P$. The Weyl group of $A_P$ is $W(A_P) = N_G(A_P)/Z_G(A_P)$. Furthermore, we set $\Gamma_M = N\Gamma \cap M$, $K_M = K \cap M$ and $X_M = M/K_M$. Observe that $K \cap M = K \cap P$.

We have $G = PK$. Therefore, any element $x \in G$ has a decomposition

$$x = namk$$

with $k\in K$, $m\in M$, $a\in A$, $n\in N$. The factor $a$ is uniquely determined by $x$. Set

$$H_P(x) = \log a .$$

The roots of $(P,A)$ will be denoted by $\Phi_P$ and $\Psi_P$ will denote the set of simple roots of $(P,A)$. For $\beta\in\Phi_P$, let

$$\mathfrak{n}_\beta = \{Y\in\mathfrak{g} \mid [H,Y] = \beta(H)Y,\ H\in\mathfrak{a}\} .$$

Then

$$\mathfrak{n} = \bigoplus_\beta \mathfrak{n}_\beta .$$

As usually, let

$$\rho_P = \frac{1}{2} \sum_{\beta\in\Phi_P} \dim(\mathfrak{n}_\beta)\beta .$$

For a given subset $F\subset\Psi_P$ we denote by $P_F$ the $\mathbb{Q}$-parabolic subgroup of $G$ associated to $F$. Note that $P\subset P_F$ and the Lie algebra $\mathfrak{a}_F$ of the split component of $P_F$ consists of all $H\in\mathfrak{a}$ such that $\alpha(H) = 0$ for all $\alpha\in F$.

Let $P = NAM$ be a rank one $\mathbb{Q}$-parabolic subgroup of $G$. Choose $H\in\mathfrak{a}$ such that $\|H\| = 1$ and $\alpha(H) > 0$, $\alpha\in\Psi_P$. There exists a unique selfadjoint element $\Omega_M\in Z(\mathfrak{m})$ such that

$$\Omega_G = H^2 - 2\rho(H)H + \Omega_M \pmod{\mathfrak{n}Z(\mathfrak{g})} \tag{1.2}$$

(c.f. [H,I,§6]). If $\Omega_G$ is defined by the admissible bilinear form $\tilde{F}$ on $\mathfrak{g}$, then $\Omega_M$ is defined by the restriction of $\tilde{F}$

to $m \times m$ .

1.4 Let $P$ be a $\mathbb{Q}$-parabolic subgroup of $G$ with unipotent radical $N_P$. Let $f$ be a complex valued locally bounded measurable function on $\Gamma\backslash G$. The constant term $f^P$ of $f$ along $P$ is defined as

$$f^P(x) = \int_{\Gamma \cap N_P\backslash N_P} f(nx)dn$$

where the measure $dn$ is normalized by the condition that the volume of $\Gamma \cap N_P\backslash N_P$ equals 1. The subspace of $L^2(\Gamma\backslash G)$ consisting of all $f$ satisfying $f^P = 0$ for all $\mathbb{Q}$-parabolic subgroups $P \neq G$ is denoted by $L^2_{cus}(\Gamma\backslash G)$. This is the space of cusp forms in $L^2(\Gamma\backslash G)$. Given a finite-dimensional unitary representation $\sigma: K \longrightarrow GL(V)$ of $K$, put

$$L^2(\Gamma\backslash G,\sigma) = (L^2(\Gamma\backslash G) \otimes V)^K .$$

Let $\tilde{E}$ be the homogeneous vector bundle over $X$ associated to $\sigma$ and put $E = \Gamma\backslash\tilde{E}$. Then $L^2(\Gamma\backslash G,\sigma)$ can be identified with the space $L^2(\Gamma\backslash X,E)$ of square integrable sections of $E$. Set

$$L^2_{cus}(\Gamma\backslash G,\sigma) = (L^2_{cus}(\Gamma\backslash G) \otimes V)^K .$$

This is the space of cusp forms in $L^2(\Gamma\backslash G,\sigma)$. We may identify $L^2_{cus}(\Gamma\backslash G,\sigma)$ with a subspace of $L^2(\Gamma\backslash X,E)$ which we denote by $L^2_{cus}(\Gamma\backslash X,E)$. Similarly we define

$$C^\infty(\Gamma\backslash G,\sigma) = (C^\infty(\Gamma\backslash G) \otimes V)^K .$$

This space can be identified with $C^\infty(\Gamma\backslash X,E)$ - the space of $C^\infty$-sections of $E$.

## 2. Eisenstein series and wave packets

For the convenience of the reader we shall recall in this section some basic facts concerning Eisenstein series and wave packets. For all details we refer to [H], [L1]. Since we are working with $\Gamma\backslash G$ in place of $G/\Gamma$, we have to change some signs and inequalities in the statements we are using from [H]. It will be clear from the context what has to be changed.

Let $P$ be a $\mathbb{Q}$-parabolic subgroup of $G$ with special split component $A$ and Langlands decomposition $P = NAM$. Let $(\sigma, V)$ be a unitary representation of $K$ in a finite dimensional Hilbert space $V$. Let $\chi\colon Z(m) \longrightarrow \mathbb{C}$ be a character of $Z(m)$ and let $\sigma_M\colon K_M \longrightarrow GL(V)$ be the restriction of $\sigma$ to $K_M = K \cap M$. Set

$$L^2_{cus}(\Gamma_M\backslash M, \sigma, \chi) = \{ \varphi \in L^2_{cus}(\Gamma_M\backslash M, \sigma_M) \mid D\varphi = \chi(D)\varphi \text{ for all } D \in Z(m) \}.$$

It is known that $L^2_{cus}(\Gamma_M\backslash M, \sigma, \chi)$ is a finite dimensional Hilbert space of automorphic forms with inner product induced from the inner product in $L^2(\Gamma_M\backslash M, \sigma_M)$. Let $\Phi \in L^2_{cus}(\Gamma_M\backslash M, \sigma, \chi)$. We extend $\Phi$ to a function $\Phi\colon (\Gamma \cap P)N_P\backslash G \longrightarrow V$ by

$$\Phi(namk) = \sigma(k)^{-1}\Phi(m) \quad (2.1)$$

Let $a^*$ be the dual Lie algebra of $a$ and let

$$(a^*)^+ = \{ \lambda \in a^* \mid \langle\lambda, \alpha\rangle > 0 \text{ for all } \alpha \in \Psi_P \}.$$

Let $\Lambda \in a^*_{\mathbb{C}}$ be such that $\mathrm{Re}(\Lambda) \in \rho_P + (a^*)^+$. Then the Eisenstein series attached to $\Phi$ is defined as

$$E(P|A,\Phi,\Lambda,x) = \sum_{\Gamma\cap P\backslash\Gamma} e^{(\Lambda+\rho_P)(H_P(\gamma x))}\Phi(\gamma x) .$$

The Eisenstein series is absolutely and uniformly convergent on compact subsets of $(\rho_P+(a^*)^+ + \sqrt{-1}\ a^*) \times G$. Let $P_i$, $i=1,2$, be two $\mathbb{Q}$-parabolic subgroups of $G$ with special split components $A_i$ and Langlands decomposition $P_i = N_iA_iM_i$, $i=1,2$. $(P_1,A_1)$ and $(P_2,A_2)$ are said to be associate if there exists $x\in G_{\mathbb{Q}}$ such that $\mathrm{Ad}(x)a_1 = a_2$. The set of all such isomorphisms is denoted by $W(a_1,a_2)$. Set $W(a) = W(a,a)$. Let $\chi\in\hat{Z}(m_1)$, $\Phi\in L^2_{cus}(\Gamma_{M_1}\backslash M_1,\sigma,\chi)$ and $\Lambda\in a^*_{1,\mathbb{C}}$ with $\mathrm{Re}(\Lambda)\in\rho_1+(a^*_1)^+$. If $(P_1,A_1)$ and $(P_2,A_2)$ are not associate and $\mathrm{rank}\,P_2 \geq \mathrm{rank}\,P_1$, then

$$E^{P_2}(P_1|A_1,\Phi,\Lambda) = 0.$$

If $(P_1,A_1)$ and $(P_2,A_2)$ are associate, then the constant term of $E(P_1|A_1,\Phi,\Lambda)$ along $P_2$ is given by

$$E^{P_2}(P_1|A_1,\Phi,\Lambda,x) = \tag{2.2}$$

$$= \sum_{w\in W(a_1,a_2)} e^{(w\Lambda+\rho_2)(H_2(x))}(c_{P_2|P_1}(w:\Lambda)\Phi)(x)$$

where $\rho_2 = \rho_{P_2}$, $H_2 = H_{P_2}$ and

$$c_{P_2|P_1}(w:\Lambda) : L^2_{cus}(\Gamma_{M_1}\backslash M_1,\sigma,\chi) \longrightarrow L^2_{cus}(\Gamma_{M_2}\backslash M_2,\sigma,{}^w\chi)$$

is a linear operator which is holomorphic for $\mathrm{Re}(\Lambda)\in\rho_1+(a^*_1)^+$. This operator is called intertwining operator.

**Lemma 2.3** There exist $C > 0$ and $H_1 \in \mathfrak{a}_1$ such that

$$\|c_{P_2|P_1}(w:\Lambda)\| \le C \frac{e^{(\mathrm{Re}(\Lambda)+\rho_1)(H_1)}}{\prod_{\alpha\in\Psi_P} \langle \mathrm{Re}(\Lambda)+\rho_1,\alpha\rangle}$$

for $\Lambda \in \rho_1 + (\mathfrak{a}_1^*)^+ + \sqrt{-1}\mathfrak{a}_1^*$ , $w \in W(\mathfrak{a}_1,\mathfrak{a}_2)$.

For the proof see Lemma 38 in [H,II].

The Eisenstein series $E(P|A,\Phi,\Lambda)$ and the intertwining operators $c_{P_2|P_1}(w:\Lambda)$ have analytic continuations to meromorphic functions of $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$ whose singularities lie along hyperplanes and they satisfy a system of functional equations.

For a given $\mathbb{Q}$-parapolic subgroup $P = NAM$ of $G$ we set

$$C^\infty((\Gamma\cap P)N\backslash G,\sigma) = (C^\infty((\Gamma\cap P)N\backslash G)\otimes V)^K .$$

Given $\chi \in \hat{Z}(\mathfrak{m})$, denote by $\mathcal{H}_{cus}(P,\sigma,\chi)$ the subspace of $C^\infty((\Gamma\cap P)N\backslash G,\sigma)$ spanned by all functions of the form $\varphi(x) = f(\exp(H_P(x))\Phi(x)$ where $f \in C_c^\infty(A)$ and $\Phi \in L^2_{cus}(\Gamma_M\backslash M,\sigma,\chi)$. For $\varphi \in \mathcal{H}_{cus}(P,\sigma,\chi)$ set

$$E(\varphi|P)(x) = \sum_{\Gamma\cap P\backslash\Gamma} \varphi(\gamma x)$$

The proofs of the following Lemmas can be found in [H,II].

**Lemma 2.4** The series $E(\varphi|P)$ converges absolutely and uniformly on compact subsets of $\Gamma\backslash G$. Moreover, for any $\varphi \in \mathcal{H}_{cus}(P,\sigma,\chi)$, one has $E(\varphi|P) \in L^2(\Gamma\backslash G,\sigma)$.

**Lemma 2.5** Let $\varphi \in H_{cus}(P,\sigma,\chi)$ and assume that $\psi \in C^\infty(\Gamma\backslash G,\sigma)$ is slowly increasing. Then

$$\int_{\Gamma\backslash G} (E(\varphi|P)(x),\psi(x))dx = \int_A e^{-2\rho(\log a)} \int_{\Gamma_M\backslash M} (\varphi(am),\psi^P(am))dmda$$

**Lemma 2.6** Let $P_1$ and $P_2$ be two $\mathbb{Q}$-parabolic subgroups of $G$, $\chi_i \in \hat{Z}(m_{P_i})$ and $\varphi_i \in H_{cus}(P_i,\sigma,\chi_i)$, $i=1,2$. If $P_1$ and $P_2$ are not associate then

$$(E(\varphi_1|P_1),E(\varphi_2|P_2)) = 0.$$

For each $\varphi \in H_{cus}(P,\sigma,\chi)$, define its Fourier transform by

$$\hat{\varphi}(\Lambda:x) = \int_A \varphi(ax)e^{-(\Lambda+\rho_P)(H_P(ax))}da .$$

Then one has

$$(2.7) \qquad E(\varphi|P) = \int_{Re(\Lambda)=\Lambda_0} E(P|A,\hat{\varphi}(\Lambda),\Lambda)\, d\Lambda_I$$

where $\Lambda_0 \in \rho+(a^*)^+$ and $\Lambda_I = Im(\Lambda)$.

**Lemma 2.8** Let $P_1$ and $P_2$ be two associate $\mathbb{Q}$-parabolic subgroups of $G$, $\chi_i \in \hat{Z}(m_{P_i})$ and $\varphi_i \in H_{cus}(P_i,\sigma,\chi_i)$, $i=1,2$. Then

$$(E(\varphi_2|P_2),E(\varphi_1|P_1)) =$$

$$= \sum_{w\in W(a_{P_1},a_{P_2})} \int_{a^*_{P_1}} ((\hat{\varphi}_2(-w\bar{\Lambda}),c_{P_2|P_1}(w:\Lambda)\hat{\varphi}_1(\Lambda))_{\Gamma_{M_2}\backslash M_2}\, d\Lambda_I$$

where $\Lambda = \Lambda_R + i\Lambda_I$ and $\Lambda_R \in \rho_{P_1}+(a^*_{P_1})^+$ .

**3. The rank one spectrum**

The main purpose of this section is to estimate the number of poles of rank one cuspidal Eisenstein series in the half-plane $\mathrm{Re}(s) > 0$. Residues of rank one cuspidal Eisenstein series at poles in $\mathrm{Re}(s) > 0$ form one part of the residual spectrum. We call the subspace of $L^2_{res}(\Gamma\backslash G)$ spanned by all these residues the rank one spectrum.

To begin with we recall some facts from [H,IV]. Let $(\sigma,V)$ be an irreducible unitary representation of K. Fix a class $\mathcal{P}$ of associate rank one $\mathbb{Q}$-parabolic subgroups of G. Let $P\in\mathcal{P}$ with Langlands decomposition $P = NAM$. The Weyl group $W(A)$ of A acts on the characters $\hat{Z}(m)$. For a given orbit $\mathcal{O}\in\hat{Z}(m)/W(A)$, put

$$L^2_{cus}(\Gamma_M\backslash M,\sigma,\mathcal{O}) = \bigoplus_{\chi\in\mathcal{O}} L^2_{cus}(\Gamma_M\backslash M,\sigma,\chi) .$$

Let $P_1,P_2\in\mathcal{P}$ with Langlands decomposition $P_i = N_iA_iM_i$, $i=1,2$. Then the orbit spaces $\hat{Z}(m_1)/W(A_1)$ and $\hat{Z}(m_2)/W(A_2)$ are in canonical one-to-one correspondence. Corresponding orbits are said to be associate.

Let $P\in\mathcal{P}$, $P = NAM$. Since $\mathrm{rank}(\mathcal{P}) = 1$, any element of $\mathcal{P}$ is conjugate either to $P$ or to the opposite group $P^- = N^-AM$. Moreover, $P$ and $P^-$ are conjugate if and only if $-1\in W(A)$. Let $P_1,\dots,P_r$ be a set of representatives for $\mathcal{P}/G_{\mathbb{Q}}$. Thus $r=1$ or 2. Let $\mathcal{P}_i = \{gP_ig^{-1} \mid g\in G_{\mathbb{Q}}\}$ $i=1,2$. Then $\mathcal{P} = \bigcup_{i=1}^{r}\mathcal{P}_i$. Let $P_{il}$, $l=1,\dots,r_i$, be a set of representatives for $\mathcal{P}_i/\Gamma$. Then

$$\{ P_{il} \mid 1\le i\le r,\ 1\le l\le r_i \}$$

is a set of representatives for $\mathcal{P}/\Gamma$. Let $y_{il}\in K$ be such that

$$P_{il} = y_{il} P_i y_{il}^{-1} \ .$$

If special split components are taken, then

$$A_{il} = y_{il} A_i y_{il}^{-1} \ .$$

Let $P_i = N_i A_i M_i$ and $P_{il} = N_{il} A_{il} M_{il}$ the corresponding Langlands decompositions. Let

$$\mathcal{O}_i = \{ \mathcal{O}_{il} \mid 1 \leq l \leq r_i \}$$

be a set of associate orbits with $\mathcal{O}_{il} \in \hat{Z}(m_{il})/W(A_{il})$. Set

$$\mathcal{E}_{cus}(\sigma, \mathcal{O}_i) = \bigoplus_l L^2_{cus}(\Gamma_{M_{il}} \backslash M_{il}, \sigma, \mathcal{O}_{il}) \ .$$

Let $\mathcal{O}_i$ and $\mathcal{O}_j$ be sets of associate orbits, $1 \leq i,j \leq r$. Let $w \in W(a_j, a_i)$ and $\Lambda_i \in a^*_{i,\mathbb{C}}$. Define

$$C_{ji}(w : \Lambda_i)$$

to be the matrix

$$\left( c_{P_{jk}|P_{il}}(y_{jk} w y_{il}^{-1} : {}^{y_{il}}\Lambda_i) \right)$$

$1 \leq l \leq r_i$, $1 \leq k \leq r_j$. $C_{ji}(w:\Lambda_i)$ maps $\mathcal{E}_{cus}(\sigma, \mathcal{O}_i)$ to $\mathcal{E}_{cus}(\sigma, \mathcal{O}_j)$. Now let $\mathcal{O} = \{ \mathcal{O}_{il} \mid 1 \leq i \leq r, \ 1 \leq l \leq r_i \}$ be a set of associate orbits. Set

$$\mathcal{E}_{cus}(\sigma, \mathcal{O}) = \bigoplus_{i=1}^{r} \mathcal{E}_{cus}(\sigma, \mathcal{O}_i) \ .$$

Let $\alpha_i$ be the unique simple root of $(P_i, A_i)$, $i=1,\ldots,r$, and put $\lambda_i = \alpha_i / |\alpha_i|$. Then $\rho_i = |\rho| \lambda_i$ (note that $|\rho_1| = |\rho_2|$ and this is denoted by $|\rho|$).

If r=1, then $W(A_1) = \{\pm 1\}$. Define

$$C(s) = C_{11}(-1:s\lambda_1)\ , \quad s\in\mathbb{C}\ .$$

If r=2, then $W(A_i) = \{1\}$ (i=1,2). Define

$$C(s) = \begin{pmatrix} 0 & C_{12}(w^{-1}:s\lambda_2) \\ C_{21}(w:s\lambda_1) & 0 \end{pmatrix}$$

where $s\in\mathbb{C}$ and w is the unique element of $W(\mathfrak{a}_1,\mathfrak{a}_2)$.

In either case

$$C(s)\ :\ \mathcal{E}_{cus}(\sigma,\mathcal{O}) \longrightarrow \mathcal{E}_{cus}(\sigma,\mathcal{O})$$

is a linear transformation that is a meromorphic function of $s\in\mathbb{C}$. C(s) satisfies the following properties

(3.1) $$C(s)C(-s) = \mathrm{Id},\quad C(s)^* = C(\bar{s}),\ s\in\mathbb{C}\ .$$

The poles of C(s) in the half-plane $\mathrm{Re}(s) \geq 0$ are all simple and contained in the interval $(0, |\rho|]$.

Let $\Phi\in\mathcal{E}_{cus}(\sigma,\mathcal{O})$, $\Phi = \{\Phi_{il}\ |\ 1 \leq i \leq r,\ 1 \leq l \leq r_i\}$. Define

$$E(\Phi,s) = \sum_{i=1}^{r}\ \sum_{l=1}^{r_i} E(P_{il}|A_{il},\Phi_{il},s({}^{y_{il}}\lambda_i)),\quad s\in\mathbb{C}.$$

The functional equation satisfied by Eisenstein series is in this case

$$E(C(s)\Phi,-s) = E(\Phi,s).$$

The poles of $E(\Phi,s)$ coincide with the poles of C(s) (c.f. [H,IV, Theorem 7 ]).

Set

$$t_{il}(x) = {}^{y_{il}}\lambda_{il}(H_{P_{il}}(x)) \ , \ x \in G,$$

$1 \le i \le r$, $1 \le l \le r_i$. It follows from (2.2) that the constant term of $E(\Phi,s)$ along $P_{il}$ is given by

$$E^{P_{il}}(\Phi,s,x) = \tag{3.2}$$
$$= e^{(s+|\rho|)t_{il}(x)}\Phi_{il}(x) + e^{(-s+|\rho|)t_{il}(x)}(C(s)\Phi)_{il}(x)$$

Here $(C(s)\Phi)_{il}$ denotes the component of $C(s)\Phi$ with respect to the orthogonal projection $E_{cus}(\sigma,\mathcal{O}) \longrightarrow L^2_{cus}(\Gamma_{M_{il}}\backslash M_{il},\sigma,\mathcal{O}_{il})$. Let $\Omega_{il} \in Z(m_{il})$ be the Casimir element. Choose $\chi_{il} \in \mathcal{O}_{il}$. Then $-\chi_{il}(\Omega_{il})$ is independent of i,l and the representative of $\mathcal{O}_{il}$. Call the common value $\mu$. It follows from formula (1.2) that

$$-\Omega E(\Phi,s) = (-s^2+|\rho|^2+\mu)E(\Phi,s) \ . \tag{3.3}$$

Our purpose is to estimate the number of poles of C(s) in the half-plane $\mathrm{Re}(s) > 0$. Given $t \in \mathbb{R}$, set

$$C_t(s) = e^{-2st|\rho|}C(s), \ s \in \mathbb{C} \ . \tag{3.4}$$

The poles of C(s) and $C_t(s)$ coincide. To begin with we shall investigate the spectral decomposition of $C_t(s)$ in a neighborhood of $\mathbb{R}^+$. By (3.1) we have $C_t(u)^* = C_t(u)$ for $u \in \mathbb{R}$, i.e., $C_t(u)$ is selfadjoint. Therefore, we can apply Rellich's theorem (c.f.[Ba, p.142 ], [K,II,§6]). Let $s_o \in \mathbb{R}^+$ and assume that $s_o$ is not a pole of C(s). Let the spectral representation of $C_t(s_o)$ be given by

$$C_t(s_o) = \sum_{i=1}^{q} \lambda_i P_i$$

where $P_i$ are the eigenprojections of $C_t(s_o)$. There exists a punctured disc $0 < |s-s_o| < \delta$ which consists only of simple points of $C_t(s)$ and the spectral representation of $C_t(s)$ takes the form

$$C_t(s) = \sum_{i=1}^{q} \sum_{j=1}^{i} \lambda_{ij}(s) P_{ij}(s), \quad 0 < |s-s_o| < \delta .$$

The eigenvalues $\lambda_{ij}(s)$ and the eigenprojections $P_{ij}(s)$ are holomorphic in $|s-s_o| < \delta$ . In particular, each eigenvalue $\lambda_{ij}(s)$ has an expansion of the form

$$\lambda_{ij}(s) = \lambda_i + \sum_{k=1}^{\infty} \lambda_{ij}^{(k)} (s-s_o)^k, \quad |s-s_o| < \delta .$$

Now assume that $s_o \in (0, |\rho|]$ is a pole of $C_t(s)$. Then $s_o$ is a simple pole of $C_t(s)$. Let

$$B = \operatorname*{Res}_{s=s_o} C_t(s).$$

**Lemma 3.5** B is positive semidefinite i.e., $(B\Phi, \Phi) \geq 0$ for all $\Phi \in E_{cus}(\sigma, \mathcal{O})$.

**Proof.** We have $B = e^{-2s_o t|\rho|} \mathrm{Res}_{s=s_o} C(s)$. It is well-known that in the $\mathbb{Q}$-rank one case $\mathrm{Res}_{s=s_o} C(s)$ is positive semidefinite (c.f. [A1], [W,§2] ). The proof extends without difficulties to our case. Q.E.D.

Put

$$B(s) = (s-s_o)C_t(s).$$

Then B(s) is holomorphic at $s=s_o$ and $B(s_o)=B$. For $u \in \mathbb{R}$, B(u) is again selfadjoint and we can apply Rellich's theorem to B(s) in the same manner as above. It follows that there is a punctured disc $0<|s-s_o|<\delta$ such that each eigenvalue $\lambda(s)$ of $C_t(s)$ has an expansion of the form

$$\lambda(s) = \frac{\mu}{s-s_o} + \sum_{j=o}^{\infty} a_j(s-s_o)^j , \quad 0<|s-s_o|<\delta ,$$

with $\mu$ an eigenvalue of B. In view of Lemma 3.5, the eigenvalues of B are non-negative. Summarizing we have proved

**Proposition 3.6** Let $u_1,\dots,u_m \in (0,|\rho|]$ be the poles of C(s) in the half-plane $\mathrm{Re}(s) \geq 0$ and let $d = \dim E_{cus}(\sigma, \mathcal{O})$. There exist real valued real analytic functions $\lambda_1(u),\dots,\lambda_d(u)$ on $\mathbb{R}^+ -$ $- \{u_1,\dots,u_m\}$ with the following properties

1) For each $u \in \mathbb{R}^+ - \{u_1,\dots,u_m\}$, $\lambda_1(u),\dots,\lambda_d(u)$ are the eigenvalues of $C_t(u)$.
2) There exists $\delta > 0$ such that, in the punctured neighborhood $0<|u-u_i|<\delta$, $\lambda_j(u)$ has an expansion of the form

$$\lambda_j(u) = \frac{\mu_{ji}}{u-u_i} + \sum_{k=0}^{\infty} a_{jk}(u-u_i)^k, \tag{3.7}$$

with $\mu_{ji} \geq 0$, $j=1,\dots,d$, $i=1,\dots,m$.

Assume that $u_1 < u_2 < \cdots < u_m$ are the poles of C(s) in the half-plane $\mathrm{Re}(s) > 0$ and set

$$B_i = \operatorname*{Res}_{s=u_i} C_t(s) \ , \ i=1,\dots,m.$$

Consider the coefficients $\mu_{ji}$, $i=1,\dots,m$, $j=1,\dots,d$, in the expansion (3.7) and set

$$n_j = \#\{\mu_{ji} \mid \mu_{ji} \neq 0, \ i=1,\dots,m\} \ ,$$

$j=1,\dots,d$. Now observe that for each $i$ $(1 \le i \le m)$, $\mu_{1i},\dots,\mu_{di}$ are the eigenvalues of $B_i$. Since by Lemma 3.5, each $B_i$ is positive semidefinite, it follows that

$$\operatorname{rank}(B_i) = \#\{\mu_{ji} \mid \mu_{ji} \neq 0, \ j=1,\dots,d\} \ .$$

Thus

$$m \le \sum_{i=1}^{m} \operatorname{rank}(B_i) = \#\{\mu_{ji} \mid \mu_{ji} \neq 0, \ i=1,\dots,m, \ j=1,\dots,d\} =$$

$$= \sum_{j=1}^{d} n_j \le d \max_j n_j \tag{3.8}$$

Assume that $n_k = \max_j n_j$ for some $k$ $(1 \le k \le d)$. If $n_k \le 1$ then $m \le d = \dim E_{cus}(\sigma, 0)$ which can be estimated using [D1]. Now suppose that $n_k > 1$ and $\{\mu_{ki} \mid \mu_{ki} \neq 0, \ i=1,\dots,m\} = \{\mu_{ki_1},\dots,\mu_{ki_p}\}$ with $i_1 < i_2 < \cdots < i_p$, $p > 1$. By Proposition 3.6, $\lambda_k(u)$ is real analytic in each interval $(u_{i_l}, u_{i_{l+1}})$, $1 \le l \le p$, and

$$\lim_{u \to u_{i_l} \pm 0} \lambda_k(u) = \pm\infty \ .$$

Let $\omega \in \mathbb{R}$ be given. Using the observations above it follows that each interval $(u_{i_l}, u_{i_{l+1}})$, $1 \le l \le p$, contains at least one point

$t_1$ such that $\lambda_k(t_1) = \omega$ . Let N(t) be the number of points $v \in (0, |\rho|]$ where $C_t(v)$ has at least one eigenvalue equal to -1. Then it follows from (3.8) that

$$m \leq d\, N(t) \quad (3.9)$$

Thus our problem is reduced to the estimation of N(t).

At this stage we need the truncation operator (c.f.[A2],[O-W]). We recall its definition. Let P = NAM be a $\mathbb{Q}$-parabolic subgroup of G. Let $\hat{\Psi}_P$ denote the dual basis of the simple roots $\Psi_P$ of (P,A). Thus

$$\langle \omega_\alpha, \beta \rangle = \delta_{\alpha\beta} \;,\; \alpha, \beta \in \Psi_P \;,\; \omega_\alpha \in \hat{\Psi}_P \;.$$

Set

$$^{+}\mathfrak{a} = \{H \in \mathfrak{a} \mid \omega_\alpha(H) > 0,\; \alpha \in \Psi_P\} \;.$$

Denote by $\chi_P$ the characteristic function of $^{+}\mathfrak{a} \subset \mathfrak{a}$ . Let V be a finite dimensional Hilbert space and $\varphi : \Gamma\backslash G \longrightarrow V$ a locally bounded measurable function. Given $H \in {}^{+}\mathfrak{a}$, set

$$\Lambda_P^H \varphi(x) = \sum_{\Gamma \cap P \backslash \Gamma} \chi_P(H_P(\gamma x) - H)\, \varphi^P(\gamma x).$$

Let $P_1, \ldots, P_\iota$ be a set of representatives for the $\Gamma$-conjugacy classes of rank one $\mathbb{Q}$-parabolic subgroups of G. Assume that the $P_{il}$, $i=1,\ldots,r$, $l=1,\ldots,r_i$, are among the $P_1,\ldots,P_\iota$ . Let $A_j$ be the special split component of $P_j$. Put

$$\mathfrak{a}_o = \bigoplus_j \mathfrak{a}_j$$

For each $\mathbb{Q}$-parabolic subgroup P of G with special split component $A_P$ there is a linear map

$$I_P : \mathfrak{a}_o \longrightarrow \mathfrak{a}_P$$

which is defined as follows: Let $P_\alpha$, $\alpha \in \Psi_P$, be the standard rank one $\mathbb{Q}$-parabolic subgroup of G associated to $F = \Psi_P - \{\alpha\}$. If $A_\alpha$ is the special split component of $P_\alpha$ then

$$\mathfrak{a}_\alpha = \bigcap_{\beta \neq \alpha} \ker(\beta).$$

For each $\alpha \in \Psi_P$, there exists $\gamma_\alpha \in \Gamma$ and $j(\alpha)$ $(1 \leq j(\alpha) \leq \iota)$ such that

$$\gamma_\alpha P_\alpha \gamma_\alpha^{-1} = P_{j(\alpha)} .$$

Denote by $H_\alpha \in \mathfrak{a}$, $\alpha \in \Psi_P$, the element defined by $\langle H_\alpha, H \rangle = \alpha(H)$, $H \in \mathfrak{a}$. Then, for $T \in \mathfrak{a}_o$,

$$I_P(T) = \sum_{\alpha \in \Psi_P} \omega_\alpha (Ad(\gamma_\alpha^{-1}) T_{j(\alpha)} + H_{P_\alpha}(\gamma_\alpha)) H_\alpha .$$

Given $T \in \mathfrak{a}_o$, set

$$\Lambda_P^T = \Lambda_P^{I_P(T)} .$$

Let $Q_1, \ldots, Q_l$ be a set of representatives for the $\Gamma$-conjugacy classes of $\mathbb{Q}$-parabolic subgroups of G. Set

$$(3.10) \qquad \Lambda^T \varphi = \sum_{i=1}^{l} (-1)^{rk(Q_i)} \Lambda_{Q_i}^T \varphi .$$

To explain some of the properties of the truncation operator $\Lambda^T$, we have to introduce a partial ordering on $\mathfrak{a}_o$. Let $Q_1,\ldots,Q_q$ be a set of representatives for the $\Gamma$-conjugacy classes of minimal $\mathbb{Q}$-parabolic subgroups of G. For each i ($1 \leq i \leq q$) there exists $g_i \in G_{\mathbb{Q}}$ such that $Q_i = g_i Q_1 g_i^{-1}$. Given $T_1, T_2 \in \mathfrak{a}_o$, write

$$T_1 \ll T_2$$

if there exists $H_o \in \mathfrak{a}_{Q_1}^+$ such that

$$\mathrm{Ad}(g_i)^{-1}(I_{Q_i}(T_2) - I_{Q_i}(T_1)) = H_o \ .$$

Now we can state the basic properties of the truncation operator.

1) There exists $T_o \in \mathfrak{a}_o$ such that, for $T \gg T_o$, $\Lambda^T \circ \Lambda^T = \Lambda^T$.

2) For $T \gg T_o$ and any $\mathbb{Q}$-parabolic subgroup P = NAM of G

$$(\Lambda^T \varphi)^P = 0, \text{ if } H_P(x) - I_P(T) \in {}^+\mathfrak{a}_P$$

and $\varphi$ is as above.

3) $\Lambda^T$ transforms sufficiently smooth slowly increasing functions into rapidly decreasing functions.

4) If $T \gg T_o$ then $\Lambda^T$ extends to an orthogonal projection on $L^2(\Gamma\backslash G) \otimes V$.

(see [A2], [O-W] for the proof of these facts).

Let $T \gg T_o$. If $\varphi \in L^2(\Gamma\backslash G,\sigma)$ then $\Lambda^T\varphi \in L^2(\Gamma\backslash G,\sigma)$ and $\Lambda^T$ induces an orthogonal projection on $L^2(\Gamma\backslash G,\sigma)$.

Next we introduce certain auxiliary operators $\Delta_T$, $T \in \mathfrak{a}_o$, acting in a Hilbert space $\mathcal{H}_T$. Let $\tilde{E}$ be the homogeneous vector bundle over X associated to $\sigma$: $K \longrightarrow GL(V)$ and let $E = \Gamma\backslash\tilde{E}$. Denote by $\nabla$ the connection on E which is obtained by pushing down the canonical invariant connection on $\tilde{E}$. Given $T \in \mathfrak{a}_o$, introduce the following subspace of the Sobolev space $H^1(\Gamma\backslash X,E)$:

$$H^1_T(\Gamma\backslash X,E) =$$

$$= \{ \varphi \in H^1(\Gamma\backslash X,E) \mid \varphi^{P_i}(a_i m_i) = 0 \text{ for } \log(a_i) > T_i,\ i=1,\dots,\iota\}$$

Let $\mathcal{H}_T$ be the closure of $H^1_T(\Gamma\backslash X,E)$ in $L^2(\Gamma\backslash X,E)$. Consider the quadratic form

$$q(\varphi) = \|\nabla\varphi\|^2 \ ,\ \varphi \in H^1_T(\Gamma\backslash X,E).$$

Since $H^1_T(\Gamma\backslash X,E)$ is a closed subspace of $H^1(\Gamma\backslash X,E)$, $q$ has an associated selfadjoint operator $\tilde{\Delta}_T$ acting in $\mathcal{H}_T$. Let

$$\Delta : C^\infty(\Gamma\backslash X,E) \longrightarrow C^\infty(\Gamma\backslash X,E)$$

be the differential operator which is induced by $-\Omega_G$ where $\Omega_G \in Z(\mathfrak{g})$ is the Casimir element. Since $(\sigma,V)$ is irreducible, there exists $\lambda_\sigma \in \mathbb{R}$ such that

$$\Delta = -\nabla^*\nabla + \lambda_\sigma \mathrm{Id} \tag{3.11}$$

(c.f. Proposition 1.1 in [M]). Set

$$\Delta_T = \tilde{\Delta}_T + \lambda_\sigma \mathrm{Id} \ . \tag{3.12}$$

Now we can continue with the estimation of $N(t)$. Let $T_\rho \in \mathfrak{a}_o$ be the element whose i-th component equals $H_{\rho_i}$ - the image of $\rho_i$ under the canonical identification of $\mathfrak{a}_i^*$ and $\mathfrak{a}_i$. Let $T_o \in \mathfrak{a}_o$ be as in 2). Then there exists $t_o$ such that

$$tT_\rho \gg T_o, \text{ if } t \geq t_o.$$

Given $P \in \mathcal{P}$, $P = NAM$, set $\lambda_P = |\alpha_P|^{-1}\alpha_P$ where $\alpha_P$ is the simple root of $(P,A)$. Observe that $|\rho_P|$ is independent of $P \in \mathcal{P}$ (c.f. [H, Lemma 81]). Call its common value $|\rho|$. Choose $t \geq t_o$. Let $\Phi \in \mathcal{E}_{cus}(\sigma, \mathcal{O})$ and $s_o \in \mathbb{C}$ such that

$$C(s_o)\Phi = -e^{2s_o|\rho|t}\,\Phi .$$

Then it follows from (3.2) that

$$(3.13) \qquad E^{P_{ij}}(\Phi, s_o, e^{tH_{ij}}m) = 0 \;, \; i=1,\dots,r, \; j=1,\dots,r_i \;,$$

where $H_{ij} = H_{\rho_{ij}}$ and $m \in M_{ij}$.

**Lemma 3.14** Let $T = tT_\rho$ . Put $\varphi = \Lambda^T E(\Phi, s_o)$. Then $\varphi \neq 0$. Moreover $\varphi$ belongs to the domain of $\Delta_T$ and satisfies

$$\Delta_T\varphi = (-s_o^2 + |\rho|^2 + \mu)\varphi$$

where $\mu$ is defined by (3.3).

**Poof.** Let $Q$ be a $\mathbb{Q}$-parabolic subgroup of G. Then $E^Q(\Phi, s_o) = 0$ unless $Q \in \mathcal{P}$ (c.f. §2). Hence

$$\Lambda^T E(\Phi, s_o) = E(\Phi, s_o) - \sum_{i=1}^{r} \sum_{j=1}^{r_i} \Lambda^T_{P_{ij}} E(\Phi, s_o).$$

Furthermore, it follows from (3.13) that

$$\chi_{P_{ij}}(\log(a_{ij}) - tH_{ij})E^{P_{ij}}(\Phi, s_o, a_{ij}m_{ij})$$

is smooth on $(A_{ij} - \{e^{tH_{ij}}\}) \times \Gamma_{M_{ij}}\backslash M_{ij}$ and continuous on $A_{ij} \times \Gamma_{M_{ij}}\backslash M_{ij}$. Hence $\Lambda^T_{P_{ij}} E(\Phi,s_o)$ belongs to $H^1_{loc}(\Gamma\backslash X,E)$ and therefore, $\Lambda^T E(\Phi,s_o)$ is in $H^1_{loc}(\Gamma\backslash X,E)$ too. But property 3) of the truncation operator implies that $\Lambda^T E(\Phi,s_o)$ is square integrable. Hence $\Lambda^T E(\Phi,s_o)\in H^1(\Gamma\backslash X,E)$. Furthermore, by property 2) satisfied by $\Lambda^T$,

$$(\Lambda^T E(\Phi,s_o))^{P_i}(x) = 0, \text{ if } H_{P_i}(x) - tT_{\rho_i}\in \mathfrak{a}_i^+,$$

$i=1,\dots,\iota$ . This shows that $\Lambda^T E(\Phi,s_o)\in H^1_T(\Gamma\backslash X,E)$. Next we have to show that $\Lambda^T E(\Phi,s_o)$ is in the domain of $\Delta_T$. The domain of $\Delta_T$ can be characterized as follows: Let $H^{-1}(\Gamma\backslash X,E)$ denote the space of all distributions in $\mathcal{D}'(\Gamma\backslash X,E)$ that extend to a continuous linear functional on $H^1(\Gamma\backslash X,E)$. The domain of $\Delta_T$ consists of all $\psi\in H^1_T(\Gamma\backslash X,E)$ such there exists a distribution $D\in H^{-1}(\Gamma\backslash X,E)$ which is orthogonal to $H^1_T(\Gamma\backslash X,E)$ and satisfies $\Delta\psi - D\in \mathcal{H}_T$. D is uniquely determined and $\Delta_T\psi = \Delta\psi - D$ . Choose $H\in \mathfrak{a}_i$ such that $\|H\| = 1$ and $\lambda_i(H) > 0$. Set $a_u = \exp(uH)$, $u\in\mathbb{R}$. Given $\psi\in C^\infty(\Gamma\backslash X,E)$, put

$$\psi^{P_i}(u,m) = \psi^{P_i}(a_u m),$$

$u\in\mathbb{R}$, $m\in M_i$. Then

$$(H^2 - 2\rho_i(H)H)\psi^{P_i}(a_u m) = (\frac{d^2}{du^2} - 2\rho_i(H)\frac{d}{du})\psi^{P_i}(u,m).$$

Let $\varphi\in C_c^\infty(\Gamma\backslash X,E)$. Employing (1.2), we obtain

$$(\Lambda^T_{P_i} E(\Phi,s_o),\Delta\varphi) =$$

$$= \int_{\Gamma\backslash G} \sum_{\Gamma\cap P_i\backslash\Gamma} \chi_{P_i}(H_{P_i}(\gamma x) - tH_{\rho_i})(E^{P_i}(\Phi,s_o,\gamma x),-\Omega\varphi(x))dx =$$

$$= \int_{\Gamma\cap P_i\backslash G} \chi_{P_i}(H_{P_i}(x) - tH_{\rho_i})(E^{P_i}(\Phi,s_o,x),-\Omega\varphi(x))\, dx =$$

$$= -\int_{t|\rho|}^{\infty} \int_{\Gamma_{M_i}\backslash M_i} (E^{P_i}(\Phi,s_o,(u,m)),(\frac{d^2}{du^2} - |\rho|\frac{d}{du} + \Omega_{M_i})\varphi^{P_i}(u,m))\cdot$$

$$\cdot e^{-(|\rho|-1)u} du dm$$

$$= (\Lambda^T_{P_i} \Delta E(\Phi,s_o),\varphi) +$$

$$+ e^{-t|\rho|^2} \int_{\Gamma_{M_i}\backslash M_i} (\frac{d}{du}E^{P_i}(\Phi,s_o,(u,m)),\varphi^{P_i}(a_u m))\Big|_{u=t|\rho|} dm .$$

Define the distribution $D\in\mathcal{D}'(\Gamma\backslash X,E)$ by

$$D(\varphi) = e^{-t|\rho|^2} \sum_{i=1}^{\iota} \int_{\Gamma_{M_i}\backslash M_i} (\frac{d}{du}E^{P_i}(\Phi,s_o,(u,m)),\varphi^{P_i}(a_u m))\Big|_{u=t|\rho|} dm .$$

Then $D\in H^{-1}(\Gamma\backslash X,E)$ and $D$ vanishes on $H^1_T(\Gamma\backslash X,E)$. Moreover

$$\Delta\Lambda^T E(\Phi,s_o) - D = \Lambda^T \Delta E(\Phi,s_o).$$

Employing again property 3) of $\Lambda^T$, it follows that $\Lambda^T\Delta E(\Phi,s_o)$ is square integrable and hence, in $\mathcal{H}_T$. Thus $\Lambda^T E(\Phi,s_o)$ belongs to the domain of $\Delta_T$ and

$$\Delta_T\Lambda^T E(\Phi,s_o) = \Lambda^T\Delta E(\Phi,s_o).$$

Employing (3.3), we obtain

$$\Delta_T \Lambda^T E(\Phi,s_o) = (-s_o^2 + |\rho|^2+\mu)\,\Lambda^T E(\Phi,s_o)$$

Finally, we observe that a direct computation shows that $D \neq 0$. This implies that $\Lambda^T E(\Phi,s_o) \neq 0$. Q.E.D.

**Corollary 3.15** Let $N_T(\lambda)$ be the number of linearly independent eigenfunctions of $\Delta_T$ with eigenvalue less than $\lambda$. Then

$$N(t) \leq N_T(|\rho|^2+\mu).$$

**Proof.** Let $s_o \in (0,|\rho|]$ and assume that $C_t(s_o)\Phi = -\Phi$ for some $\Phi \in \mathcal{E}_{cus}(\sigma,\mathcal{O})$, $\Phi \neq 0$. Then condition (3.13) holds for $E(\Phi,s_o)$ and the Corollary follows from Lemma 3.14. Q.E.D.

It remains to estimate $N_T(\lambda)$. For this purpose we shall use a covering of $\Gamma\backslash X$ by special neighborhoods constructed in [B-S]. We start with the description of these neighborhoods. For details we refer to [B-S], [Z]. Let $P$ be a $\mathbb{Q}$-parabolic subgroup of $G$ with special split component $A$ and corresponding Langlands decomposition $P = NAM$. There is a canonical isomorphism

$$\mu_P : N \times X_M \times A \xrightarrow{\sim} X ,$$

where $X_M = M/K_M$. The map $\mu_P$ commutes with $P$ where the action of $P$ on $N \times X_M \times A$ is defined by

$$p\cdot(n_1,z,a_1) = (nman_1a^{-1}m^{-1},mz,aa_1)$$

where $p = nma$, $a\in A$, $m\in M$, $n\in N$. Set

$$e(P) = N \times X_M \quad\text{and}\quad e'(P) = \Gamma\cap P\backslash e(P).$$

There is a canonical fibration

$$\pi_P : e'(P) \longrightarrow \Gamma_M \backslash X_M$$

with fibre $\Gamma \cap N \backslash N$. Given $\tau \in \mathfrak{a}$, put

$$A_\tau = \{a \in A \mid \alpha(\log a) > \alpha(\tau),\ \alpha \in \Psi_P\} .$$

If $\tilde{Y}$ is an open subset of $e(P)$ and $\tau \in \mathfrak{a}$, put

$$\tilde{W}(\tilde{Y},\tau) = \mu_P(\tilde{Y} \times A_\tau).$$

Now assume that $Y \subset e'(P)$ is an open subset and $\tilde{Y}$ its inverse image under the canonical projection $e(P) \longrightarrow e'(P)$. Then $\tilde{W}(\tilde{Y},\tau)$ is $\Gamma \cap P$-invariant and we put

$$W(Y,\tau) = \Gamma \cap P \backslash \tilde{W}(\tilde{Y},\tau) .$$

**Lemma 3.16** Let $Y$ be a relatively compact open subset of $e'(P)$. Then if $\tau \in \mathfrak{a}$ is sufficiently large, the equivalence relations defined on $\tilde{W}(\tilde{Y},\tau)$ by $\Gamma$ and $\Gamma \cap P$ are the same. For such $\tau$, we have $W(Y,\tau) = \pi(\tilde{W}(\tilde{Y},\tau))$ where $\pi : X \longrightarrow \Gamma\backslash X$ is the canonical projection and $\mu_P$ induces an isomorphism

$$\mu_P' : Y \times A_\tau \xrightarrow{\sim} W(Y,\tau).$$

The proof of this Lemma follows from a modification of (10.3) in [B-S].

An open set in $\Gamma\backslash X$ of the form $W(Y_P,\tau_P)$ with $Y_P \subset e'(P)$ a relatively compact open subset and $\tau_P \in \mathfrak{a}_P$ is called a special neighborhood. Note that for $P=G$, $W(Y_P, \tau_P)=Y_P$ is a relatively compact open subset of $\Gamma\backslash X$. Let $\omega_M \subset \Gamma_M\backslash X_M$ be a relatively compact open subset and put $Y = \pi_P^{-1}(\omega_M)$. Then $Y = \Gamma\cap P\backslash(N\times\Gamma_M\omega_M)$ and $\tilde{W}(\tilde{Y},\tau)$ is N-invariant. Therefore, the cuspidal condition makes sense on $W(Y,\tau)$. Indeed, let

$$U = N\times A_\tau\times(\Gamma_M\omega_M K_M).$$

Then U is invariant under left multiplication by $\Gamma\cap P$ and right multiplication by $K_M$ and $W(Y,\tau) = \Gamma\cap P\backslash U/K_M$. Thus, any section of E over $W(Y,\tau)$ can be identified with a map $\varphi: U \longrightarrow V$ satisfying $\varphi(\gamma xk) = \sigma(k)^{-1}\varphi(x)$, $\gamma\in\Gamma\cap P$, $k\in K_M$, Given $F\subset\Psi_P$ and $\varphi\in L^2(W(Y,\tau),E)$,

$$\varphi^{P_F}(x) = \int_{\Gamma\cap N_F\backslash N_F}\varphi(nx)dn$$

is well defined and belongs to $L^2(W(Y,\tau),E)$. Let $W=W(Y,\tau)$ and set

$$L^2_{cus}(W,E) = \{\ \varphi\in L^2(W,E)\ |\ \varphi^{P_F} = 0 \text{ for all } F\subset\Psi_P\ \}.$$

Let $\Delta_W$ be the selfadjoint operator in $L^2(W,E)$ which is associated to the quadratic form $\varphi\longrightarrow\|\nabla\varphi\|^2$ acting in the Sobolev space $H^1(W,E)$. In other words, $\Delta_W$ is the selfadjoint extension of $\nabla^*\nabla$ acting on $C_c^\infty(W,E)$ which is obtained by imposing Neumann boundary conditions on $\partial W$. It is clear that $L^2_{cus}(W,E)$ is an invariant subspace for $\Delta_W$. Furthermore, we have

**Proposition 3.17** 1) $\Delta_W$ has pure point spectrum in $L^2_{cus}(W,E)$ and a compact resolvent when restricted to this subspace.
2) Let $N_W(\lambda)$ denote the number of linearly independent cuspidal eigenfunctions of $\Delta_W$ with eigenvalue less than $\lambda$. There exists a constant $C > 0$ such that

$$N_W(\lambda) \leq C(1 + \lambda^{n/2}) \ , \ \lambda \geq 0,$$

where $n = \dim X$.

**Proof.** In the case when $P$ is a minimal $\mathbb{Q}$-parabolic subgroup of $G$ and $\omega_M = \Gamma_M \backslash X_M$, this is Corollary 7.6 in [D1]. A straight forward extension of his method gives the proof in general.Q.E.D.

Next we shall construct a covering of $\Gamma\backslash X$ by special neighborhoods and apply modified Neumann bracketing to reduce the estimation of $N_T(\lambda)$ to Proposition 3.17. As above, let $Q_i = N_i A_i M_i$ $(1 \leq i \leq l)$ be a set of representatives for the $\Gamma$-conjugacy classes of $\mathbb{Q}$-parabolic subgroups of $G$. Furthermore, if $W = W(Y_P, \tau_P)$ is a special neighborhood with respect to some $\mathbb{Q}$-parabilic subgroup $P$ of $G$, we set

$$H^1_{cus}(W,E) = L^2_{cus}(W,E) \cap H^1(W,E)$$

where $H^1(W,E)$ denotes the Sobolev space.

**Proposition 3.18** Let $T = tT_\rho \in \mathfrak{a}_0$ with $t \geq t_0$ as above. There exist $\tau_i \in \mathfrak{a}_i$ and relatively compact open subsets $\omega_{M_i} \subset \Gamma_{M_i} \backslash X_{M_i}$, $1 \leq i \leq l$, such that the following conditions are satisfied:
1) Let $Y_i = \pi_{Q_i}^{-1}(\omega_{M_i})$. Then the canonical map $W(Y_i, \tau_i) \longrightarrow \Gamma\backslash X$ is injective and $\{W(Y_i, \tau_i) \mid i = 1, \ldots, l\}$ is a covering of $\Gamma\backslash X$.

2) Set $W_i=W(Y_i,\tau_i)$, $i=1,\dots,l$. The map $\varphi \longmapsto (\varphi|W_1,\dots,\varphi|W_l)$ defines an embedding

$$H^1_T(\Gamma\backslash X,E) \subset \bigoplus_{i=1}^{l} H^1_{cus}(W_i,E)$$

**Proof.** Let $P=NAM$ be any $\mathbb{Q}$-parabolic subgroup of G with special split component A. Given $\alpha\in\Psi_P$, let $P_\alpha = N_\alpha A_\alpha M_\alpha$ denote the rank one $\mathbb{Q}$-parabolic subgroup of G associated to $\Psi_P-\{\alpha\}$. There exists $\gamma\in\Gamma$ and i ($1 \le i \le l$) such that

$$P_\alpha = {}^\gamma P_i. \tag{3.19}$$

Let $\varphi\in H^1_T(\Gamma\backslash X,E)$. Then $\varphi^{P_\alpha}(x) = \varphi^{P_i}(\gamma^{-1}x)$. Now observe that

$$H_{P_i}(\gamma^{-1}x) = Ad(\gamma^{-1})H_{{}^\gamma P_i}(x) + H_{P_i}(\gamma^{-1}).$$

By assumption $\varphi^{P_i}(\gamma^{-1}x) = 0$ if $H_{P_i}(\gamma^{-1}x) > tH_{\rho_i}$. Hence

$$\varphi^{P_\alpha}(x) = 0 \text{ if } H_{P_\alpha}(x) > tH_{\rho_\alpha} + H_{P_\alpha}(\gamma) \tag{3.20}$$

Now let $F\subset \Psi_P$ be any subset with $\alpha\in F$ and denote by $P_F\subset P$ the $\mathbb{Q}$-parabolic subgroup of G associated to $\Psi_P-F$. Let $P_F=N_FA_FM_F$ be the Langlands decomposition. Then

$$\varphi^{P_F}(x) = \int_{(N_F\cap\Gamma)N_\alpha\backslash N_F} \varphi^{P_\alpha}(\bar{n}x)\, d\bar{n}\ .$$

Now observe that $\mathfrak{n}_F = \mathfrak{n}_\alpha \oplus \mathfrak{n}_\alpha^\perp$ and $\mathfrak{n}_\alpha^\perp \subset \mathfrak{m}_\alpha$. Therefore any $n\in N_F$ can be written as $n=n_1n_2$ with $n_1\in N_\alpha$ and $n_2\in M_\alpha$. This implies

$$H_{P_\alpha}(nx) = H_{P_\alpha}(x) \quad \text{for } n\in N_F.$$

Thus, by (3.20),

$$(3.21) \qquad \varphi^{P}F(x) = 0 \text{ if } H_{P_\alpha}(x) > tH_{\rho_\alpha} + H_{P_\alpha}(\gamma) .$$

For each $\alpha\in\Psi_P$, let $\gamma_\alpha$ be determined by (3.19). Furthermore, observe that

$$a_P = \bigoplus_{\alpha\in\Psi_P} a_\alpha \ , \ a_\alpha = \bigcap_{\beta\neq\alpha} \ker(\beta).$$

Let $\tau_P\in a_P$ be the element whose component in $a_\alpha$ is $tH_{\rho_\alpha} + H_{P_\alpha}(\gamma_\alpha)$. Then, for any $F\subset\Psi_P$, it follows from (3.17) that

$$(3.22) \qquad \varphi^{P}F(x) = 0 \text{ if } \exp(H_P(x))\in A_{\tau_P} .$$

Set $R = \{Q_1,\dots,Q_1\}$. For each $Q\in R$ we shall denote by $\tau_Q^o\in a_Q$ the element constructed above. Now we construct a covering $\{W(Y_Q,\tau_Q)\,|\,Q\in R\}$ of $\Gamma\backslash X$ recursively as in [Z,(3.6)]. Moreover, we can assume that for each $Q\in R$, $\tau_Q$ is such that

$$\alpha(\tau_Q) > \alpha(\tau_Q^o) \text{ for all } \alpha\in\Psi_Q.$$

Employing (3.22), it follows that this covering satisfies 1) and 2). Q.E.D.

Now we are ready to prove our main result

**Theorem 3.23.** Let $T = tT_\rho$ with $t \geq t_o$ as above.

1) $\Delta_T$ has a compact resolvent.

2) Let $n=\dim X$. There exists a constant $C > 0$ such that

$$N_T(\lambda) \leq C(1 + \lambda^{n/2}), \ \lambda \geq 0 .$$

**Proof.** Let $\{W_i \mid i=1,\dots,l\}$ be a covering of $\Gamma\backslash X$ by special neighborhoods satisfying 2) of Proposition 3.18. By Proposition 3.17, each embedding

$$H^1_{cus}(W_i,E) \longrightarrow L^2(W_i,E)$$

$(1 \le i \le l)$ is compact. Employing Proposition 3.18, 2), it follows that the embedding

$$H^1_T(\Gamma\backslash X,E) \longrightarrow L^2(\Gamma\backslash X,E)$$

is compact. Therefore the resolvent of $\tilde{\Delta}_T$ is compact which proves 1).

By (3.12), it is sufficient to estimate the number of eigenvalues of $\tilde{\Delta}_T$. Let $\lambda_j$ denote the j-th eigenvalue of $\tilde{\Delta}_T$. We apply the mini-max principle in the form

$$\lambda_i = \min_{V} \max_{\varphi \in V} \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2}$$

where V runs over all subspaces of $H^1_T(\Gamma\backslash X,E)$ of dimension j (c.f. [F-S]). Now observe that there is a constant $C > 0$ such that

$$(3.24) \qquad \frac{\sum_{i=1}^{l} \|\nabla\varphi|W_i\|^2}{\sum_{i=1}^{l} \|\varphi|W_i\|^2} \le C \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2}$$

for all $\varphi \in H^1_T(\Gamma\backslash X,E)$. Let $\tilde{\lambda}_j$ be the j-th eigenvalue of the operator $\Delta_{W_1} \oplus \cdots \oplus \Delta_{W_l}$. Then

$$\tilde{\lambda}_j = \min_{\tilde{V}} \max_{(\varphi_i)\in\tilde{V}} \frac{\sum_{i=1}^{l} \|\nabla\varphi_i\|^2}{\sum_{i=1}^{l} \|\varphi_i\|^2}$$

where $\tilde{V}$ runs now over all subspaces of $\bigoplus_{i=1}^{l} H^1_{cus}(W_i,E)$ of dimension j. Put $\tilde{V}= J(V)$ where

$$J : H^1_T(\Gamma\backslash X,E) \longrightarrow \bigoplus_{i=1}^{l} H^1_{cus}(W_i,E)$$

is the map $\varphi \longmapsto (\varphi|W_1,\ldots,\varphi|W_l)$. Then $\tilde{V}$ is an j-dimensional subspace of $\bigoplus_{i=1}^{l} H^1_{cus}(W_i,E)$ and employing (3.24), we get

$$\tilde{\lambda}_j \le C\lambda_j \ .$$

Combined with Proposition 3.17, this implies

$$N_T(\lambda) \le \sum_{i=1}^{l} N_{W_i}(\lambda-\lambda_\sigma) \le C_1(1 + \lambda^{n/2}) \ .$$

Q.E.D.

**Corollary 3.25** Let $m(\sigma,\mathcal{O})$ be the number of poles of the intertwining operator $C(s) : \mathcal{E}_{cus}(\sigma,\mathcal{O}) \longrightarrow \mathcal{E}_{cus}(\sigma,\mathcal{O})$ in the half-plane $Re(s) > 0$. There exists a constant $C > 0$ independent of $\mathcal{O}$ such that

$$m(\sigma,\mathcal{O}) \le C(1 + \mu^n)$$

where $n=\dim X$.

**Proof.** Using (3.9), Lemma 3.15 and Theorem 3.23, it follows that

$$m(\sigma,\mathcal{O}) \le C\dim(\mathcal{E}_{cus}(\sigma,\mathcal{O}))(1+\mu^{n/2}).$$

Employing Theorem 9.1 of [D1] , we can estimate $\dim\mathcal{E}_{cus}(\sigma,\mathcal{O})$ by $C(1+\mu^{n/2})$. This implies our result. Q.E.D.

**Corollary 3.26** The number of poles, counted to multiplicity, of $\det C(s)$ in $\mathrm{Re}(s)>0$ is bounded by

$$C_1(1+\mu^{3n/2})$$

where $C_1>0$ is independent of $\mathcal{O}$ .

**Proof.** Let $s_o$, $\mathrm{Re}(s_o)>0$, be a pole of $\det C(s)$. Then $s_o$ is a pole of $C(s)$. Since $s_o$ is a simple pole of $C(s)$, the order of $\det C(s)$ at $s_o$ does not exceed $d=\dim\mathcal{E}_{cus}(\sigma,\mathcal{O})$. Applying again Theorem 9.1 of [D1] to estimate d, we get the desired result. Q.E.D.

## 4. Analytic continuation of rank one cuspidal Eisenstein series

In this section we develop a new method of analytic continuation of rank one cuspidal Eisenstein series. This method is an extension of the method used by Colin de Verdiere [Co] in the case of $SL(2,\mathbb{R})$.

Let $(\sigma,V)$ be a fixed irreducible unitary representation of $K$ and $P=NAM$ a rank one $\mathbb{Q}$-parabolic subgroup of $G$ with special split component $A$. We employ the notation of §3. Let $\alpha$ be the simple root of $(P,A)$ and put $\lambda = \alpha/|\alpha|$. We identify $\mathfrak{a}$ with $\mathbb{R}$ via the map $\lambda: \mathfrak{a} \longrightarrow \mathbb{R}$. Fix $u_0 \in \mathbb{R}$ sufficiently large and choose $f \in C^\infty(\mathbb{R})$ such that $f(u)=0$ for $u \le u_0$ and $f(u)=1$ for $u \ge u_0+1$. Let $\Phi \in L^2_{cus}(\Gamma_M \backslash M,\sigma,\chi)$ and put $\mu=\chi(\Omega_M)$. For $s \in \mathbb{C}$, put

$$(4.1) \qquad \Theta(\Phi,s,x) = \sum_{\Gamma \cap P \backslash \Gamma} f(H_P(\gamma x)) e^{(s\lambda+\rho)(H_P(\gamma x))} \Phi(\gamma x).$$

**Lemma 4.2** For each $x \in G$, the sum (4.1) is finite.

**Proof.** This follows from the analogous statement of Lemma 4.2 in [O-W].

In particular, for each $x \in G$, $\Theta(\Phi,s,x)$ is an entire function of $s \in \mathbb{C}$. In the following two Lemmas we establish some elementary properties of $\Theta(\Phi,s)$ that we need for the first step of the analytic continuation.

**Lemma 4.3** For each $s \in \mathbb{C}$,

$$(\Omega + (-s^2 + |\rho|^2 + \mu))\Theta(\Phi,s,x)$$

is square integrable.

**Proof.** By Lemma 4.2, we can switch differentiation and summation in (4.1). If we use (1.2), it follows that

$$(4.4)\qquad \begin{aligned}&(\Omega + (-s^2+|\rho|^2+\mu))(f(H_P(x))e^{(s\lambda+\rho)(H_P(x))}\Phi(x)) = \\ &= h(H_P(x))e^{(s\lambda+\rho)(H_P(x))}\Phi(x).\end{aligned}$$

with $h\in C^\infty(\mathbb{R})$ and $\operatorname{supp} h \subset (u_o,u_o+1)$. The Lemma now follows from Lemma 2.4 Q.E.D.

**Lemma 4.5** Let $\operatorname{Re}(s) > |\rho|$ . Then

$$\Theta(\Phi,s) - E(P|A,\Phi,s)$$

is square integrable.

**Proof.** Set $g=1-f$. Then $g(u)=0$ for $u \geq u_o+1$ and $g(u)=1$ for $u \leq u_o$. Set

$$E^{(1)}(\Phi,s,x) = \sum_{\Gamma\cap P\backslash\Gamma} g(H_P(\gamma x))e^{(s\lambda+\rho)(H_P(\gamma x))}\Phi(\gamma x).$$

Then

$$\Theta(\Phi,s,x) - E(\Phi,s,x) = E^{(1)}(\Phi,s,x) .$$

For $\operatorname{Re}(s) > |\rho|$ , the series converges absolutely and uniformly on compact sets. This follows from Lemma 24 in [H,II,§2]. Choose a sequence $\{g_n\}_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ with $g_n \xrightarrow[n\to\infty]{} g$ in the $C^\infty$-topology. Put

$$E_n^{(1)}(\Phi,s,x) = \sum_{\Gamma\cap P\backslash\Gamma} g_n(H_P(\gamma x))e^{(s\lambda+\rho)(H_P(\gamma x))}\Phi(\gamma x).$$

Lemma 2.4 implies that $E_n^{(1)}(\Phi,s)$ is square integrable. For $z\in\mathbb{C}$ let

$$\hat{g}_n(s,z) = \int_{\mathbb{R}} g_n(u)e^{(s-z)u}du.$$

We may regard $W(\alpha)$ as a subset of $\{\pm 1\}$. Employing Lemma 2.8, we get

$$\|E_n^{(1)}(\Phi,s)\|^2 =$$

$$= \sum_{w\in W(\alpha)} \int_{c-i\infty}^{c+i\infty} \hat{g}_n(s,-wz)\overline{\hat{g}_n(s,z)}(\Phi,C(w:z)\Phi)_{\Gamma_M\backslash M}\, d|z|$$

where $c>|\rho|$. By Lemma 2.3, there exists $C>0$ such that

$$\|C(w:z)\| \le C$$

for $Re(z)=c>|\rho|$. Hence

$$\|E_n^{(1)}(\Phi,s)\|^2 \le C_1 \|\hat{g}_n(s)\|^2 = \frac{C_1}{4\pi}\int_{\infty}^{\infty}|g_n(u)|^2 e^{2Re(s)u}du \tag{4.6}$$

$$\le C_2$$

where $C_2$ is independent of n. It is easy to see that for each $x\in G$, $E_n^{(1)}(\Phi,s,x) \longrightarrow E^{(1)}(\Phi,s,x)$ as $n \longrightarrow \infty$. Combined with (4.6), it follows from Fatou's Lemma that $E^{(1)}(\Phi,s)$ is square integrable. Q.E.D.

Let E and $\Delta$ have the same meaning as in §3. If we consider $\Delta$ as an operator in $L^2(\Gamma\backslash X,E)$ with domain $C_c^\infty(\Gamma\backslash X,E)$ then $\Delta$ is symmetric and therefore, essentially selfadjoint (c.f. Corollary 1.2 in [Mo]). Let $\overline{\Delta}$ denote the unique selfadjoint extension of $\Delta$

in $L^2(\Gamma\backslash X,E)$. It follows from (3.12) that $\bar{\Delta}$ is bounded from below. Therefore, the spectrum $\mathrm{Spec}(\bar{\Delta})$ of $\bar{\Delta}$ is contained in a half line $[c,\infty)$, $c > -\infty$. By Lemma 4.3, $(\Delta - (-s^2+|\rho|^2+\mu))\theta(\Phi,s)$ is square integrable and therefore, we can apply to it the resolvent $(\bar{\Delta} - \lambda \mathrm{Id})^{-1}$. The first step in the analytic continuation of rank one cuspidal Eisenstein series is the following

**Proposition 4.7** Let $\Phi \in L^2_{cus}(\Gamma_M\backslash M,\sigma,\chi)$ and assume that $s\in\mathbb{C}$ is such that $-s^2+|\rho|^2+\mu \notin \mathrm{Spec}(\bar{\Delta})$. Then

$$E(P|A,\Phi,s) = \theta(\Phi,s) - \tag{4.8}$$
$$- (\bar{\Delta} - (-s^2+|\rho|^2+\mu))^{-1}(\Delta-(-s^2+|\rho|^2+\mu))\theta(\Phi,s).$$

**Proof.** Denote the right hand side by $\tilde{E}(\Phi,s)$. By definition, it satisfies $(\Delta - (-s^2+|\rho|+\mu))\tilde{E}(\Phi,s)=0$. By (3.3), $E(P|A,\Phi,s)$ satisfies the same differential equation. On the other hand, by Lemma 4.5, $E(P|A,\Phi,s) - \tilde{E}(\Phi,s)$ is square integrable for $\mathrm{Re}(s) > |\rho|$. Since $\bar{\Delta}$ is selfadjoint, it follows that $E(P|A,\Phi,s) = \tilde{E}(\Phi,s)$ for $\mathrm{Re}(s) > |\rho|$. The Lemma follows by uniqueness of analytic continuation. Q.E.D.

Befor we can continue we have to modify the operator $\Delta_T$ introduced in §3. Let $\mathcal{P}$ be that class of associate rank one $\mathbb{Q}$-parabolic subgroup of G which contains P. As in §3, let $P_{ij}$, $i=1,\dots,r$, $j=1,\dots,r_i$, be a set of representatives for the $\Gamma$-conjugacy classes in $\mathcal{P}$ and let $\mathcal{O} = \{ \mathcal{O}_{ij} | 1 \le i \le r,\ 1 \le j \le r_i\}$ be a set of associate orbits. Given $t\in\mathbb{R}$, let

$$H^1_t(\Gamma\backslash X,E;\mathcal{O}) \subset H^1(\Gamma\backslash X,E)$$

be the subspace consisting of all $\varphi\epsilon H^1(\Gamma\backslash X,E)$ satisfying:

1) If Q is a $\mathbb{Q}$-parabolic subgroup of G and $Q\notin P$, then $\varphi^Q=0$.

2) For all $a\epsilon A_{ij}$, $\varphi^{P_{ij}}(a\cdot)\epsilon L^2_{cus}(\Gamma_{M_{ij}}\backslash M_{ij},\sigma,\mathcal{O}_{ij})$, $i=1,\ldots,r$, $j=1,\ldots,r_i$.

3) For all $m\epsilon M_{ij}$, $\varphi^{P_{ij}}(am)=0$ if $\log(a) > tH_{\rho_{ij}}$, $i=1,\ldots,r$, $j=1,\ldots,r_i$.

Denote by $H_t(\mathcal{O})$ the closure in $L^2(\Gamma\backslash X,E)$ of $H^1_t(\Gamma\backslash X,E;\mathcal{O})$. The quadratic form

$$q(\varphi) = \|\nabla\varphi\|^2 \ , \ \varphi\epsilon H^1_t(\Gamma\backslash X,E;\mathcal{O}),$$

is closed and therefore, it has an associated selfadjoint operator $\tilde{\Delta}_t$ acting in $H_t(\mathcal{O})$. Set

$$\Delta_t = \tilde{\Delta}_t + \lambda_\sigma \mathrm{Id}$$

where $\lambda_\sigma$ is determined by (3.11). Let $t \geq t_o$ and $T=tT_\rho$ (c.f. §3). Since $H^1_t(\Gamma\backslash X,E;\mathcal{O}) \subset H^1_T(\Gamma\backslash X,E)$, the proof of Theorem 3.19 extends to $\Delta_t$ and gives

**Lemma 4.9** 1) $\Delta_t$ has a compact resolvent.

2) Let $N_t(\lambda)$ denote the number of linearly independent eigenfunctions of $\Delta_t$ with eigenvalue less than $\lambda$. There exists a constant $C>0$ such that

$$N_t(\lambda) \leq C(1 + \lambda^{n/2}), \ \lambda \geq 0,$$

$n = \dim X$.

The next step is to replace $\bar{\Delta}$ by $\Delta_t$ in (4.8). This is justified by the following Lemma:

**Lemma 4.10** There exists $t_o \in \mathbb{R}$ such that

$$(\Delta - (-s^2 + |\rho|^2 + \mu))\Theta(\Phi, s) \in \mathcal{H}_t(\mathcal{O})$$

for $t \geq t_o$.

**Proof.** Let $h \in C^\infty(\mathbb{R})$ be determined by (4.4). Then $\operatorname{supp} h \subset (u_o, u_o+1)$. Let

$$\Psi_s(x) = h(H_P(x)) e^{(s\lambda+\rho)(H_P(x))} \Phi(x).$$

Then $\Psi_s \in \mathcal{H}_{cus}(P, \sigma, \chi)$ and

(4.11) $$(\Delta - (-s^2 + |\rho|^2 + \mu))\Theta(\Phi, s) = E(\Psi_s | P).$$

By Lemma 2.4, $E(\Psi_s | P)$ is square integrable. Furthermore, if Q is any $\mathbb{Q}$-parabolic subgroup of G then (2.7) and the description of the constant terms of Eisenstein series (see §2) gives

$$E^Q(\Psi_s | P) = 0 \text{ if } Q \not\supset P.$$

Now set $P' = P_{ij}$ for some i,j $(1 \leq i \leq r,\ 1 \leq j \leq r_i)$. Let A' be the special split component of P' and $P' = N'A'M'$ the corresponding Langlands decomposition. Set $\mathcal{O}' = \mathcal{O}_{ij}$. It follows from (2.7) that

$$E(\Psi_s | P) = \int_{c-i\infty}^{c+i\infty} \hat{h}(z-s) E(P | A, \Phi, z)\, d|z|$$

where $c > |\rho|$ and $\hat{h}(w) = \int_{\mathbb{R}} h(u) e^{-wu} du$, $w \in \mathbb{C}$.

Employing (2.2), we obtain

$$E^{P'}(\Psi_s|P)(a'm') =$$

$$= \sum_{w\in W(a,a')} \int_{c-i\infty}^{c+i\infty} \hat{h}(z-s)e^{(z(w\lambda)+\rho')(\log a')}(c_{P'|P}(w:z)\Phi)(m')d|z|\cdot$$

Hence for $a'\in A$ fixed, $E^{P'}(\Psi_s|P)(a'\cdot)$ belongs to $L^2_{cus}(\Gamma_{M'}\backslash M',\sigma,\mathcal{O}')$. To establish condition 3) we shall compute $E^{P'}(\Psi_s|P)$ along lines similar to [H,II,§5]. Recall that $P'$ is conjugate either to P or $P^- = N^-AM$ - the opposite group to P. Assume that $P'={}^yP$, $y\in G_{\mathbb{Q}}$. The other case is similar. Using computations similar to [H,II,§4] and Lemma 33 in [H,II], it follows that

$$E^{P'}(\Psi_s|P)(x) = \sum_{\Gamma\cap P\backslash\Gamma/\Gamma\cap N'} \Phi_{s,\gamma}(x)$$

with

$$\Phi_{s,\gamma}(x) = \int_{N'\cap{}^{\gamma^{-1}}N\backslash N'} \Psi_s(\gamma n'x)\, dn'$$

Let $P_o=N_oA_oM_o$ be a minimal $\mathbb{Q}$-parabolic subgroup of G with special split component $A_o$ such that $P\supset P_o$, $A_o\supset A$. Write $y^{-1}\gamma^{-1}=n_o\omega p_o$ where $n_o\in N_{o,\mathbb{Q}}$, $\omega\in N(A_o)_{\mathbb{Q}}$ and $p_o\in P_{o,\mathbb{Q}}$. Then $N\cap{}^{\omega}P\subset{}^{\omega}N$ and

$$\Phi_{s,\gamma}(x) = \int_{N\cap{}^{\omega}N\backslash N} \Psi_s(\gamma^{yn_o}nx)\, dn.$$

Moreover, for $a'\in A'$, $m'\in M'$, $\gamma^{yn_o}na'm' = p_o^{-1}\omega^{-1}nn_o^{-1}ay^{-1}m'$ where $a=y^{-1}a'y\in A$. Let $n_o=n_1n_2$ with $n_1\in M\cap N_{o,\mathbb{Q}}$, $n_2\in N_{\mathbb{Q}}$. Then

$$\Phi_{s,\gamma}(a'm') = \int_{N \cap {}^{\omega}N\backslash N} \Psi_s(p_o^{-1}\omega^{-1}nn_1^{-1}ay^{-1}m')\, dn$$

Now observe that

$$p_o^{-1}\omega^{-1}nn_1^{-1}ay^{-1}m' \in N^{(\omega^{-1})}ap_o^{-1}\omega^{-1}\,{}^{(a^{-1})}nn_1^{-1}y^{-1}m'.$$

Set $\varkappa(a) = \det_{n \cap \omega_n \backslash n}(\mathrm{Ad}(a))$. Then

$$(4.12) \qquad \Phi_{s,\gamma}(a'm') = \varkappa(a) \int_{N \cap {}^{\omega}N\backslash N} \Psi_s({}^{(\omega^{-1})}ap_o^{-1}\omega^{-1}nn_1^{-1}y^{-1}m')\, dn\ .$$

Choose $k \in K$ so that $p_o^{-1}\omega^{-1}nk^{-1} \in P$. Then

$$H_P(p_o^{-1}\omega^{-1}nn_1^{-1}y^{-1}m') = H_P(kn_1^{-1}y^{-1}m') + H_P(p_o^{-1}\omega^{-1}n)$$

Furthermore, $m' = ymy^{-1}$ with $m \in M$, i.e., $y^{-1}m' = my^{-1}$.

Let ${}^*Q_1,\dots,{}^*Q_q$ be a set of representatives for the $\Gamma_M$-conjugacy classes of minimal $\mathbb{Q}$-parabolic subgroups of M. Denote by $Q_i \subset P$ the associated $\mathbb{Q}$-parabolic subgroups of G. Then $Q_i$, $1 \le i \le q$, are minimal $\mathbb{Q}$-parabolic subgroups of G. Let ${}^*Q_i = {}^*N_i{}^*A_i{}^*M_i$ be the Langlands decomposition with respect to the special split component ${}^*A_i$ of ${}^*Q_i$. Then $Q_i = N_iA_iM_i$ with $M_i = {}^*M_i$, $A_i = {}^*A_iA$ and $N_i = {}^*N_iN$ is the Langlands decomposition of $Q_i$. Let ${}^*S_i$ be a Siegel domain in M with respect to ${}^*Q_i$. Then ${}^*S_i$ is contained in a Siegel domain $S_i$ in G with respect to $Q_i$. Now observe that results analogous to II, §1 in [H] are true in our case. We only have to replace inf by sup, $-\infty$ by $\infty$ and reverse inequalities. In particular, Corollary 2 to Lemma 21 in [H] implies in our case that

$$\sup_{m \in {}^*S_i} \lambda(H_P(kn_1^{-1}my^{-1})) < \infty.$$

There exist Siegel domains ${}^*S_i$, $i=1,\dots,q$, so that

$$M = \bigcup_{i=1}^{q} \Gamma_M {}^*S_i .$$

Therefore, if $S \subset M$ is a fundamental domain for $\Gamma_M$, we get

$$\sup_{m \in S} \lambda(H_P(kn_1^{-1}my^{-1})) < \infty .$$

Now consider $H_P(p_o^{-1}\omega^{-1}n)$. Let **G** be the reductive algebraic group so that **G**(ℝ)=G. We may assume that **G** is connected. It follows from §12 in [B-T] that for some multiple $\Lambda=q\lambda$ ($q\in\mathbb{Z}$, $q\geq 1$), there exists a finite-dimensional irreducible rational representation $(\pi,V)$ of **G** with the following properties: There exists a non-zero $v\in V_{\mathbb{Q}}$ with $\pi(p)v=v$ for $p\in N_oM_o$ and $\pi(a)v=e^{\Lambda(\log a)}v$ for $a\in A_o$. Choose a scalar product on V so that the operators $\pi(a)$, $a\in A_o$, are selfadjoint. Since $G=KP$, there exist constants $C_2 \geq C_1 > 0$ such that

$$(4.13) \qquad C_1 e^{-\Lambda(H_P(x))} \leq \|\pi(x^{-1})v\| \leq C_2 e^{-\Lambda(H_P(x))}, \quad x\in G.$$

Put $x=p_o^{-1}\omega^{-1}n$. Then $x^{-1}=n_1y^{-1}\omega^{-1}$ with $n_1=n^{-1}n_o^{-1}\in N_o$. Since $N_o$ is defined over ℚ, there exists a basis $v_1,\dots,v_h \in V_{\mathbb{Q}}$ such that

$$(4.14) \qquad \pi(n)v_i - v_i \in \sum_{j>i} \mathbb{R}v_j , \quad n\in N_o,$$

(c.f. Corollary 15.5 in [B2]). Let $L \subset V_{\mathbb{Q}}$ be the lattice generated by $v_1,\dots,v_h$. By Proposition 10.13 of [R], there exists a subgroup $\Gamma_1$ of $G_{\mathbb{Z}}$ of finite index such that $\pi(\Gamma_1)L=L$. Since $\Gamma$ is commensurable with $\Gamma_1$ and $y\in G_{\mathbb{Q}}$, there exists $b\in\mathbb{N}$ such that $\pi(y^{-1})\pi(\Gamma)L \subset b^{-1}L$. Therefore, by (4.14) it follows that there exists a constant $C_3 > 0$ such that

$$\|\pi(ny^{-1}\gamma)v\| \geq C_3$$

for $n \in N_o$ and $\gamma \in \Gamma$. Combined with (4.13), we get

$$H_P(\gamma yn) \leq C_4$$

for all $n \in N_o$ and $\gamma \in \Gamma$.

Putting our results together, it follows that there exists $C > 0$ and a fundamental domain $S' \subset M'$ for $\Gamma_{M'}$ such that

$$H_P(p_o^{-1}\omega^{-1}nn_1^{-1}y^{-1}m') \leq C \tag{4.15}$$

for $m' \in S'$, $n \in N'$ and $\gamma \in \Gamma$. The restriction of $\omega$ to A belongs to W(A). Thus $\omega|A = \pm 1$. Assume that $\omega|A = 1$. Then $P' = \gamma^{-1}P\gamma$ and $\Phi_{s,\gamma}(x) = \Psi_s(\gamma x)$. This shows that $\Phi_{s,\gamma}(a'm') = 0$ if $\log(a') > C$. Note that there is a single class $\bar{\gamma} \in \Gamma \cap P\backslash\Gamma/\Gamma \cap N$ with $P' = \gamma^{-1}P\gamma$. Now assume that $\omega|A = -1$. Then $H_P(\omega a) \longrightarrow -\infty$ if $\log(a) \longrightarrow \infty$. Then (4.12) together with (4.15) implies that there exists $C_1$ with $\Phi_{s,\gamma}(a'm') = 0$ for $m' \in S'$, $\gamma \in \Gamma$ and $\lambda(\log a') > C_1$. The definition of $\Phi_{s,\gamma}$ implies that this holds for $m' \in M'$. Q.E.D.

Choose $t \geq t_o$ as in Lemma 4.10. Given $\Phi \in L^2_{cus}(\Gamma_M\backslash M,\sigma,\chi)$ and $s \in \mathbb{C}$, put

$$\begin{aligned} F(\Phi,s) &= \Theta(\Phi,s) - \\ &- (\Delta_t - (-s^2 + |\rho|^2 + \mu))^{-1}((\Delta - (-s^2 + |\rho|^2 + \mu))\Theta(\Phi,s)). \end{aligned} \tag{4.16}$$

By Lemma 4.10, the right hand side is well-defined. Moreover, Lemma 4.9 shows that $F(\Phi,s)$ is a meromorphic function of $s \in \mathbb{C}$. We shall now investigate the properties of $F(\Phi,s)$. By definition,

$F(\Phi,s) - \Theta(\Phi,s) \in L^2(\Gamma\backslash X,E)$. This fact implies that $F(\Phi,s)$ is a distributional section of E. Now observe that the description of the domain of $\Delta_t$ is similar to that of $\Delta_T$. In particular, it implies that

$$(\Delta - (-s^2+|\rho|^2+\mu))F(\Phi,s) = S$$

where $S \in H^{-1}(\Gamma\backslash X,E)$ and S is orthogonal to $H^1_t(\Gamma\backslash X,E)$.

**Lemma 4.17** Let $Q=N_QA_QM_Q$ be a $\mathbb{Q}$-parabolic subgroup of G, $\chi_Q \in \hat{Z}(m_Q)$ and $\varphi \in H_{cus}(Q,\sigma,\chi_Q)$. Assume that either $Q \notin P$ or $Q=P_{ij}$ for some i,j ($1 \le i \le r$, $1 \le j \le r_i$) and $\chi_Q \notin O_{ij}$. Then $S(E(\varphi|Q))=0$.

**Proof.** Using (4.11) and a simple approximation argument, we get

$$(\Theta(\Phi,s),(\Delta - (-\bar{s}^2+|\rho|^2+\mu))E(\varphi|Q)) = (E(\Psi_s|P),E(\varphi|Q)).$$

If $Q \notin P$, the right hand side vanishes by Lemma 2.6. If $Q=P_{ij}$ for some i,j ($1 \le i \le r$, $1 \le j \le r_i$), the right hand side vanishes by Lemma 2.8 and the assumption on $\varphi$. Put $\psi=F(\Phi,s) - \Theta(\Phi,s)$. Then $\psi \in H_t(O)$. In particular, $\psi^Q(a\cdot)$ is square integrable for all Q and Lemma 2.5 gives

$$(\psi,(\Delta - (-\bar{s}^2+|\rho|^2+\mu))E(\varphi|Q)) =$$

$$= - \int_A e^{-2\rho(\log a)} \int_{\Gamma_{M_Q}\backslash M_Q} (\psi^Q(am),(\Omega + (-\bar{s}^2+|\rho|^2+\mu))\varphi(am))\, dmda .$$

If $Q \notin P$ we have $\psi^Q=0$ and the right hand side vanishes. If $Q=P_{ij}$, then $\psi^Q(a\cdot) \perp H_{cus}(Q,\sigma,\chi_Q)$ and the right hand side vanishes too. Q.E.D.

Let $H^{\tilde{1}}(\Gamma\backslash X,E;\mathcal{O})$ be the subspace of $H^1(\Gamma\backslash X,E)$ consisting of all $\varphi$ which satisfy the first two of the conditions defining $H^1_t(\Gamma\backslash X,E;\mathcal{O})$. It remains to determine S on $H^1(\Gamma\backslash X,E;\mathcal{O})$. For this purpose we modify the truncation operator $\Lambda^T$. Let $\xi\in C^\infty(\mathbb{R})$ be such that $\xi(u)=1$ for $u \geq 0$ and $\xi(u)=0$ for $u \leq -1$. Let $P_1,\dots,P_h$ be a set of representatives for the Γ-conjugacy classes in P. Given $\varphi\in L^2(\Gamma\backslash X,E)$ and $t\in\mathbb{R}$, set

$$\Lambda^t_{P_i,\xi}\varphi(x) = \sum_{\Gamma\cap P_i\backslash\Gamma} \xi(\lambda_{P_i}(H_{P_i}(\gamma x)) - t|\rho|)\varphi^{P_i}(\gamma x),$$

$i=1,\dots,h$. Let

$$\Lambda^t_\xi\varphi = \varphi - \sum_{i=1}^{h} \Lambda^t_{P_i,\xi}\varphi$$

**Lemma 4.18** There exists $t_1\in\mathbb{R}$ such that for $t \geq t_1$ and $\varphi\in H^1(\Gamma\backslash X,E;\mathcal{O})$,

$$\Lambda^t_\xi\varphi\in H^1_t(\Gamma\backslash X,E;\mathcal{O})$$

**Proof.** Let $T=tT_\rho$, $t \geq t_o$ (c.f. §3) and $\varphi\in H^1(\Gamma\backslash X,E;\mathcal{O})$. We may assume that $\varphi$ is smooth. We have

$$\Lambda^t_\xi\varphi = \Lambda^T\varphi - \sum_{i=1}^{h} \Lambda^t_{P_i,\xi_o}\varphi$$

where $\xi_o = \xi - \chi_{[o,\infty)}$ and $\Lambda^t_{P_i,\xi_o}\varphi$ is defined in the same manner as $\Lambda^t_{P_i,\xi}\varphi$ . T is chosen so that $\Lambda^T\varphi$ is square integrable. Now consinder $\Lambda^t_{P_i,\xi_o}\varphi$ . Note that $\operatorname{supp}(\xi_o)\subset(-1,0)$. Let $\{\xi_n\}_{n\in\mathbb{N}} \subset C^\infty_c((-1,0))$ be a sequence with $\xi_n(u)\longrightarrow\xi(u)$ for all $u\in(-1,0)$ and $\|\xi_n - \xi_o\|_{L^2}\longrightarrow 0$ as $n\longrightarrow\infty$. Let i $(1 \leq i \leq h)$ be fixed and set

$$\Psi_n(x) = \xi_n(H_{P_i}(x))\varphi^{P_i}(x) .$$

By definition of $H^1(\Gamma\backslash X,E;\mathcal{O})$, we have $\varphi^{P_i}(a\cdot)\in L^2_{cus}(\Gamma_{M_i}\backslash M_i,\sigma,\mathcal{O}_i)$ for $a\in A_i$ fixed. Hence $\Psi_n\in\bigoplus_{\chi\in\mathcal{O}_i}\mathcal{H}_{cus}(P_i,\sigma,\chi)$. By Lemma 2.4, $E(\Psi_n|P_i)\in L^2(\Gamma\backslash X,E)$ and, using Lemma 2.8, it follows that $\|E(\Psi_n|P_i)\|\leq C$ independent of n. Furthermore, for any compact set $\omega\subset G$, there are only finitely many $\gamma\in\Gamma\cap P_i\backslash\Gamma$ such that $H_{P_i}(\gamma x) > tH_{\rho_i}$ for $x\in\omega$. This is simply the analogous statement of Lemma 4.2 in [O-W] in our case. Using this fact, it follows that $E(\Psi_n|P_i)(x)\longrightarrow \Lambda^t_{P_i,\xi_o}\varphi(x)$ as $n\longrightarrow\infty$ for all $x\in G$. Therefore $\Lambda^t_{P_i,\xi_o}\varphi\in L^2(\Gamma\backslash X,E)$ by Fatou's Lemma. Hence $\Lambda^t_\xi\varphi\in L^2(\Gamma\backslash X,E)$. The same argument shows that $\nabla\Lambda^t_\xi\varphi\in L^2(\Gamma\backslash X,E)$. Thus $\Lambda^t_\xi\varphi\in H^1(\Gamma\backslash X,E)$. Next consider the constant terms of $\Lambda^t_{P_i,\xi}\varphi$. Let $\{g_n\}\subset C^\infty_c(\mathbb{R})$ be a sequence with $g_n\longrightarrow\xi$ in the $C^\infty$-topology. Set

$$\Phi_n(x) = g_n(H_{P_i}(x))\varphi^{P_i}(x) .$$

As above, we have $\Phi_n\in\bigoplus_{\chi\in\mathcal{O}_i}\mathcal{H}_{cus}(P_i,\sigma,\chi)$. Using Lemma 2.4, (2.7) and (2.2), it follows that $E(\Phi_n|P_i)\in H^1(\Gamma\backslash X,E;\mathcal{O})$. On the other hand, employing again the analogous statement of Lemma 4.2 in [O-W], we see that $E(\Phi_n|P_i)(x)\longrightarrow\Lambda^t_{P_i,\xi}\varphi(x)$ as $n\longrightarrow\infty$, uniformly on compact subsets of G. Hence $\Lambda^t_\xi\varphi\in H^1(\Gamma\backslash X,E;\mathcal{O})$. Furthermore, by property 2) satisfied by the truncation operator and the choice of T, we have

$$(\Lambda^T\varphi)^{P_i}(x) = 0 \quad \text{if } \lambda_{P_i}(H_{P_i}(x)) > t|\rho|,\ i=1,\dots,h.$$

Finally, employing arguments similar to those of the proof of Lemma 4.10 combined with a simple approximation argument, it follows that there exists $t' \in \mathbb{R}$ such that for $t \geq t'$,

$$(\Lambda^t_{P_i,\xi_o}\varphi)^{P_j}(x) = 0 \quad \text{if } \lambda_{P_j}(H_{P_j}(x)) > t|\rho|, \; i,j=1,\ldots,h.$$

$t_o$ and $t'$ are independent of $\varphi$ . Hence $\Lambda^t_\xi\varphi \in H^1_t(\Gamma\backslash X,E;\mathcal{O})$. Q.E.D.

Let $t_2 = \max\{t_o,t_1\}$ and $t \geq t_2$. Let $\varphi \in H^1(\Gamma\backslash X,E;\mathcal{O})$. Since S is orthogonal to $H^1_t(\Gamma\backslash X,E;\mathcal{O})$, it follows from Lemma 4.18 that

$$(4.19) \qquad S(\varphi) = \sum_{i=1}^{h} S(\Lambda^t_{P_i,\xi}\varphi) .$$

Next we investigate the constant term $F^{P_i}(\Phi,s)$, $i=1,\ldots,h$. It follows from the definition of $F(\Phi,s)$ that for $a \in A_i$ fixed, the section $F^{P_i}(\Phi,s,(a,\cdot)) - \Theta^{P_i}(\Phi,s,(a,\cdot))$ belongs to $L^2_{cus}(\Gamma_{M_i}\backslash M_i,\sigma,\mathcal{O}_i)$. Furthermore, let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \mathrm{Re}(s)$. Then $f(u)e^{(s-z)u}$ is a rapidly decreasing function of $u \in \mathbb{R}$. Then

$$(4.20) \qquad \Theta(\Phi,s) = \int_{c-i\infty}^{c+i\infty} \hat{f}(z-s)E(P|A,\Phi,z)\; d|z|$$

with $c > \mathrm{Re}(s)$. The proof is similar to the proof of Lemma 28 in [H,II,§3] . Using this formula combined with (2.2), we obtain $\Theta^{P_i}(\Phi,s,(a\cdot)) \in L^2_{cus}(\Gamma_{M_i}\backslash M_i,\sigma,\chi_i)$. Hence

$$(4.21) \qquad F^{P_i}(\Phi,s,(a\cdot)) \in L^2_{cus}(\Gamma_{M_i}\backslash M_i,\sigma,\mathcal{O}_i), \; i=1,\ldots,h.$$

Let $g \in C^\infty(\mathbb{R})$ with $\mathrm{supp}\, g \subset (t-1,t)$ and let $\Psi \in L^2_{cus}(\Gamma_{M_i}\backslash M_i,\sigma,\chi_i)$, $\chi_i \in \mathcal{O}_i$. Set

$$\psi(x) = g(H_{P_i}(x))\Psi(x).$$

It follows in the same way as above that $E(\psi|P_i)\in H_t^1(\Gamma\backslash X,E;\mathcal{O})$. Hence $S(E(\psi|P_i))=0$, i.e.,

$$(F(\Phi,s),(\Delta-(-\bar{s}^2+|\rho|^2+\mu))E(\psi|P_i)) = 0.$$

In view of (4.21), we can apply Lemma 2.5 which implies

$$0 = \int_{A_i} e^{-2\rho(\log a)} \int_{\Gamma_{M_i}\backslash M_i} (F^{P_i}(\Phi,s,am),(\Omega + (-\bar{s}^2+|\rho|^2+\mu))\psi(am))\cdot dm\, da \;. \tag{4.22}$$

Let $H\in\mathfrak{a}_i$ such that $\lambda_i(H) > 0$ and $\|H\| = 1$. Set

$$g_i(s,u) = \int_{\Gamma_{M_i}\backslash M_i} (F^{P_i}(\Phi,s,e^{uH}m),\Psi(m))\, dm \;.$$

Using (1.2), (4.22) and elliptic regularity, it follows that $g_i(s,u)$ is a smooth function of $u\in(t-1,t)$ and satisfies

$$\left(-\frac{d^2}{du^2} + 2|\rho|\frac{d}{du}\right)g_i(s,u) = (-s^2+|\rho|^2)g_i(s,u) \;.$$

Hence

$$g_i(s,u) = C_1(s)e^{(s+|\rho|)u} + C_2(s)e^{-(s-|\rho|)u},\; u\in(t-1,t),$$

and $C_1(s)$, $C_2(s)$ are meromorphic functions of $s\in\mathbb{C}$. This implies that there exist linear operators

$$A_{P_i|P}(s),\; B_{P_i|P}(s) \;:\; L^2_{cus}(\Gamma_M\backslash M,\sigma,\chi) \longrightarrow L^2_{cus}(\Gamma_{M_i}\backslash M_i,\sigma,\mathcal{O}_i)$$

which are meromorphic functions of $s\in\mathbb{C}$ such that

$$F^{P_i}(\Phi,s,x) = \tag{4.23}$$

$$= e^{(s\lambda_i+\rho_i)(H_i(x))}(A_{P_i|P}(s)\Phi)(x) + e^{(-s\lambda_i+\rho_i)(H_i(x))}(B_{P_i|P}(s)\Phi)(x)$$

where $H_i(x)=H_{P_i}(x)$ and $\lambda_i(H_i(x))\in(t-1,t)$. Denote by $F_i(\Phi,s)$ the element in $C^\infty(\Gamma \cap P_i\backslash G,\sigma)$ which is defined by the right hand side of (4.23). Let $t \geq t_2$ and put

$$\begin{aligned} G(\Phi,s,x) &= F(\Phi,s,x) - \Lambda^t_\xi F(\Phi,s)(x) + \\ &+ \sum_{i=1}^{h} \sum_{\Gamma\cap P_i\backslash\Gamma} \xi(\lambda_{P_i}(H_{P_i}(\gamma x)) - t|\rho|)F_i(\Phi,s,\gamma x), \quad x\in G. \end{aligned} \tag{4.24}$$

$G(\Phi,s)$ is a meromorphic function of $s\in\mathbb{C}$. Moreover we have

**Proposition 4.25** $G(\Phi,s)$ belongs to $C^\infty(\Gamma\backslash X,E)$ and it satisfies

$$(\Delta - (-s^2+|\rho|^2+\mu))G(\Phi,s) = 0.$$

**Proof.** Let $Q=N_QA_QM_Q$ be a $\mathbb{Q}$-parabolic subgroup of G, $\chi_Q\in\hat{Z}(m_Q)$ and and $\varphi\in\mathcal{H}_{cus}(P,\sigma,\chi_Q)$. If $Q\notin\mathcal{P}$ or $Q=P_i$ for some i $(1\leq i\leq h)$ and $\chi_Q\notin\mathcal{O}_i$, then it follows from Lemma 4.17 and (4.23) that

$$(G(\Phi,s),(\Delta - (-\bar{s}^2+|\rho|^2+\mu))E(\varphi|Q)) = 0 \ .$$

Now assume that $Q=P_j$ for some j $(1\leq j\leq h)$ and $\chi_Q\in\mathcal{O}_j$. Let $\psi = E(\varphi|Q)$. Then $\psi\in H^1(\Gamma\backslash X,E;\mathcal{O})$. Furthermore, set

$$\xi_i(x) = \xi(\lambda_{P_i}(H_{P_i}(x)) - t|\rho|), \ i=1,\dots,h.$$

If we apply (4.19) and Lemma 2.5, we obtain

$(G(\Phi,s),(\Delta - (-\bar{s}^2+|\rho|^2+\mu))\psi) =$

$$= \sum_{i=1}^{h} \int_{A_i} e^{-2\rho(\log a_i)} \int_{\Gamma_{M_i}\backslash M_i} (F^{P_i}(\Phi,s,a_im_i),[\Delta,\xi_i]\psi^{P_i}(a_im_i))dm_i da_i$$

$$- \sum_{i=1}^{h} \int_{A_i} e^{-2\rho(\log a_i)} \int_{\Gamma_{M_i}\backslash M_i} (F_i(\Phi,s,a_im_i),[\Delta,\xi_i]\psi^{P_i}(a_im_i))dm_i da_i \ .$$

Now observe that $[\Delta,\xi_i]\psi^{P_i}(a_im_i)=0$ unless $\lambda_i(\log a_i)\in(t-1,t)$. But $F^{P_i}(\Phi,s,a_im_i) = F_i(\Phi,s,a_im_i)$ for $\lambda_i(\log a_i)\in(t-1,t)$. Thus

$$(G(\Phi,s),(\Delta - (-\bar{s}^2+|\rho|^2+\mu))\psi) = 0.$$

But it follows from Theorem 4.6 of [Ca] that each $\psi\in C_c^\infty(\Gamma\backslash X,E)$ can be approximated in the $C^\infty$-topology by linear combinations of wave packets $E(\varphi|Q)$. Hence $(\Delta - (-s^2+|\rho|^2+\mu))G(\Phi,s)=0$ in the sense of distributions. Then elliptic regularity implies that $G(\Phi,s)$ is a smooth section of E. Q.E.D.

Given an orbit $\mathcal{O}\in\hat{Z}(\mathfrak{m})/W(A)$ and $\Phi\in L^2_{cus}(\Gamma_M\backslash M,\sigma,\mathcal{O})$, we define $G(\Phi,s)$ in the obvious way. Let $\Phi\in\mathcal{E}_{cus}(\sigma,\mathcal{O})$ with $\Phi=\{\Phi_{ij}|\ 1\le i\le r, 1\le j\le r_i\}$ and $\Phi_{ij}\in L^2_{cus}(\Gamma_{M_{ij}}\backslash M_{ij},\sigma,\mathcal{O}_{ij})$. Set

$$G(\Phi,s) = \sum_{i=1}^{r} \sum_{j=1}^{r_i} G(\Phi_{ij},s) \ .$$

$G(\Phi,s)$ is a meromorphic function of $s\in\mathbb{C}$. For each $s\in\mathbb{C}$ which is not a pole, $G(\Phi,s)\in C^\infty(\Gamma\backslash X,E)$ and it satisfies

$$(\Delta - (-s^2+|\rho|^2+\mu))G(\Phi,s) = 0. \tag{4.26}$$

Concerning the constamt terms, we have

**Lemma 4.27** Let Q be a $\mathbb{Q}$-parabolic subgroup of G.

1) If $Q \notin P$ then $G^Q(\Phi,s) = 0$.

2) There exist linear operators $A(s), B(s) : \mathcal{E}_{cus}(\sigma,\mathcal{O}) \longrightarrow \mathcal{E}_{cus}(\sigma,\mathcal{O})$ which are meromorphic functions of $s \in \mathbb{C}$ such that

$$G^{P_{ij}}(\Phi,s,x) =$$

$$= e^{(s+|\rho|)t_{ij}(x)}(A(s)\Phi)_{ij}(x) + e^{(-s+|\rho|)t_{ij}(x)}(B(s)\Phi)_{ij}(x) \ ,$$

$i=1,\ldots,r$, $j=1,\ldots,r_i$, in the notation of §3.

**Proof.** 1) follows immediately from the definition of $G(\Phi,s)$ and the properties of $F(\Phi,s)$. To prove 2), we observe that by definition, $G^{P_{ij}}(\Phi,s,a\cdot) \in \mathcal{E}_{cus}(\sigma,\mathcal{O})$ and it satisfies

$$(\Delta - (-s^2+|\rho|^2+\mu))G^{P_{ij}}(\Phi,s) = 0.$$

Using (1.2), the result follows. Q.E.D.

**Lemma 4.28** The operator $A(s) : \mathcal{E}_{cus}(\sigma,\mathcal{O}) \longrightarrow \mathcal{E}_{cus}(\sigma,\mathcal{O})$ is invertible as a meromorphic function.

**Proof.** Assume that $\det(A(s)) \equiv 0$. Thus, for each $s \in \mathbb{C}$ which is not a pole of $A(s)$, there exists $\Phi \in \mathcal{E}_{cus}(\sigma,\mathcal{O})$, $\Phi \neq 0$, such that $A(s)\Phi = 0$. Assume that $Re(s) > |\rho|$ . We claim that $G(\Phi,s)$ is square integrable. To see this consider $G(\Phi_{ij},s)$. Using (4.24) and the definition of $F(\Phi_{ij},s)$, it follows that

$$G(\Phi_{ij},s) = G_1(\Phi_{ij},s) + G_2(\Phi_{ij},s)$$

where $G_2(\Phi_{ij},s)$ is square integrable and $G_1(\Phi_{ij},s)$ is smooth and satisfies the following property: There exists $r\in\mathbb{R}$ such that for all $D\in U(\mathfrak{g})$, $DG_1(\Phi_{ij},s)$ is slowly increasing with exponent of growth r. Let $T=tT_\rho$ with $t\geq t_0$. Then $\Lambda^T G_1(\Phi_{ij},s)$ is rapidly decreasing (c.f. Theorem 5.2 of [O-W]) and therefore square integrable. Since $\Lambda^T$ extends to an orthogonal projection of $L^2(\Gamma\backslash X,E)$, $\Lambda^T G_1(\Phi_{ij},s)$ is square integrable too. Thus $\Lambda^T G(\Phi,s)$ is square integrable. On the other hand, by Lemma 4.27, $G(\Phi,s)$ - - $\Lambda^T G(\Phi,s)$ is the sum of

$$\sum_{\Gamma\cap P_{ij}\backslash\Gamma} \chi_{P_{ij}}(H_{P_{ij}}(\gamma x) - tH_{\rho_{ij}})G^{P_{ij}}(\Phi,s,\gamma x) \tag{4.29}$$

$i=1,\ldots,r$, $j=1,\ldots,r_i$. Using again Lemma 4.27, we have

$$G^{P_{ij}}(\Phi,s,x) = e^{(-s+|\rho|)t_{ij}(x)}(B(s)\Phi)_{ij}(x) \;.$$

If we apply Lemma 2.8 and a simple approximation argument, it follows that the terms (4.29) are square integrable for $\mathrm{Re}(s) > |\rho|$. Since $\bar{\Delta}$ is selfadjoint, it follows from (4.26) that $G(\Phi,s)=0$ for $\mathrm{Re}(s)>|\rho|$, $s\neq\bar{s}$. By analytic continuation this holds for all s. Let

$$F(\Phi,s) = \sum_{i,j} F(\Phi_{ij},s) \;.$$

It follows from the definition of $G(\Phi,s)$ that

$$G(\Phi,s) = \Lambda^T F(\Phi,s) + \sum_{i,j} R_{ij}(\Phi,s)$$

where

$$R_{ij}(\Phi,s) = \sum_{\Gamma\cap P_{ij}\backslash\Gamma} \chi_{P_{ij}}(\lambda_{ij}(H_{P_{ij}}(\gamma x) - t|\rho|)F_{ij}(\Phi,s,\gamma x)$$

and $F_{ij}(\Phi,s)$ is defined in the same way as $F_i(\Phi,s)$ above. Employing Lemma 2.5, we get

$$(\Lambda^T F(\Phi,s),\Lambda^T R_{ij}(\Phi,s)) = (\Lambda^T F(\Phi,s),R_{ij}(\Phi,s)) = 0.$$

Hence

$$\|\Lambda^T F(\Phi,s)\|^2 = (\Lambda^T F(\Phi,s),G(\Phi,s)) = 0$$

and therefore, $\Lambda^T F(\Phi,s) = 0$. Set

$$\Theta(\Phi,s) = \sum_{i,j} \Theta(\Phi_{ij},s).$$

In view of (4.16), we get $\Lambda^T\Theta(\Phi,s)=0$. Let s be fixed and chose $c > \mathrm{Re}(s)$. By (4.20)

$$\Lambda^T\Theta(\Phi,s) = \int_{c-i\infty}^{c+i\infty} \hat{f}(z-s)\Lambda^T E(\Phi,z)\, d|z|.$$

If we make use of the scalar product formula for truncated Eisenstein series in [L1,p.135], we get

$$\|\Lambda^T\Theta(\Phi,s)\|^2 =$$

$$= \int_{c-i\infty}^{c+i\infty}\int_{c-i\infty}^{c+i\infty} \hat{f}(z_1-s)\overline{\hat{f}(z_2-s)}\ \frac{1}{z_1+\bar{z}_2}\ (e^{(z_1+\bar{z}_2)t|\rho|}\ \|\Phi\|^2 -$$

$$- e^{-(z_1+\bar{z}_2)t|\rho|}(C(z_1)\Phi,C(z_2)\Phi)) +$$

$$+ \frac{1}{z_1-\bar{z}_2}\left(e^{(z_1-\bar{z}_2)t|\rho|}(\Phi,C(z_2)\Phi) - e^{(\bar{z}_2-z_1)t|\rho|}(C(z_1)\Phi,\Phi)\right) \quad dz_1dz_2.$$

Using Lemma 2.3, it follows that the right hand side is non-zero if t is sufficiently large. But the right hand side is real analytic in t and therefore, it vanishes at most at a discrete set of points. Moreover, if $T_i=t_iT_\rho$, $i=1,2$, and $t_1 > t_2$ then $||\Lambda^{T_1}\Theta(\Phi,s)|| \geq ||\Lambda^{T_2}\Theta(\Phi,s)||$. Thus $\Lambda^T\Theta(\Phi,s) \neq 0$ unless $\Phi=0$. This is a contradiction to our assumption that $\det(A(s)) \equiv 0$. Q.E.D.

We can now state the main result of this section.

**Theorem 4.30** Let $\Phi\in\mathcal{E}_{cus}(\sigma,\mathcal{O})$. Then

$$E(\Phi,s) = G(A(s)^{-1}\Phi,s)$$

as meromorphic functions of $s\in\mathbb{C}$. The intertwining operator $C(s)$ is given by $C(s) = B(s)A(s)^{-1}$.

**Proof.** Put

$$R(\Phi,s) = E(\Phi,s) - G(A(s)^{-1}\Phi,s) .$$

Let $\mathrm{Re}(s) > |\rho|$. We claim that $R(\Phi,s)$ is square integrable. This can be seen as follows. In the proof of Lemma 4.28 we observed that $\Lambda^TG(A(s)^{-1}\Phi,s)$ is square integrable. $\Lambda^TE(\Phi,s)$ is also square integrable. Hence $\Lambda^TR(\Phi,s)$ is square integrable. Employing the description of the constant terms of $E(\Phi,s)$ (c.f. §2) combined with Lemma 4.27, we get

$$R(\Phi,s) - \Lambda^TR(\Phi,s) = \sum_{i,j} \tilde{R}_{ij}(\Phi,s)$$

where

$$\tilde{R}_{ij}(\Phi,s) = \sum_{\Gamma\cap P_{ij}\backslash\Gamma} \chi_{P_{ij}}(H_{P_{ij}}(\gamma x) - tH_{\rho_{ij}})R^{P_{ij}}(\Phi,s.\ x).$$

If we make use of (3.3), Lemma 4.27 and Lemma 2.8, it follows that $\tilde{R}_{ij}(\Phi,s)$ is square integrable for $Re(s)>|\rho|$, $i=1,\dots,r$, $j=1,\dots,r_i$. This shows that $R(\Phi,s)$ is square integrable for $Re(s)>|\rho|$. Now observe that $\Delta R(\Phi,s) = (-s^2+|\rho|^2+\mu)R(\Phi,s)$. Since $\bar{\Delta}$ is selfadjoint, we get $R(\Phi,s)=0$ for $Re(s)>|\rho|$. Since $R(\Phi,s)$ is a meromorphic function of $s\in\mathbb{C}$, it vanishes for all $s\in\mathbb{C}$. This gives the equation claimed in the Theorem. If we compare the constant terms of both sides of this equation and use Lemma 4.27, we get $C(s)=B(s)A(s)^{-1}$. Q.E.D.

**Remark.** Theorem 4.30 gives a new construction of the analytic continuation of rank one cuspidal Eisenstein series.

## 5. The order of growth of det C(s)

The main purpose of this section is to prove that the determinant of the intertwining operator C(s) is a meromorphic function of order $\leq n+2$ where $n=\dim X$.

Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of the selfadjoint operator $\Delta_t$ introduced in §4. For simplicity we shall assume that zero is not an eigenvalue of $\Delta_t$. According to Lemma 4.9 we have

$$\#\{\lambda_i \mid \lambda_i \leq \lambda\} \leq C(1+\lambda^{n/2}) \ , \ \lambda \geq 0,$$

for some constant $C>0$ and $n=\dim X$. This implies that

$$(5.1) \qquad \sum_{j=1}^{\infty} |\lambda_j|^{-k} < \infty$$

for $k > n/2$. As usually, for $p \in \mathbb{N}$, let

$$E(u,p) = (1-u)\exp(u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}) \ , \ u \in \mathbb{C}.$$

Put $p=[n/2]$ . Then the infinite product

$$\tilde{P}(z) = \prod_{j=1}^{\infty} E(\frac{z}{\lambda_j}, p)$$

converges uniformly on compact subsets of $\mathbb{C}$ and $\tilde{P}(z)$ is an entire function of order $n/2$ whose zeros are $\lambda_1, \lambda_2, \ldots$ (c.f. [Bo, pp. 18-19] ). For $s \in \mathbb{C}$ put

$$P(s) = \tilde{P}(-s^2+|\rho|^2+\mu).$$

Now observe that in view of Lemma 4.9, $(\Delta_t - z\mathrm{Id})^{-1}$ is a meromorphic function of $z \in \mathbb{C}$ with simple poles at $z=\lambda_1, \lambda_2, \ldots$

Let $\Phi\in\mathcal{E}_{cus}(\sigma,\mathcal{O})$. Using (4.16) and (4.24), it follows that $P(s)G(\Phi,s,x)$ is an entire function of $s\in\mathbb{C}$. Therefore, $P(s)A(s)\Phi$ and $P(s)B(s)\Phi$ are also entire functions of $s\in\mathbb{C}$ and we shall now estimate the order of growth of $|P(s)|\,\|A(s)\|$ and $|P(s)|\,\|B(s)\|$. First we need an auxiliary Lemma. For each $j\in\mathbb{N}$, put

$$\tilde{P}_j(z) = \prod_{k\neq j} E(\frac{z}{\lambda_k},p) .$$

**Lemma 5.2** There exists a constant $C > 0$ such that

$$|\tilde{P}_j(z)| \le e^{C|z|^{n/2+1}} , \quad z\in\mathbb{C},\ j\in\mathbb{N}.$$

**Proof.** We have

$$\log|\tilde{P}_j(z)| = \Big(\sum_{\substack{|\lambda_k|\le 2|z|\\ k\neq j}} + \sum_{\substack{|\lambda_k|>2|z|\\ k\neq j}}\Big) \log|E(\frac{z}{\lambda_k},p)| = S_1 + S_2.$$

To estimate $S_1$ observe that $|z|/|\lambda_k| \ge 1/2$ and therefore

$$\left(\frac{|z|}{|\lambda_k|}\right)^l \le 2^{p-1}\left(\frac{|z|}{|\lambda_k|}\right)^p , \quad 0 \le l \le p$$

and

$$\log|1 - \frac{z}{\lambda_k}| \le 1 + \frac{|z|}{|\lambda_k|} \le 1 + 2^{p-1}\left(\frac{|z|}{|\lambda_k|}\right)^p .$$

Hence

$$\log|E(\frac{z}{\lambda_k},p)| \le 2^{p+1}\left(\frac{|z|}{|\lambda_k|}\right)^p . \tag{5.3}$$

Let $\varepsilon > 0$ and $p_1 = n/2$. Using (5.3), we get

$$S_1 \le 2^{p+1} |z|^p \sum_{|\lambda_k| \le 2|z|} |\lambda_k|^{-p} \le C|z|^{n/2+\varepsilon} \sum_{|\lambda_k| \le 2|z|} |\lambda_k|^{-p_1-\varepsilon} \le$$

$$\le C(\varepsilon)|z|^{n/2+\varepsilon}$$

where $C(\varepsilon)$ depends on $\varepsilon$ and p. Now consider $S_2$. In this case $|z|/|\lambda_k| < 1/2$. Using 2.6.3 in [Bo], we get

$$\log \left|E(\frac{z}{\lambda_k},p)\right| \le 2\left|\frac{z}{\lambda_k}\right|^{p+1} .$$

If $p=n/2$, this implies

$$S_2 \le 2|z|^{p+1} \sum_{2|z| < |\lambda_k|} |\lambda_k|^{-p-1} = C_1 |z|^{n/2+1} .$$

If $p=(n-1)/2$, we get $S_2 \le C_2|z|^{(n+1)/2}$. Thus $\log|\tilde{P}_j(z)| \le C|z|^{n/2+1}$, $z\in\mathbb{C}$, $j\in\mathbb{N}$. Q.E.D.

Let $\Phi\in \mathcal{E}_{cus}(\sigma,\mathcal{O})$, $||\Phi|| = 1$ , and set

$$(5.4) \qquad H(\Phi,s) = (\Delta - (-s^2+|\rho|^2+\mu))\Theta(\Phi,s), \quad s\in\mathbb{C}.$$

**Lemma 5.5** There exist constants $C_1,C_2> 0$ such that for $\varphi\in L^2(\Gamma\backslash X,E)$ and $s\in\mathbb{C}$,

$$|P(s)|\ |((\Delta_t-(-s^2+|\rho|^2+\mu))^{-1}(H(\Phi,s)),\varphi)| \le C_1\exp(C_2|s|^{n+2})||\varphi||$$

**Proof.** First we observe that by Lemma 4.10, $\Delta H(\Phi,s)\in \mathcal{H}_t(\mathcal{O})$. Hence $H(\Phi,s)\in H^1_t(\Gamma\backslash X,E;\mathcal{O})$. From the description of the domain of $\Delta_t$ it follows then that $H(\Phi,s)$ belongs to the domain of $\Delta_t$ and $\Delta_t H(\Phi,s) = \Delta H(\Phi,s)$. Iterating this argument it follows that for each $l \in \mathbb{N}$, $H(\Phi,s)$ is in the domain of $\Delta_t^l$ and $\Delta_t^l H(\Phi,s) = \Delta^l H(\Phi,s)$. Now let $\{\Phi_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions of $\Delta_t$ corresponding to the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ Using the observation above, we get

$$(5.6) \qquad \lambda_j^l (H(\Phi,s),\Phi_j) = (\Delta^l H(\Phi,s),\Phi_j) \ , \ j,l \in \mathbb{N}.$$

Let $\varphi \in L^2(\Gamma\backslash X,E)$. Then by (5.6),

$$((\Delta_t - (-s^2+|\rho|^2+\mu))^{-1}(H(\Phi,s)),\varphi) = \sum_{j=1}^{\infty} \frac{(\Delta^n H(\Phi,s),\Phi_j)(\Phi_j,\varphi)}{\lambda_j^n(\lambda_j - (-s^2+|\rho|^2+\mu))}$$

Employing Lemma 5.2 and (5.1) it follows that the right hand side, multiplied by $P(s)$, can be estimated by

$$C_1 \, \|\Delta^n H(\Phi,s)\| \, \|\varphi\| \exp(C_2|s|^{n+2}) \ .$$

Now apply (1.2) to estimate $\|\Delta^n H(\Phi,s)\|$ and the result follows. Q.E.D.

Let $\Psi \in L^2_{cus}(\Gamma_{M_{ij}}\backslash M_{ij},\sigma,\mathcal{O}_{ij})$ $(1 \leq i \leq r,\ 1 \leq j \leq r_i)$, $\|\Psi\|=1$, and let $g \in C^\infty(\mathbb{R})$ with $\operatorname{supp} g \subset (t-1,t)$. Set

$$\psi(x) = g(H_{P_{ij}}(x))\Psi(x).$$

**Lemma 5.7** There exist constants $C_1,C_2 > 0$ such that

$$|P(s)|\,|(G(\Phi,s),E(\psi|P_{il}))| \leq C_1 \exp(C_2|s|^{n+2})\|h\|_{L^2} \ , s \in \mathbb{C} \ .$$

$i=1,\ldots,r$, $l=1,\ldots,r_i$.

**Proof.** If we use Lemma 2.5 together with (4.20) and (2.2), we obtain

$$(\Theta(\Phi_{jk},s),E(\psi|P_{il})) =$$

$$= \sum_{w\in W(a_{jk},a_{il})} \int_{c-i\infty}^{c+i\infty} \hat{f}(z-s)\overline{\hat{g}(-w\bar{z}+|\rho|)(c_{P_{il}|P_{jk}}(w:z)\Phi_{jk},\Psi)}\, d|z|$$

where c is any real number with $c > \mathrm{Re}(s)$. Using Lemma 2.3, one can estimate the right hand side by $C_3\exp(C_4|s|)\|h\|_{L^2}$. Thus (4.16) together with Lemma 5.5 implies

$$|P(s)|\,|(F(\Phi,s),E(\psi|P_{il}))| \le C_1\exp(C_2|s|^{n+2})(\|h\|_{L^2} + \|E(\psi|P_{il})\|),$$

for $s\in\mathbb{C}$. Making use of Lemma 2.8 and Lemma 2.3 it is easy to see that $\|E(\psi|P_{il})\| \le C_5\|h\|_{L^2}$. Furthermore, if we apply Lemma 2.5, then it follows from (4.24) that

$$(F(\Phi,s),E(\psi|P_{il})) = (G(\Phi,s),E(\psi|P_{il})).$$

Q.E.D.

Using again Lemma 2.5 combined with Lemma 4.27, we obtain

$$(G(\Phi,s),E(\psi|P_{il})) =$$

$$= \int_{\mathbb{R}} e^{(s-|\rho|)u}g(u)du\,((A(s)\Phi)_{il},\Psi) + \tag{5.8}$$

$$+ \int_{\mathbb{R}} e^{(-s-|\rho|)u}g(u)du\,((B(s)\Phi)_{il},\Psi)\ .$$

Now we make a particular choice for g. Let $h \in C^\infty(\mathbb{R})$ with supp h contained in $(t-1,t)$ and set $g(u)=e^{(s+|\rho|)u}\frac{d}{du}(e^{-2su}h(u))$. Then the second integral involving g vanishes and the first one equals $2s \int_{\mathbb{R}} h(u)du$. Furthermore, $\|g\|_{L^2} \le Ce^{c|s|}(\|h\|_{L^2} + \|h'\|_{L^2})$. Assume that $h \geq 0$, $h \neq 0$. Using Lemma 5.7 together with (5.8) we get an estimate for $|P(s)|\,|((A(s)\Phi)_{i1},\Psi)|$. In the same way one can estimate $|P(s)|\,|((B(s)\Phi)_{i1},\Psi)|$. Summarizing our results, we have seen that there exist constants $C,c > 0$ such that for all $\Phi,\Psi \in E_{cus}(\sigma,\mathcal{O})$ with $\|\Phi\|=1$, $\|\Psi\|=1$, we have

$$|P(s)|\,|(A(s)\Phi,\Psi)| \le C\exp(c|s|^{n+2})$$

(5.9)

$$|P(s)|\,|(B(s)\Phi,\Psi)| \le C\exp(c|s|^{n+2})$$

for $s\in\mathbb{C}$. This implies the following

**Theorem 5.10** Let $C(s) : E_{cus}(\sigma,\mathcal{O}) \longrightarrow E_{cus}(\sigma,\mathcal{O})$ be the intertwining operator. There exist entire functions $F_1(s)$ and $F_2(s)$ of order $\le n+2$ such that

$$\det C(s) = \frac{F_1(s)}{F_2(s)} \quad , \ s\in\mathbb{C}.$$

**Proof.** By Theorem 4.30, we have $\det C(s) = \det B(s)(\det A(s))^{-1}$. Set $F_1(s) = P(s)^n \det B(s)$ and $F_2(s) = P(s)^n \det A(s)$. It follows from (5.9) that $F_1(s)$ and $F_2(s)$ are entire functions of order $\le n+2$. Q.E.D.

## 6. Factorization of det C(s)

We keep the notation of the previous sections. In view of Theorem 5.10 we can apply Hadamard's factorization theorem to factorize det C(s). This however, needs some additional preparation.

**Lemma 6.1** Let $d=\dim \mathcal{E}_{cus}(\sigma,\mathcal{O})$ and set $q_1=e^{2(t_o+1)d}$. Then

$$\lim_{\sigma\to\infty} q_1^{-\sigma}|\det C(\sigma+i\tau)| = 0$$

for all $\tau\in\mathbb{R}$.

**Proof.** According to Proposition 4.7,

$$E(\Phi,s) = \Theta(\Phi,s) - (\bar{\Delta} - (-s^2+|\rho|^2+\mu))^{-1}(H(\Phi,s))$$

for $-s^2+|\rho|^2+\mu\notin \operatorname{Spec}(\bar{\Delta})$. Here $H(\Phi,s)$ is defined by (5.4). If we follow the proof of Lemma 4.10, then we see that there exists $t_o\in\mathbb{R}$ independent of the orbit $\mathcal{O}$ such that

$$\Theta^{P_{il}}(\Phi,s,x) = e^{(s+|\rho|)t_{il}(x)}\Phi_{il}(x)$$

for $\Phi\in\mathcal{E}_{cus}(\sigma,\mathcal{O})$ and $H_{P_{il}}(x) > t_o H_{\rho_{il}}$, $i=1,\dots,r$, $l=1,\dots,r_i$. Using this fact together with (3.2), we see by comparing the constant terms that

$$-e^{(-s+|\rho|)t_{il}(x)}(C(s)\Phi)_{il}(x)$$

is the constant term of $(\bar{\Delta} - (-s^2+|\rho|^2+\mu))^{-1}(H(\Phi,s))$ along $P_{il}$ for $H_{P_{il}}(x) > t_o H_{\rho_{il}}$. Now observe that

$$\|(\bar{\Delta} - (-s^2+|\rho|^2+\mu))^{-1}\| = \operatorname{dist}(-s^2+|\rho|^2+\mu,\operatorname{Spec}(\bar{\Delta}))^{-1}$$

(c.f. [K,V,§3.8]). But $\mathrm{Spec}(\bar{\Delta}) \subset [c,\infty)$, $c > -\infty$. Hence for $\mathrm{Re}(s) \geq C_1$,

$$(6.2) \qquad \|(\bar{\Delta} - (-s^2+|\rho|^2+\mu))^{-1}(H(\Phi,s))\| \leq C_2|s|^{-2}\|H(\Phi,s)\|$$

for some constants $C_1$, $C_2$. Using (4.11), Lemma 2.8 and Lemma 2.3, a simple estimation gives

$$(6.3) \qquad \|H(\Phi,s)\| \leq C_3|s|e^{(u_o+1)\mathrm{Re}(s)}, \quad s\in\mathbb{C}.$$

Let $g\in C^\infty(\mathbb{R})$ with $\mathrm{supp}\, g \subset (t_o,t_o+1)$ and $\Psi\in L^2_{cus}(\Gamma_{M_{i1}}\backslash M_{i1},\sigma,\mathcal{O}_{i1})$. Set $\psi(x)=g(H_{P_{i1}}(x))\Psi(x)$. Using the observation above concerning the constant term of $(\bar{\Delta} - (-s^2+|\rho|^2+\mu))^{-1}(H(\Phi,s))$, it follows from Lemma 2.5, combined with (6.2) and (6.3) that

$$\left|\int_{\mathbb{R}} g(u)e^{-(s+|\rho|)u}du\,((C(s)\Phi)_{i1},\Psi)\right| =$$

$$= |((\bar{\Delta} - (-s^2+|\rho|^2+\mu))^{-1}(H(\Phi,s)),E(\psi|P_{i1}))| \leq$$

$$\leq C_4|s|^{-1}e^{(u_o+1)\mathrm{Re}(s)}\|E(\psi|P_{i1})\| \quad ,$$

$\mathrm{Re}(s) \geq C_1$. We may assume that $t_o \geq u_o$. Let $\Psi\in\mathcal{E}_{cus}(\sigma,\mathcal{O})$, $\|\Psi\|=1$. Then this inequality implies that

$$|(C(s)\Phi,\Psi)| \leq C_5|s|^{-1}e^{2(t_o+1)\mathrm{Re}(s)}, \quad \mathrm{Re}(s) \geq C_1.$$

Hence

$$|\det C(s)| \leq C_6|s|^{-d}\,e^{2d(t_o+1)\mathrm{Re}(s)}, \quad \mathrm{Re}(s) \geq C_1.$$

This implies the Lemma. Q.E.D.

Let $\sigma_1,\ldots,\sigma_l \in (0,|\rho|]$ be the poles of $\det C(s)$ in $\mathrm{Re}(s) \geq 0$ and let $q_1$ be as in Lemma 6.1. Set

$$\xi(s) = q_1^{-s} \prod_{i=1}^{l} \frac{(s-\sigma_i)}{(s+\sigma_i)} \det C(s) \quad . \tag{6.4}$$

Then $\xi(s)$ has the following properties:

1) $\xi(s)\xi(-s) = 1$, $s \in \mathbb{C}$.

2) $|\xi(s)| = 1$ for $\mathrm{Re}(s)=0$.

3) $\xi(s)$ is holomorphic in the half-plane $\mathrm{Re}(s) > 0$ and satisfies $|\xi(s)| \leq 1$ for $\mathrm{Re}(s) \geq 0$.

1) and 2) follow from (3.1). 3) is a consequence of 2), Lemma 6.1 and the maximum principle. Consider the series

$$\sum_{\eta} \frac{\mathrm{Re}(\eta)}{|\eta|^2} \tag{6.5}$$

where $\eta$ runs over all zeros, counted to multiplicity, of $\xi(s)$ in the half-plane $\mathrm{Re}(s) > 0$. Then we have

**Lemma 6.6** The series (6.5) converges.

**Proof.** By 3), $\xi(s)$ is analytic in the half-plane $\mathrm{Re}(s) > 0$ and is continuous and bounded in the half-plane $\mathrm{Re}(s) \geq 0$. The convergence follows from Carleman's theorem [T,§3.71] . Q.E.D.

Now observe that by Theorem 5.10,

$$\xi(s) = \frac{H_1(s)}{H_2(s)} \;, \quad s \in \mathbb{C} \;,$$

where $H_1(s)$ and $H_2(s)$ are entire functions of order $\leq n+2$. Let

$\eta$ be a zero of $\xi(s)$. Then it follows from (3.1) that $\bar{\eta}$ is a zero and $-\eta$, $-\bar{\eta}$ are poles of $\xi(s)$. By Hadamard's factorization theorem we get

$$\xi(s) = e^{P(s)} \frac{\prod_{\eta} E(\frac{s}{\eta},n+2)\, E(\frac{s}{\bar{\eta}},n+2)}{\prod_{\eta} E(\frac{s}{-\eta},n+2) E(\frac{s}{-\bar{\eta}},n+2)} \tag{6.7}$$

where $\eta$ runs over half the zeros of $\xi(s)$ in $\mathrm{Re}(s) > 0$ and we have chosen one representative for each pair $\{\eta,\bar{\eta}\}$ of zeros. $P(s)$ is a polynomial in $s$ of order $\leq n+2$. Now consider the expression

$$I_k(\eta) = \frac{1}{\eta^k} + \frac{1}{\bar{\eta}^k} - \frac{1}{(-\eta)^k} - \frac{1}{(-\bar{\eta})^k}$$

for $1 \leq k \leq n+2$, $\eta \in \mathbb{C}$. If $k$ is even then $I_k = 0$. Assume that $k$ is odd. Put $\eta = |\eta| e^{i\vartheta}$. Then

$$I_k = \frac{4}{|\eta|^k} \cos(k\vartheta) \; .$$

For $k$ odd there exists a constant $C(k)$ such that $|\cos(k\vartheta)| \leq$ $\leq C(k)\, |\cos\vartheta|$. Hence by Lemma 6.6,

$$\sum_{\eta} |I_k(\eta)| \leq C_1(k) \sum_{\eta} |I_1(\eta)| = C_1(k) \sum_{\eta} \frac{\mathrm{Re}(\eta)}{|\eta|^2} < \infty \; .$$

Therefore, the exponential factors in (6.7) can be combined to give

$$\xi(s) = e^{Q(s)} \prod_{\eta} \frac{(s-\eta)(s-\bar{\eta})}{(s+\eta)(s+\bar{\eta})} \; .$$

$Q(s)$ is a polynomial of degree $\leq n+2$. The infinite product can be rewritten as

$$\prod_{\eta} \left(1 - 4s \frac{\mathrm{Re}(\eta)}{(s+\eta)(s+\bar{\eta})}\right)$$

and by Lemma 6.6, this product is absolutely convergent.

Now consider $Q(s)$. The equation $\xi(i\lambda)\xi(-i\lambda)=1, \lambda\in\mathbb{R}$, implies $Q(i\lambda) + Q(-i\lambda) = 2\pi i l$ for some $l\in\mathbb{Z}$. Thus

$$Q(s) = \sum_{k=o}^{[\frac{n+2}{2}]} a_k s^{2k+1} + \pi i l .$$

Moreover by (3.1), $\overline{\xi(s)} = \xi(\bar{s})$. Therefore $a_k\in\mathbb{R}$. Let $k_o$ be the largest $k$ such that $a_k \neq 0$. Assume that $k_o > 0$. If $a_{k_o} > 0$, we get

$$\xi(\sigma) \sim \exp(a_{k_o}\sigma^{2k_o+1})$$

for $\sigma\in\mathbb{R}$ and $\sigma \longrightarrow \infty$ . This contrdicts the fact that $|\xi(s)| \leq 1$ in $\mathrm{Re}(s) \geq 0$. Now assume that $a_{k_o} < 0$. Then we can choose $s$ in the half-plane $\mathrm{Re}(s) > 0$ so that $\mathrm{Re}(s^{2k_o+1}) < 0$ and tends to $-\infty$ as $s \longrightarrow \infty$. Again, we get $|\xi(s)| \longrightarrow \infty$. Thus $Q(s)=as + \pi i l$, $a\in\mathbb{R}$, $a < 0$. Using (6.4), we obtain

**Theorem 6.8** Let $\sigma_1,\ldots,\sigma_l\in(0,|\rho|]$ denote the poles of $\det C(s)$ in the half-plane $\mathrm{Re}(s) \geq 0$ and let $\eta$ run over all poles, counted to multiplicity, of $\det C(s)$ in the half-plane $\mathrm{Re}(s) < 0$. Finally, let $q=q_1e^a$. Then

$$\det C(s) = \det C(0)\, q^s \prod_{j=1}^{l} \frac{s+\sigma_j}{s-\sigma_j} \prod_{\eta} \frac{s+\bar{\eta}}{s-\eta} , \quad s\in\mathbb{C} .$$

Using Theorem 6.8 we can compute the logarithmic derivative of det C(s):

$$\frac{d}{ds}\log\det C(s) = \log q - \sum_{j=1}^{1}\frac{2\sigma_j}{s^2-\sigma_j^2} - \sum_{\eta}\frac{2\mathrm{Re}(\eta)}{(s-\eta)(s+\bar{\eta})} .$$

Now put $s=i\lambda$ , $\lambda\in\mathbb{R}$. Then

$$\frac{d}{ds}\log\det C(i\lambda) = \log q + \sum_{j=1}^{1}\frac{2\sigma_j}{\lambda^2+\sigma_j^2} + \tag{6.9}$$

$$+ \sum_{\eta}\frac{2\mathrm{Re}(\eta)}{\mathrm{Re}(\eta)^2+(\lambda-\mathrm{Im}(\eta))^2}$$

## 7. Estimation of the number of poles of det C(s)

Let the notation be the same as in §3. Our purpose in this section is to obtain an estimate, which is uniform with respect to $\mathcal{O}$, of the number of poles of det C(s) in a finite region.
As above, let $\mu=\mu(\mathcal{O})$ be the common eigenvalue $-\chi(\Omega_{M_{il}})$, $\chi\in\mathcal{O}_{il}$, $i=1,\dots,r$, $l=1,\dots,r_i$. First we prove

**Theorem 7.1** There exists a constant $C>0$ which is independent of $\mathcal{O}$ such that

$$\left|\int_{-\Lambda}^{\Lambda}\frac{d}{ds}\log\det C(i\lambda)\,d\lambda\right|\le C(1+(\Lambda^2+|\rho|^2+\mu)^{n/2})(1+\mu^{n/2}),\quad \Lambda\in\mathbb{R}.$$

**Proof.** Let $t_o$ be as in §4 and $t\ge t_o$. Set

$$C_t(s)=e^{-2st|\rho|}C(s).$$

In view of (3.1), $C_t(s)$ is unitary for Re(s)=0 and hence, can be diagonalized. Moreover, $C_t(s)$ is holomorphic in a neighborhood of Re(s)=0. Therefore we can apply Rellich's theorem [Ba,p.142] which implies that there exist real valued real analytic functions $\beta_1(\lambda),\dots,\beta_d(\lambda)$ of $\lambda\in\mathbb{R}$ such that $e^{i\beta_1(\lambda)},\dots,e^{i\beta_d(\lambda)}$ are the eigenvalues of $C_t(i\lambda)$. Each $\beta_j(\lambda)$ is only determined up to $2\pi\mathbb{Z}$. Moreover, the functional equation (3.1) implies $C_t(0)^2=\mathrm{Id}$. Hence $\beta_j(0)=\pi l$, $l\in\mathbb{Z}$, $j=1,\dots,d$. Put

$$\tilde\beta_j(\lambda)=\int_0^{\infty}\beta_j'(u)\,du,\ j=1,\dots,d.$$

Then we can choose either $\beta_j=\tilde\beta_j$ or $\beta_j=\tilde\beta_j+\pi$ and we get

$$(7.2)\qquad \left| \int_{-\Lambda}^{\Lambda} \frac{d}{ds} \log \det C_t(i\lambda)\, d\lambda \right| \le 2 \sum_{j=1}^{d} |\tilde{\beta}_j(\Lambda)| \le 2d \max_j |\tilde{\beta}_j(\Lambda)| .$$

Let $n_j(\Lambda)$ be the number of points $w \in [0,\Lambda]$ such that $e^{i\beta_j(w)} = -1$, i.e., $\beta_j(w) = (2k+1)\pi$ for some $k \in \mathbb{Z}$. Obviously, we have

$$|\tilde{\beta}_j(\Lambda)| \le 4\pi n_j(\Lambda) \ , \ j=1,\dots,d \ .$$

Let $n(\Lambda)$ be the number of points $w \in [0,\Lambda]$ such that $C_t(iw)$ has at least one eigenvalue equal to $-1$. Then $n_j(\Lambda) \le n(\Lambda)$, $j=1,\dots,d$, and by (7.2), it is sufficient to estimate $n(\Lambda)$. Let $w \in [0,\Lambda]$ and $\Phi \in \mathcal{E}_{cus}(\sigma,\mathcal{O})$, $\Phi \neq 0$, and assume that $C_t(iw)\Phi = -\Phi$, i.e., $C(s)\Phi = -e^{2iwt|\rho|}\Phi$ . Set $T = tT_\rho$ . Using Lemma 3.14 and Theorem 3.23, we obtain

$$n(\Lambda) \le N_T(\Lambda^2 + |\rho|^2 + \mu) \le C(1 + (\Lambda^2 + |\rho|^2 + \mu)^{n/2}) \ .$$

Furthermore, $d = \dim \mathcal{E}_{cus}(\sigma,\mathcal{O})$ can be estimated by Theorem 9.1 of [D1]. Then (7.2) implies our result. Q.E.D.

Now we can estimate the number of poles of $\det C(s)$ in the half-plane $Re(s) < 0$ . First we consider poles on the real line. Observe that for $\sigma \in \mathbb{R}^+$,

$$\int_{-1}^{1} \frac{\sigma}{\sigma^2 + \lambda^2}\, d\lambda \le \int_{-\infty}^{\infty} \frac{1}{1+\lambda^2}\, d\lambda \ .$$

In Theorem 7.1 we put $\Lambda = 1$ and then insert (6.9). If we make use of Corollary 3.26, we get

$$(7.3) \qquad \left| 2 \log q + \int_{-1}^{1} \sum_{\eta} \frac{2\mathrm{Re}(\eta)}{\mathrm{Re}(\eta)^2 + (\lambda - \mathrm{Im}(\eta))^2} \, d\lambda \right| \le C(1+\mu^{3n/2})$$

with C independent of $\mathcal{O}$. We distinguish two cases:

a) $q \ge 1$. By definiton of q, we have

$$0 \le \log q \le 2(t_o+1) \dim E_{cus}(\sigma, \mathcal{O}) .$$

Using Theorem 9.1 in [D1], it follows that

$$\log q \le C_1(1+\mu^{n/2})$$

with $C_1$ independent of $\mathcal{O}$. Since $\mathrm{Re}(\eta) < 0$, (7.3) implies

$$(7.4) \qquad \int_{-1}^{1} \sum_{\eta} \frac{|\mathrm{Re}(\eta)|}{\mathrm{Re}(\eta)^2 + (\lambda - \mathrm{Im}(\eta))^2} \, d\lambda \le C_2(1+\mu^{3n/2}) .$$

b) $q < 1$. Then $\log q < 0$. On the other hand, all terms in the series on the left hand side of (7.3) are negative. Hence we get (7.4) in this case too.

Let $c > 0$ and denote by $N(c, \mathcal{O})$ the number of poles, counted to multiplicity, of $\det C(s)$ in $[-c, 0)$. Then

$$N(c, \mathcal{O}) \int_{-1/c}^{1/c} \frac{1}{1+\lambda^2} \, d\lambda \le \sum_{\substack{-c \le \lambda < 0 \\ \sigma \text{ pole}}} \int_{-1}^{1} \frac{|\sigma|}{\sigma^2 + \lambda^2} \, d\lambda \le$$

$$\le \int_{-1}^{1} \sum_{\eta} \frac{|\mathrm{Re}(\eta)|}{\mathrm{Re}(\eta)^2 + (\lambda - \mathrm{Im}(\eta))^2} \, d\lambda .$$

Using (7.4) and Corollary 3.25, we get

**Theorem 7.5** Let $c > 0$. There exists $C > 0$ independent of $\mathcal{O}$ such that the number of poles, counted to multiplicity, of $\det C(s)$ in $[-c, |\rho|]$ is bounded by $C(1 + \mu^{3n/2})$.

Choose an orthonormal basis $\Phi_1, \ldots, \Phi_d$ in $\mathcal{E}_{cus}(\sigma, \mathcal{O})$ and set $C_{ij}(s) = (C(s)\Phi_i, \Phi_j)$. Let $s_o$ be a pole of $C(s)$ and let $\nu_{ij}(s_o)$ be the order of $C_{ij}(s)$ in $s_o$. Set

$$\nu(s_o) = \max_{i,j} \nu_{ij}(s_o) .$$

If $s_o$ is not a pole of $C(s)$ we set $\nu(s_o) = 0$.

**Corollary 7.6** Let $c > 0$. There exists $C > 0$ independent of $\mathcal{O}$ such that

$$\sum_{-c \le s_o \le |\rho|} \nu(s_o) \le C(1 + \mu^{3n/2}).$$

**Proof.** We write

$$\sum_{-c \le s_o \le |\rho|} \nu(s_o) = \sum_{-c \le s_o \le 0} \nu(s_o) + \sum_{0 \le s_o \le |\rho|} \nu(s_o) .$$

Since $\nu(s_o) \le 1$ for $s_o \ge 0$, the second sum equals the number of poles of $C(s)$ in $[0, |\rho|]$ which can be estimated by Corollary 3.25. Now assume that $s_o$ with $\mathrm{Re}(s_o) < 0$ is a pole of $C(s)$. We distinguish two cases:

a) $C(s)$ is holomorphic at $-s_o$.

By (3.1) we have

(7.7) $$C(s) = C(-s)^{-1} = (\det C(-s))^{-1} D(-s)$$

and $D(-s)$ is obtained from $C(-s)$ via Kramer's rule. Then $D(-s)$ is holomorphic at $s_o$. Hence $\det C(-s)$ has to have a zero of order $\geq \nu(s_o)$ at $s_o$. Again, by (3.1),

$$\det C(s) = (\det C(-s))^{-1} \tag{7.8}$$

and therefore, $\det C(s)$ has a pole of order $\geq \nu(s_o)$ at $s_o$.

b) $-s_o$ is a pole of $C(s)$.

Since $\mathrm{Re}(-s_o) > 0$, it follows that $-s_o \in (0,|\rho|]$ and $\nu(-s_o) \leq 1$. Hence each $(D(s)\Phi_i,\Phi_j)$ $(1 \leq i,j \leq d)$ has at most a pole of order $d-1$ at $-s_o$. Assume that $\nu(s_o) \geq d$. By (7.7), it follows that $\det C(-s)$ has to have a zero of order $\geq \nu(s_o)-d+1$ at $s_o$. By (7.8), $\det C(s)$ has a pole of order $\geq \nu(s_o)-d+1$ at $s_o$. Our result follows now from Theorem 7.5, Corollary 3.25 and Theorem 9.1 of [D1]. Q.E.D.

The same method can be used to estimate the number of poles of $\det C(s)$ in a circle of radius $\Lambda$. If we repeat the arguments above with the integral over $[-1,1]$ replaced by the integral over $[-2\Lambda,2\Lambda]$, we get

**Theorem 7.9** There exists a constant $C > 0$ which is independent of $\mathcal{O}$ such that

$$\sum_{|\eta| \leq \Lambda} 1 \leq C(1 + \Lambda^n + \mu^{3n/2}), \quad \Lambda \geq 0,$$

where $\eta$ runs through the poles, counted to multiplicity, of $\det C(s)$.

## 8. The trace class conjecture

We shall now prove Theorem 0.1 of the introduction. The proof will follow from Theorem 7.5 and the description of the residual spectrum by Langlands [L1].

As mentioned in the introduction, the discrete spectrum $L^2_d(\Gamma\backslash G,\sigma)$ decomposes in the direct sum of the space of cusp forms $L^2_{cus}(\Gamma\backslash G,\sigma)$ and its orthogonal complement $L^2_{res}(\Gamma\backslash G,\sigma)$ - the residual spectrum and, in view of [D1], it is sufficient to prove Theorem 0.1 for eigenfunctions in $L^2_{res}(\Gamma\backslash G,\sigma)$. For this purpose we have to recall the description of $L^2_{res}(\Gamma\backslash G,\sigma)$ obtained by Langlands in [L1, Ch.7]. It follows from his theory of Eisenstein systems that $L^2_{res}(\Gamma\backslash G,\sigma)$ is spanned by "iterated residues" of cuspidal Eisenstein series. We shall now explain this in more detail.

Let P=NAM be a $\mathbb{Q}$-parabolic subgroup of G. If $\alpha\in\Phi_P$, denote by $\check{\alpha} = 2H_\alpha/\alpha(H_\alpha)$ the co-root associated to $\alpha$. Given $\alpha\in\Phi_P$ and $c\in\mathbb{R}$, we set

$$H(\alpha,c) = \{\Lambda\in\mathfrak{a}^*_{\mathbb{C}} \mid \Lambda(\check{\alpha})=c\}.$$

An affine subspace $H\subset\mathfrak{a}^*_{\mathbb{C}}$ is called admissible if $H$ is the intersection of such hyperplanes. Suppose that $H_1\supset H_2$ are two admissible affine subspaces of $\mathfrak{a}^*_{\mathbb{C}}$ and $H_2$ is of codimension one in $H_1$. Let $F(\Lambda)$ be a meromorphic function on $H_1$ whose singularities lie along hyperplanes which are admissible as subspaces of $\mathfrak{a}^*_{\mathbb{C}}$. Choose a real unit vector $\Lambda_0$ in $H_1$ normal to $H_2$. Then we can define a meromorphic function $\mathrm{Res}_{H_2}F$ on $H_2$ by

$$\mathrm{Res}_{H_2} F(\Lambda) = \frac{\delta}{2\pi i} \int_0^1 F(\Lambda + \delta e^{2\pi i\vartheta}\Lambda_o)\, d(e^{2\pi i\vartheta})$$

if $\delta$ is so small that $F(\Lambda + z\Lambda_o)$ has no singularities for $0 < |z| < 2\delta$. The singularities of $\mathrm{Res}_{H_2} F$ lie on the intersections with $H_2$ of the singular hyperplanes of F different from $H_2$. Now consider a complete flag

$$\mathfrak{a}^*_{\mathbb{C}} = H_p \supset H_{p-1} \supset \cdots \supset H_1 \supset H_o = \{\Lambda\}$$

of affine admissible subspaces of $\mathfrak{a}^*_{\mathbb{C}}$ and let $\Lambda_i \in H_i$ be a real unit vector which is normal to $H_{i-1}$, $i=1,\dots,p$. We call $F = \{H_i, \Lambda_i\}$ an admissible flag. Let F be a meromorphic function on $\mathfrak{a}^*_{\mathbb{C}}$ whose singularities lie along admissible hyperplanes of $\mathfrak{a}^*_{\mathbb{C}}$. Then we define inductively $F_i$ by

$$F_p = F,\ F_i = \mathrm{Res}_{H_i} F_{i+1},\ i=o,\dots,p-1.$$

Set

$$\mathrm{Res}_F F = F_o .$$

Now let $\chi \in \hat{Z}(\mathfrak{m})$ and $\Phi \in L^2_{cus}(\Gamma_M \backslash M, \sigma, \chi)$. The singularities of the Eisenstein series $E(P|A,\Phi,\Lambda)$ lie along hyperplanes of $\mathfrak{a}^*_{\mathbb{C}}$ which are defined by equations of the form $\Lambda(\check{\alpha}) = w$, $w \in \mathbb{C}$, $\alpha \in \Phi_P$. Let $H(\alpha_i, c_i)$, $i=o,\dots,p-1$, be a set of real singular hyperplanes of $E(P|A,\Phi,\Lambda)$ with $\bigcap_i H(\alpha_i, c_i) = \{\Lambda_o\}$. Set $H_i = \bigcap_{j \geq i} H(\alpha_j, c_j)$, $i=0,\dots,p-1$, and $H_p = \mathfrak{a}^*_{\mathbb{C}}$. Choose real unit vectors $\Lambda_i \in H_i$ normal to $H_{i-1}$. Then $F = \{H_i, \Lambda_i\}$ is an admissible flag. Furthermore, let $\varphi \in C^\infty_c(\mathfrak{a})$ and let $\hat{\varphi}(\Lambda)$ be its Fourier transform. $\hat{\varphi}(\Lambda)$ is holomorphic on $\mathfrak{a}^*_{\mathbb{C}}$.

Put

$$(8.1) \qquad \psi = \mathrm{Res}_F[E(P|A,\Phi,\Lambda)\hat{\varphi}(\Lambda)] \ .$$

It is clear that $\psi$ depends only on the derivatives of $\hat{\varphi}$ at $\Lambda_o$. Let $C(\mathfrak{a}^*)$ be the positive cone in $\mathfrak{a}^*$ spanned by the simple roots of (P,A). If $\Lambda_o \in C(\mathfrak{a}^*)$ then $\psi$ is square integrable and satisfies

$$\Omega\psi = (\|\Lambda_o\|^2 - \|\rho_P\|^2 + \chi(\Omega_M))\psi \ .$$

$L^2_{res}(\Gamma\backslash G,\sigma)$ is spanned by all the $\psi$ obtained in this way where P runs over a set of representatives of the $\Gamma$-conjugacy classes of $\mathbb{Q}$-parabolic subgroups of G. For a given P, $\chi$ runs over $\hat{Z}(\mathfrak{m}_P)$, and $\Phi$ over $L^2_{cus}(\Gamma_{M_P}\backslash M_P,\sigma,\chi)$. Furthermore, if $\psi \in L^2_{res}(\Gamma\backslash G,\sigma)$ is defined by (8.1) then $\|\Lambda_o\|^2 \le \|\rho_P\|^2$. Finally, observe that the dimension of $L^2_{cus}(\Gamma_{M_P}\backslash M_P,\sigma,\chi)$ can be estimated by Theorem 9.1 in [D1]. Therefore, the proof of Theorem 0.1 is reduced to the following problem: For a given cuspidal Eisenstein series $E(P|A,\Phi,\Lambda)$ we have to estimate the number of its singular hyperplanes, counted to multiplicity, which are real and intersect a given compact set containing the origin. Using the scalar product formula for truncated Eisenstein series ([L2,§9],[O-W,p.487]), it follows that it is sufficient to estimate the corresponding number of singular hyperplanes of the intertwining operators $c_{P_2|P_1}(w:\Lambda)$, $w \in W(\mathfrak{a}_{P_1},\mathfrak{a}_{P_2})$, for any pair $P_1,P_2$ of associate $\mathbb{Q}$-parabolic subgroups of G.

To proceed we have to recall some facts from [H,V]. Let $\mathcal{P}$ be a class of associate $\mathbb{Q}$-parabolic subgroups of G. Let P=NAM be any element in $\mathcal{P}$. Denote by $\mathcal{C}$ the set of Weyl chambers in $\mathfrak{a}$. There is a one-to-one correspondence between $\mathcal{P}/G_{\mathbb{Q}}$ and $\mathcal{C}/W(A)$

(c.f. [H,V,§4]). To each $C \in \mathcal{C}$ one can associate a unique $P_C \in \mathcal{P}$ with $P_C = N_C AM$. Let $\mathcal{C} = \{C_1, \dots, C_r\}$ and set $P_i = P_{C_i}$, $i = 1, \dots, r$. This set contains a set of representatives for $\mathcal{P}/G_{\mathbb{Q}}$. For each conjugacy class $\mathcal{P}_i = \{gP_ig^{-1} \mid g \in G_{\mathbb{Q}}\}$ we choose a set of representatives $P_{ik}$ $(1 \le k \le r_i)$ for the $\Gamma$-conjugacy classes in $\mathcal{P}_i$ and choose $y_{ik}$ in $G_{\mathbb{Q}}$ such that $P_{ik} = y_{ik} P_i y_{ik}^{-1}$. Let $A_i$ be a split component of $P_i$, $1 \le i \le r$, and $A_{ik} = y_{ik} A_i y_{ik}^{-1}$. Then $A_{ik}$ is a split component of $P_{ik}$. Let $P_{ik} = N_{ik} A_{ik} M_{ik}$ be the corresponding Langlands decomposition. Let $\mathcal{O} = \{\mathcal{O}_{ik} \mid i = 1, \dots, r,\ k = 1, \dots, r_i\}$ be a set of associate orbits where $\mathcal{O}_{ik} \in \hat{Z}(\mathfrak{m}_{ik})/W(A_{ik})$. Set

$$L_{ik} = \bigoplus_{\chi \in \mathcal{O}_{ik}} L_{cus}(\Gamma_{M_{ik}} \backslash M_{ik}, \sigma, \chi), \quad L_i = \bigoplus_{k=1}^{r_i} L_{ik} .$$

Given $w \in W(\mathfrak{a}_i, \mathfrak{a}_j)$ and $\Lambda \in \mathfrak{a}^*_{i,\mathbb{C}}$, the intertwining operator

$$C_{ji}(w:\Lambda): L_i \longrightarrow L_j$$

is defined by

$$(\psi, C_{ji}(w:\Lambda)\varphi)_{L_i} = (\psi, c_{P_{jl}|P_{ik}}(y_{jl} w y_{ik} : {}^{y_{ik}}\Lambda)\varphi)_{L_{jl}}$$

for $\varphi \in L_{ik}$, $\psi \in L_{jl}$.

As explained in V,§4 of [H], the functional equation implies that there exists k $(1 \le k \le r)$ such that

$$C_{ji}(w:\Lambda) = C_{jk}(1:w\Lambda) C_{ki}(w:\Lambda)$$

and $C_{ki}(w:\Lambda)$ is entire. Hence it is sufficient to consider $C_{ji}(1:\Lambda)$. Furthermore, by Lemma 117 in [H,V,§4] there exists a sequence $i = i_1, \dots, i_p = j$, $1 \le i_l \le r$, such that the chambers $C_{i_l}$

and $C_{i_{l+1}}$ are adjacent for all $l=1,\dots,p-1$ and

(8.2) $$C_{ji}(1:\Lambda) = C_{i_p i_{p-1}}(1:\Lambda)\cdots C_{i_3 i_2}(1:\Lambda) C_{i_2 i_1}(1:\Lambda).$$

Hence our problem is reduced to the investigation of $C_{ji}(1:\Lambda)$ for adjacent chambers $C_i$ and $C_j$. This is done in the proof of Lemma 116 in [H,V,§4] . We recall the main facts. Assume that $i=1$ and $j=2$. Since $C_1$ and $C_2$ are adjacent, there exists a $\mathbb{Q}$-parabolic subgroup $(P',A')$ of G which dominates $(P_1,A)$ and $(P_2,A)$ and whose rank equals $\mathrm{rank}(P_1)-1$. Set $(P'_{1k},A'_{1k}) = {}^{y_{1k}}(P',A')$ and $(P'_{2l},A'_{2l}) = {}^{y_{2l}}(P',A')$ $(1\le k\le r_1,\ 1\le l\le r_2)$. We may assume that there exists $\gamma\in\Gamma$ such that $P'_{1k} = {}^{\gamma}P'_{2l}$. Otherwise one has $c_{P_{2l}|P_{1k}}(y_{2l}y_{1k}^{-1}:{}^{y_{1k}}\Lambda)=0$. Let $u\in(N'_{2l})_{\mathbb{Q}}$ be such that ${}^{\gamma u}A'_{2l} = A'_{1k}$. Let $w\in W(\mathfrak{a}_{1k},\mathfrak{a}_{2l})$ be given by $w=\mathrm{Ad}(y_{2l}y_{1k}^{-1})$ on $\mathfrak{a}_{1k}$ and let $\Lambda_o = {}^{y_{1k}}\Lambda$ . Then

(8.3) $$c_{P_{2l}|P_{1k}}(w:\Lambda_o)=\exp(-{}^{y_{2l}}(\Lambda+\rho_2)(H_{2l}(\gamma))\ \tau_\gamma^{-1} c_{{}^{\gamma}P_{2l}|P_{1k}}(\gamma uw:\Lambda_o)$$

where $\tau_\gamma$ is defined by $(\tau_\gamma\varphi)(x)=\varphi(\gamma x)$. Let $({}^*P_1,{}^*A_1) = (M'_{1k}\cap P_{1k}, M'_{1k}\cap A_{1k})$ and $({}^*P_2,{}^*A_2) = (M'_{1k}\cap {}^{\gamma u}P_{2l}, M'_{1k}\cap {}^{\gamma u}A_{2l})$. If ${}^*P_i={}^*N_i{}^*A_i{}^*M_i$ is the Langlands decomposition of ${}^*P_i$ with respect to ${}^*A_i$, $i=1,2$, then ${}^*M_1=M_{1k}$ and ${}^*M_2={}^{u}M_{2l}$. Moreover, $\mathfrak{a}_{1k}=\mathfrak{a}'_{1k}\oplus {}^*\mathfrak{a}_1$. Let $w_o=\gamma uw$. Then $w_o=1$ on $\mathfrak{a}'_{1k}$. Denote by ${}^*w_o$ the restriction of $w_o$ to ${}^*\mathfrak{a}_1$ and by ${}^*\Lambda_o$ the restriction of $\Lambda_o$ to ${}^*\mathfrak{a}_1$. Then

(8.4) $$c_{{}^{\gamma}P_{2l}|P_{1k}}(\gamma uw:\Lambda_o) = c_{{}^*P_2|{}^*P_1}({}^*w_o:{}^*\Lambda_o).$$

Now observe that $*P_1$ and $*P_2$ are $\mathbb{Q}$-parabolic subgroups of $M'_{1k}$ of rank one. Therefore we can apply Corollary 7.6 to estimate the real poles of the right hand side in a finite interval $[-c,|\rho|]$. Then (8.2) together with (8.3) and (8.4) leads to

**Proposition 8.5** Let $B_R \subset \mathfrak{a}^*_{i,\mathbb{C}}$ be the ball of radius R with center at the origin and let $N_{ji}(R,\mathcal{O})$ be the number of singular hyperplanes, counted to multiplicity, of $C_{ji}(1:\Lambda)$ which are real and intersect $B_R$. There exists a constant $C > 0$ which is independent of $\mathcal{O}$ such that

$$N_{ji}(R,\mathcal{O}) \leq C(1 + \lambda^{2n}).$$

This completes the proof of Theorem 0.1.

At the end of this section we shall explain how one can derive the adèlic version of Corollary 0.2 from our results.

Let G now denote a reductive linear algebraic group defined over $\mathbb{Q}$. For a given place v of $\mathbb{Q}$ we shall write $G(\mathbb{Q}_v)$ for the group of $\mathbb{Q}_v$-rational points of G. In particular, $G(\mathbb{R})$ is now the reductive Lie group which we denoted by G before. Let A be the ring of adèles of $\mathbb{Q}$ and let G(A) be the corresponding adèle-valued group. If f stands for the set of finite places of $\mathbb{Q}$ and $A^f$ is the corresponding ring of finite adèles, then

$$G(A) = G(\mathbb{R})G(A^f).$$

Let $P_o$ be a fixed minimal parabolic subgroup of G, defined over $\mathbb{Q}$. At any finite place v, define $K_v$ to be $G(\mathbb{Z}_v)$ if $G(\mathbb{Q}_v) = P_o(\mathbb{Q}_v)G(\mathbb{Z}_v)$. In this case $K_v$ is a maximal compact subgroup of

$G(\mathbb{Q}_v)$. This covers almost all v. For the remaining v let $K_v$ be any open compact subgroup of $G(\mathbb{Q}_v)$ such that $G(\mathbb{Q}_v)=P_o(\mathbb{Q}_v)K_v$. If $v=\infty$, we let $K_\infty$ be a maximal compact subgroup of $G(\mathbb{R})$ such that the Lie algebras of $K_\infty$ and $A_p(\mathbb{R})$ are orthogonal under the Killing form. Then

$$K = \prod_v K_v$$

is a maximal compact subgroup of $G(\mathbf{A})$.

Let $K^f$ be any open compact subgroup of $G(\mathbf{A}^f)$. It follows from [B1,§5] that $G(\mathbf{A})$ is the disjoint union of finitely many double cosets $G(\mathbb{Q})x_i G(\mathbb{R})K^f$, $1 \leq i \leq l$. Put

$$\Gamma_i = G(\mathbb{Q}) \cap x_i G(\mathbb{R}) K^f x_i^{-1} \quad , \; i=1,\ldots,l \; .$$

Then $\Gamma_i$ is an arithmetic subgroup of $G(\mathbb{R})$ and

(8.6) $$G(\mathbb{Q})\backslash G(\mathbf{A})/K^f = \coprod_{i=1}^{l} (\Gamma_i\backslash G(\mathbb{R}))x_i \; .$$

This allows us to apply our results to the adèlic case.

Let Z be the center of G and $Z(\mathbb{R})^o$ the connected component of 1 in $Z(\mathbb{R})$. It follows from (8.6) that

$$L^2(Z(\mathbb{R})^o G(\mathbb{Q})\backslash G(\mathbf{A}))^{K^f} \simeq \bigoplus_{i=1}^{l} L^2(Z(\mathbb{R})^o \Gamma_i\backslash G(\mathbb{R}))$$

as $G(\mathbb{R})$-modules. Furthermore, if $L^2_d(Z(\mathbb{R})^o G(\mathbb{Q})\backslash G(\mathbf{A}))$ is the discrete spectrum of the right regular representation R of $G(\mathbf{A})$ on $L^2(Z(\mathbb{R})^o G(\mathbb{Q})\backslash G(\mathbf{A}))$ then

(8.7) $$L^2_d(Z(\mathbb{R})^o G(\mathbb{Q})\backslash G(\mathbb{A}))^{K^f} \simeq \bigoplus_{i=1}^{l} L^2_d(Z(\mathbb{R})^o \Gamma_i\backslash G(\mathbb{R}))$$

as $G(\mathbb{R})$-modules. Let

$$h = \prod_v h_v$$

be a function on $G(\mathbb{A})$ which satisfies the following properties:

1) $h \in C_c^\infty(G(\mathbb{R}))$

2) For v finite, $h_v$ is locally constant with compact support.

3) For almost all places v, $h_v$ is the characteristic function of $G(\mathbb{Z}_v)$.

The linear combinations of these functions are usually denoted by $C_c^\infty(G(\mathbb{A}))$. Assume in addition that h is K-finite. Then there exists an open compact subgroup $K^f$ of $G(\mathbb{A}^f)$ such that h is invariant under $K^f$. Hence R(h) maps $L^2(Z(\mathbb{R})^o G(\mathbb{Q})\backslash G(\mathbb{A}))$ into the subspace of $K^f$-invariant functions. Let $R^d(h)$ be the restriction of R(h) to the discrete spectrum. It follows from (8.7) that on the $K^f$-invariant subspace, $R^d(h)$ corresponds under the isomorphism (8.7) to $\bigoplus_{i=1}^{l} R^d_{\Gamma_i}(h_\infty)$. Using Corollary 0.2, we get

**Corollary 8.8** For each K-finite function $h \in C_c^\infty(G(\mathbb{A}))$, the operator $R^d(h)$ is of the trace class.

In the same way one can prove that $R^d_\chi(h)$ is of the trace class for any character $\chi$ of $Z(\mathbb{R})^o$. Here $R_\chi$ is the right regular regular representation twisted by the character $\chi$.

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