GEOMETRIC RIGIDITY FOR THE CLASS $\mathcal{S}$ OF TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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ABSTRACT. We consider all the transcendental meromorphic functions from the class $\mathcal{S}$ whose Julia set is a Jordan curve. We show that then the Julia set is either a straight line or its Hausdorff dimension is strictly larger than 1.

1. Introduction

Suppose $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a meromorphic function and that f is a Jordan curve. In 1919 Fatou proved in [3] that if f is rational then either it is a circle/line or it has no tangents on a dense subset. Later on it has been proved ([2], [9], comp. [8]) that the second alternative in the hyperbolic case is much stronger: the Hausdorff dimension of $J(f)$ is strictly larger than 1 (thus by the topological exactness every non-empty open subset of $J(f)$ has Hausdorff dimension larger that 1. The case when parabolic point is allowed was covered in [10]. Relaxing the Jordan curve hypothesis further results have been obtained in [8], [10], [11] and [4]. All of this in the rational case. In the landscape of transcendental functions an analogous result have been proved in [5]. For the class $\mathcal{S}$ it gives our result under additional assumption that there are no rationally indifferent periodic points. In the current paper we prove the straight line/fractal dichotomy in its fullest strength for the whole class $\mathcal{S}$. We do not consider cases. We do not use either any knowledge about the dynamics of inner functions. Instead, we associate to our transcendental map a conformal iterated function system, in the sense of [6] and [7], and apply the results proved there. Beyond the class $\mathcal{S}$ the theorem in general fails. D. Hamilton in [4] has constructed meromorphic functions that are not in the class $\mathcal{B}$, whose Julia sets are rectifiable Jordan curves, but do not form straight lines.

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2. The Theorem

Theorem 2.1. *Suppose $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a transcendental meromorphic function in class $\mathcal{S}$. If the Julia set $J(f)$ of f is a Jordan curve, then it is either a straight line or its Hausdorff dimension is strictly larger than* 1.

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Proof. Let A_0 and A_1 be the two connected components of $\hat{\mathbb{C}} \setminus J(f)$. Since our function is in the class $\mathcal{S}$ for each $i = 0, 1$ there exists $a_i \in \overline{A}_i \setminus \{\infty\}$ such that $f^2(a_i) = a_i$ and either a_i is an attracting fixed point for f^2 or a_i is a rationally indifferent fixed point for f^2 and A_i is its basin of immediate attraction. In order to work with Euclidean derivatives we change the system of coordinates by a Möbius transformation so that ∞ is sent to a finite point and, moreover, the image of the whole Julia set is contained in the complex plane $\mathbb{C}$. If one of the points a_0 or a_1 is a parabolic point, denote it by ω. Otherwise, let ω be an arbitrary periodic point in $J(f)$. Let A be one (arbitrary) of the sets A_0 or A_1. Replaceing f by its sufficiently high iterate, we may assume without loss of generality that $f(\omega) = \omega$ and $f(A) = A$. For every $k \geq 1$ denote by $\hat{\gamma}_k$ the only arc in $J(f)$ containing ω and with endpoints in $f^{-k}(\omega) \setminus \{\omega\}$. Fix $k \geq 1$ so large that

$$\bigcap_{j=0}^{\infty} f^{-j}(\hat{\gamma}_k) = \{\omega\}. \tag{2.1}$$

Set

$$\gamma = \overline{J(f) \setminus \hat{\gamma}_k}.$$

It follows from our assumptions and Theorem (ii) in [1] that there exists a closed topological disk X contained in $\mathbb{C}$ with the following properties:

(a) $\gamma \subset X$
(b) The boundary of X is a piecewise smooth Jordan curve without cusps containing both endpoints of γ.
(c) There exists an open simply connected set V disjoint from the postcritical set of f.
(d) If f_*^{-n} is a holomorphic inverse branch of f^n defined on V such that $f_*^{-n}(X) \cap \mathrm{Int}X \neq \emptyset$, then $f_*^{-n}(X) \subset X$ and $f_*^{-n}(V) \subset V$.

We now form an iterated function system in the sense of [7]. It is defined to consist of all holomorphic inverse branches $f_*^{-n} : V \to \mathbb{C}$ such that $f_*^{-n}(X) \cap \mathrm{Int}X \neq \emptyset$ and $f^k(f_*^{-n}(X)) \cap \mathrm{Int}X \neq \emptyset$ for all $k = 1, 2, \ldots, n-1$. We parameterize all such inverse branches by a countable alphabet I and denote them by ϕ_i, $i \in I$. It follows immediately from this definition that $\phi_i(\mathrm{Int}X) \cap \phi_j(\mathrm{Int}X) = \emptyset$ whenever $i \neq j$. Along with properties (a)-(d) this implies that $S = \{\phi_i : X \to X\}_{i \in I}$ is a conformal iterated function system in the sense of [7]. Let J^* be the limit set of S. Using (2.1) note that

$$J^* = \gamma \setminus \bigcup_{n=0}^{\infty} f^{-n}(\{\omega\} \cup E),$$

where E is the countable set of essential singularities of f (for original f the set E is a singleton, say e, but for an iterate f^k, the entire set $\{e\} \cup f^{-1}(\{e\}) \cup f^{-2}(\{e\}) \cup \ldots f^{-(k-1)}(e)$ consists of essential singularities of f^k). In particular

$$\mathrm{HD}(J^*) = \mathrm{HD}(\gamma) = \mathrm{HD}(J(f) := h.$$

Now suppose that $\mathrm{HD}(J(f) = 1$. So, $\mathrm{HD}(J^*) = 1$, and it follows from Theorem 4.5.1 and Theorem 4.5.11 in [7] that $\mathrm{H}^1(J^*) < +\infty$. Since $\mathrm{H}^1(J^*) = \mathrm{H}^1(\gamma) > 0$, it follows from Theorem 4.5.10 in [7] (with $d = 1$ and X replaced by γ) that $m := (\mathrm{H}^1(\gamma))^{-1}\mathrm{H}^1|_\gamma$ is a 1-conformal measure for the system S, meaning that

$$m(\phi_i(A) = \int_A |\phi_i'| dm$$

for every Borel set $A \subset \gamma$ and

$$m(\phi_i(\gamma) \cap \phi_j(\gamma)) = 0$$

whenever $i \neq j$. Theorem 4.4.7 in [7] yields then (it is in fact much stronger than we need) the existence of a unique Borel probability measure μ on γ with the following properties:

(e) $\mu(J^*) = 1$,
(f) μ and m are equivalent with positive and continuous Radon-Nikodym derivatives.
(g) (invariance) For every Borel set $A \subset \gamma$, $\sum_{i \in I} \mu(\phi_i(A)) = \mu(A)$.

Now consider two Riemann mappings $R_0 : \overline{\mathbb{D}^1} \to \overline{A_0}$ and $R_1 : \hat{\mathbb{C}} \setminus \mathbb{D}^1 \to \{A_1\}$ such that $R_0(1) = R_1(1) = \omega$ (since $J(f)$ is a Jordan curve, R_0 and R_1 are uniquely defined respectively on the on closed disks $\overline{\mathbb{D}^1}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}^1$ due to Caratheodeory's theorem). Define two continuous maps

$$g_0 := R_0^{-1} \circ f \circ R_0 : \overline{\mathbb{D}^1} \setminus R_0^{-1}(E) \to \overline{\mathbb{D}^1} \quad \text{and} \quad g_1 := R_1^{-1} \circ f \circ R_1 : (\hat{\mathbb{C}} \setminus \mathbb{D}^1) \setminus R_1^{-1}(E) \to \hat{\mathbb{C}} \setminus \mathbb{D}^1.$$

Thus, the Schwartz Reflection Principle allows us to extend g_0 and g_1 respectively to $\mathbb{C} \setminus R_0^{-1}(E)$ and $\mathbb{C} \setminus R_1^{-1}(E)$. Then the iterated function system S lifts up to the two respective systems $S_0 = \{\phi_i^0\}_{i \in I}$ and $S_1 = \{\phi_i^1\}_{i \in I}$ formed by respective inverse branches of iterates of g_0 and g_1. Fix $j \in \{0, 1\}$. Note that the normalized Lebesgue measure λ_j on $R_j^{-1}(\gamma)$ is a conformal measure for the system S_j. Again, by Theorem 4.4.7 in [7], this system has a unique invariant measure μ_j equivalent to λ_j. But $\mu \circ R_j$ is also S_j-invariant and, by Riesz Theorem, is equivalent to λ_j. Thus, $\mu_j = \mu \circ R_j$. Hence,

$$\mu_1 = \mu \circ R_1 = \mu \circ R_0 \circ (R_0^{-1} \circ R_1) = \mu_0 \circ R_0^{-1} \circ R_1 = \mu_0 \circ (R_1^{-1} \circ R_0)^{-1}. \tag{2.2}$$

For evry $z \in S^1$ put

$$D_j(z) = \frac{d\mu_i}{d\lambda}(z).$$

In view of Theorem 6.1.3 from [7] the function $z \mapsto D_j(z)$ has a real-analytic extension onto a neighborhood of $R_j^{-1}(\gamma)$ in $\mathbb{C}$. Let

$$F_j(z) = \int_1^z D_j(t) d\lambda(t),$$

where the integration is taken along the unit circle arc from 1 to z against the Lebesgue measure λ on S^1. Formula (2.2) and $R_1^{-1} \circ R_0(1) = 1$ then give for every $z \in R_1^{-1}(\gamma)$ that

$$F_0(z) = F_1(R_1^{-1} \circ R_0(z)).$$

Since both functions F_1 and F_0 are invertible (as D_j is positive on $R_j^{-1}(\gamma)$), we conclude that $R_1^{-1} \circ R_0 = F_1^{-1} \circ F_0$ is real analytic on $R_0^{-1}(\gamma)$. Thus, $R_1^{-1} \circ R_0$ has a holomorphic extension ψ on an open neighborhood U of $R_0^{-1}(\gamma)$ in $\mathbb{C}$. The formula

$$T(z) = \begin{cases} R_0(z) & \text{if } z \in \overline{\mathbb{D}^1} \cap U \\ & \text{if } z \in (\hat{\mathbb{C}} \setminus \mathbb{D}^1) \cap U \end{cases}$$

thus defines a holomorphic map from U into $\mathbb{C}$ mapping $R_0^{-1}(\gamma)$ onto γ. Therefore, γ is a real-analytic curve, and topological exactness of $f : J(f) \to J(f)$ implies that $J(f)$ itself is a real-analytic curve. So, by the Schwartz Reflection Principle R_0 extends to an entire bijective map of $\mathbb{C}$ onto $\mathbb{C}$. Thus, R_0 is an affine map ($z \mapsto az + b$), and $J(f) = R_0(S^1)$ is a geometric circle. We are done. □

Note that because of Theorem in [1] (where the hypothesis of having two completely invariant domains can be weakened by requiring that the second iterate has two completely invariant domains) our assumption that the Julia set $J(f)$ is a Jordan curve is equivalent to require that f^2 has two completely invariant domains.

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