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Updown Categories: Commutation Conditions and Algebraic Structures

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Abstract

In earlier work the author introduced the notion of an updown category, which can be regarded as a graded poset with multiplicities and automorphisms. An updown category $\mathcal{C}$ naturally has associated linear operators U and D on the graded vector space $\Bbbk(\operatorname{Ob}\mathcal{C})$. In Stanley's differential posets, the commutator $[D, U]$ is a constant multiple of the identity. We consider various "commutation conditions" weaker than this: in particular, the "weak commutation condition" that every element of $\operatorname{Ob}\mathcal{C}$ is an eigenvector for $[D, U]$. We also show how a Hopf algebra structure on $\Bbbk(\operatorname{Ob}\mathcal{C})$ can provide a way of showing that $\mathcal{C}$ satisfies the weak commutation condition. We illustrate with various examples, including updown categories of integer partitions, integer compositions, planar rooted trees, and rooted trees.

1 Introduction

In [16], Stanley introduced the idea of a differential poset. This is a graded, locally finite poset P (with $\hat{0}$) such that the operators U, D defined on the free vector space $\Bbbk P$ generated by P ($\Bbbk$ a field of characteristic 0) by

$$Up = \sum_{q \text{ covers } p} q \quad \text{and} \quad Dp = \sum_{p \text{ covers } q} q \tag{1}$$

satisfy $[U, D] = rI$ for some integer r. The principal example is Young's lattice of integer partitions (with $r = 1$). Stanley developed an extensive theory enumerating paths in the Hasse diagram of a differential poset. As he showed in [17], most of these results generalize to sequentially differential posets, which are defined like differential posets except that instead of $[U, D] = rI$ one assumes the restriction $[U, D]_i$ of $[U, D]$ to rank-i elements is $r_i I_i$, where $r_0, r_1, \ldots$ is a sequence of integers.

In [8] the author considered the graded poset $\mathcal{T}$ of rooted trees, which have a structure similar to Stanley's sequentially differential posets. Here one has $[U, D]_i = (i + 1)I_i$, provided equations (1) defining U and D are replaced by

$$Up = \sum_{q \text{ covers } p} u(p; q)q \quad \text{and} \quad Dp = \sum_{p \text{ covers } q} d(q; p)q, \tag{2}$$

where $u(p; q)$ is the number of vertices of the rooted tree p to which a new edge and terminal vertex can be added to get q, and $d(p; q)$ is the number of different edges of q that, when deleted, leave p; as shown in [8], $u(p; q) \neq d(p; q)$ in general. In [8] most of Stanley's enumerative results in [17] were carried over to $\mathcal{T}$.

In [10] the author generalized a graded poset to an "updown category". Here one has a category $\mathcal{C}$, whose object set is graded, and associated to each pair $c, c' \in \operatorname{Ob}\mathcal{C}$ with $|c'| = |c| + 1$ are a pair of nonnegative integers $u(c; c')$ and $d(c; c')$. The set $\mathcal{C}$ is naturally a poset, with $u(c; c')$ and $d(c; c')$ nonzero if and only if c precedes c' in the partial order. There are operators U and D on the free vector space $\Bbbk(\operatorname{Ob}\mathcal{C})$ defined by equations (2). We present these definitions, along with those of the generating functions associated with an updown category, in §2 below. We also describe the "even covering conditions" introduced in [10].

While nothing is assumed about the commutator $[U, D]$ as part of the definition of an updown category, in §3 below we consider a variety of "commutation conditions" ranging from the absolute commutation condition (i.e., $[U, D] = rI$) to the weak commutation condition (i.e., all elements of $\operatorname{Ob}\mathcal{C}$ are eigenvectors of $[U, D]$). Our theory is somewhat similar to Fomin's theory of duality of graded graphs [3, 4], but is both more restrictive and more general: more restrictive in that the functions $u(p; q)$ and $d(p; q)$ must give rise to the same partial order, i.e., for any pair p, q we have $u(p; q) = 0$ if and only if $d(p; q) = 0$; and more general in that we discuss weaker commutation conditions than he does. An updown category with the sequential commutation condition (the restriction of $[U, D]$ to rank-i elements is a scalar multiple of the identity for all i) has essentially the properties of a sequentially differential poset. We explore some consequences of the weak commutation condition, and also prove (Theorem 3.2) that under appropriate hypotheses the weak commutation condition together with the even covering properties implies the sequential commutation condition.

In §4 we consider updown categories with a compatible Hopf algebra structure: i.e., the vector space $\Bbbk(\operatorname{Ob}\mathcal{C})$ has a Hopf algebra structure related to the operators U and D. We also consider Hopf algebras for which there is a "B_+ operator", which occurs naturally in the case of Hopf algebras of trees. Results of this section (particularly Theorems 4.2 and 4.4) provide a practical method for showing that various updown categories satisfy the weak commutation condition.

In §5 we offer eleven examples. These include updown categories whose objects are the subsets of a finite set, monomials, necklaces, integer partitions, integer compositions, planar rooted trees, and rooted trees. For each example we determine which commutation condition it satisfies.

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2 Updown categories

We recall from [10] the following definitions.

Definition 2.1. An updown category is a small category $\mathcal{C}$ with a rank functor $|\cdot| : \mathcal{C} \to \mathbb{N}$ (where $\mathbb{N}$ is the ordered set of natural numbers regarded as a category) such that

A1. Each rank $\mathcal{C}_n = \{p \in \operatorname{Ob}\mathcal{C} : |p| = n\}$ is finite.

A2. The zeroth rank $\mathcal{C}_0$ consists of a single object $\hat{0}$, and $\operatorname{Hom}(\hat{0}, p)$ is nonempty for all objects p of $\mathcal{C}$.

A3. For objects p, p' of $\mathcal{C}$, $\operatorname{Hom}(p, p')$ is always finite, and $\operatorname{Hom}(p, p') = \emptyset$ unless $|p| < |p'|$ or $p = p'$. In the latter case, $\operatorname{Hom}(p, p)$ is a group, denoted $\operatorname{Aut}(p)$.

A4. Any morphism $p \to p'$, where $|p'| = |p| + k$, factors as a composition $p = p_0 \to p_1 \to \cdots \to p_k = p'$, where $|p_{i+1}| = |p_i| + 1$;

A5. If $|p'| = |p| + 1$, the actions of $\operatorname{Aut}(p)$ and $\operatorname{Aut}(p')$ on $\operatorname{Hom}(p, p')$ (by precomposition and postcomposition respectively) are free.

Definition 2.2. For any two objects p, p' of an updown category $\mathcal{C}$ with $|p'| = |p| + 1$,

$$u(p; p') = |\operatorname{Hom}(p, p')/\operatorname{Aut}(p')| = \frac{|\operatorname{Hom}(p, p')|}{|\operatorname{Aut}(p')|}$$

and

$$d(p; p') = |\operatorname{Hom}(p, p')/\operatorname{Aut}(p)| = \frac{|\operatorname{Hom}(p, p')|}{|\operatorname{Aut}(p)|}.$$

It follows immediately from these definitions that

$$u(p; p')|\operatorname{Aut}(p')| = d(p; p')|\operatorname{Aut}(p)|. \tag{3}$$

If $u(c; c') = d(c; c')$ for every pair $c, c' \in \operatorname{Ob}\mathcal{C}$ with $|c'| = |c| + 1$, we say $\mathcal{C}$ is univalent. If in fact $u(c; c') = d(c; c')$ is either 1 or 0 for any such pair, we say $\mathcal{C}$ is unital. If $|\mathcal{C}_i| = 1$

for all i, we call $\mathcal{C}$ an infinite chain: if there is some n such that $|\mathcal{C}_i| = 1$ for $0 \le i \le n$ and $\mathcal{C}_i = \emptyset$ for $i > n$, we call $\mathcal{C}$ a finite chain.

As mentioned in the introduction, we have operators U and D on $\Bbbk(\operatorname{Ob}\mathcal{C})$ given by equations (2). These two operators are adjoint with respect to the inner product $\langle , \rangle$ on $\Bbbk(\operatorname{Ob}\mathcal{C})$ defined by

$$\langle p, p' \rangle = \begin{cases} |\operatorname{Aut} p|, & \text{if } p' = p, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

The definitions of $u(c; c')$ and $d(c; c')$ can be extended to any pair with $|c'| \ge |c|$ by

$$u(c; c') = \frac{\langle U^{|c'|-|c|}(c), c' \rangle}{|\operatorname{Aut}(c')|}, \quad d(c; c') = \frac{\langle U^{|c'|-|c|}(c), c' \rangle}{|\operatorname{Aut}(c)|}.$$

Then equation (3) still holds, as do the relations

$$U^k(c) = \sum_{|c'|=|c|+k} u(c; c')c', \quad D^k(c) = \sum_{|c'|=|c|+k} d(c; c')c'.$$

If $\mathcal{C}$ and $\mathcal{D}$ are updown categories, then so is their product $\mathcal{C} \times \mathcal{D}$ (see [10, Prop. 2.1]).

We define a partial order on $\operatorname{Ob}\mathcal{C}$ by setting $c \preceq c'$ if and only if $u(c; c') \neq 0$ (or equivalently $d(c; c') \neq 0$. If c' covers c in this partial order (i.e., $c \preceq c'$ and $|c'| = |c| + 1$) we write $c \lhd c'$.

Definition 2.3. For any updown category $\mathcal{C}$, the object generating function is

$$O_{\mathcal{C}}(t) = \sum_{n \ge 0} \sum_{p \in \mathcal{C}_n} \frac{t^n}{|\operatorname{Aut}(p)|}$$

and the morphism generating function is

$$M_{\mathcal{C}}(t) = \sum_{n \ge 0} \sum_{p \in \mathcal{C}_n} \sum_{q \in \mathcal{C}_{n+1}} \frac{u(p; q) t^{2n+1}}{|\operatorname{Aut}(p)|}.$$

Both generating functions can be expressed in terms of the formal series

$$S_{\mathcal{C}}(t) = \sum_{p \in \operatorname{Ob}\mathcal{C}} \frac{p t^{|p|}}{|\operatorname{Aut}(p)|} \in \Bbbk(\operatorname{Ob}\mathcal{C})[[t]]$$

and the inner product (4) as follows:

$$\langle S_{\mathcal{C}}(t), S_{\mathcal{C}}(t) \rangle = O_{\mathcal{C}}(t^2) \tag{5}$$

and

$$\langle U S_{\mathcal{C}}(t), S_{\mathcal{C}}(t) \rangle = \langle S_{\mathcal{C}}(t), D S_{\mathcal{C}}(t) \rangle = M_{\mathcal{C}}(t). \tag{6}$$

Following [10], we say $\mathcal{C}$ is evenly up-covered if there is a sequence $u_0, u_1, \dots$ of integers such that

$$\sum_{c' \in \mathcal{C}_{n+1}} u(c, c') = u_n$$

for all $c \in \mathcal{C}_n$, and evenly down-covered if there is a sequence $d_1, d_2, \dots$ of integers with

$$\sum_{c' \in \mathcal{C}_{n-1}} d(c'; c) = d_n$$

for all $c \in \mathcal{C}_n$. If $\mathcal{C}$ is evenly down-covered with $d_n = n$, we call $\mathcal{C}$ factorial. In [10] the following results are proved.

Theorem 2.1. *If $\mathcal{C}$ and $\mathcal{D}$ are factorial updown categories, so is $\mathcal{C} \times \mathcal{D}$.*

Theorem 2.2. *If $\mathcal{C}$ and $\mathcal{D}$ are updown categories, their generating functions and those of their product $\mathcal{C} \times \mathcal{D}$ are related by*

$$O_{\mathcal{C}\times\mathcal{D}}(t) = O_{\mathcal{C}}(t) O_{\mathcal{D}}(t)$$

and

$$M_{\mathcal{C}\times\mathcal{D}}(t) = M_{\mathcal{C}}(t) O_{\mathcal{D}}(t^2) + O_{\mathcal{C}}(t) M_{\mathcal{D}}(t^2).$$

Theorem 2.3. *Let $\mathcal{C}$ be an updown category with $O_{\mathcal{C}}(t) = \sum_{n\ge 0} a_n t^n$.*

1. *If $\mathcal{C}$ is evenly up-covered, then*

$$M_{\mathcal{C}}(t) = \sum_{n\ge 0} a_n u_n t^{2n+1}.$$

2. *If $\mathcal{C}$ is evenly down-covered, then*

$$M_{\mathcal{C}}(t) = \sum_{n\ge 1} a_n d_n t^{2n-1}.$$

In particular, if $\mathcal{C}$ is factorial then $M_{\mathcal{C}}(t) = tO'_{\mathcal{C}}(t^2)$.

If $\mathcal{C}$ is both evenly up-covered and evenly down-covered, we can equate the two expressions for $M_{\mathcal{C}}(t)$ in the preceding result to get

$$a_n u_n = a_{n+1} d_{n+1} \quad \text{for all } n \ge 0. \tag{7}$$

3 Commutation Conditions

We shall consider various conditions on the commutator of the operators D and U introduced in §2. In what follows we write P_i for the restriction of the operator P to rank i, so $[D,U]_i = D_{i+1}U_i - U_{i-1}D_i$.

Definition 3.1. Let $\mathcal{C}$ be an updown category, with operators D and U as defined above. We write I for the identity operator on $\Bbbk(\operatorname{Ob}\mathcal{C})$.

1. If $[D,U] = rI$, where r is a scalar, then $\mathcal{C}$ satisfies the absolute commutation condition (ACC) with constant r.

2. If $[D,U]_i = (ai+b)I_i$ for constants a, b then $\mathcal{C}$ satisfies the linear commutation condition (LCC) with slope a and intercept b.

3. If $[D,U]_i = r_i I_i$ for some sequence of scalars $\{r_0, r_1, \dots, \}$, then $\mathcal{C}$ satisfies the sequential commutation condition (SCC).

4. If every element of $\operatorname{Ob}\mathcal{C}$ is an eigenvector for $[D,U]$, then $\mathcal{C}$ satisfies the weak commutation condition (WCC).

Evidently ACC $\implies$ LCC $\implies$ SCC $\implies$ WCC. Of course the SCC coincides with the WCC if $\mathcal{C}$ is a chain. We can rephrase the preceding definition as follows. The updown category $\mathcal{C}$ satisfies the WCC if there is a function $\epsilon : \operatorname{Ob}\mathcal{C} \to \Bbbk$ such that $(DU - UD)(c) = \epsilon(c)c$ for all $c \in \operatorname{Ob}\mathcal{C}$. Then $\mathcal{C}$ satisfies the SCC if $\epsilon(c)$ is a function of $|c|$, the LCC if $\epsilon(c)$ is a linear function of $|c|$, and the ACC if $\epsilon(c)$ is independent of c. We have the following result about products; cf. Lemma 2.2.3 of [4].

Proposition 3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be updown categories.*

1. *If $\mathcal{C}$ satisfies the ACC with constant r and D satisfies the ACC with constant s, then $\mathcal{C} \times \mathcal{D}$ satisfies the ACC with constant $r+s$.*

2. *If $\mathcal{C}$ and $\mathcal{D}$ satisfy the LCC with slope a, then so does $\mathcal{C} \times \mathcal{D}$.*

3. *If $\mathcal{C}$ and $\mathcal{D}$ satisfy the WCC, then so does $\mathcal{C} \times \mathcal{D}$.*

Proof. Since any element of $\mathcal{C} \times \mathcal{D}$ covering $(c,d) \in \operatorname{Ob}(\mathcal{C} \times \mathcal{D})$ must have the form (c',d) with c' covering c or (c,d') with d' covering d, we have

$$U(c,d) = \sum_{|c'|=|c|+1} u(c;c')(c',d) + \sum_{|d'|=|d|+1} u(d;d')(c,d') = (Uc,d) + (c,Ud), \qquad (8)$$

and similarly for D. If $\mathcal{C}$ and $\mathcal{D}$ satisfy the WCC, we can calculate that

$$(DU - UD)(c,d) = ((DU - UD)c, d) + (c, (DU - UD)d) = (\epsilon(c) + \epsilon(d))(c,d),$$

from which all three parts follow easily. $\square$

Note that $\hat{0}$ is an eigenvector for $[D,U]$ in any updown category $\mathcal{C}$, since

$$[D,U]\hat{0} = DU\hat{0} = \sum_{c\in\mathcal{C}_1} u(\hat{0};c)Dc = \left(\sum_{c\in\mathcal{C}_1} u(\hat{0};c)d(\hat{0};c)\right)\hat{0}.$$

Henceforth we write $\epsilon(\hat{0})$ for $\sum_{c\in\mathcal{C}_1} u(\hat{0};c)d(\hat{0};c)$, whether $\mathcal{C}$ satisfies the WCC or not. Our axioms require that $\epsilon(\hat{0}) \geq |\mathcal{C}_1|$.

Assuming the WCC, we can define a new generating function as follows.

Definition 3.2. If $\mathcal{C}$ is an updown category satisfying the WCC, the generating function $O^{\epsilon}_{\mathcal{C}}(t)$ is defined by

$$O^{\epsilon}_{\mathcal{C}}(t) = \sum_{p\in\operatorname{Ob}\mathcal{C}} \frac{\epsilon(p)t^{|p|}}{|\operatorname{Aut}(p)|}.$$

We record some immediate consequence of our definitions in the next result.

Proposition 3.2. *Let $\mathcal{C}$ be an updown category satisfying the WCC.*

1. $\langle [D,U]S_{\mathcal{C}}(t), S_{\mathcal{C}}(t)\rangle = O^{\epsilon}_{\mathcal{C}}(t^2)$.
2. *If $\mathcal{C}$ satisfies the ACC with constant r, then $O^{\epsilon}_{\mathcal{C}}(t) = rO_{\mathcal{C}}(t)$.*
3. *If $\mathcal{C}$ satisfies the LCC with slope a and intercept b, then $O^{\epsilon}_{\mathcal{C}}(t) = atO'_{\mathcal{C}}(t) + bO_{\mathcal{C}}(t)$.*

If an updown category $\mathcal{C}$ is evenly down-covered, we have

$$US_{\mathcal{C}}(t) = \sum_{c\in\operatorname{Ob}\mathcal{C}}\sum_{c'\rhd c} \frac{u(c;c')c't^{|c|}}{|\operatorname{Aut} c|} = \sum_{c\in\operatorname{Ob}\mathcal{C}}\sum_{c'\rhd c} \frac{d(c;c')c't^{|c|}}{|\operatorname{Aut} c'|}$$
$$= \sum_{c'\in\operatorname{Ob}\mathcal{C}} \frac{c't^{|c'|-1}}{|\operatorname{Aut} c'|}\sum_{c\lhd c'} d(c;c') = \sum_{c\in\operatorname{Ob}\mathcal{C}} \frac{cd_{|c|}t^{|c|-1}}{|\operatorname{Aut} c|}$$

so that

$$\langle US_{\mathcal{C}}(t), US_{\mathcal{C}}(t)\rangle = \sum_{c\in\operatorname{Ob}\mathcal{C}} \frac{t^{2|c|-2}d^2_{|c|}}{|\operatorname{Aut} c|} = \sum_{n\geq 0} t^{2n}d^2_{n+1}a_{n+1}, \tag{9}$$

where $O_{\mathcal{C}}(t) = \sum_{n\geq 0} a_n t^n$. Similarly, if $\mathcal{C}$ is evenly up-covered, we have

$$\langle DS_{\mathcal{C}}(t), DS_{\mathcal{C}}(t)\rangle = \sum_{n\geq 1} t^{2n}u^2_{n-1}a_{n-1}. \tag{10}$$

This gives us the following result.

Proposition 3.3. *Suppose the updown category $\mathcal{C}$ is evenly up-covered, evenly down-covered, and satisfies the WCC. Then*

$$O^{\epsilon}_{\mathcal{C}}(t) = d_1u_0a_0 + \sum_{n\geq 1}(d_{n+1}u_n - u_{n-1}d_n)a_nt^n,$$

where $O_{\mathcal{C}}(t) = \sum_{n\geq 0} a_n t^n$.

Proof. Subtract equation (10) from equation (9) to get

$$\langle US_{\mathcal{C}}(t), US_{\mathcal{C}}(t)\rangle - \langle DS_{\mathcal{C}}(t), DS_{\mathcal{C}}(t)\rangle = d_1^2 a_1 + \sum_{n\geq 1} t^{2n}(d_{n+1}^2 a_{n+1} - u_{n-1}^2 a_{n-1}).$$

The left-hand side is $\langle [D,U]S_{\mathcal{C}}(t), S_{\mathcal{C}}(t)\rangle$, so by the first part of Proposition 3.2 we have

$$O_{\mathcal{C}}^{\epsilon}(t^2) = d_1^2 a_1 + \sum_{n\geq 1} t^{2n}(d_{n+1}^2 a_{n+1} - u_{n-1}^2 a_{n-1}).$$

Now replace t^2 by t and use equation (7) to obtain the conclusion. □

The following result generalizes Proposition 2.4 of [8].

Theorem 3.1. *Let $\mathcal{C}$ be an updown category satisfying the WCC, and define $\epsilon : \operatorname{Ob}\mathcal{C} \to \Bbbk$ as above. Then for objects c_1, c_2 of $\mathcal{C}$,*

$$\langle U(c_1), U(c_2)\rangle - \langle D(c_1), D(c_2)\rangle = \begin{cases} 0, & \text{if } c_1 \neq c_2; \\ \epsilon(c)|\operatorname{Aut}(c)|, & \text{if } c_1 = c_2 = c. \end{cases}$$

Proof. Since U and D are adjoint, the left-hand side is $\langle [D,U]c_1, c_2\rangle$. □

Remark. The second alternative of this result can be written

$$\sum_{c'\rhd c} u(c;c')^2 |\operatorname{Aut}(c')| - \sum_{c''\lhd c} d(c'';c)^2 |\operatorname{Aut}(c'')| = \epsilon(c)|\operatorname{Aut}(c)|,$$

or, dividing by $|\operatorname{Aut}(c)|$ and using equation (3),

$$\sum_{c'\rhd c} u(c;c')d(c;c') - \sum_{c''\lhd c} u(c'';c)d(c'';c) = \epsilon(c). \qquad (11)$$

Equation (11) has the following consequence.

Corollary 3.1. *If the updown category $\mathcal{C}$ satisfies the WCC, then*

$$\sum_{c\in\mathcal{C}_n} \sum_{c'\in\mathcal{C}_{n+1}} u(c;c')d(c;c') = \sum_{|c|\leq n} \epsilon(c)$$

Proof. We use induction on n. Equation (11) with $|c| = 0$ gives the result for $n = 1$. Now suppose the conclusion holds for $n < k$, and use equation (11) with $|c| = k$ to get

$$\sum_{c'\in\mathcal{C}_{k+1}} u(c;c')d(c;c') = \sum_{c''\in\mathcal{C}_{k-1}} u(c'';c)d(c'';c) + \epsilon(c).$$

Sum this on $c \in \mathcal{C}_k$ and apply the induction hypothesis to obtain the conclusion for $n = k$. □

If $\mathcal{C}$ is unital, we have the following result.

Corollary 3.2. *Suppose $\mathcal{C}$ is a unital updown category that satisfies the WCC. Then*

$$M_{\mathcal{C}}(t) = \frac{t}{1-t^2} O^{\epsilon}_{\mathcal{C}}(t^2).$$

Proof. Since $\mathcal{C}$ is unital, the left-hand side of the preceding result is

$$\sum_{|c|=n} \sum_{|c'|=n+1} u(c;c')^2 = \sum_{|c|=n} \sum_{|c'|=n+1} u(c;c')$$

Now

$$US_{\mathcal{C}}(t) = \sum_{n\geq 0} \sum_{|c|=n} Uct^n = \sum_{n\geq 0} \sum_{|c|=n} \sum_{|c'|=n+1} u(c;c')c't^n,$$

so we have

$$M_{\mathcal{C}}(t) = \langle US_{\mathcal{C}}(t), S_{\mathcal{C}}(t)\rangle = \sum_{n\geq 0} \sum_{|c|=n} \sum_{|c'|=n+1} u(c;c')t^{2n+1} = \sum_{n\geq 0} \sum_{|c|\leq n} \epsilon(c)t^{2n+1}$$

and the result follows by equation (6) and Definition 3.2. □

The first alternative of Theorem 3.1 has interesting implications in the cases where the induced poset is a lattice. This includes Examples 1, 4, 7 and 8 in §5 below.

Corollary 3.3. *Let $\mathcal{C}$ be an updown category whose induced poset* $(\operatorname{Ob}\mathcal{C}, \preceq)$ *is a lattice. If $\mathcal{C}$ satisfies the WCC, then* $(\operatorname{Ob}\mathcal{C}, \preceq)$ *is modular.*

Proof. It is enough to show that, for $x, y \in \operatorname{Ob}\mathcal{C}$ with $x \neq y$, x and y cover $x \wedge y$ iff $x \vee y$ covers both x and y (see [1, §17]). Necessarily $x \neq y$ satisfying either condition must be in the same rank. For such x and y,

$$\langle Dx, Dy\rangle = \begin{cases} d(x\wedge y; x)d(x\wedge y; y)|\operatorname{Aut}(x\wedge y)|, & \text{if } x \text{ and } y \text{ cover } x\wedge y,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\langle Ux, Uy\rangle = \begin{cases} u(x; x\vee y)u(y; x\vee y)|\operatorname{Aut}(x\vee y)|, & \text{if } x\vee y \text{ covers } x \text{ and } y,\\ 0, & \text{otherwise.}\end{cases}$$

But Theorem 3.1 implies $\langle Ux, Uy\rangle = \langle Dx, Dy\rangle$, so the required equivalence follows. □

If $\mathcal{C}$ is unital, this result can be strengthened as follows (cf. Proposition 1.3 of [16] and Proposition 2.2 of [17]).

Corollary 3.4. *Suppose $\mathcal{C}$ is a unital updown category. If* $(\operatorname{Ob}\mathcal{C}, \preceq)$ *is a lattice, then $\mathcal{C}$ satisfies the WCC iff* $(\operatorname{Ob}\mathcal{C}, \preceq)$ *is modular.*

Proof. In view of the preceding result, it suffices to show that $\mathcal{C}$ satisfies the WCC when $(\operatorname{Ob}\mathcal{C}, \preceq)$ is modular. In this case we have, for $c \in \mathcal{C}_n$,

$$DUc = D(\sum_{c' \rhd c} c') = |C^+(c)|c + \sum_{\{d \in \mathcal{C}_n :\ |c \vee d| = n+1\}} d$$

and

$$UDc = U(\sum_{c'' \lhd c} c'') = |C^-(c)|c + \sum_{\{d \in \mathcal{C}_n :\ |c \wedge d| = n-1\}} d,$$

where $C^+(c)$ is the set of elements that cover c and $C^-(c)$ is the set of elements c covers. Now modularity implies $|c \wedge d| + |c \vee d| = 2n$ for $c, d \in \mathcal{C}_n$, so $|c \wedge d| = n-1$ iff $|c \vee d| = n+1$. It follows that

$$(DU - UD)c = (|C^+(c)| - |C^-(c)|)c,$$

so $\mathcal{C}$ satisfies the WCC. □

The commutation conditions do not force $(\operatorname{Ob}\mathcal{C}, \preceq)$ to be a lattice, even if $\mathcal{C}$ is unital: see Example 3 below. Examples 9, 10 and 11 also satisfy the WCC although the induced poset is not a lattice. Whether or not the induced poset is a lattice, in the unital case the even covering properties allow us to deduce the SCC from the WCC as follows.

Theorem 3.2. *Suppose the unital updown category $\mathcal{C}$ is evenly up-covered, evenly down-covered, and satisfies the WCC. Then $\mathcal{C}$ satisfies the SCC with $r_n = u_n - d_n$. Further, the numbers u_n and d_n must satisfy the relations*

$$u_n - d_n = u_n d_{n+1} - u_{n-1} d_n, \quad n \geq 1. \tag{12}$$

Proof. For $c \in \mathcal{C}_n$ we can write

$$U(c) = \sum_{i=1}^{u_n} c(i)$$

and thus

$$DU(c) = u_n c + \sum_{i=1}^{u_n} \sum_{j=1}^{d_{n+1}-1} c(i,j)$$

for elements $c(i,j) \in \mathcal{C}_n - \{c\}$. Similarly, if $c \neq \hat{0}$ we have

$$UD(c) = d_n c + \sum_{i=1}^{d_n} \sum_{j=1}^{u_{n-1}-1} c'(i,j)$$

for $c'(i,j) \in \mathcal{C}_n - \{c\}$. The WCC implies that all the elements $c(i,j)$ and $c'(i,j)$ cancel in the difference $DU(c) - UD(c)$, so $(DU - UC)(c) = (u_n - d_n)c$. But then $\mathcal{C}$ satisfies the SCC with $r_n = u_n - d_n$, and equation (12) follows by comparing Proposition 3.3 with $O^{\epsilon}_{\mathcal{C}}(t) = \sum_{n \geq 0} r_n a_n t^n$. □

Remarks. 1. The hypotheses imply $d_1 = 1$ and $u_0 = |\mathcal{C}_1|$. If in addition $d_n = 1$ for all n, then equation (12) implies $u_{n-1} = 1$ for all n, and thus that $\mathcal{C}$ is a chain.
2. If in addition to the hypotheses $\mathcal{C}$ is factorial, then equation (12) implies $u_n = |\mathcal{C}_1| - n$. Hence $\mathcal{C}_k = \emptyset$ for $k > |\mathcal{C}_1|$, and $r_n = |\mathcal{C}_1| - 2n$ for $0 \le n \le |\mathcal{C}_1|$. From equation (7),

$$|\mathcal{C}_n| = a_n = \binom{|\mathcal{C}_1|}{n}, \quad 0 \le n \le |\mathcal{C}_1|.$$

This is realized by $\mathcal{A}^{|\mathcal{C}_1|}$ in Example 1 below.
3. Another solution of equation (12) is

$$d_n = \frac{q^n - 1}{q - 1} \quad \text{and} \quad u_n = \frac{q^{N-n} - 1}{q - 1}, \quad 0 \le n \le N.$$

If q is a prime power, this is realized by Example 2 below.
4. For an updown category $\mathfrak{W}$ that satisfies the hypotheses of the theorem, is not a chain, and has $|\mathfrak{W}_n| \neq 0$ for all $n \ge 0$, see Example 3 below.

In an updown category satisfying the SCC, we can obtain the kinds of results proved by Stanley for sequentially differential posets [17] and by Fomin for **r**-graded graphs in [4]. For example, we have the following result by essentially the same proof as Theorem 2.3 of [17] (see also Proposition 2.7 of [8]).

Theorem 3.3. *Let $\mathcal{C}$ be an updown category satisfying the SCC, and let $p \in \mathcal{C}_k$. Call a word $w = w_1 w_2 \cdots w_s$ in U and D a valid p-word if the number of U's minus the number of D's in w is k, and, for each $1 \le i \le s$, the number of D's in $w_i \cdots w_s$ does not exceed the number of U's. For such a word w, let $S = \{i : w_i = D\}$ and*

$$c_i = |\{j : j > i, w_j = U\}| - |\{j : j \ge i, w_j = D\}|, \quad i \in S.$$

Then for any valid p-word w,

$$\langle w\hat{0}, p\rangle = d(\hat{0}; p) \prod_{i \in S} (r_0 + r_1 + \cdots + r_{c_i}).$$

This result has the following corollary (cf. [8, Proposition 2.8] and [4, Theorem 1.5.2]).

Corollary 3.5. *Let $\mathcal{C}$ be an updown category satisfying the SCC, and let $p \in \mathcal{C}_k$. Then for nonnegative a,*

$$\sum_{|q|=k+a} d(p; q) u(\hat{0}; q) = u(\hat{0}; p) \prod_{i=0}^{a-1} (r_0 + r_1 + \cdots + r_{k+i}).$$

Proof. Set $w = D^a U^{a+k}$ in the preceding result to get

$$\langle D^a U^{a+k} \hat{0}, p\rangle = d(\hat{0}; p) \prod_{i=0}^{a-1} (r_0 + r_1 + \cdots + r_{k+i}).$$

Expand out the left-hand side to get

$$\sum_{|q|=k+a} u(p;q)d(\hat{0};q) = d(\hat{0};p)\prod_{i=0}^{a-1}(r_0 + r_1 + \cdots + r_{k+i}).$$

Now use equation (3) and divide by $|\operatorname{Aut} p|/|\operatorname{Aut}\hat{0}|$ to obtain the conclusion. ☐

In particular, taking $p = \hat{0}$ in this result gives

$$\sum_{|q|=a} d(\hat{0};q)u(\hat{0};q) = \prod_{i=0}^{a-1}(r_0 + r_1 + \cdots + r_i).$$

4 Algebra Structures on $\Bbbk(\operatorname{Ob}\mathcal{C})$

In this section we consider some algebraic structures on $\Bbbk(\operatorname{Ob}\mathcal{C})$ and their implications for the commutation conditions.

Definition 4.1. We say $\mathcal{C}$ has a Pieri algebra structure if $\Bbbk(\operatorname{Ob}\mathcal{C})$ has a graded product

$$\Bbbk\mathcal{C}_n \otimes \Bbbk\mathcal{C}_m \to \Bbbk\mathcal{C}_{n+m}$$

with unit element $\hat{0} \in \mathcal{C}_0$, such that $U(c) = U(\hat{0})c$ for all $c \in \operatorname{Ob}\mathcal{C}$.

Pieri algebra structures are compatible with products.

Proposition 4.1. *If $\mathcal{C}$ and $\mathcal{D}$ have Pieri algebra structures, then so does $\mathcal{C} \times \mathcal{D}$.*

Proof. Identifying $\Bbbk(\operatorname{Ob}(\mathcal{C} \times \mathcal{D}))$ with $\Bbbk(\operatorname{Ob}\mathcal{C}) \otimes \Bbbk(\operatorname{Ob}\mathcal{D})$, we can define a product by

$$(c_1 \otimes d_1)(c_2 \otimes d_2) = c_1c_2 \otimes d_1d_2. \tag{13}$$

Under this identification equation (8) becomes

$$U_{\mathcal{C}\times\mathcal{D}} = U_{\mathcal{C}} \otimes \operatorname{id} + \operatorname{id} \otimes U_{\mathcal{D}},$$

from which we see that equation (13) provides a Pieri algebra structure for $\mathcal{C} \times \mathcal{D}$:

$$U_{\mathcal{C}\times\mathcal{D}}(c \otimes d) = U_{\mathcal{C}}(c) \otimes d + c \otimes U_{\mathcal{D}}(d) = U_{\mathcal{C}}(\hat{0}_{\mathcal{C}})c \otimes d + c \otimes U_{\mathcal{D}}(\hat{0}_{\mathcal{D}})d = U_{\mathcal{C}\times\mathcal{D}}(\hat{0}_{\mathcal{C}\times\mathcal{D}})(c \otimes d).$$

☐

If $\mathcal{C}$ has a Pieri algebra structure and additionally $\Bbbk(\operatorname{Ob}\mathcal{C})$ admits a coproduct

$$\Delta : \Bbbk\mathcal{C}_n \to \bigoplus_{a+b=n} \Bbbk\mathcal{C}_a \otimes \Bbbk\mathcal{C}_b$$

such that $\Bbbk(\operatorname{Ob}\mathcal{C})$ is a graded connected Hopf algebra, then there is a second multiplication $\circ$ given by

$$\langle u \circ v, w\rangle = \langle u \otimes v, \Delta(w)\rangle.$$

The following result gives some properties of $\circ$.

Proposition 4.2. *Let $\mathcal{C}$ be an updown category with a Pieri algebra structure such that $\Bbbk \operatorname{Ob} \mathcal{C}$ is a graded connected Hopf algebra.*

1. *The element $\frac{\hat{0}}{|\operatorname{Aut} \hat{0}|}$ acts as an identity for $\circ$.*
2. *The operator D is a derivation for $\circ$.*

Proof. For the first part, note that

$$\langle \hat{0} \circ v, w \rangle = \langle \hat{0} \otimes v, \Delta(w) \rangle = \langle \hat{0} \otimes v, \hat{0} \otimes w \rangle = \langle \hat{0}, \hat{0} \rangle \langle v, w \rangle = |\operatorname{Aut} \hat{0}| \langle v, w \rangle,$$

so $\hat{0} \circ v = |\operatorname{Aut} \hat{0}| v$. Similarly, $v \circ \hat{0} = |\operatorname{Aut} \hat{0}| v$.

For the second part, it suffices to show

$$\langle D(u \circ v), w \rangle = \langle D(u) \circ v + u \circ D(v), w \rangle \tag{14}$$

for all w. The left-hand side of equation (14) is

$$\begin{aligned} \langle u \circ v, U(w) \rangle = \langle u \otimes v, \Delta(U(\hat{0}) w) \rangle = \langle u \otimes v, \Delta U(\hat{0}) \Delta(w) \rangle = \\ = \sum_w (\langle u, U(w') \rangle \langle v, w'' \rangle + \langle u, w' \rangle \langle v, U(w'') \rangle), \end{aligned}$$

where $\Delta(w) = \sum_w w' \otimes w''$, since $U(\hat{0})$ is primitive. On the other hand, the right-hand side of (14) is

$$\langle D(u) \otimes v, \Delta(w) \rangle + \langle u \otimes D(v), \Delta(w) \rangle = \sum_w \langle u, U(w') \rangle \langle v, w'' \rangle + \sum_w \langle u, w' \rangle \langle v, U(w'') \rangle,$$

and equation (14) follows. □

If it happens that the "new" multiplication $\circ$ coincides with the old one, we have the following result.

Theorem 4.1. *If $\mathcal{C}$ has a Pieri algebra structure and $\Bbbk(\operatorname{Ob} \mathcal{C})$ is a graded connected Hopf algebra such that $u \circ v = uv$ (i.e., $\Bbbk(\operatorname{Ob} \mathcal{C})$ is self-dual via the inner product $\langle , \rangle$), then $\mathcal{C}$ satisfies the ACC (with constant $\epsilon(\hat{0}) \in \Bbbk$).*

Proof. For any $p \in \operatorname{Ob} \mathcal{C}$,

$$[D, U](p) = D(U(\hat{0}) \circ p) - U(\hat{0}) \circ D(p) = DU(\hat{0}) \circ p + U(\hat{0}) \circ D(p) - U(\hat{0}) \circ D(p) = \epsilon(\hat{0}) p.$$

□

As we shall see, this result applies to Examples 4 and 7 in §5 below.

Now suppose U satisfies

$$U(p \circ q) = U(p) \circ q + p \circ U(q) - p \circ U(\hat{0}) \circ q \tag{15}$$

for all $p, q \in \operatorname{Ob} \mathcal{C}$. We say U is a pre-derivation for $\circ$ in this case: note that U is a derivation exactly when $U(\hat{0}) = 0$ (and any derivation is a pre-derivation). Note also that equation (15) is satsified in the case where $U(c) = U(\hat{0}) \circ c$ for all $c \in \operatorname{Ob} \mathcal{C}$. A straightforward calculation establishes the following.

Proposition 4.3. *The commutator of pre-derivations is a pre-derivation.*

Definition 4.2. We say that the updown category $\mathcal{C}$ has a compatible Hopf algebra structure if $\Bbbk(\operatorname{Ob}\mathcal{C})$ is a graded connected Hopf algebra whose product gives $\mathcal{C}$ a Pieri multiplication and whose coproduct makes U a pre-derivation for $\circ$ as defined above.

An updown category with a compatible Hopf algebra structure need not satisfy the WCC (see Example 5 below), but we have the following result.

Theorem 4.2. *If $\mathcal{C}$ has a compatible Hopf algebra structure and $(\Bbbk(\operatorname{Ob}\mathcal{C}), \circ)$ is generated as an algebra by eigenvectors of $[D, U]$, then $\mathcal{C}$ satisfies the WCC.*

Proof. Suppose and $p, q \in \operatorname{Ob}\mathcal{C}$ are eigenvectors of $[D, U]$ with eigenvalues $\epsilon(p)$, $\epsilon(q)$ respectively. Since $[D, U]$ is a pre-derivation,

$$[D,U](p\circ q) = [D,U](p)\circ q + p\circ [D,U](q) - p\circ [D,U](\hat{0})\circ q = (\epsilon(p)+\epsilon(q)-\epsilon(\hat{0}))p\circ q. \quad (16)$$

□

We say $\mathcal{C}$ admits a B_+ operator if $k(\operatorname{Ob}\mathcal{C})$ has a Hopf algebra structure such that

$$m^* B_+(u) = B_+(u)\otimes \hat{0} + (\operatorname{id}\otimes B_+)m^*(u) \quad (17)$$

where $B_+ : \Bbbk(\operatorname{Ob}\mathcal{C}) \to \Bbbk(\operatorname{Ob}\mathcal{C})$ is an injective linear function that increases degree by 1, and m^* is the dual comultiplication defined by

$$\langle uv, w\rangle = \langle u\otimes v, m^*(w)\rangle$$

for $u, v, w \in \Bbbk(\operatorname{Ob}\mathcal{C})$. Then we have the following result.

Proposition 4.4. *If $\mathcal{C}$ admits a B_+ operator, then $DB_+(u) = B_+D(u)$ for $|u| > 0$.*

Proof. Writing $m^*(u) = \sum_u u' \otimes u''$, we have

$$\begin{aligned}
\langle DB_+(u), w\rangle &= \langle B_+(u), U(\hat{0})w\rangle = \langle m^*B_+(u), U(\hat{0})\otimes w\rangle \\
&= \langle B_+(u)\otimes \hat{0} + (\operatorname{id}\otimes B_+)m^*(u), U(\hat{0})\otimes w\rangle \\
&= \langle B_+(u)\otimes \hat{0} + \sum_u u'\otimes B_+u'', U(\hat{0})\otimes w\rangle \\
&= \langle B_+(u), U(\hat{0})\rangle\langle \hat{0}, w\rangle + \sum_u \langle u', U(\hat{0})\rangle\langle B_+(u''), w\rangle
\end{aligned}$$

while

$$\begin{aligned}
\langle B_+D(u), w\rangle = \langle D(u), B_+^*(w)\rangle = \langle u, UB_+^*(w)\rangle &= \langle m^*u, U(\hat{0})\otimes B_+^*(w)\rangle \\
&= \sum_u \langle u', U(\hat{0})\rangle\langle B_+(u''), w\rangle.
\end{aligned}$$

The two expressions agree unless $|w| = |u| = 0$. □

Examples of updown categories admitting a B_+ operator occur in Examples 4, 10, and 11 below. The next result distinguishes the first of these examples from the other two.

Theorem 4.3. *Suppose the Pieri algebra structure on $\mathcal{C}$ is commutative and $|\mathcal{C}_1| = 1$. If $\mathcal{C}$ admits a B_+ operator, then there is a scalar $t \neq 0$ with $B_+D(u) = DB_+(u) = tu$ for all $|u| > 0$. It follows that $\mathcal{C}$ is an infinite chain.*

Proof. Let $\mathcal{C}_1 = \{a\}$. Then the hypotheses mean that for any $u \in \mathcal{C}_n$, $n \geq 1$, there exists $u_1 \in \Bbbk\mathcal{C}_{n-1}$ so that

$$m^*(u) = u \otimes \hat{0} + u_1 \otimes a + \cdots + a \otimes u_1 + \hat{0} \otimes u. \tag{18}$$

There must be nonzero scalars r and s so that $U(\hat{0}) = ra$ and $B_+(\hat{0}) = sa$. For any $w \in \operatorname{Ob}\mathcal{C}$,

$$\langle w, D(u)\rangle = \langle U(w), u\rangle = \langle raw, u\rangle = \langle ra \otimes w, m^*(u)\rangle = \langle ra, a\rangle\langle w, u_1\rangle = r|\operatorname{Aut} a|\langle w, u_1\rangle,$$

so $Du = r|\operatorname{Aut} a|u_1$. Now apply the identity (17) to equation (18) to get

$$m^*B_+(u) = B_+(u) \otimes \hat{0} + u \otimes sa + \cdots + a \otimes B_+(u_1) + \hat{0} \otimes B_+(u),$$

and the cocommutativity of m^* implies

$$su = B_+(u_1) = \frac{1}{r|\operatorname{Aut} a|} B_+D(u),$$

i.e., $B_+D(u) = DB_+(u) = rs|\operatorname{Aut} a|u$.

To see that $\mathcal{C}$ must be an infinite chain, suppose first that $|\mathcal{C}_n| > 1$ for some n: we can assume n minimal. Let u_1, u_2 be distinct elements of $\mathcal{C}_n$. Then if v is the unique object in $\mathcal{C}_{n-1}$, we have the contradiction

$$0 = \langle tu_1, tu_2\rangle = \langle B_+D(u_1), B_+D(u_2)\rangle = d(v; u_1)d(v; u_2)\langle B_+(v), B_+(v)\rangle \neq 0.$$

That $\mathcal{C}$ is infinite follows from the fact that $B_+^k\hat{0}$ is nonzero for all k. ☐

Definition 4.3. Suppose $\Bbbk(\operatorname{Ob}\mathcal{C})$ has a compatible Hopf algebra structure and admits a B_+ operator. We say $\mathcal{C}$ has a unilateral arboreal structure if in addition

$$UB_+(p) = B_+U(p) + U(\hat{0}) \circ B_+(p)$$

for all $p \in \operatorname{Ob}\mathcal{C}$, and a bilateral arboreal structure if

$$UB_+(p) = B_+U(p) + U(\hat{0}) \circ B_+(p) + B_+(p) \circ U(\hat{0}).$$

Theorem 4.4. *Suppose $\mathcal{C}$ has a (unilateral or bilateral) arboreal structure such that $\Bbbk(\operatorname{Ob}\mathcal{C})$ can be generated via the product $\circ$ and the operation B_+ from eigenvectors of $[D, U]$. Then $\mathcal{C}$ satsifies the WCC.*

Proof. It suffices to show that $p \circ q$ and $B_+(p)$ are eigenvectors of $[D,U]$ whenever p and q are. The first statement follows from equation (16) above. For the second, we note that in the unilateral arboreal case,

$$\begin{aligned}
&[D,U]B_+(p) = DUB_+(p) - UDB_+(p) \\
&= DB_+U(p) + D(U(\hat{0}) \circ B_+(p)) - UB_+D(p) \\
&= B_+DU(p) + DU(\hat{0}) \circ B_+(p) + U(\hat{0}) \circ DB_+(p) - B_+UD(p) - U(\hat{0}) \circ DB_+(p) \\
&= B_+[D,U]p + \epsilon(\hat{0})|\operatorname{Aut}\hat{0}|B_+(p).
\end{aligned}$$

Hence if p is an eigenvector of $[D,U]$ we have

$$[D,U]B_+(p) = (\epsilon(p) + \epsilon(\hat{0})|\operatorname{Aut}\hat{0}|)B_+(p). \tag{19}$$

A similar calculation shows that

$$[D,U]B_+(p) = (\epsilon(p) + 2\epsilon(\hat{0})|\operatorname{Aut}\hat{0}|)B_+(p) \tag{20}$$

in the bilateral arboreal case. □

5 Examples

In this section we present eleven examples of updown categories. Eight of them appear in the last section of [10]. For the convenience of the reader we have included a cross-reference to [10] at the beginning of each example where it applies.

Example 1. (Subsets of a finite set; [10, Example 1]) First, let $\mathcal{A}$ be an updown category such that $\mathcal{A}_0 = \{\hat{0}\}$, $\mathcal{A}_1 = \{\hat{1}\}$, $\mathcal{A}_n = \emptyset$ for $n \neq 0, 1$, and $\operatorname{Hom}(\hat{0}, \hat{1})$ has a single element. The groups $\operatorname{Aut}(\hat{0})$ and $\operatorname{Aut}(\hat{1})$ are trivial since they act freely on the one-element set $\operatorname{Hom}(\hat{0}, \hat{1})$. The object and morphism generating functions are evidently

$$O_{\mathcal{A}}(t) = 1 + t \quad \text{and} \quad M_{\mathcal{A}}(t) = t.$$

We also have

$$(DU - UD)\hat{0} = D\hat{1} = \hat{0}$$

and

$$(DU - UD)\hat{1} = -U\hat{0} = -\hat{1},$$

so $\mathcal{A}$ satisfies the LCC with slope -2. Also, $\mathcal{A}$ has a Pieri multiplication if we identify $\hat{1}$ with x in the algebra $\Bbbk[x]/x^2$, but this cannot be improved to a compatible Hopf algebra structure: for then x would have to be primitive, but this would force $0 = \Delta(x^2) = 2x \otimes x$.

There is an identification of objects of $\mathcal{A}^n$ with subsets of $\{1, 2, \ldots, n\}$: an n-tuple $(c_1, \ldots, c_n)$ corresponds to the set $\{i : c_i = \hat{1}\}$. The induced partial order is inclusion of sets, and $\mathcal{A}^n$ is unital. It is also factorial and evenly up-covered. From Theorem 2.2, the generating functions are

$$O_{\mathcal{A}^n}(t) = (1+t)^n \quad \text{and} \quad M_{\mathcal{A}^n}(t) = nt(1+t^2)^{n-1}.$$

By Proposition 3.1, $\mathcal{A}^n$ satisfies the LCC with slope -2; in fact, $(DU - UD)p = (n - 2|p|)p$ for $p \in \mathrm{Ob}\,\mathcal{A}^n$ by Remark 2 following Theorem 3.2. Cf. Example 2.5(b) of [17] and Example 6.2.6 of [2].

Example 2. Let q be a prime power, $\mathbb{F}_q$ the finite field with q elements, and $\mathcal{V}$ the category of subspaces of the N-dimensional vector space $\mathbb{F}_q^N$ over $\mathbb{F}_q$ with inclusions as morphisms. Then $\mathcal{V}$ is a unital updown category, with rank given by dimension and $\hat{0}$ the zero subspace.

The object generating function is

$$O_{\mathcal{V}}(t) = \sum_{k=0}^{n} \begin{bmatrix} N \\ k \end{bmatrix}_q t^k,$$

where

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \frac{(q^N - 1)(q^{N-1} - 1) \cdots (q^{N-k+1} - 1)}{(q^k - 1) \cdots (q - 1)}$$

is the Gaussian binomial coefficient. Any $V \in \mathcal{V}_k$ has $\left[\begin{smallmatrix} k \\ k-1 \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]_q$ subspaces of dimension $k-1$, so $\mathcal{V}$ is evenly down-covered with $d_k = \left[\begin{smallmatrix} k \\ 1 \end{smallmatrix}\right]_q$. By Theorem 2.3,

$$M_{\mathcal{V}}(t) = \sum_{k=1}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q \begin{bmatrix} k \\ 1 \end{bmatrix}_q t^{2k-1} = \sum_{k=1}^{n} \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} N \\ 1 \end{bmatrix}_q t^{2k-1} = t \begin{bmatrix} N \\ 1 \end{bmatrix}_q \sum_{k=0}^{n-1} \begin{bmatrix} N-1 \\ k \end{bmatrix}_q t^{2k}.$$

Note that these generating functions are the "q-versions" of those of Example 1.

Since the induced poset $(\mathrm{Ob}\,\mathcal{V}, \preceq)$ is a modular lattice, $\mathcal{V}$ satisfies the WCC by Corollary 3.4. In fact, $\mathcal{V}$ is also evenly up-covered (with $u_k = \left[\begin{smallmatrix} N-k \\ N-k-1 \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} N-k \\ 1 \end{smallmatrix}\right]_q$), so Theorem 3.2 implies that $\mathcal{V}$ satisfies the SCC with

$$r_k = \begin{bmatrix} N-k \\ 1 \end{bmatrix}_q - \begin{bmatrix} k \\ 1 \end{bmatrix}_q = \frac{q^{N-k} - q^k}{q - 1}.$$

Cf. Example 2.5(d) of [17] and Example 6.2.6 of [2].

Example 3. Fix a positive integer n, and let $\mathfrak{W}$ be the unital updown category with $\mathfrak{W}_0 = \{\hat{0}\}$ and $\mathfrak{W}_i = \{a_i, b_i, c_i\}$ for $i \geq 1$. The nontrivial covering relations are:

$$a_{i+1} \rhd a_i, b_i; \quad b_{i+1} \rhd a_i, c_i; \quad \text{and} \quad c_{i+1} \rhd b_i, c_i \quad \text{for } i \geq 1.$$

Then $\mathfrak{W}$ is an updown category with object generating function

$$O_{\mathfrak{W}}(t) = 1 + 3t + 3t^2 + \cdots = \frac{1 + 2t}{1 - t}.$$

By direct computation it is easy to show that $\mathfrak{W}$ satisfies the SCC with $r_0 = 3$, $r_1 = 1$, and $r_i = 0$ for $i \geq 2$. Hence

$$O_{\mathfrak{W}}^{\epsilon}(t) = 3 + 3t = 3(1 + t)$$

and so by Corollary 3.2 we have

$$M_{\mathfrak{W}}(t) = \frac{t}{1-t^2} O^{\epsilon}_{\mathfrak{W}}(t^2) = \frac{3t(1+t^2)}{1-t^2}. \tag{21}$$

Also, $\mathfrak{W}$ is evenly down-covered with $d_1 = 1$ and $d_i = 2$ for $i \geq 2$, Thus equation (21) also follows from Theorem 2.3. The induced poset $(\operatorname{Ob}\mathfrak{W}, \preceq)$ is not a lattice since a_1 and c_2 have no least upper bound.

Since $\mathfrak{W}$ is also evenly up-covered (with $u_0 = 3$ and $u_i = 2$ for $i \geq 1$), Theorem 3.2 applies. In fact, any updown category satisfying the hypotheses of Theorem 3.2 that is like $\mathfrak{W}$ in having N elements in rank i for $i \geq 1$, and $u_i = d_{i+1} = k$ for $i \geq 1$, must have $N = k^2 - k + 1$ by equation (12).

Example 4. (Monomials; [10, Example 3]) Let $\mathcal{S}$ be the category with $\mathcal{S}_n = \{[n]\}$, where $[n] = \{1, 2, \ldots, n\}$ (and $[0] = \emptyset$), and let $\operatorname{Hom}([m],[n])$ be the set of injective functions from $[m]$ to $[n]$. Then $\mathcal{S}$ is an updown category with $\operatorname{Aut}[n]$ the symmetric group on n letters. Since $\operatorname{Hom}([n],[n{+}1])$ has $(n{+}1)!$ elements, $u([n];[n{+}1]) = 1$ and $d([n];[n{+}1]) = n+1$. Clearly $\mathcal{S}$ is evenly up-covered and factorial, so

$$O_{\mathcal{S}}(t) = e^t \quad \text{and} \quad M_{\mathcal{S}}(t) = te^{t^2}$$

by Theorem 2.3.

We can give $\mathcal{S}$ a Pieri algebra structure by identifying $[n]$ with x^n in the polynomial algebra $\Bbbk[x]$. Then U is multiplication by x and $D = \frac{d}{dx}$. If we declare x primitive, then $\Bbbk[x]$ is a compatible self-dual Hopf algebra structure for $\mathcal{S}$ (note $\langle x^n, x^m \rangle = n!\delta_{n,m}$). Hence $\mathcal{S}$ satisfies the ACC with constant 1 by Theorem 4.1. Also, $\mathcal{S}$ admits the B_+ operator

$$B_+(t^n) = \frac{t^{n+1}}{n+1}.$$

Objects of the updown category $\mathcal{S}^n$ can be identified with monomials in n commuting indeterminates $x_1, \ldots, x_n$. By Theorem 2.2, the generating functions are

$$O_{\mathcal{S}^n}(t) = e^{nt} \quad \text{and} \quad M_{\mathcal{S}^n}(t) = nte^{nt^2}.$$

There is a Pieri algebra structure on $\mathcal{S}^n$ in which U is multiplication by $x_1 + \cdots + x_n$ and

$$D = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}.$$

Making the x_i primitive gives a compatible self-dual Hopf algebra structure for $\mathcal{S}^n$, so Theorem 4.1 says $\mathcal{S}^n$ satisfies the ACC with constant $D(x_1 + \cdots + x_n) = n$. (Of course this already follows from Proposition 3.1.) Cf. Example 2.2.2 of [4].

Example 5. For a fixed positive integer c, let $\mathcal{F}$ be the updown category with $\mathcal{F}_n = \{f : [n] \to [c]\}$. Note that an element of $\mathcal{F}_n$ can be thought of as a monomial in the noncommuting indeterminates $X_1, \ldots, X_c$. A morphism from $f \in \mathcal{F}_n$ to $g \in \mathcal{F}_m$, $m \geq n$, is an order-preserving injection $\iota : [n] \to [m]$ such that $f = g\iota$ as functions on $[n]$. In terms of monomials, in the case $c = 2$

$$U(X_1^2X_2) = 3X_1^3X_2 + X_1^2X_2X_1 + X_2X_1^2X_2 + X_1X_2X_1X_2 + 2X_1X_2^2.$$

In fact, it is easy to see that $\Bbbk(\operatorname{Ob}\mathcal{F})$ can be identified with the underlying vector space of the noncommutative polynomial algebra $\Bbbk\langle X_1, \ldots, X_c\rangle$, and we can give it a Pieri algebra structure by multiplying elements according to shuffle product. We have

$$U(w) = (X_1 + \cdots + X_c) \sqcup\!\sqcup\, w$$

for any word w in $X_1, \ldots, X_n$. Evidently $\mathcal{F}$ is univalent, so

$$O_{\mathcal{F}}(t) = \sum_{n\geq 0} |\mathcal{F}_n| t^n = \sum_{n\geq 0} c^n t^n = \frac{1}{1 - ct}. \tag{22}$$

Now $\mathcal{F}$ has a compatible Hopf algebra structure, with product $\sqcup\!\sqcup$ and the "deconcatenation" coproduct, e.g.,

$$\Delta(X_1X_2^2) = X_1X_2^2 \otimes 1 + X_1X_2 \otimes X_2 + X_1 \otimes X_2^2 + 1 \otimes X_1X_2^2.$$

With this coproduct, the second multiplication defined in §4 is just concatenation, e.g., $X_1X_2 \circ X_2X_1^2 = X_1X_2^2X_1^2$. By Proposition 4.2, D is the derivation of $\Bbbk\langle X_1, \ldots, X_c\rangle$ sending each X_i to 1, i.e.,

$$D = \frac{\partial}{\partial X_1} + \cdots + \frac{\partial}{\partial X_c}.$$

Hence $\mathcal{F}$ is factorial, so it follows from Theorem 2.3 and equation (22) that

$$M_{\mathcal{F}}(t) = tO'_{\mathcal{F}}(t^2) = \frac{ct}{(1 - ct^2)^2}.$$

If $c > 1$, then $\mathcal{F}$ does not satisfy the WCC: it fails to hold for the generators X_i. Nevertheless, there is a formula for $[D, U]$. For $1 \leq i \leq c$ and w a word in $X_1, \ldots, X_c$, let

$$D_i w = \frac{\partial w}{\partial X_i} \quad \text{and} \quad U_i w = X_i \sqcup\!\sqcup\, w.$$

Then $[D_i, U_i](w) = (|w| + |w|_i + 1)w$, where $|w|_i$ is the X_i-degree of w, and for $j \neq i$ $[D_i, U_j]$ is the derivation that takes x_i to x_j and all x_p with $p \neq i$ to 0. We have

$$[D, U]w = ((c + 1)|w| + c)w + \sum_{i\neq j} [D_i, U_j]w.$$

Example 6. (Necklaces; [10, Example 5]) For a fixed positive integer c, let $\mathcal{N}_m$ be the set of m-bead necklaces with beads of c possible colors. A morphism from $f \in \mathcal{N}_m$ to $g \in \mathcal{N}_n$, $m \leq n$, is an injective function sending each bead of f to a bead of g of the same color and preserving the cyclic order. Then $u(p; q)$ is the number of ways to insert a bead into necklace p to get necklace q, and $d(p;q)$ is the number of different beads of q that can be deleted to give p. Evidently $\mathcal{N}$ is factorial and also evenly up-covered with $u_0 = c$ and $u_m = mc$ for $m \geq 1$. From Theorem 2.3,

$$O_{\mathcal{N}}(t) = 1 - \log(1 - ct) \quad \text{and} \quad M_{\mathcal{N}}(t) = \frac{ct}{1 - ct^2}.$$

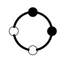
There is a Pieri algebra structure on $\mathcal{N}$ given by associating certain polynomials in $\Bbbk\langle X_1, \ldots, X_c\rangle$ with necklaces. Let $\mathcal{A} = \Bbbk\langle X_1, \ldots, X_c\rangle$, and define a linear function $R : \mathcal{A} \to \mathcal{A}$ by $R(1) = 1$, $R(X_i) = X_i$ for any i, and

$$R(X_{i_1} X_{i_2} \cdots X_{i_k}) = X_{i_k} X_{i_1} X_{i_2} \cdots X_{i_{k-1}}$$

for any monomial in $\mathcal{A}$. Let $\mathcal{A}^R$ be the R-invariant polynomials in $\mathcal{A}$, and define the linear function $P : \mathcal{A} \to \mathcal{A}^R$ by $P(1) = 1$, $P(X_i) = X_i$, and

$$P(w) = w + R(w) + \cdots + R^{k-1}(w)$$

for $w = X_{i_1} \cdots X_{i_k}$. For any necklace we have an associated polynomial in $\mathcal{A}^R$ that uses X_i to represent the ith color, e.g., for $c = 2$

$P(X_1^2 X_2^2) = X_1^2 X_2^2 + X_2 X_1^2 X_2 + X_2^2 X_1^2 + X_1 X_2^2 X_1$ represents the necklace  .

Then it can be shown that $P(w_1) \sqcup\!\sqcup P(w_2) \in \mathcal{A}^R$ for any monomials $w_1, w_2 \in \mathcal{A}$, so the shuffle product on $\mathcal{A}$ restricts to $\mathcal{A}^R$. Further, this is a Pieri algebra structure for $\mathcal{N}$, since if $p \in \mathcal{N}_m$ is represented by $P(w)$, then $U(p) \in \mathcal{N}_{m+1}$ is represented by

$$(X_1 + X_2 + \cdots + X_c) \sqcup\!\sqcup P(w).$$

For $c \geq 2$, $\mathcal{N}$ does not satisfy the WCC. There is a formula for $[D, U]p$ similar to that of the last example, but involving cyclic derivatives in the sense of [15]. Let $C_i = T_i P$, where

$$T_i(w) = \begin{cases} w', & \text{if } w = X_i w', \\ 0, & \text{otherwise.} \end{cases}$$

If the necklace p is represented by $P(w)$, then Dp is represented by $P(C_1 + \cdots + C_c)w$ and $[D, U]p$ is represented by

$$(c + 1)|w|P(w) + \sum_{j \neq i} P(X_j C_i(w)).$$

The induced poset $(\operatorname{Ob}\mathcal{N}, \preceq)$ also fails to be a lattice when $c \geq 2$.

Example 7. (Integer partitions with unit weights; [10, Example 6]) Let $\mathcal{Y}$ be the category with $\operatorname{Ob}\mathcal{Y}$ the set of integer partitions, i.e., finite sequences $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k.$$

The rank of a partition is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$; we write $\ell(\lambda)$ for the length (number of parts) of λ. The set of morphisms $\mathrm{Hom}(\lambda, \mu)$ contains a single element if and only if $\lambda_i \leq \mu_i$ for all i. Then $\mathcal{Y}$ is a unital updown category. By identifying λ with the Schur symmetric function s_λ in the algebra Sym of symmetric functions, we have a Pieri algebra structure for $\mathcal{Y}$ since

$$s_1^k s_\lambda = \sum_{|\mu|=|\lambda|+k} u(\lambda; \mu) s_\mu$$

for any partition μ (for definitions see [13]). Now the standard inner product on the algebra of symmetric functions [13, I,§5] makes the Schur functions an orthonormal basis. Further, the usual Hopf algebra structure on Sym (see [5]) is known to be self-dual via this inner product, so Theorem 4.1 shows that $\mathcal{Y}$ satisfies the ACC with constant $D(s_1) = 1$ (cf. [16, Corollary 1.4]).

Since $\mathcal{Y}_n$ is the set of partitions of n, the object generating function

$$O_{\mathcal{Y}}(t) = \sum_{n \geq 0} |\mathcal{Y}_n| t^n = \frac{1}{(1-t)(1-t^2)(1-t^3)\cdots}$$

is familiar. Since $\mathcal{Y}$ satisfies the ACC with constant 1, $O^{\mathfrak{e}}_{\mathcal{Y}}(t) = O_{\mathcal{Y}}(t)$ by the second part of Proposition 3.2 and thus

$$M_{\mathcal{Y}}(t) = \frac{t}{1-t^2} O_{\mathcal{Y}}(t^2) = \frac{t}{(1-t^2)^2(1-t^4)(1-t^6)\cdots}$$

by Corollary 3.2. Also, since the induced poset $(\mathrm{Ob}\,\mathcal{Y}, \preceq)$ is a lattice, Corollary 3.3 requires it to be modular: in fact, $(\mathrm{Ob}\,\mathcal{Y}, \preceq)$ is distributive (see the introduction of [16]).

Besides providing the motivating example of theory of differential posets in [16], $\mathcal{Y}$ appears as Example 1.6.8 of [4].

Example 8. (Integer partitions with non-univalent weights; [10, Example 7]). Let $\mathcal{K}$ be the category with $\mathrm{Ob}\,\mathcal{K}$ the set of integer partitions, and $\mathrm{Hom}(\lambda, \mu)$ defined as follows. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_m)$, always written in decreasing order. Then a morphism from λ to μ is an injective function $f : [n] \to [m]$ such that $\lambda_i \leq \mu_j$ whenever $f(i) = j$.

The category $\mathcal{K}$ has the same objects as $\mathcal{Y}$, but it also has the nontrivial automorphism groups: for a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mathrm{Aut}(\lambda)$ is the group of permutations of $\{1, \ldots, k\}$ exchanging parts of λ of equal size. Thus

$$|\mathrm{Aut}\,\lambda| = m_1(\lambda)! m_2(\lambda)! \cdots ,$$

where $m_i(\lambda)$ is the number of parts of size i in λ. For partitions λ, μ with $|\mu| = |\lambda| + 1$, the multiplicities $u(\lambda, \mu)$ and $d(\lambda, \mu)$ are nonzero if μ comes from λ by increasing a part of size k in λ to a part of size $k+1$ in μ: in this case $u(\lambda, \mu) = m_k(\lambda)$ and $d(\lambda, \mu) = m_{k+1}(\mu)$, where we allow the case $k = 0$ and make the convention $m_0(\lambda) = 1$. The weights $d(\lambda; \mu)$ appear implicitly in [12] and explicitly in [11], where they are referred to as "Kingman's branching": see especially [11, Fig. 4]. As noted in [11], the $d(\lambda;\mu)$ appear in the multiplication rule for monomial symmetric functions in Sym:

$$m_1^k m_\lambda = \sum_{|\mu|=|\lambda|+k} d(\lambda;\mu) m_\mu,$$

where m_λ is the monomial symmetric function associated with λ. Then

$$\widetilde{m}_1^k \widetilde{m}_\lambda = \sum_{|\mu|=|\lambda|+k} u(\lambda;\mu) \widetilde{m}_\mu,$$

where $\widetilde{m}_\lambda = |\operatorname{Aut}\lambda| m_\lambda$, so identifying λ with $\widetilde{m}_\lambda$ gives a Pieri algebra structure for $\mathcal{K}$. From [10] the object and morphism generating functions are

$$O_{\mathcal{K}}(t) = \exp\left(\frac{t}{1-t}\right) \quad \text{and} \quad M_{\mathcal{K}}(t) = \frac{t}{1-t^2} \exp\left(\frac{t^2}{1-t^2}\right).$$

The updown category $\mathcal{K}$ satisfies the WCC with

$$[D,U](\lambda) = (1 + m_1(\lambda))\lambda \tag{23}$$

for all partitions λ. To prove this we use Theorem 4.2, since $\Bbbk(\operatorname{Ob}\mathcal{K})$ has a compatible Hopf structure given by identifying the partition λ with $\widetilde{m}_\lambda$ as defined above: we again use the usual Hopf algebra structure on the set Sym of symmetric functions. Then the second multiplication is given by the union operation on partitions, e.g., $(2,1)\circ(3,1,1) = (3,2,1,1,1)$, and on the generators (n) we have

$$[D,U](n) = \begin{cases} (n), & \text{if } n > 1, \\ 2(1), & \text{if } n = 1. \end{cases}$$

Equation (23) follows by induction on the number of parts of λ, using equation (16).

Example 9. (Integer compositions; [10, Example 8]) Let $\mathcal{C}_n$ be the set of integer compositions of n, i.e. sequences $I = (i_1, \dots, i_p)$ of positive integers with $a_1 + \cdots + a_m = n$; as with partitions we write $\ell(I)$ for the length of I. A morphism from $(i_1, \dots, i_p) \in \mathcal{C}_n$ to $(j_1, \dots, j_q) \in \mathcal{C}_m$ is an order-preserving injective function $f : [p] \to [q]$ such that $i_a \le j_{f(a)}$ for all $a \in [p]$. Then $\mathcal{C}$ is a univalent updown category. To get a Pieri algebra structure for $\mathcal{C}$ one can use the ring QSym of quasi-symmetric functions (for definitions see [14, Sect. 9.4]): if M_I is the monomial quasi-symmetric function associated with I, then

$$M_1^k M_I = \sum_{|J|=|I|+k} u(I;J) M_J.$$

The object generating function is

$$O_{\mathcal{C}}(t) = \sum_{n\ge 0} |\mathcal{C}_n| t^n = 1 + \sum_{n\ge 1} 2^{n-1} t^n = \frac{1-t}{1-2t}.$$

From [10] the morphism generating function is

$$M_{\mathcal{C}}(t) = \frac{t - t^3}{(1 - 2t^2)^2}.$$

The updown category $\mathcal{C}$ satisfies the WCC with

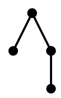
$$\epsilon(I) = \ell(I) + 2m_1(I) + 1, \tag{24}$$

where $m_1(I)$ is the number of 1's in I. This can be proved using Theorem 4.2. First, note QSym is a Hopf algebra with coproduct

$$\Delta(M_K) = \sum_{I \sqcup J = K} M_I \otimes M_J$$

where $I \sqcup J$ is a the juxtaposition of compositions I and J. Then the dualized coproduct is just $M_I \circ M_J = M_{I \sqcup J}$, for which the length-1 elements $M_{(n)}$ generate: further, equation (24) holds for these elements. Equation (24) in general then follows by induction on length using equation (16).

Example 10. (Planar rooted trees; [10, Example 9]) Let $\mathcal{P}_n$ be the set of functions $f : [2n] \to \{\langle, \rangle\}$ with $\operatorname{card} f^{-1}(\langle) = \operatorname{card} f^{-1}(\rangle) = n$ and $\operatorname{card} f^{-1}(\langle) \cap [i] \geq \operatorname{card} f^{-1}(\rangle) \cap [i]$ for all $1 \leq i \leq 2n$. We declare $\operatorname{Aut}(f)$ to be trivial for all objects f of $\mathcal{P}$, and define a morphism from $f \in \mathcal{P}_n$ to $g \in \mathcal{P}_{n+1}$ to be an injective, order-preserving function $h : [2n] \to [2n+2]$ such that the two values of $[2n+2]$ not in the image of h are consecutive, and $f(i) = gh(i)$ for $1 \leq i \leq 2n$. Then $\mathcal{P}$ is a univalent updown category. We can regard elements of $\mathcal{P}_n$ as balanced bracket arrangements, and also as planar rooted trees, e.g.

$\langle\langle\rangle\langle\langle\rangle\rangle\rangle$ is identified with  .

Thinking of $\mathcal{P}$ as the updown category of planar rooted trees, the rank is the count of non-root vertices. (The empty bracket arrangement $\emptyset \in \mathcal{P}_0$ is identified with the tree $\bullet$ consisting of the root vertex.)

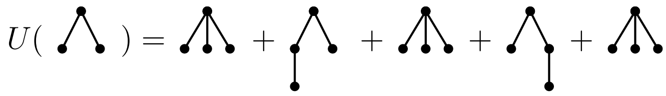
It is well known that balanced bracket arrangements, or equivalently planar rooted trees, are enumerated by Catalan numbers:

$$O_{\mathcal{P}}(t) = \sum_{n \geq 0} |\mathcal{P}_n| t^n = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^n = \frac{1 - \sqrt{1 - 4t}}{2t}.$$

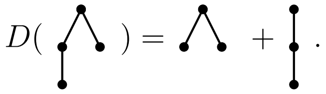
Now there are $2n+1$ possibilities for order-preserving injections $[2n] \to [2n+2]$ that miss two consecutive values, so $\mathcal{P}$ is evenly up-covered with $u_n = 2n + 1$ and by Theorem 2.3 the morphism generating function is

$$M_{\mathcal{P}}(t) = \sum_{n \geq 0} \frac{2n+1}{n+1} \binom{2n}{n} t^{2n+1} = \sum_{n \geq 0} \binom{2n+1}{n+1} t^{2n+1} = \frac{1 - \sqrt{1 - 4t^2}}{2t\sqrt{1 - 4t^2}}.$$

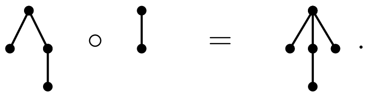
In the tree language, $U(t)$ is the sum of the $2|t|+1$ planar rooted trees obtained by attaching a new edge and terminal vertex at every possible position of t, and $D(t)$ as the sum of all trees obtained by deleting a terminal edge of t. For example,



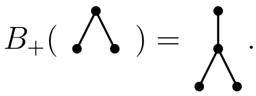
and



The updown category $\mathcal{P}$ satisfies the WCC, as can be shown from Theorem 4.4. First note that $\Bbbk(\mathrm{Ob}\,\mathcal{P})$ has a Hopf algebra structure as described in [9]. For this Hopf algebra structure the dual multiplication $\circ$ is juxtaposition of balanced bracket arrangements, or equivalently joining two planar rooted trees at the root:



Then U is a pre-derivation for $\circ$, so $\mathcal{P}$ has a compatible the Hopf algebra structure. The unary operation $B_+ : \mathcal{P}_n \to \mathcal{P}_{n+1}$ encloses a balanced bracket operation in an outer pair of delimiters, or equivalently adds a new root vertex at the top of a planar rooted tree:



This B_+ satisfies the identity (17), and indeed $\mathcal{P}$ has a bilateral arboreal structure. Hence $\mathcal{P}$ satisfies the WCC by Theorem 4.4. In fact, the eigenvalues are given by

$$\epsilon(t) = 2|t| + \tau(t) + 1, \tag{25}$$

where $\tau(t)$ is the number of terminal vertices of t (and $\tau(\bullet) = 0$), for any planar rooted tree t. Equation (16) shows that equation (25) holds for $t_1 \circ t_2$ whenever it holds for t_1 and t_2, and equation (20) shows that it holds for $B_+(t_1)$ whenever it holds for t_1. Now any planar rooted tree t with $|t| > 0$ can be written as either $t_1 \circ t_2$ or $B_+(t_1)$ with $|t_1|, |t_2| < |t|$, so equation (25) follows by induction on $|t|$ starting with the unique element of $\mathcal{P}_1$.

Example 11. (Rooted trees; [10, Example 10]) Let $\mathcal{T}_n$ consist of partially ordered sets P such that (i) P has $n+1$ elements; (ii) P has a greatest element; and (iii) for any $v \in P$, the set of elements of P exceeding v forms a chain. The Hasse diagram of such a poset P is a tree with the greatest element (the root vertex) at the top. A morphism of $\mathcal{T}$ from $P \in \mathcal{T}_m$ to $Q \in \mathcal{T}_n$ is an injective order-preserving function $f : P \to Q$ that sends the root of P to the root of Q, and which preserves covering relations (i.e., if $v \lhd w$ in the partial order on P, then $f(v) \lhd f(w)$ in the partial order on Q). Then $\mathcal{T}$ is an updown category.

As shown in [10], the categorical definition above gives

$$u(P;Q) = n(P;Q) \quad \text{and} \quad d(P;Q) = m(P;Q),$$

where n and m are the multiplicities introduced in [8]. Since $u(P;Q)$ can be interpreted as the number of vertices of P to which a new terminal edge can be attached to get Q, it follows that $\mathfrak{T}$ is evenly up-covered with

$$\sum_{Q\in\mathfrak{T}_{n+1}} u(P;Q) = |P| + 1 = n + 1 \quad \text{for all } P \in \mathfrak{T}_n.$$

It is also shown in [10] that the object and morphism generating functions are

$$O_{\mathfrak{T}}(t) = \sum_{n\geq 0} \frac{(n+1)^n}{(n+1)!} t^n \quad \text{and} \quad M_{\mathfrak{T}}(t) = \sum_{n\geq 0} \frac{(n+1)^n}{n!} t^{2n+1}.$$

The operators U and D on $\Bbbk(\mathrm{Ob}\,\mathfrak{T})$ appear in §2 of [8] as $\mathfrak{N}$ and $\mathfrak{P}$ respectively. As is proved there (Proposition 2.2), $\mathfrak{T}$ satisfies the LCC with $\epsilon(P) = |P| + 1$ (Note that the grading in [8] differs by 1 from the one used here). That $\mathfrak{T}$ satisfies the LCC can also be proved along the lines of the previous example using the Grossman-Larson Hopf algebra structure on $\Bbbk(\mathrm{Ob}\,\mathfrak{T})$ [7]: this Hopf algebra structure gives $\mathfrak{T}$ a unilateral arboreal structure.

From Corollary 3.1 it follows that

$$\sum_{P\in\mathfrak{T}_n} \sum_{Q\in\mathfrak{T}_{n+1}} u(P;Q)d(P;Q) = \sum_{k=0}^{n} (k+1)|\mathfrak{T}_k|,$$

which may be compared with the identity

$$\sum_{P\in\mathfrak{T}_n} \sum_{Q\in\mathfrak{T}_{n+1}} u(P;Q) = (n+1)|\mathfrak{T}_n|$$

that follows from $\mathfrak{T}$ being evenly up-covered.

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