Change of polarization and Hodge numbers of moduli spaces of torsion free sheaves on surfaces

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0. Introduction

Let S be a smooth projective surface over the complex numbers, $c_1 \in Pic(S)$ and $c_2 \in H^4(S,\mathbb{Z})$. If L is an ample divisor on S we can study the moduli space $M_L(c_1,c_2)$ of L-semistable torsion free sheaves $\mathcal{E}$ on S of rank 2 with $det(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$ and the open subscheme $M_L^0(c_1,c_2) \subset M_L(c_1,c_2)$ parametrizing locally free sheaves. In [Q1]-[Q3] Qin studies the change of $M_L^0(c_1,c_2)$ when L varies and partially also that of $M_L(c_1,c_2)$ and gives a number of applications. It turns out that the ample cone of S has a chamber structure such that $M_L(c_1,c_2)$ only depends on the chamber of L, and the change of $M_L(c_1,c_2)$, when L passes through the wall between two chambers, can be controlled. In particular in [Q2] these results are used to determine the Picard group of $M_L(\sigma,c_2)$ for S a ruled surface with effective anticanonical bundle and σ the section with σ^2 minimal.

We first extend the approach of Qin to torsion free sheaves. We also look at the connection to the moduli space $Spl(c_1,c_2)$ of simple sheaves and its possible non-separated structure. Then we apply our results to the Hodge numbers of $M_L(c_1,c_2)$ for S a surface with $-K_S$ effective, c_1 not divisible by 2 in $Num(S)$ and L not lying on a wall (then $M_L(c_1,c_2)$ is smooth and projective of dimension $4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$ or empty). Our tool for computing the Hodge numbers are virtual Hodge polynomials (see e.g. [Ch]). We first obtain a simple formula for the change of the Hodge numbers of $M_L(c_1,c_2)$ when L passes through a wall (thm. 3.4). For a $K3$-surface or an abelian surface it follows that they are independent of the chamber of L.

Finally, if S is a ruled surface with $-K_S$ effective and $c_1 \cdot f$ is odd for a fibre f, we compute the Hodge numbers of $M_L(c_1,c_2)$ (thm. 4.4). While the precise formula is quite complicated, for c_2 big enough about 3/8 of the Hodge numbers are independent of c_2 and L and given by a quite simple power series (thm. 4.5).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Background material

(a) Notation and generalities

In this paper let S be a projective surface over $\mathbb{C}$. We denote by $NS(S)$ the Neron-Severi group of S, i.e. the image of $Pic(S) \to H^2(S,\mathbb{Z})$ and by $Pic^0(S)$ its kernel. Let $Num(S) := Pic(S)/_{\equiv}$ where $\equiv$ denotes numerical equivalence.

Proposition 1.1. *(Serre duality and Hirzebruch-Riemann-Roch for extension groups) ([Mu2], see prop. 1.7 in [Q2]). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be torsion free sheaves on S. Then*

1. $\mathrm{Ext}^i(\mathcal{F}_1,\mathcal{F}_2)$ *is canonically dual to* $\mathrm{Ext}^{2-i}(\mathcal{F}_2,\mathcal{F}_1\otimes K_S)$
2. $\sum_i(-1)^i\mathrm{Ext}^i(\mathcal{F}_1,\mathcal{F}_2)$ *is the part in $H^4(X,\mathbb{Z})$ of $ch(\mathcal{F}_1)^*ch(\mathcal{F}_2)td(T_S)$ where $*$ acts on $H^{2i}(S,\mathbb{Z})$ by multiplication with $(-1)^i$.*

Let $c_1 \in Pic(S)$, $c_2 \in \mathbb{Z}$ (which we identify with $H^4(X,\mathbb{Z})$). Let $Spl(c_1,c_2)$ be the moduli space of simple torsion-free sheaves $\mathcal{E}$ on S of rank 2 with $det(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$. This is a locally Hausdorff analytic space of finite dimension ([K-O], [No]). In general it is however not separated and not neccessarily a scheme. Let L be a polarization of S. We mostly consider stability and semistability in the sense of Gieseker and Maruyama. So we write L-(semi)stable instead of Gieseker (semi)stable with respect L and L-slope (semi)stable instead of (semi)stable with respect to L in the sense of Mumford-Takemoto. Let $M_L(c_1,c_2)$ be the moduli space of L-semistable torsion-free sheaves $\mathcal{E}$ on S of rank 2 with $det(\mathcal{E}) = c_1$ and $c_2(\mathcal{E}) = c_2$ and $M_L^s(c_1,c_2)$ its open subscheme of stable sheaves, which is also an open subscheme of $Spl(c_1,c_2)$.

(b) Hodge numbers of Hilbert schemes

For a scheme X over $\mathbb{C}$ let $h^{p,q}(X) = dimH^q(X,\Omega_X^p)$ and

$$h(X:x,y) = \sum_{p,q}(-1)^{p+q}h^{p,q}(X)x^py^q$$

the Hodge polynomial. Let $Hilb^n(S)$ be the Hilbert scheme of zero-dimensional subschemes of length n on S. In [Göl] its Betti numbers were computed and in [G-S] its Hodge numbers using perverse sheaves and mixed Hodge modules. The result is

$$\sum_{n\geq 0} h(Hilb^n(S):x,y)t^n = \prod_{k=1}^{\infty}\prod_{p,q=0}^{2}(1-x^{p+k}y^{q+k})^{(-1)^{p+q+1}h^{p,q}(S)}.$$

Using virtual Hodge polynomials (see below) this was proven independently in [Ch] together with a formula for the Hodge numbers of the variety of pairs of subschemes $Z_n \subset Z_{n+1}$ of lengths n and $n+1$.

(c) Virtual Hodge polynomials

Virtual Hodge polynomials were introduced in [D-K] and brought to my attention by Cheah ([Ch]). They can be viewed as a tool for computing the Hodge numbers of smooth projective varieties by reducing to simpler varieties. I review some of the results and notations about virtual Hodge polynomials from pages 2-3 of [Ch].

Definition 1.2. Let X be a complex variety. Then by [De] the cohomology $H^k_c(X,\mathbb{Q})$ with compact support carries a natural mixed Hodge structure. If X is smooth and projective this Hodge structure coincides with the classical one. Following [Ch] we put

$$e^{p,q}(X) := \sum_k (-1)^k h^{p,q}(H^k_c(X,\mathbb{Q})),$$
$$e(X:x,y) := \sum_{p,q} e^{p,q}(X) x^p y^q.$$

By [D-K] and Ch] these virtual Hodge polynomials have the following properties:

(1) If X is a smooth projective variety, then $e(X:x,y) = h(X:x,y)$.
(2) For $Y \subset X$ Zariski-closed and $U = X \setminus Y$, $e(X:x,y) = e(U:x,y) + e(Y:x,y)$.
(3) For $f: Y \longrightarrow X$ a Zariski-locally trivial fibre bundle with fibre F, $e(Y:x,y) = e(X:x,y)e(F:x,y)$.
(4) If $f: X \longrightarrow Y$ is a bijective morphism, then $e(X:x,y) = e(Y:x,y)$.

2. Walls and chambers for torsion-free sheaves

In this section we review and extend some results of Qin about the change of moduli spaces of torsion-free sheaves when the polarization varies.

Definition 2.1. (see [Q3] Def I.2.1.5) Let C_S be the ample cone in $Num(S) \otimes \mathbb{R}$. For $\xi \in Num(S)$ let

$$W^\xi := C_S \cap \{x \in Num(S) \otimes \mathbb{R} \mid x \cdot \xi = 0\}.$$

W^ξ is called the wall of type (c_1, c_2) determined by ξ if and only if there exists $G \in Pic(S)$ with $G \equiv \xi$ such that $G + c_1$ is divisible by 2 in $Pic(S)$ and $c_1^2 - 4c_2 \le G^2 < 0$. W^ξ is nonempty if there is a polarisation L with $L\xi = 0$. Let $W(c_1, c_2)$ be the union of the walls of type (c_1, c_2). A chamber of type (c_1, c_2) is a connected component of $C_S \setminus W(c_1, c_2)$. In future we dwrite wall and chamber instead of wall and chamber of type (c_1, c_2). We say that W^ξ is a face of a chamber $\mathcal{C}$ if the closure $\overline{\mathcal{C}}$ contains a nonempty open subset of W^ξ. It is clear that two different chambers $\mathcal{C}_1$, $\mathcal{C}_2$ can have at most one common face.

Lemma 2.2. *Let $\mathcal{E}$ be a torsion free sheaf of rank 2 on S with $det(\mathcal{E}) = c_1$, $c_2(\mathcal{E}) = c_2$, which is L_1-semistable and L_2-unstable for two polarizations L_1, L_2 not on a wall.*

(1) *$\mathcal{E}$ is L_1-slope stable and L_2-slope unstable.*

(2) *There is a nontrivial extension*

$$0 \longrightarrow \mathcal{I}_{Z_1}(F) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z_2}(c_1 - F) \longrightarrow 0, \qquad (*)$$

where $\xi \equiv (2F - c_1)$ determines a nonempty wall with $\xi L_1 < 0 < \xi L_2$ and $Z_1 \in Hilb^n(S), Z_2 \in Hilb^m(S)$ with $n + m = (4c_2 - c_1^2 + \xi^2)$.

Proof. This result is essentially shown in the proof of ([Q2] lemma 2.1) for S a ruled surface and c_1 the class of a section. The proof only uses that c_1 is not divisible by 2 in $Num(S)$ in order to exclude $F \equiv c_1 - F$. We assume therefore $F \equiv c_1 - F$. As $\mathcal{E}$ is L_1-semistable and L_2-unstable, it also sits in an extension

$$0 \longrightarrow \mathcal{I}_{W_1}(F + G) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{W_2}(c_1 - F - G) \longrightarrow 0,$$

with $L_1 G \le 0 < L_2 G$. One of the induced maps $\mathcal{I}_{Z_1}(F) \to \mathcal{I}_{W_1}(F + G)$, $\mathcal{I}_{Z_1}(F) \to \mathcal{I}_{W_2}(c_1 - F - G)$ has to be injective, so either G or $c_1 - 2F - G$ is effective, a contradiction to $L_1 G \le 0 < L_2 G$ and $c_1 - 2F \equiv 0$. □

For the rest of section 2 and section 3 we assume that $\xi \equiv 2F - c_1$ determines a nonempty wall of type (c_1, c_2)

Lemma 2.3. *Let $\mathcal{E}$ be given by a non-trivial extension $(*)$. Then*

(1) $\mathrm{Hom}(\mathcal{I}_{Z_1}(F), \mathcal{E}) = \mathbb{C}$.

(2) *$\mathcal{E}$ is simple.*

(3) *$\mathcal{I}_{Z_1}(F)$ is the unique subsheaf of $\mathcal{E}$ of the form $\mathcal{I}_{W_1}(G)$ with torsion-free quotient and $2G - c_1 \equiv \xi$.*

Proof. As $(2F - c_1)^2 < 0$ and $L(2F - c_1) = 0$ for some polarization L, neither $2F - c_1$ nor $c_1 - 2F$ can be effective. Thus $\mathrm{Hom}(\mathcal{I}_{Z_1}(F), \mathcal{I}_{Z_2}(c_1 - F)) = 0$,

$\mathrm{Hom}(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Z_1}(F)) = 0$. So (1) follows by applying $\mathrm{Hom}(\mathcal{I}_{Z_1}(F), \cdot\,)$ to $(*)$. By applying $\mathrm{Hom}(\mathcal{I}_{Z_2}(c_1 - F), \cdot\,)$ to $(*)$ and using that the extension $(*)$ is nontrivial, we get $\mathrm{Hom}(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{E}) = 0$. So $\mathcal{E}$ is simple by the sequence $0 \to \mathrm{Hom}(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{E}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \to \mathrm{Hom}(\mathcal{I}_{Z_1}(F), \mathcal{E})$.

(3) Assume we have a sequence

$$0 \longrightarrow \mathcal{I}_{W_1}(G) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{Z_2}(c_1 - G) \longrightarrow 0,$$

where $2G - c_1 \equiv \xi$. As there are polarizations L_1, L_2 with $L_1\xi < 0 < L_2\xi$, neither $c_1 - F - G$ nor $F + G - c_1$ can be effective. Therefore the induced maps $\mathcal{I}_{W_1}(G) \to \mathcal{I}_{Z_2}(c_1 - F)$, $\mathcal{I}_{Z_1}(F) \to \mathcal{I}_{W_2}(c_1 - G)$ are zero. So $\mathcal{I}_{W_1}(G) \to \mathcal{I}_{Z_1}(F)$ and $\mathcal{I}_{Z_1}(F) \to \mathcal{I}_{W_1}(G)$ are injective and $F = G$, $W_1 = Z_1$. □

Definition 2.4. Let $E_\xi^{n,m}$ be the set of sheaves lying in nontrivial extensions $(*)$ with $len(Z_1) = n, len(Z_2) = m$, where $m + n = c_2 - (c_1^2 - \xi^2)/4$. By Lemma 2.3, $E_\xi^{n,m}$ is a subset of $Spl(c_1, c_2)$. We put $E_\xi = \bigcup_{n+m=c_2-(c_1^2-\xi^2)/4} E_\xi^{n,m}$. By lemma 2.3 this is a disjoint union.

For the rest of sections 2 and 3 when writing L_1, L_2 we will always assume that L_1, L_2 are polarizations in chambers with W^ξ as common face and $\xi L_1 < 0 < \xi L_2$.

Proposition 2.5. *Let $\mathcal{E} \in E_\xi^{n,m}$. Then $\mathcal{E}$ is L_2-slope unstable, and the following are equivalent:*

1. *$\mathcal{E}$ is not L_1-slope stable.*
2. *$\mathcal{E}$ is L-slope unstable with respect to any polarization $L \notin W^\xi$.*
3. *The extension class of $(*)$ lies in the kernel of the natural map $\mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Z_1}(F)) \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{O}(F))$.*
4. *$\mathcal{E} \in E_{-\xi}^{n+m-r,r}$ for some $r < n$.*

Proof. The L_2-slope unstability and the implications (4)⇒(2)⇒(1) are obvious.

(1)⇒(4): Assume $\mathcal{E}$ is not L_1-slope stable. Then we have an exact sequence

$$0 \longrightarrow \mathcal{I}_{W_1}(G) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{W_2}(c_1 - G) \longrightarrow 0 \qquad (**)$$

with $L_1 G \geq L_1(c_1 - G)$. If the induced map $\mathcal{I}_{W_1}(G) \to \mathcal{I}_{Z_1}(F)$ was an injection, we would get the contradiction $L_1 G \leq L_1 F < L_1(c_1 - F) \leq L_1(c_1 - G)$. So $\mathcal{I}_{W_1}(G) \to \mathcal{I}_{Z_2}(c_1 - F)$ is an injection, and $c_1 - F - G$ is effective. Assume $c_1 - F - G$ is strictly effective, and let L be a polarization with $L\xi = 0$. Then $0 < L(c_1 - F - G) = Lc_1/2 - LG$, so $L(2G - c_1) < 0 \leq L_1(2G - c_1)$. By (**) we have $(2G - c_1)^2 \geq c_1^2 - 4c_2$. By $L(2G - c_1) < 0 \leq (2G - c_1)L_1$ there is a polarization M with $M(2G - c_1) = 0$. Thus by the Hodge index theorem and using $L(2G - c_1) < 0$, we get $(2G - c_1)^2 < 0$. So $\eta \equiv 2G - c_1$ defines a nonempty wall. As L_1 does not lie on a wall, W^η lies strictly between L_1 and L, a contradiction. So $G = c_1 - F$, and we have a diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{I}_{W_1}(c_1 - F) & & & & \\
 & & & & \downarrow & \searrow \beta & & & \\
0 & \rightarrow & \mathcal{I}_{Z_1}(F) & \rightarrow & \mathcal{E} & \rightarrow & \mathcal{I}_{Z_2}(c_1 - F) & \rightarrow & 0 \\
 & & & \searrow \alpha & \downarrow & & & & \\
 & & & & \mathcal{I}_{W_2}(F) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0. & & & &
\end{array}
$$

As $c_1 - 2F$ is neither effective nor anti-effective, α and β are injective. As $\mathcal{E}$ is simple, the vertical extension cannot be split. Furthermore $len(Z_1) + len(Z_2) = len(W_1) + len(W_2)$ and, by the injectivity of α (and the fact that $(*)$ is not split), $len(W_2) < len(Z_1)$.
(4)$\Rightarrow$(3): Let $\bar{\mathcal{E}} := (\mathcal{E} \oplus \mathcal{I}_{W_2}(F))/\mathcal{I}_{Z_1}(F)$ (the embedding $\mathcal{I}_{Z_1}(F) \rightarrow \mathcal{E} \oplus \mathcal{I}_{W_2}(F)$ is given by $(*)$ and the standard injection $\mathcal{I}_{Z_1}(F) \rightarrow \mathcal{I}_{W_2}(F)$). Then the projection $\mathcal{E} \rightarrow \mathcal{I}_{W_2}(F)$ and the identity on $\mathcal{I}_{W_2}(F)$ give a map $\bar{\mathcal{E}} \rightarrow \mathcal{I}_{W_2}(F)$ splitting the sequence

$$0 \longrightarrow \mathcal{I}_{W_2}(F) \longrightarrow \bar{\mathcal{E}} \longrightarrow \mathcal{I}_{Z_2}(c_1 - F) \longrightarrow 0$$

induced from $(*)$. Therefore the extension class of $(*)$ lies in $ker\big[\mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Z_1}(F)) \rightarrow \mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{W_2}(F))\big]$, and (3) follows.
(3)$\Rightarrow$(4): Assume $(\mathcal{E} \oplus \mathcal{O}(F))/\mathcal{I}_{Z_1}(F) = \mathcal{O}(F) \oplus \mathcal{I}_{Z_2}(c_1 - F)$. Let $\mathcal{I}_{W_2}(F)$ and $\mathcal{I}_{W_1}(c_1 - F)$ be image and kernel of the composition $\mathcal{E} \rightarrow \mathcal{O}(F) \oplus \mathcal{I}_{Z_2}(c_1 - F) \rightarrow \mathcal{O}(F)$. Then

$$0 \longrightarrow \mathcal{I}_{W_1}(c_1 - F) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_{W_2}(F) \longrightarrow 0,$$

does not split because $\mathcal{E}$ is simple. $\square$

Remark 2.6. Every $\mathcal{E} \in E_\xi^{0,m}$ (in particular each locally free sheaf in E_ξ) is L_1-slope stable and L_2-slope unstable. If $E_\xi^{n,m} \neq \emptyset$ for $n > 0$, then $E_\xi^{n,m} \cap E_\xi^{n+m-r,r} \neq \emptyset$ for each $r < n$. In particular there are $\mathcal{E} \in E_\xi^{n,m}$, which are L-slope unstable for every $L \notin W^\xi$. So prop. 2.5 shows an important difference between locally free sheaves and torsion free sheaves.

Proof. The first sentence is obvious. Let $\mathcal{E} \in E_\xi^{n,m}$ be given by an extension $(*)$, where Z_1 does not intersect Z_2. Let $Y_1 \subsetneq Z_1$ be a subscheme of length r. By the proof of proposition 2.5, $\mathcal{E} \in E_\xi^{n+m-r,r}$ if the extension class of $(*)$ lies in $ker\big[\mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Z_1}(F)) \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Y_1}(F))\big]$ and not in $ker\big[\mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Z_1}(F)) \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Y_2}(F))\big]$ for any scheme Y_2 with $Y_1 \subsetneq Y_2 \subsetneq Z_1$. By the sequence $0 \longrightarrow \mathcal{I}_{Z_1}(F) \longrightarrow \mathcal{I}_{Y_i}(F) \longrightarrow \mathcal{I}_{Y_i/Z_1}(F) \longrightarrow 0$ and the fact that $2F - c_1$ is not effective these kernels are isomorphic to $\mathrm{Hom}(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Y_i/Z_1}(F)) \simeq \mathbb{C}^{n-len(Y_i)}$. □

Definition 2.7. Let $V_\xi^{n,m} \subset E_\xi^{n,m}$ be the set of all torsion free sheaves $\mathcal{E}$ sitting in extensions $(*)$ whose extension class does not lie in $ker\big[\mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Z_1}(F)) \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{W_1}(F))\big]$. We put $V_\xi = \bigcup_{n+m=(4c_2-c_1^2+\xi^2)/4} V_\xi^{n,m}$.

Lemma 2.8. *Assume ξ, η define the same wall and $V_\xi^{n,m} \cap V_\eta^{l,s} \neq \emptyset$. Then $\xi = \eta$ and $n = l$.*

Proof. Let $\mathcal{E} \in V_\xi^{n,m} \cap V_\eta^{l,s}$. Let L be a polarization in a chamber having W^ξ as a face with $L\xi < 0$. Then by proposition 2.5, $\mathcal{E}$ is L-slope stable and therefore $\eta L < 0$. $\mathcal{E}$ fits into sequences $(*)$, $(**)$ with $(2F - c_1) \equiv \xi$, $(2G - c_1) \equiv \eta$. Then, as in the proof of ([Q3] prop. II.1.2.5), $c_1 - F - G$ cannot be effective. Therefore the sequences $(*)$, $(**)$ induce injections $\mathcal{I}_{Z_1}(F) \to \mathcal{I}_{W_1}(G)$, $\mathcal{I}_{W_1}(G) \to \mathcal{I}_{Z_1}(F)$. □

Theorem 2.9.

(1) *For L not on a wall, $M_L(c_1, c_2)$ only depends on the chamber of L, and $M_L(c_1, c_2) \setminus M_L^s(c_1, c_2)$ is independent of L.*

(2) *As subsets of $Spl(c_1, c_2)$ we have a decomposition*

$$M_{L_1}^s(c_1, c_2) = \left(M_{L_2}^s(c_1, c_2) \setminus \left(\coprod_\eta \coprod_{n,m} V_{-\eta}^{n,m} \right) \right) \sqcup \left(\coprod_\eta \coprod_{n,m} V_\eta^{n,m} \right),$$

where η runs over the classes in $Num(S)$ with $\eta L_1 < 0$ defining the wall $W^\eta = W^\xi$ and $n + m = (4c_2 - c_1^2 + \eta^2)/4$. Furthermore $V_\eta^{n,m} = E_\eta^{n,m} \setminus E_\eta^{n,m} \cap E_{-\eta}$, $V_{-\eta}^{n,m} = E_{-\eta}^{n,m} \setminus E_{-\eta}^{n,m} \cap E_\eta$.

Proof. (1) and (2) follow from lemma 2.2. The decomposition follows from lemma 2.2, lemma 2.3 and proposition 2.5. Lemma 2.8 implies that the union is disjoint. The identity $V_\eta^{n,m} = E_\eta^{n,m} \setminus E_\eta^{n,m} \cap E_{-\eta}$ follows from proposition 2.5. □

Remark 2.10. We see from theorem 2.9 and remark 2.6 that theorem 2.6 and corollary 2.7 of [Q2] are imprecise. With S, L, L_0, σ, ζ_1 as in [Q2] the correct result for thm. 2.6 is $M_L(\sigma, c_2) = (M_{L_0}(\sigma, c_2) \setminus E_{-\zeta_1}^{0,1}) \sqcup E_{\zeta_1}^{0,1} \sqcup (E_{\zeta_1}^{1,0} \setminus E_{-\zeta_1}^{1,0})$.

Assume now that the Picard number $\rho(S)$ of S is at least 2.

Proposition 2.11.

(1) *There is a integer k such that for each $c_2 > k$ there exists a component M of $Spl(c_1, c_2)$ containing L_1-slope stable sheaves $\mathcal{E}$ for L_1 lying in one chamber and sheaves $\mathcal{F}$ which are L-slope unstable for each L not lying on a wall.*

(2) *In particular for $c_2 > k$ and c_1 not divisible by 2 in $Num(S)$, $Spl(c_1, c_2)$ is not separated.*

Proof. (1) By $\rho(S) \geq 2$ we find $F \in Pic(S)$ with $2F - c_1 \not\equiv 0$ and $(2F - c_1)L = 0$ for an ample divisor L. Let $\xi \equiv 2F - c_1$, and $l := (4c_2 - c_1^2 + \xi^2)/4$ and choose c_2 big enough, such that $l \geq h^0(S, c_1 - 2F + K_S) + 2$. Then ξ defines a nonempty wall. Let $Z_2 \in Hilb^{l-1}(S)$, then $H^1(S, \mathcal{I}_{Z_2}(c_1 + K_S - 2F)) = \mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{O}(F))^* \neq 0$ by the cohomology sequence of

$$0 \to \mathcal{I}_{Z_2}(c_1 + K_S - 2F) \to \mathcal{O}(c_1 + K_S - 2F) \to \mathcal{O}_{Z_2}(c_1 + K_S - 2F) \to 0.$$

Let $x \in S \setminus Z_2$. Applying $\mathrm{Hom}(\mathcal{I}_{Z_2}(c_1 - F), \cdot\,)$ to $0 \to \mathcal{I}_x(F) \to \mathcal{O}(F) \to \mathcal{O}_x(F) \to 0$, we see that $\mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), I_x(F)) \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{O}(F))$ is surjective but not injective. Thus (1) follows by prop. 2.5 for M the component of $Spl(c_1, c_2)$ containing $E_\xi^{1,l-1}$.

(2) If c_1 is not divisible by 2 in $Num(S)$, $M_{L_1}^s(c_1, c_2) = M_{L_1}(c_1, c_2)$ is an open and projective subscheme of $Spl(c_1, c_2)$, intersecting M; so if M were separated it would contain M, which contradicts (1). □

3. The case of effective anticanonical divisor

Now let S be a surface with $-K_S$ effective. For a simple torsion free sheaf $\mathcal{E}$ on S we have $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E})_0 = 0$, where the index 0 refers to the derived functor of the trace-free homomorphisms. Thus $M_L^s(c_1,c_2)$ is smooth of dimension $4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$ or empty for each polarisation L.

Definition 3.1. Let $T_{n,m} := Pic^0(S) \times Hilb^n(S) \times Hilb^n(S)$ and let $\mathcal{P}$ be the pullback of the Poincaré line bundle from $S \times Pic^0(S)$ to $S \times T_{n,m}$. Let $\mathcal{I}_{Z_n(S)}$ be the ideal sheaf of the universal subscheme $Z_n(S)$ in $S \times Hilb^n(S)$. Let π, p_S, q_1, q_2 be the projections of $S \times T_{n,m}$ to $T_{n,m}$, S, $S \times Hilb^n(S)$ and $S \times Hilb^m(S)$ respectively. Let $\mathcal{V}_1 := q_1^*(\mathcal{I}_{Z_n(S)}) \otimes p_S^*(F) \otimes \mathcal{P}^{\otimes 2}$ and $\mathcal{V}_2 := q_2^*(\mathcal{I}_{Z_m(S)}) \otimes p_S^*(c_1 - F)$. We put $\mathcal{E}_\xi^{n,m} := \mathrm{Ext}^1_\pi(\mathcal{V}_2, \mathcal{V}_1)$, where $\mathrm{Ext}^i_\pi(\mathcal{V}_2, \cdot\,)$ is the right derived functor of $\mathrm{Hom}_\pi(\mathcal{V}_2, \cdot\,) := \pi_* \mathcal{H}om(\mathcal{V}_2, \cdot\,)$.

Lemma 3.2.

(1) *There is an isomorphism* $\mathrm{Ext}^1(\mathcal{V}_2, \mathcal{V}_1) \simeq H^0(S \times T_{n,m}, \mathcal{E}_\xi^{n,m})$.

(2) $\mathcal{E}_\xi^{n,m}$ *is locally free of rank* $-\xi(\xi - K_S)/2 + n + m - \chi(\mathcal{O}_S)$.

(3) *Over* $S \times \mathbb{P}(\mathcal{E}_\xi^{n,m})$ *we have a tautological extension*

$$0 \longrightarrow p^*(\mathcal{V}_1) \longrightarrow \mathcal{V} \longrightarrow p^*(\mathcal{V}_2) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_\xi^{n,m})}(-1) \longrightarrow 0,$$

where $p : S \times \mathbb{P}(\mathcal{E}_\xi^{n,m}) \to S \times T_{n,m}$ *is the projection, such that for each* $t \in \mathbb{P}(\mathcal{E}_\xi^{n,m})$ *the restriction to* $S \times \{t\}$ *is isomorphic to the extension corresponding to* t.

(4) *There is a natural bijective morphism* $\nu_{\xi,n,m} : \mathbb{P}(\mathcal{E}_\xi^{n,m}) \longrightarrow E_\xi^{n,m}$.

Proof. For $t \in T$ the fibres $(\mathcal{V}_2)_t$, $(\mathcal{V}_1)_t$ are $\mathcal{I}_{Z_2}(c_1 - G)$, $\mathcal{I}_{Z_1}(G)$ for suitable $G \in Pic(S)$ with $2G - c_1 \equiv \xi$. As $2G - c_1$ is not effective, $\mathrm{Hom}((\mathcal{V}_2)_t, (\mathcal{V}_1)_t) = 0$ and as $-K_S$ is effective and $c_1 - 2G$ is not effective, $\mathrm{Ext}^2((\mathcal{V}_2)_t, (\mathcal{V}_1)_t) = 0$ by Serre duality. So $\mathrm{Hom}_\pi(\mathcal{V}_2, \mathcal{V}_1) = 0$, $\mathrm{Ext}^1_\pi(\mathcal{V}_2, \mathcal{V}_1)$ is locally free and its rank is given by Riemann Roch (prop. 1.1). (1) and (3) now follow from the degeneration of the spectral sequence $H^i(\mathrm{Ext}^j_\pi(\mathcal{V}_2, \mathcal{V}_1)) \Rightarrow \mathrm{Ext}^{i+j}(\mathcal{V}_2, \mathcal{V}_1)$ see ([H-S],[Q2], [OG]).

(4) By Kodaira classification surfaces S with $-K_S$ effective have torsion-free $H^2(S, \mathbb{Z})$. Therefore $Num(S) = NS(S)$, and by (3) there is a natural surjective morphism $\nu_{\xi,n,m} : \mathbb{P}(\mathcal{E}_\xi^{n,m}) \to E_\xi^{n,m}$. By lemma 2.3 it is also injective. □

Remark 3.3. Let $u : T_{n,m} \to T_{0,m}$ be the projection. Then there is a natural map $\mathcal{E}_\xi^{n,m} \to u^*(\mathcal{E}_\xi^{0,m})$ (which fibrewise is the natural map $\mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{I}_{Z_1}(F)) \to \mathrm{Ext}^1(\mathcal{I}_{Z_2}(c_1 - F), \mathcal{O}(F))$). It gives a section s of $u^*(\mathcal{E}_\xi^{0,m}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_\xi^{n,m})}(1)$ whose zero locus is $\nu_{\xi,n,m}^{-1}(E_\xi^{n,m} \cap E_{-\xi})$ by proposition 2.5. In particular this is a closed subscheme.

Theorem 3.4.

(1) $e(M_{L_1}(c_1,c_2):x,y) = e(M_{L_2}(c_1,c_2):x,y) + ((1-x)(1-y))^{q(S)}$

$$\cdot\left(\sum_\eta h(Hilb^{[l_\eta]}(S\sqcup S):x,y)(xy)^{l_\eta-\eta(\eta+K_S)/2-\chi(\mathcal{O}_S)}\frac{1-(xy)^{\eta K_S}}{1-xy}\right),$$

where η runs over the classes in $Num(S)$ determining the wall $W^\eta = W^\xi$ with $\eta L_1 < 0$ and $l_\eta := (4c_2 - c_1^2 + \eta^2)/4$.

(2) *If c_1 is not divisible by 2 in $Num(S)$ (or more generally if $M_{L_1}(c_1,c_2)$ and $M_{L_2}(c_1,c_2)$ are smooth), then the same holds for $h(M_{L_1}(c_1,c_2):x,y)$ and $h(M_{L_2}(c_1,c_2):x,y)$ instead of $e(M_{L_1}(c_1,c_2):x,y)$ and $e(M_{L_2}(c_1,c_2):x,y)$.*

Proof. If c_1 is not divisible by 2 in $Num(S)$, then for L not lying on a wall $M_L(c_1,c_2) = M_L^s(c_1,c_2)$ is smooth and projective, so (2) follows from (1).

Property (2) of the virtual Hodge polynomials and thm. 2.9 give

$$e(M_{L_1}(c_1,c_2):x,y) = e(M_{L_0}(c_1,c_2):x,y) + \sum_\eta \big(e(V_\eta:x,y) - e(V_{-\eta}:x,y)\big).$$

By remark 3.3 $E_\eta \cap E_{-\eta}$ is a closed subscheme of E_η, so

$$\begin{aligned} &e(V_\eta:x,y) - e(V_{-\eta}:x,y) - \big(e(E_\eta:x,y) - e(E_{-\eta}:x,y)\big) \\ &\quad = e(E_\eta\cap E_{-\eta}:x,y) - e(E_\eta\cap E_{-\eta}:x,y) = 0. \end{aligned}$$

By lemma 2.3 $E_\eta = \coprod_{n+m=l_\eta} E_\eta^{n,m}$, and using also properties (2),(3) and (4) we get

$$e(E_\eta^{n,m}:x,y) = h(E_\eta^{n,m}:x,y) = h(Pic^0(S)\times Hilb^n(S)\times Hilb^m(S)\times \mathbb{P}_w:x,y),$$

where $w+1 = -\eta(\eta-K_S)/2 + l_\eta - \chi(\mathcal{O}_S)$ is the rank of $\mathrm{Ext}^1_\pi(\mathcal{V}_2,\mathcal{V}_1)$. We see that

$$\sum_{n+m=l_\eta} h(Hilb^n(S):x,y)h(Hilb^m(S):x,y) = h\big(Hilb^{l_\eta}(S\sqcup S):x,y\big).$$

So (2) follows by thm 2.9. □

Corollary 3.5. *If S is a $K3$ surface or an abelian surface, and c_1 is not divisible by 2 in $NS(S)$, then the Hodge numbers of $M_L(c_1,c_2)$ are independent of the polarization L as long as L does not lie on a wall.*

Proof. As K_S is trivial in this case, this follows immediately from theorem 3.4. □

4. Hodge numbers of moduli spaces of stable sheaves on ruled surfaces

Let S be a ruled surface with $-K_S$ effective over a curve C of genus g with projection $p : S \longrightarrow C$. Let f be a fibre of p and σ the section with σ^2 minimal. We put $e = -\sigma^2$; then $K_S \equiv -2\sigma + (2g-2-e)f$. Let $c_1 \in Pic(S)$ with $c_1 \cdot f$ odd. By normalizing we assume in future that $c_1 \equiv \sigma + \epsilon f$ with $\epsilon \in \{0,1\}$. We want to compute the Hodge numbers of $M_L(c_1, c_2)$ for a polarization L not lying on a wall. In the case $c_1 = \sigma$ the Picard group $Pic(M_L(c_1, c_2))$ was determined in [Q2] and in the case $c_1 f$ odd and $g = 0$ it was determined in [Na].

Remark 4.1. It is well-known that $NS(S)$ is a free abelian group generated by the classes of σ and f. If $A \equiv \alpha\sigma + \beta f$ is an effective divisor, then $\alpha \geq 0$, $\beta \geq 0$ if $e \geq 0$ and $-e\alpha + 2\beta \geq 0$ if $e < 0$. So the effectiveness of $-K_S$ implies $e \geq 0$ and $2g - 2 \leq e$ or $g = -e = 1$ (see [Q2]).

For $L \equiv \alpha\sigma + \beta f$ we put $r_L = \beta/\alpha$ following [Q2]. Then L is ample if and only if $\alpha > 0$ and $r_L > e$ in case $e \geq 0$ or $\alpha > 0$ and $r_L > e/2$ in case $e < 0$. We also see that $L \cdot M = 0$ if and only if $r_L + r_M = e$.

Remark 4.2. A wall of type (c_1, c_2) is W^ξ for $\xi \equiv (2\alpha+1)\sigma + (2\beta+\epsilon)f$, where α and β are integers such that $-4c_2 + c_1^2 \leq \xi^2 < 0$. We can assume that $\alpha \geq 0$; then this is equivalent to

(1) $\beta < 0$ if $e \geq 0$, $-(2\beta + \epsilon) > \alpha + 1/2$ if $e < 0$ (and therefore $e = -1$).
(2) $l_{\alpha,\beta} := c_2 - \alpha(\alpha+1)e + (2\alpha+1)\beta + \alpha\epsilon \geq 0$.

Lemma 4.3. *(see [Q2] prop.2.3)* $M_L(c_1, c_2) = \emptyset$ *for* $r_L > 2c_2 + e - \epsilon$.

Theorem 4.4. *Let L be a polarization not lying on a wall; let*

$$W(L) := \left\{(\alpha,\beta) \in \mathbb{Z}^2 \;\middle|\; \alpha \geq 0, e - r_L > \frac{2\beta+\epsilon}{2\alpha+1}\right\},$$

$$f_L(x,y,t) := \sum_{(\alpha,\beta)\in W(L)} \Big((xy)^{\alpha((2\alpha+1)e-4\beta-2\epsilon+2\chi(\mathcal{O}_S))} - (xy)^{(\alpha+1)(2\alpha+1)e-4\beta-2\epsilon-2\chi(\mathcal{O}_S))}\Big) t^{(\alpha^2+\alpha)e-(2\alpha+1)\beta-\epsilon\alpha}.$$

Then

$$\sum_m h(M_L(c_1,m))t^m = \frac{f_{L,c_1}(x,y,t)}{(1-x)^g(1-y)^g(1-xy)} \cdot \prod_{k>0} \frac{(1-x^{2k-2}y^{2k-1}t^k)^{2g}(1-x^{2k-1}y^{2k-2}t^k)^{2g}}{(1-x^{2k-1}y^{2k-1}t^k)^2(1-x^{2k}y^{2k}t^k)^2(1-x^{2k+1}y^{2k+1}t^k)^2}.$$

Proof. As $M_{L_0}(c_1, c_2) = \emptyset$ for $r_{L_0} > 2c_2 + e - \epsilon$, we can compute $h(M_L(c_1, c_2) : x, y)$ by summing up the changes for all walls betweem L_0 and L, i.e. for all $\xi := (2\alpha + 1)\sigma + (2\beta + \epsilon)f$ with $\alpha > 0$, $e - r_L > \frac{2\beta+\epsilon}{2\alpha+1} \geq -2c_2 + \epsilon$ and $l_{\alpha,\beta} \geq 0$.

We first want to see that $l_{\alpha,\beta} \geq 0$ implies $2\beta + \epsilon \geq (-2c_2 + \epsilon)(2\alpha + 1)$. Using

$$2\beta + \epsilon \leq \begin{cases} -1 & \text{if } e \geq 0, \\ -(\alpha + 1) & \text{if } e = -1 \end{cases}$$

(see remark 4.1), $l_{\alpha,\beta} \geq 0$ implies $c_2 > 0$. Now assume $2\beta + \epsilon < (-2c_2+\epsilon)(2\alpha+1)$. If $\alpha = 0$, then $l_{\alpha,\beta} \leq c_2 + \beta < 0$; and if $\alpha > 0$, then $l_{\alpha,\beta} < c_2 - \alpha(\alpha+1)e - \alpha(2\alpha+1)(2c_2 - \epsilon) + \beta$, and by $c_2 \geq 1$, $\beta < 0$, $e \geq -1$ this is < 0. By

$$-\xi(\xi + K_S) - \chi(\mathcal{O}_S) = \alpha\big((2\alpha + 1)e - (4\beta + 2\epsilon) + 2\chi(\mathcal{O}_S)\big),$$
$$-\xi(\xi - K_S) - \chi(\mathcal{O}_S) = (\alpha + 1)\big((2\alpha + 1)e - (4\beta + 2\epsilon) - 2\chi(\mathcal{O}_S)\big),$$

theorem 3.4 and remark 4.2 we get

$$h(M_L(c_1,c_2) : x, y) = \frac{(1-x)^g(1-y)^g}{1-xy} \sum_{(\alpha,\beta)} \Big((xy)^{\alpha((2\alpha+1)e-4\beta-2\epsilon+2\chi(\mathcal{O}_S))} - (xy)^{(\alpha+1)((2\alpha+1)e-4\beta-2\epsilon-2\chi(\mathcal{O}_S))}\Big) h\big(Hilb^{[l_{\alpha,\beta}]}(S \sqcup S) : x, y\big)(xy)^{l_{\alpha,\beta}},$$

where (α, β) runs over the set $\{(\alpha, \beta) \in W(L) \mid l_{\alpha\beta} \geq 0\}$.

By rem. 4.2(2) we can express c_2 in terms of α, β, $l_{\alpha,\beta}$ and see that, given $(\alpha, \beta) \in W(L)$, letting c_2 run through all possible values is equivalent to letting $l_{\alpha,\beta}$ run through all nonnegative integers. Finally we use the formula

$$\sum_{m\geq 0} h\big(Hilb^m(S \sqcup S) : x, y\big)(xyt)^m = \Big(\sum_{n\geq 0} h(Hilb^n(S) : x, y)(xyt)^n\Big)^2$$
$$= \prod_{k>0} \frac{(1 - x^{2k-1}y^{2k}t^k)^{2g}(1 - x^{2k}y^{2k-1}t^k)^{2g}}{(1 - x^{2k-1}y^{2k-1}t^k)^2(1 - x^{2k}y^{2k}t^k)^2(1 - x^{2k+2}y^{2k+2}t^k)^2}.$$

□

Unfortunately the formula for the Hodge numbers of $M_L(c_1, c_1)$ is not very simple. However it turns out that for c_2 large enough about the first 3/8 of the Hodge numbers are independent of L and given by a quite simple formula.

Theorem 4.5.

$$h(M_L(c_1, c_2) : x, y) \equiv \frac{1 - xy}{(1-x)^g(1-y)^g} \prod_{k\geq 1} \frac{(1 - x^{2k-2}y^{2k-1})^{2g}(1 - x^{2k-1}y^{2k-2})^{2g}}{(1 - x^k y^k)^4}$$

modulo $(xy)^{c_2-w}$, *where*

$$w = \begin{cases} [1/(2r_L)+1] & \textit{if } S = \mathbb{P}_1 \times \mathbb{P}_1,\ \epsilon = 1 \textit{ and } r_L \le 1/3; \\ [r_L + \epsilon - e/2] & \textit{otherwise,} \end{cases}$$

and $[a]$ *denotes the largest integer* $\le a$.

Proof. Let f_m be the coefficient of t^m in $f_L(x,y,t)$.
Claim: $f_m \equiv 1$ *modulo* $(xy)^{m-w}$.
Proof of the Claim: Let $(\alpha,\beta) \in W(L)$. For $\alpha = 0$ we get

$$(xy)^{\alpha((2\alpha+1)e-4\beta-2\epsilon+2\chi(\mathcal{O}_S))}t^{(\alpha^2+\alpha)e-(2\alpha+1)\beta-\epsilon\alpha} = t^{-\beta},$$

where $-\beta$ can run over all integers bigger then $r_L + \epsilon - e/2$. Therefore by thm. 4.4 it is enough to prove

(1) If $\alpha > 0$ then $g_1(\alpha,\beta) := \alpha^2 e - (2\alpha-1)\beta - \alpha\epsilon + 2\alpha\chi(\mathcal{O}_S) \ge -w$,
(2) $g_2(\alpha,\beta) := (\alpha+1)^2 e - (2\alpha+3)\beta - (\alpha+2)\epsilon - (2\alpha+2)\chi(\mathcal{O}_S) \ge -w$.

(1) If $e \ge 0$, then $e \ge -2\chi(\mathcal{O}_S)$ (rem. 4.1), therefore $g_1(\alpha,\beta) \ge -(2\alpha-1)\beta - \alpha\epsilon > 0$. If $e < 0$, then $e = -1$, $\chi(\mathcal{O}_S) = 0$ and $-2\beta \ge (\alpha+1)+\epsilon$, therefore

$$g_1(\alpha,\beta) \ge -\alpha^2 + (\alpha - 1/2)(\alpha+1) - \epsilon/2 > -1.$$

(2) If $e > 0$ or $\chi(\mathcal{O}_S) \ge 0$ or $\epsilon = 0$, then

$$g_2(\alpha,\beta) \ge (\alpha+1)^2 e + (2\alpha+3) - (\alpha+2)\epsilon - (2\alpha+2)\chi(\mathcal{O}_S) \ge 0.$$

If $e = -1$, then $g_2(\alpha,\beta) \ge -(\alpha+1)^2 + (\alpha+3/2)(\alpha+1) - \epsilon/2 \ge 0$. If $e = 0$ and $\chi(\mathcal{O}_S) = -1$ and $\epsilon = 1$, then $g_2(\alpha,\beta) = -(2\alpha+3)\beta - (3\alpha+4)$. So if $\beta < -1$, then $g_2(\alpha,\beta) > 0$, and if $\beta = -1$, then $g_2(\alpha,\beta) = -(\alpha+1)$ and $r_L = 1/(2\alpha+1)$. So the claim follows.

By thm. 4.4 $h(M_L(c_1,c_2):x,y)$ is the coefficient of t^{c_2} of $k(x,y,xyt)f_L(x,y,t)$, for a power series $k(x,y,z) = \sum k_n(x,y)z^n$. So we get

$$\begin{aligned} h(M_L(c_1,c_2):x,y) &= \sum_{m \le c_2} f_{c_2-m} k_m(x,y)(xy)^m \\ &\equiv \sum_{m \le c_2} (xy)^m k_m(x,y) \quad \text{modulo } (xy)^{c_2-w} \\ &\equiv k(x,y,xy) \quad \text{modulo } (xy)^{c_2+1}. \end{aligned}$$

So we obtain our result by replacing $f_L(x,y)$ by 1, putting $t=1$ in the formula of thm. 4.4 and an easy calculation. □

Instead of fixing the determinant $det(\mathcal{E})$ we can also consider $M_L(C_1,c_2)$ the moduli space of torsion free sheaves with topological first Chern class $C_1 \in NS(S)$. For ξ determining a wall of type (c_1,c_2) (where the cohomology class of c_1 is C_1) let $\widetilde{E}_\xi^{n,m}$ be the set of sheaves lying in extensions

$$0 \to \mathcal{I}_{Z_1}(F) \to \mathcal{E} \to \mathcal{I}_{Z_2}(c_1-G) \to 0,$$

with $len(Z_1)=n$, $len(Z_2)=m$, $F+G-c_1 \equiv \xi$ and $F-G\equiv 0$ and let $\widetilde{V}_\xi^{n,m}$ be the subset of $\widetilde{E}_\xi^{n,m}$ where $(\mathcal{E}\oplus\mathcal{O}(F))/\mathcal{I}_{Z_1}(F) \neq \mathcal{O}(F)\oplus\mathcal{I}_{Z_2}(c_1-G)$. Then, after making the obvious changes, the results of chapters 2 and 3 all hold with $M_L(c_1,c_2)$, $E_\xi^{n,m}$ and $V_\xi^{n,m}$ replaced by $M_L(C_1,c_2)$, $\widetilde{E}_\xi^{n,m}$ and $\widetilde{V}_\xi^{n,m}$. In the modification of lemma 3.2 $\widetilde{E}_\xi^{n,m}$ is bijective to a projective bundle over $Pic^0(S)\times Pic^0(S)\times Hilb^n(S)\times Hilb^m(S)$, and therefore in thm 3.4 the factor $((1-x)(1-y))^{q(S)}$ is replaced by $((1-x)(1-y))^{2q(S)}$. So the formulas of thm. 4.4 and thm. 4.5 hold for $M_L(C_1,c_2)$ without the factor $(1-x)^g(1-y)^g$ in the denominator.

By [E-S2] and [B] under the assumptions of thm 4.4 the cohomology ring $H^*(M_L(C_1,c_2),\mathbb{Q})$ is generated by the Künneth components $c_i(\mathcal{F})/1$, $c_i(\mathcal{F})/f$, $c_i(\mathcal{F})/\sigma$, $c_i(\mathcal{F})/pt$ of the Chern classes of any universal sheaf $\mathcal{F}$ over $S\times M_L(C_1,c_2)$ (pt is the class of a point). If M is the pullback of a line bundle on $M_L(C_1,c_2)$, then also $\mathcal{F}\otimes M$ is a universal sheaf, and its Künneth components generate $H^*(M_L(C_1,c_2),\mathbb{Q})$. So $c_1(\mathcal{F})/1$ lies in the space generated by $c_1(\mathcal{F})/1+2c_1(M)$, $c_2(\mathcal{F})/\sigma+c_1(M)$, $c_2(\mathcal{F})/f+c_1(M)$, $c_3(\mathcal{F})/pt$ for all $M\in Pic(M_L(c_1,c_2))$, and thus for all $M\in Pic(M_L(c_1,c_2))\otimes\mathbb{Q}$. So we can put $M=-\frac{1}{2}(det(\mathcal{F})/1)$ and see that the generator $c_1(\mathcal{F})/1$ is redundant. Then thm. 4.5 can be reformulated:

Corollary 4.6. *There is no relation between the (graded commutative) generators $c_{j_1}(\mathcal{F})/1$, $c_{j_2}(\mathcal{F})/f$, $c_{j_3}(\mathcal{F})/\sigma$, $c_{j_4}(\mathcal{F})/pt$ ($j_i\geq 2$ for $i=1,\ldots,4$) in dimension lower then $2c_2-2w$*

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