ON THE THEORY OF GRADED STRUCTURES

by

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MPI/SFB 85-17

ON THE THEORY OF GRADED STRUCTURES

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INTRODUCTION

Typical objects in commutative algebra are the graded rings associated to ideals, since they are suitable to describe algebraic geometric objects such as tangent and normal cones. A graded ring $G_I(A)$ is by definition the ring associated to the filtration on the commutative ring A given by $\{I^n\}_{n\in\mathbb{N}}$ and if the Krull intersection theorem applies to A, for instance in the local case, there is a well-defined function which associates to every element $a \in A-\{0\}$ the integer $v_I(a) = \max\{n/a \in I^n\}$; then one deduces a function $F : A \longrightarrow G_I(A)$ which is defined by $F(a) = \bar{a} \in I^{v_I(a)}/I^{v_I(a)+1}$ and $F(0) = 0$.

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*) This work was done while the author was visiting the MPI (Max-Planck-Institut für Mathematik) in Bonn, during the winter-semester 1984-85.

If J is an ideal contained in I, then the kernel $F(J)$ of the surjective homomorphism $G_I(A) \longrightarrow G_{I/J}(A/J)$ is the homogeneous ideal generated by $\{F(a)/a \in J\}$. Now, if A is noetherian also $G_I(A)$ is noetherian, hence $F(J)$ is finitely generated and a set $\{f_1,\ldots f_r\}$ of elements of J such that $F(J) = (F(f_1),\ldots,F(f_r))$ is termed an I-standard base of J. Sometimes $G_I(A)$ is known; for instance if (A,m,k) is a local regular ring of dimension d and $I = m$, then $G_I(A) \simeq k[x_1,\ldots,x_d]$, so that the knowledge of $G_{I/J}(A/J)$ is equivalent to that of an I-standard base of J.

Standard bases arose in [4] as a tool in the process of desingularizing an algebraic variety, but only much later some attempts were made to get control on their explicit computation. For instance in [7] and [8] criteria for both detecting and computing standard bases were given, based on the fact that $\{f_1,\ldots,f_r\}$ is an I-standard base of J if and only if the homogeneous syzygies of $F(f_1),\ldots,F(f_r)$ can be lifted to syzygies of $f_1,\ldots f_r$.

It was in the middle sixties that Hironaka used for the first time that notion and it was more or less at the same time that Buchberger introduced in his Ph.D. thesis the concept of Gröbner base (G-base); the purpose was to give an explicit algorithm for computing a base of the k-vector-space A/I, where $A = k[x_1,\ldots,x_n]$ and I is an ideal of height n. But it was only in the late seventies that, together with the strong development of computer algebra, the notion of G-base started playing a pivotal role in the most essential computations on the polynomial rings (see [1] for more informations on this aspect and for a wide bibliography). Let me recall the definition of G-base. Given $A = k[x_1,\dots,x_n]$ and a total ordering $<$ on $\mathbb{Z}^n$ such that $(\mathbb{Z}^n, <)$ is a totally ordered group, if we denote by T the set of terms (i.e. monomials with coefficients 1) of A , then the natural injection of T into $\mathbb{Z}^n$ endows T with a structure of totally ordered semigroup; if moreover every element of $T-\{0\}$ is positive, then $<$ is called a term-ordering. Once such a term-ordering is given, to every polynomial $f \in A$ we may associate its maximum term $M(f)$ and a G-base of an ideal J with respect to the given ordering is a set $\{f_1,\dots,f_r\}$ of nonzero elements of J such that every element f of J can be written as $f = \Sigma a_i f_i$ with either $a_i = 0$ or $M(f) \geq M(a_i)M(f_i)$. This property turns out to be equivalent to: for every element f of J , $M(f)$ is multiple of some $M(f_i)$, equivalently the ideal generated by $\{M(f)/f \in I\}$ is also generated by $\{M(f_1),\dots,M(f_r)\}$.

Here we see an analogy with the concept of standard base; moreover all the techniques for constructing G-bases are based on the notion of "critical pairs" which enables to construct the syzygies of the maximum terms, and this is a second analogy. It should also be mentioned that in recent works (see [6]) Mora could successfully use similar techniques to those of G-bases to construct algorithms for computing some standard bases.

Taking these analogies as a leading theme, the present paper is aimed to providing a unified frame for both the theories which underlie the notions of standard bases and G-bases. This is achieved by introducing the concept of graded structure on a commutative ring, the category of "modules" over a graded structure and the notion of distinguished base of a module. All the results given in this general setup specialize to old and new results concerning graded rings associated to ideals and the polynomial ring so that a link is constructed between known results on one theory and new results on the other one (a typical example of that is Corollary 3.12) and of course they have a wider range of applications; moreover our theory provides a theoretic background for many ideas which are developing nowadays in computer algebra and it also gives new tools to work with, for instance in connection to the given classification of all the term-orderings (see Section 2).

Now let me put the accent on another important remark. For the purpose of computing invariants and operations of ideals in the polynomial ring, the notion of term-ordering and of G-base are so important since among the possible graduations on the polynomial ring A , those ones associated to term-orderings are extremal in the sense that they split A into a direct sum of one-dimensional vector spaces. This fact has the disadvantage of producing G-bases with a possibly large number of elements, but on the other hand it has the great advantage of allowing to work with syzygies of terms, which are certainly the most trivial to be computed.

The remark that term-orderings give rise to the most refined graduations, which are therefore suitable to be compared to other graduations, inspires the definition of double structures and of "modules" over a double structure. It turns out that under mild assumptions the two graded objects associated to a module over a double structure have the "same" Hilbert functions; this fact of course specializes again to new and old results and among these it should be mentioned the famous theorem of Macaulay (see [5]).

We turn now to a description of the contents of the four sections, with some highlights on their main features.

Section 1 starts with the definition of graded structure over a commutative ring A ; modules over a graded structure and morphisms are defined in such a way that a suitable category is constructed. Then we define the v-filtered structures on A , the category of modules on them and we prove the equivalence of the two categories (Theorem 1.2). Some basic examples are discussed and it is shown that an assumption of noetherianity implies that the ordered group over which the graded objects are graded has to be isomorphic to $\mathbb{Z}^n$. Finally the technical but fundamental notion of Krull module is discussed; it allows to define quotients and it will be very essential in the following. After the proof that the ordered groups associated to noetherian structures are isomorphic to $\mathbb{Z}^n$ it is natural to look for a description of the orderings on $\mathbb{Z}^n$ and this is achieved in section 2 (Theorem 2.5). This should be suitable for applications in computer algebra and it has as a first consequence the fact that all the finite modules over a noetherian structure are Krull modules if moreover Γ° is positive ; here Γ° denotes the semigroup of the elements $\gamma \in \mathbb{Z}^n$ such that the graded object associated to the structure is nonzero at γ .

The last two sections are the heart of the work. In section 3 it is introduced the notion of d-base, which specializes to standard bases, G-bases, Macaulay bases and so on, and it is shown that over Krull modules the notion of d-base is equivalent to another important notion, which allows to compute equations in the graded objects. As a consequence it is shown that finite Krull modules over noetherian structures have free resolutions in their category (Theorem 3.5) and then the connection between d-bases and syzygies is analized (Theorem 3.6 and its corollaries). As an application at the end of section 3 it is investigated the relationship between d-bases and regular sequences; Corollary 3.12 specializes to the main result of [9] and also to new results, when applied to other d-bases.

The fourth and final section is devoted to the study of double structures on a ring A and of modules over a double structure. It is shown that the d-bases behave very well in double structures (Proposition 4.6) and it is proved as a main result (Theorem 4.9) that the two associated graded objects to a module over a double structure have the same component of degree 0 , say G_0 ; moreover they can be given a graduation over the same group Δ in such a way that for every $\delta \in \Delta$ the two homogeneous components of degree δ have the same image in the Grothendieck groups of finitely generated G_0-modules.

Of course the main application is to Hilbert functions and this is illustrated by the description of two typical situations.

It is a pleasure for me to express my gratitude to G. Valla, T. Mora and in particular to U. Orbanz for valuable conversations on the subject of the present paper, and to the Max-Planck-Institut for the hospitality while this work was in progress.

§1. GRADED STRUCTURES ON COMMUTATIVE RINGS

Let A be a commutative ring with 1; let Γ be a totally ordered group, whose ordering is denoted by $<$; let G be a Γ-graded commutative ring with 1. Then let $v : A-\{0\} \longrightarrow \Gamma$ be a function such that Γ is generated by $\mathrm{Im}(v)$; let $F : A \longrightarrow G$ be a function and assume that the following properties hold

A1 $F(a) \in G_{v(a)}$ for every $a \neq 0$

A2 $F(a) = 0$ if and only if $a = 0$

A3 $\mathrm{Im}(F) = \bigcup_{\gamma\in\Gamma} G_\gamma$

A4 $v(ab) \leq v(a)+v(b)$ for every a,b such that $ab \neq 0$

A5 $v(ab) = v(a)+v(b) \Rightarrow F(ab) = F(a)F(b)$ for every a,b such that $ab \neq 0$.

A6 $v(ab) < v(a)+v(b) \Rightarrow F(a)F(b) = 0$ for every a,b such that $ab \neq 0$; $ab = 0 \Rightarrow F(a)F(b) = 0$

A7 $v(a-b) \leq \mathrm{Max}(v(a),v(b))$ for every a,b such that $a \neq 0$, $b \neq 0$, $a \neq b$

A8 $v(a-b) = v(a) = v(b) \Rightarrow F(a-b) = F(a)-F(b)$ for every a,b such that $a \neq 0$, $b \neq 0$, $a \neq b$

A9 $v(a-b) < v(a) = v(b) \Rightarrow F(a) = F(b)$ for every a,b such that $a \neq 0$, $b \neq 0$, $a \neq b$.

Moreover, if A and G are k-algebras over a field k , assume

A10 $v(a) = 0$ and $F(a) = a$ for every $a \in k-\{0\}$.

DEFINITION 1: The quintuple (A,Γ,v,G,F) with the properties A1,...,A9 (A1,...,A10) is denoted by $\mathcal{A}$ and called a graded structure (a graded k-structure) on A . If $\mathcal{A} = (A,\Gamma,v,G,F)$, $\mathcal{A}' = (A,\Gamma,v,G',F')$ are graded structures on A , they are said to be equivalent if there exists a Γ-graded isomorphism $\alpha : G \longrightarrow G'$ such that $\alpha \circ F = F'$.

REMARK 1. Let $\mathbb{Z}_q$ be the characteristic ring of A and denote by $(\mathbb{Z}_q)^*$ the group of the invertible elements of $\mathbb{Z}_q$. Then properties A1,...,A9 already imply that $v(a) = 0$ and $F(a) = a\cdot 1_G$ for every $a \in (\mathbb{Z}_q)^*$.

Namely by A3 there exists $u \in A$ such that $F(u) = 1_G$; then $0 \neq F(1_A) = 1_G\cdot F(1_A) = F(u)\cdot F(1_A) = F(u\cdot 1_A) = 1_G$ (we used A5 and A6). So $F(1_A) = 1_G$ and since $1_G \in G_0$, $v(1_A) = 0$. Now every element $a \in (\mathbb{Z}_q)^*$ is a sum of 1_A's , hence $v(a) \leq 0$ by A7 ; but if $v(a) < 0$ then $0 = v(1_A) = v(a\cdot a^{-1}) \leq v(a)+v(a^{-1}) < 0$ a contradiction. So $v(a) = 0$ and by A8 we get $F(a) = a\cdot F(1_A) = a\cdot 1_G$.

REMARK 2. As before it is easy to see that properties A1,...,A10 imply that $v(ca) = v(a)$ and $F(ca) = cF(a)$ for every $a \neq 0$ and every $c \in k-\{0\}$.

<u>REMARK 3</u>. In A5,A6,A8,A9 also $\Leftarrow$ holds. It is an easy checking; let us check $\Leftarrow$ in A5 . By A4 $v(ab) \leq v(a) + v(b)$, but if $v(ab) < v(a) + v(b)$ then $F(a) \cdot F(b) = 0$ by A6 ; but $F(ab) = F(a) \cdot F(b)$ by assumption, hence $ab = 0$ by A2 a contradiction.

<u>REMARK 4</u>. Properties A1,...,A9 imply that $v(a) < v(b) \Rightarrow v(b-a) = v(b)$ and $F(b-a) = F(b)$ for every a,b such that $ab \neq 0$.
Namely $v(b-a) \leq v(b)$ by A7 , but if $v(b-a) < v(b)$ then $v(b) = v(b-a+a) \leq \mathrm{Max}(v(b-a), v(a)) < v(b)$, a contradiction. Now use A9.

<u>EXAMPLE 1</u>. Let (R,m) be a local ring, I an ideal and consider the order function v_I with respect to I , i.e. if $x \in R$, $x \neq 0$, $v_I(x) = n$ if $x \in I^n - I^{n+1}$. Let $\Gamma = \mathbb{Z}$, $v(x) = -v_I(x)$, $G = \bigoplus_{n\in\mathbb{N}} (gr_I(R))_{-n}$ where $(gr_I(R))_{-n} = I^n/I^{n+1}$ and define $F : R \longrightarrow G$ by the following rule: $F(0) = 0$, $F(x) = \bar{x} \in I^n/I^{n+1}$ where $-n = v(x)$. Then $(R,\mathbb{Z},v,G,F)$ is a graded structure on R .

<u>EXAMPLE 2</u>. Let $A = k[x_1,...,x_n]$. Let $\Gamma = \mathbb{Z}$, $G = A$ graded by the total degree, where $\deg(x_i) = q_i \in \mathbb{N}^+$ and consider $d : A-\{0\} \longrightarrow \mathbb{Z}$, the "total degree" function. Now, if $f \in A$ $f = f_{d_1}+...+f_{d_n}$ where f_{d_i} is a homogeneous nonzero polynomial of degree d_i and $d_1<..<d_n$, then let $F(f) = f_{d_n}$.

Then $(A,\mathbb{Z},d,A,F)$ is a graded structure on A.

EXAMPLE 3. As before let $A = k[x_1,\ldots,x_n]$. We give an order $<$ to the group $\mathbb{Z}^n$ in such a way that $(\mathbb{Z}^n,<)$ is an ordered group (we shall see later a classification of all such orderings; see Theorem 2.5) . Let $\Gamma = (\mathbb{Z}^n,<)$. Now, given a monomial $M = c\cdot x_1^{r_1}\ldots x_n^{r_n}$, we put $v(M) = (r_1,\ldots,r_n) \in \Gamma$ and, given a polynomial f , we may write it as a sum of nonzero monomials $f = M_1+\ldots+M_r$ in such a way that $v(M_1) < \ldots < v(M_r)$. So we get a Γ-graded ring structure on A and now let $v(F) = v(M_r)$ and define $F : A \longrightarrow A$ by $F(f) = M_r$.

Then (A,Γ,v,A,F) is a graded structure on A.

EXAMPLE 4. Let $A = k[x,y,y^{-1}]$. We give an order to the group $\mathbb{Z}^2$ in the following way: $(a,b) > 0 \Leftrightarrow \sqrt{2}\,a + b > 0$; so $(\mathbb{Z}^2,<)$ is an ordered group (see the remark after Corollary 2.6). If $M = c\cdot x^a y^b$ we put $v(M) = (a,b)$ and going on as in Example 3, we get a graded structure on A . Here, of course, $\mathrm{Im}(v)$ is the half-plane $a \geq 0$ and we want to remark that in A it is possible to find a sequence of monomials $\{M_n\}_{n\in\mathbb{N}}$ such that $M_n \neq 0$ for every n and $\inf v(M_n) = 0$.

Namely, it is possible to choose integers a_n,b_n with $a_n > 0$ and $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = -\sqrt{2}$ (see for instance Hardy-Wright "The theory of numbers" Thm. 36 p. 30) and then we put $M_n = x^{a_n} y^{b_n}$.

If we allow "negative" valuations on "positive" monomials the same phenomenon can occur also on $k[x,y]$. Namely define $(a,b) > 0 \Leftrightarrow \sqrt{2}\,a - b > 0$ and go on as before.

Let now $\mathcal{A} = (A,\Gamma,v,G,F)$ be a graded structure and let M be an A-module, T a Γ-graded G-module. Then let $w : M-\{0\} \longrightarrow \Gamma$ and $\emptyset : M \longrightarrow T$ be functions and assume that the following properties hold

M1 $\emptyset(m) \in T_{w(m)}$ for every $m \neq 0$

M2 $\emptyset(m) = 0$ if and only if $m = 0$

M3 $\mathrm{Im}(\emptyset) = \bigcup_{\gamma \in \Gamma} T_\gamma$

M4 $w(am) \leq v(a)+w(m)$ for every a,m such that $am \neq 0$

M5 $w(am) = v(a)+w(m) \Rightarrow \emptyset(am) = F(a) \cdot \emptyset(m)$ for every a,m such that $am \neq 0$

M6 $w(am) < v(a)+w(m) \Rightarrow F(a) \cdot \emptyset(m) = 0$ for every a,m such that $am \neq 0$; $am = 0 \Rightarrow F(a)\emptyset(m) = 0$

M7 $w(m-n) \leq \mathrm{Max}(w(m), w(n))$ for every m,n such that $m \neq 0$, $n \neq 0$, $m \neq n$

M8 $w(m-n) = w(m) = w(n) \Rightarrow \emptyset(m-n) = \emptyset(m) - \emptyset(n)$ for every m,n such that $m \neq 0$, $n \neq 0$, $m \neq n$

M9 $w(m-n) < w(m) = w(n) \Rightarrow \emptyset(m) = \emptyset(n)$ for every m,n such that $m \neq 0$, $n \neq 0$, $m \neq n$.

<u>DEFINITION 2</u>. The quintuple $(M,\Gamma,w,T,\emptyset)$ with the properties M1,...,M9 is denoted by $\mathcal{M}$ and called an $\mathcal{A}$-module. With the same meaning we say that on the A-module M there is a graded $\mathcal{A}$-structure. The notion of equivalence given for

the graded structures on A easily extends to modules.

<u>REMARK</u>. It is clear that Remarks 2,3,4 after the definition of $\mathcal{A}$ have analogous ones with respect to $\mathcal{M}$.

<u>DEFINITION 3</u>. Let $\mathcal{A}$ and $\mathcal{M}$ be as before and let $N \subset M$ be a submodule; let $(\emptyset(N))$ denote the sub G-module of T , which is defined by $(\emptyset(N)) = \bigoplus_{\gamma} \emptyset(N)_{\gamma}$ where $\emptyset(N)_{\gamma} = \{\emptyset(n)/w(n) = \gamma,\ n \in N\} \cup \{0\}$. Then $\mathcal{N} = (N,\Gamma,w',(\emptyset(N)),\emptyset')$, where $w' = w|_{N-\{0\}}$ and $\emptyset' = \emptyset|_N$, is a graded $\mathcal{A}$-structure on N , which is called the induced structure on N or the submodule of $\mathcal{M}$ associated to N .

<u>DEFINITION 4</u>. Let $\mathcal{M} = (M,\Gamma,w,T,\emptyset)$, $\mathcal{M}' = (M',\Gamma,w',T',\emptyset')$ be $\mathcal{A}$-modules, $\lambda : M \longrightarrow M'$ an A-homomorphism and $\Lambda : T \longrightarrow T'$ a graded G-homomorphism such that $w'(\lambda(m)) \leq w(m)$ for every m such that $\lambda(m) \neq 0$. Moreover assume that

$$\Lambda(\emptyset(m)) = \begin{cases} \emptyset'(\lambda(m)) & \text{if } w'(\lambda(m)) = w(m) \\ 0 & \text{if } w'(\lambda(m)) < w(m) \end{cases}$$

Then (λ,Λ) is said to be an $\mathcal{A}$-morphism.

Of course this notion is suitable to define a notion of $\mathcal{A}$-morphism between two equivalence classes of $\mathcal{A}$-modules, according to Definition 1. Therefore, given a graded structure $\mathcal{A}$, we have described a category.

DEFINITION 5. We denote by $\mathcal{G}_A$ the category whose objects are the equivalence classes of $\mathcal{A}$-modules and whose maps are equivalence classes of $\mathcal{A}$-morphisms.

At this point we are going to define another category.

DEFINITION 6. Let A be a commutative ring with 1 ; let Γ be a totally ordered group, whose ordering is denoted by $<$; let $\mathcal{F}_A = \{F_\gamma A\}_{\gamma\in\Gamma}$ be a set of additive subgroups of A with the following properties

a) $F_\gamma A \subseteq F_{\gamma'} A$ if $\gamma<\gamma'$

b) $F_\gamma A \cdot F_{\gamma'} A \subseteq F_{\gamma+\gamma'} A$

c) For every $a \in A$, $a \neq 0$, there exists a minimum γ such that $a \in F_\gamma A$ (If A is a k-algebra over a field k , then for every $c \in k-\{0\}$ the minimum is 0 by definition).

Then $\mathcal{F}_A$ is said to be a valued filtration of groups on A and the triple $(A,\Gamma,\mathcal{F}_A)$ is denoted by $\mathcal{A}^*$ and called a v-filtered structure on A .

DEFINITION 7. Let now $\mathcal{A}^* = (A,\Gamma,\mathcal{F}_A)$ be a v-filtered structure on A ; let M be an A-module and let $\mathcal{F}_M = \{F_\gamma M\}_{\gamma\in\Gamma}$ be a set of additive subgroups of M with the following properties

a') $F_\gamma M \subseteq F_{\gamma'} M$ if $\gamma < \gamma'$

b') $F_\gamma A \cdot F_{\gamma'} M \subseteq F_{\gamma+\gamma'} M$

c') For every $m \in M$, $m \neq 0$, there exists a minimum γ such that $m \in F_\gamma M$.

Then F_M is said to be a valued A-filtration of groups on M and the triple (M,Γ,F_M) is denoted by M^* and called an A^*-module or a v-filtered A-structure on M .

<u>REMARK</u>. The property c') implies that $\bigcap_\gamma F_\gamma M = 0$ and $\bigcup_\gamma F_\gamma M = M$.

<u>DEFINITION 8</u>. Let $M^* = (M,\Gamma,F_M)$, $M'^* = (M',\Gamma,F_{M'})$ be A^*-modules and let $\lambda : M \longrightarrow M'$ be an A-homomorphism such that $\lambda(F_\gamma M) \subseteq F_\gamma M'$. Then λ is said to be an A^*-morphism.

<u>DEFINITION 9</u>. We denote by G_{A*} the category whose objects are the A^*-modules and whose maps are the A^*-morphisms.

<u>LEMMA 1.1</u>. <u>Let</u> A <u>and</u> Γ <u>be as before; then every graded structure</u> A <u>on</u> A <u>gives rise to a</u> v-<u>filtered structure</u> A^* <u>on</u> A . <u>Conversely every</u> v-<u>filtered structure</u> A' <u>on</u> A <u>gives rise to a graded structure</u> $(A')_\circ$ <u>on</u> A . <u>Moreover</u>

a) $((A')_\circ)^* = A'$

b) $(A^*)_\circ$ <u>is equivalent to</u> A .

<u>PROOF</u>. Let $A = (A,\Gamma,v,G,F)$ and define $F_\gamma A = \{a \in A / v(a) \leq \gamma\} \cup \{0\}$; then it is easily seen that $F_A = \{F_\gamma A\}_{\gamma\in\Gamma}$ is a valued filtration on A , therefore $A^* = (A,\Gamma,F_A)$ is a v-filtered structure on A . Let now $A' = (A,\Gamma,F_A)$, where $F_A = \{F_\gamma A\}_{\gamma\in\Gamma}$, be a v-filtered structure on A and if $a \neq 0$ let us denote by $v(a)$ the minimum γ

such that $a \in F_\gamma A$. We denote now by $F^\circ_\gamma A = \bigcup_{\gamma' < \gamma} F_{\gamma'} A$ and we put $gr_F(A) = \bigoplus_\gamma F_\gamma A / F^\circ_\gamma A$. By using a),b) it is easy to see that $gr_F(A)$ is a Γ-graded ring. Now, if $a \in A$ we put $in(a) = 0$ if $a = 0$ and $in(a) = \bar{a} \in F_{v(a)} A / F^\circ_{v(a)} A$ if $a \neq 0$ and then it is easy to check that $(A')_\circ = (A, \Gamma, v, gr_F(A), in)$ is a graded structure on A . Now a) is straightforward; let us prove b); we have to produce an isomorphism $\alpha : gr_F(A) \longrightarrow G$ such that $\alpha \circ in = F$. If we are given a homogeneous element in $gr_F(A)$, then either it is zero and then α sends it to zero, or it is of type $in(a)$ where $v(a) = \gamma$. Then $\alpha(in(a)) = F(a)$ and we extend it by linearity. To see that it is well-defined one uses A9; to see that it is a group homomorphism one uses A8; to see that it is a ring homomorphism one uses A5, A6; to see that it is surjective one uses A3; to see that it is injective one uses A2 and to see that it is graded one uses A1. Moreover A10 takes care of the situation when we are dealing with k-algebras.

THEOREM 1.2. Let A and Γ be as before and let be given a graded structure $\mathcal{A}$ on A and a v-filtered structure $\mathcal{A}^*$ on A , which correspond each other according to Lemma 1.1. Let now $\mathcal{G}_\mathcal{A}$ denote as before the category of equivalence classes of $\mathcal{A}$-modules and $\mathcal{G}_{\mathcal{A}^*}$ the category of $\mathcal{A}^*$-modules. Then $\mathcal{G}_\mathcal{A}$ and $\mathcal{G}_{\mathcal{A}^*}$ are isomorphic.

PROOF. We define two functors $\mathcal{F} : \mathcal{G}_A \longrightarrow \mathcal{G}_{A^*}$, $\mathcal{F}^* : \mathcal{G}_{A^*} \longrightarrow \mathcal{G}_A$.

DEFINITION of $\mathcal{F}$: given an A-module M , we define $\mathcal{F}(M)$ to be $\{(M,\Gamma,F_M)$ where $F_M = \{F_\gamma M\}_{\gamma\in\Gamma}$ and $F_\gamma M = \{m \in M / w(m) \leq \gamma\} \cup \{0\}$. If $(\lambda,\Lambda) : M \longrightarrow M'$ is a morphism of A-modules then λ is easily seen to define a morphism of A^*-modules between $\mathcal{F}(M)$ and $\mathcal{F}(M')$.

DEFINITION of $\mathcal{F}^*$: given an A^*-module M^*, the definition of $\mathcal{F}^*M^*$ is parallel to the definition of $(A')_\circ$ in the proof of Lemma 1.1. If $\lambda : M^* \longrightarrow M'^*$ is an A^*-morphism, then we have to define $\Lambda : gr_F(M) \longrightarrow gr_{F'}(M')$ and we do it in the following way; $\Lambda(0) = 0$ and if $x \in (gr_F(M))_\gamma$ $x \neq 0$ then $x = \overline{m}$ with $m \in F_\gamma M$; then we put $\Lambda(x) = \overline{\lambda(m)}$ in $F_\gamma M/F^\circ_\gamma M$. To conclude it is now a matter of easy checking.

In the following we feel free of interchanging the roles of $\mathcal{G}_A$ and $\mathcal{G}_{A^*}$ and we use only the symbol $\mathcal{G}_A$.

EXAMPLE 5. Let $A = k[x_1,\dots,x_n]$, $\Gamma = \mathbb{Z}$ and $F_A = \{F_{-p}A\}$ where $F_{-p}A = \{f(x_1,\dots,x_n)/f(x_1^{q_1},\dots,x_n^{q_n}) \in (x_1,\dots,x_n)^p\}$ where $Q = (q_1,\dots,q_n)$ is a fixed n-uple of positive integers. Then $(A,\mathbb{Z},F_A)$ is a v-filtered structure on A .

<u>EXAMPLE 6</u>. Let $A = k[x,y]$, $\Gamma = \mathbb{Z}$, $\mathcal{F}_A = \{F_{-p}A\}$ where $F_{-p}A = (x,y^p)$. Here c) fails, hence this is not a valued filtration.

<u>EXAMPLE 7</u>. Let $\mathcal{A} = (A,\Gamma,v,G,F)$ be a graded structure, let $\gamma \in \Gamma$ and let $\mathcal{A}(-\gamma)$ denote the quintuple $(A,\Gamma,v_\gamma,G(-\gamma),F_\gamma)$ where v_γ is defined by $v_\gamma(a) = v(a)+\gamma$, $G(-\gamma)$ is the graded G-module defined by $G(-\gamma)_{\gamma'} = G_{\gamma'-\gamma}$ and $F_\gamma(a) = F(a)$. Then $\mathcal{A}(-\gamma)$ is an $\mathcal{A}$-module for every $\gamma \in \Gamma$. Moreover if $a \in A$ is different from zero, then the multiplication by a is an $\mathcal{A}$-morphism of $\mathcal{A}(-\gamma)$ in $\mathcal{A}(-\gamma+v(a))$ (here we used the terminology of v-filtered structures).

<u>BASIC EXAMPLE 8</u>. Let $\mathcal{A} = (A,\Gamma,v,G,F)$ be a graded structure and $\mathcal{M} = (M,\Gamma,w,T,\emptyset)$ an $\mathcal{A}$-module. Let us choose $\{m_1,\ldots,m_r\}$ to be a set of nonzero elements of M ; let A^r be the free module of rank r over A , whose canonical base we denote by $(e_1,\ldots,e_r)$. Let $w_i = w(m_i)$ and let $w^+ : A^r-\{0\} \longrightarrow \Gamma$ be defined in the following way

$$w^+(a_1,\ldots,a_r) = \max_{a_i \neq 0} \{v(a_i) + w_i\}$$

To w^+ we associate a filtration on A^r as in the definition of $\mathcal{F}$ (see the proof of Theorem 1.2); this turns out to be a valued filtration. Then we get an associated graded

G-module, which turns out to be $\overset{r}{\underset{i=1}{\oplus}} G(-w_i)$ and a map $F^+ : A^r \longrightarrow \overset{r}{\underset{i=1}{\oplus}} G(-w_i)$ which is defined by $F^+(a_1,\ldots,a_r) = (\tilde{F}(a_1),\ldots,\tilde{F}(a_r))$ where

$$\tilde{F}(a_i) = \begin{cases} 0 \text{ if } a_i = 0 \text{ or } v(a_i) + w_i < w^+(a_1,\ldots,a_r) \\ F(a_i) \text{ if } v(a_i) + w_i = w^+(a_1,\ldots,a_r) \end{cases}$$

We denote by $\mathcal{L}(w_1,\ldots,w_r)$ or by $\overset{r}{\underset{i=1}{\oplus}} \mathcal{A}(-w_i))$ the $\mathcal{A}$-module $(A^r,\Gamma,w^+,\overset{r}{\underset{i=1}{\oplus}} G(-w_i),F^+)$. Then the A-homomorphism $\lambda : A^r \longrightarrow M$ defined by $\lambda(e_i) = m_i$ is an $\mathcal{A}$-morphism of $\mathcal{L}(w_1,\ldots,w_r)$ in $\mathcal{M}$ and the corresponding graded G-homomorphism $\Lambda : \overset{r}{\underset{i=1}{\oplus}} G(-w_i) \longrightarrow T$ is defined by $\Lambda(e_i) = \emptyset(m_i)$ (here $(e_1,\ldots,e_r)$ is the canonical base of $\overset{r}{\underset{i=1}{\oplus}} G(-w_i))$. This morphism $(\lambda,\Lambda) : \mathcal{L}(w_1,\ldots,w_r) \longrightarrow \mathcal{M}$ will be refered to as the canonical morphism associated to $m_1,\ldots,m_r$.

<u>DEFINITION 10</u>. An $\mathcal{A}$-module $\mathcal{L}$ is said to be finite free if it is isomorphic to an $\mathcal{A}$-module of type $\mathcal{L}(w_1,\ldots,w_r)$.

<u>DEFINITION 11</u>. A graded structure $\mathcal{A} = (A,\Gamma,v,G,F)$ is called noetherian if A and G are noetherian. An $\mathcal{A}$-module $\mathcal{M} = (M,\Gamma,w,T,\emptyset)$ is called finite if M is a finite A-module and T a finite graded G-module.

Henceforth, given a graded structure $\mathcal{A} = (A,\Gamma,v,G,F)$, we <u>shall denote by</u> $\Gamma°(\mathcal{A}) = \operatorname{Im} v = \{\gamma \in \Gamma / G_\gamma \neq 0\}$, <u>so that</u>

$\Gamma^\circ(A)$ <u>is a generating subset of</u> Γ as we stated at the very beginning. In the same way, if M is an A-module, we define $\Gamma^\circ(M)$. Of course, if Γ is a finitely generated group and Γ° is a subsemigroup, Γ° need not be finitely generated; however we get

<u>PROPOSITION 1.3.</u> <u>Let</u> $A = (A,\Gamma,v,G,F)$ <u>be a noetherian structure. Then</u>

a) $\Gamma^\circ(A)$ <u>generates a finitely generated semigroup</u>

b) <u>If moreover</u> G <u>is an integral domain, then</u> $\Gamma^\circ(A)$ <u>is a finitely generated semigroup</u>.

c) Γ <u>is a finitely generated torsion-free group, hence isomorphic to</u> $\mathbb{Z}^n$ <u>for a suitable</u> n .

<u>PROOF</u>. By [3] G is a finitely generated G_0-algebra. So let $G = G_0[x_1,\dots,x_r]$ where x_i is homogeneous of degree γ_i and different from zero. Then clearly $\Gamma^\circ(A) \subseteq \langle \gamma_1,\dots,\gamma_r \rangle$, (Here $\langle \dots \rangle$ denotes the semigroup generated by ...) and if G is an integral domain $\Gamma^\circ(A) = \langle\gamma_1,\dots,\gamma_r\rangle$. Now Γ is also generated by $\{\gamma_1,\dots,\gamma_r\}$ and since it is totally ordered, it is torsion-free.

We observe that if A is noetherian and Γ° is finitely generated, then G need not be noetherian as the following example shows.

<u>EXAMPLE 9</u>. Let $A = k[x,y,z]$, $\Gamma = \mathbb{Z}^2$ ordered by $(a,b) > 0$ if either $a+b > 0$ or $a+b = 0$ and $b > 0$, and consider the following filtration:

$F_{p,q}A = 0$ if either $p < 0$ or $q < 0$

$F_{0,0}A = k$

$F_{p,q}A =$ k-vector space generated by $\{x^a y^b z^c / a+b+c \leq p+q$, $b \leq q\}$ if $q > 0$, $p \geq 0$

$F_{p,q}A =$ k-vector space generated by $\{x^a y^b z^c / a+b+c \leq p$, $a+b \leq p-1\}$ if $q = 0$, $p > 0$.

Then we get a valued filtration on A and $\Gamma^\circ = \mathbb{N}^2$ generated by $\{(0,1),(1,0)\}$, but the associated graded ring is not noetherian. Namely $v(x) = (0,1), v(x^2) = (1,1),\ldots,$ $v(x^n) = (n-1,1)$ hence the initial forms of x^n are part of a minimal set of generators of G as a k-algebra.

In the next section we shall describe the orderings on $\mathbb{Z}^n$, but now let us investigate another fundamental aspect of the theory.

If A is a graded structure, $M = (M,\Gamma,F_M)$ an A-module and $N = (N,\Gamma,F_N)$ a sub A-module of M , then $F_N = \{F_\gamma N\}_{\gamma\in\Gamma}$ where $F_\gamma N = F_\gamma M \cap N$ and the natural filtration on M/N is $F_{M/N} = \{F_\gamma(M/N)\}_{\gamma\in\Gamma}$ where $F_\gamma(M/N) = (F_\gamma M+N)/N$; however this need not be a valued filtration, as the following easy example shows.

EXAMPLE 10. Let $A = k[x]$, $\Gamma = \mathbb{Z}$, $v(a) = -v_{(x)}(a)$ (see Ex 1). This gives a valued filtration on A, hence a graded structure on A. Let $I = (x-x^2)$, $\bar{A} = A/I$ and on $\bar{A}$ we consider the induced filtration as we explained before. It is clear that $\bar{x} \in A/I$ belongs to $F_n\bar{A}$ for $n < 0$; hence $F_{\bar{A}}$ is not a valued filtration.

In order to overcome this difficulty and for many other purposes, which will become clear later on, we introduce the following

DEFINITION 12. Let A be a noetherian structure and M a finite A-module on M. Assume that for every finite free A-module $L = (L,\ldots)$, every morphism $\lambda : L \longrightarrow M$, every $F_\gamma L$ with $\gamma \in \Gamma \cup \{-\infty\}$ (here we use the convention: $F_{-\infty}L = (0)$), every strictly decreasing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of elements of $\Gamma^\circ(M)$ and every submodule N of M, we have

$$\bigcap_{n \in \mathbb{N}} (\lambda(F_\gamma L) + N + F_{\gamma_n} M) = \lambda(F_\gamma L) + N \quad .$$

Then M is said to be a Krull module. Moreover a noetherian structure A such that every finite free A-module is a Krull A-module is said to be a strong Krull structure.

We are going to use the symbol M/N for the triple $(M/N,\Gamma,F_{M/N})$ as decribed before.

PROPOSITION 1.4. Let A be a noetherian structure, M a Krull A-module and let N be a submodule of M. Then

1) N is a Krull module.
2) The filtration induced on M/N is valued, hence M/N is an A-module and $\Gamma^\circ(M/N) \subseteq \Gamma^\circ(M)$
3) M/N is a Krull A-module.
4) If A is a strong Krull structure and I a submodule of A, then A/I is a strong Krull structure.

PROOF. 1) Obvious

2) Let $x \in M-N$ and let $\gamma_1 = w(x)$; then $x \in F_{\gamma_1}M+N$. If $x \notin F^\circ_{\gamma_1}M+N$ then $\bar{w}(\bar{x}) = \gamma_1$; if $x \in F^\circ_{\gamma_1}M+N$ then $x = x_2+n$ with $\gamma_2 = w(x_2) < \gamma_1$ and so on. If this procedure does not stop, we get a strictly decreasing sequence $\{\gamma_n\}$ of elements of $\Gamma^\circ(M)$ such that $x \in \bigcap_n (N+F_{\gamma_n}M) = N$, a contradiction. Therefore the procedure stops after a finite number of steps and it yields the valuation of $\bar{x}$. Let now $\gamma = \bar{w}(\bar{x})$; then $x \in (F_\gamma M+N) - (F^\circ_\gamma M+N)$ hence $x = y+n$ with $w(y) = \gamma$ and this proves that $\Gamma^\circ(M/N) \subseteq \Gamma^\circ(M)$.

3) Let $L = (L,\dots)$ be a finite free module, let $\lambda : L \longrightarrow M/N$ be a morphism and let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence of elements of $\Gamma^\circ(M/N)$, which is contained in $\Gamma^\circ(M)$ by 2). Let $p : M \longrightarrow M/N$ be the projection and let $\alpha : L \longrightarrow M$ be such that $\lambda = p \circ \alpha$ (this is possible since L is free). Finally let N' be

a submodule of M such that $N \subseteq N'$. We get

$$\bigcap_n (\lambda(F_\gamma L) + N'/N + F_{\gamma_n}(M/N)) =$$

$$= \bigcap_n (p(\alpha(F_\gamma L)) + p(N') + p(F_{\gamma_n} M)) =$$

$$= p(\bigcap_n(\alpha(F_\gamma L) + N' + F_{\gamma_n} M)) = p(\alpha(F_\gamma L) + N') =$$

$$= \lambda(F_\gamma L) + N'/N .$$

4) Every finite free module over A/I is a quotients of a finite free module over A , so we can apply 3).

<u>PROPOSITION 1.5.</u> <u>Let</u> $A = (A,\Gamma,v,G,F)$ <u>be a Krull-module over itself and assume that</u> G <u>is noetherian</u>. <u>Then</u> A <u>is noetherian</u>.

<u>PROOF</u>. Let I be an ideal of A ; then there exist $a_1,\ldots,a_r \in I$ such that $F(I)$ is generated by $\{F(a_1),\ldots,F(a_r)\}$. Let $J = (a_1,\ldots,a_r) \subseteq I$ and let $x \in I$ with $v(x) = \gamma$. Then $F(x) = \Sigma R_i F(a_i)$ with $R_i = 0$ or $R_i = F(r_i)$ with $v(r_i) + v(a_i) = \gamma$. Then by A8,A9 we get $F(x)=F(\Sigma r_i a_i)$, hence $v(x-\Sigma r_i a_i) = \gamma_2 < \gamma$. Now we replace x by $x-\Sigma r_i a_i$ and we go on with this procedure: If we get 0 after a finite number of steps, we are done. Otherwise we get a strictly decreasing sequence $\{\gamma_n\}_{n+\mathbb{N}}$ of elements of $\Gamma^\circ(A)$ such that $x \in \bigcap_n (J+F_{\gamma_n} A)$, so that $x \in J$ by the Krull-assumption.

LEMMA 1.6. *Let* $\mathcal{A}$ *be a graded structure and* $\mathcal{M}$ *an* $\mathcal{A}$-*module. Assume that* $\Gamma^\circ(\mathcal{A}) \leq 0$; *then*

a) $F_\gamma A$ *is an ideal for every* $\gamma \in \Gamma$.

b) $F_\gamma M$ *is a submodule of* M *for every* $\gamma \in \Gamma$.

PROOF. It is straightforward.

PROPOSITION 1.7. *Let* $\mathcal{A} = (A,\Gamma,v,G,F)$ *be a graded structure with* A *local noetherian,* $\Gamma^\circ(\mathcal{A}) = -\mathbb{N}$, $F_{-n}A = I^n$, *where* I *is an ideal. Then*

a) *Every finite* $\mathcal{A}$-*module* $\mathcal{M} = (M,\ldots)$ *such that* $\mathcal{F}_M$ *is cofinal to* $\mathcal{F}'_M = \{F_{-p}M\}_{p\in\mathbb{N}}$ *where* $F_{-p}M = I^pM$, *is a Krull module*.

b) $\mathcal{A}$ *is a strong Krull structure*.

PROOF. We have seen that $G \simeq \bigoplus_n I^n/I^{n+1}$, which is well-known to be noetherian. Moreover the Krull property for A is nothing but the standard Krull intersection property for ideals of A (use Lemma 1.6). The same argument extends to modules with the given property, in particular to finite free modules and we get also b).

§2. TYPES OF ORDERINGS ON $\mathbb{Z}^n$.

Within this section we adopt the convention that, given a group G, an "ordering $<$ on G " means a "total ordering $<$ on G such that $(G,<)$ is an ordered group".

DEFINITION 1. An ordering on a group G is said to be continuous with respect to a given topology on G , if for every $p \in G$ such that there exists a neighborhood (nbh.) U_p with $U_p \subseteq G^+$, then $p \in G^+$ and the same for G^- .

In this section we consider $\mathbb{Z}^n, \mathbb{Q}^n, \mathbb{R}^n$ as additive groups and topological properties are understood as properties of the euclidean topology.

LEMMA 2.1. a) Every ordering on $\mathbb{Z}^n$ extends uniquely to an ordering on $\mathbb{Q}^n$.
b) Every ordering on $\mathbb{Q}^n$ is such that $(\mathbb{Q}^n)^+$ and $(\mathbb{Q}^n)^-$ are convex sets; in particular it is continuous.

PROOF. a) If $p \in \mathbb{Q}^n$ we take an m in $\mathbb{N}^+$ such that $mp \in \mathbb{Z}^n$ and of course we say that p is positive (negative) if mp is positive (negative).
b) If $p,q \in (\mathbb{Q}^n)^+$ and $p < q$, then the segment $\overline{pq}$ is given by $p+(1-t)(q-p)$ $0 \leq t \leq 1$, $t \in \mathbb{Q}$, hence, again by clearing positive denominators, we see that it is contained in $(\mathbb{Q}^n)^+$.

<u>REMARK</u>. On $\mathbb{R}^n$ there are many noncontinuous orderings. For instance let us take $n = 1$ and consider a base E of $\mathbb{R}$ as a $\mathbb{Q}$-vector space; then we give a total order to E and if $r \in \mathbb{R}$ $r = \Sigma_i \lambda_i e_i$ only a finite number of e_i's is involved, hence we may associate to r the "first" nonzero coordinate; call it $\lambda(r)$ and say that $r > 0$ iff $\lambda(r) > 0$.

<u>DEFINITION 2</u>. Given an ordering on $\mathbb{Q}^n$, we denote by $V(\mathbb{Q}^n)$ or simply by V the set of points $p \in \mathbb{R}^n$ s.t. for every nbh. U_p of p , both $U_p \cap (\mathbb{Q}^n)^+$ and $U_p \cap (\mathbb{Q}^n)^-$ are nonempty.

<u>LEMMA 2.2</u>. V is a subvectorspace of $\mathbb{R}^n$ of dimension $n-1$.

<u>PROOF</u>. To show that V is a subvectorspace is an easy excercise. Now let us consider the function $s : \mathbb{R}^n - V \longrightarrow \{-1,1\}$ defined by $s(p) = 1$ if there exists a nbh. U_p of p such that $U_p \cap \mathbb{Q}^n \subset (\mathbb{Q}^n)^+$; $s(p) = -1$ if there exists a nbh. U_p of p such that $U_p \cap \mathbb{Q}^n \subset (\mathbb{Q}^n)^-$. Now s is continuous if we endow $\{-1,1\}$ with the discrete topology. If $\dim V < n-1$, then $\mathbb{R}^n - V$ is connected hence $\mathrm{Im}(s) = \{1\}$ say. This implies that in $\mathbb{Q}^n$ we can find antipodal points inside $(\mathbb{Q}^n)^+$, a contradiction. If $\dim V = n$, then $V = \mathbb{R}^n$; however if $\{e_1,\ldots,e_n\}$ is the canonical base of $\mathbb{R}^n$ and e_i^* denotes the vector of the set $\{e_i,-e_i\}$ which is in $(\mathbb{Q}^n)^+$, then $e_1^*,\ldots,e_n^*$

generate an n-dimensional polyhedron which is in $(\mathbb{Q}^n)^+$ by Lemma 2.1 b), a contradiction.

PROPOSITION 2.3. Every ordering on $\mathbb{Q}^n$ extends to a continuous ordering on $\mathbb{R}^n$ (The extension is not necessarily unique).

PROOF. Given an ordering on $\mathbb{Q}^n$ we get the set V of definition 2, which is a subvectorspace of $\mathbb{R}^n$ by Lemma 2.2, hence we may choose a vector $v_1 \in \mathbb{R}^n$ which is orthogonal to V and inside $s^{-1}\{1\}$ (see the proof of Lemma 2.2). Now the extension of the ordering to $\mathbb{R}^n - V$ is uniquely determined by the requirement that it is continuous and it can be expressed by saying that for every $v \in \mathbb{R}^n - V$ $v > 0$ iff $v \cdot v_1 > 0$, where "." denotes the usual scalar product. We denote by $V_{\mathbb{Q}}$ the sub-$\mathbb{Q}$-vectorspace $V \cap \mathbb{Q}^n$ of $\mathbb{Q}^n$ and observe that $\dim_{\mathbb{Q}} V_{\mathbb{Q}} \leq \dim_{\mathbb{R}} V = n-1$.
If $\dim V_{\mathbb{Q}} = n-1$ then we are exactly in the same situation as before, but with dimension one less.
If $\dim V_{\mathbb{Q}} = d < n-1$, then we denote by $\overline{V}_{\mathbb{Q}}$ the $\mathbb{R}$-vectorspace generated by $V_{\mathbb{Q}}$ and we choose an orthogonal base $\{v_2,\dots,v_{n-d}\}$ of the space $\overline{V}_{\mathbb{Q}}^{\perp} \cap V$. Of course $(V-V_{\mathbb{Q}}) \cap \mathbb{Q}^n = \emptyset$ hence we can say that for every $v \in V-\overline{V}_{\mathbb{Q}}$, $v > 0$ iff the first nonzero coordinate of the vector $(v\cdot v_2,\dots,v\cdot v_{n-d})$ is positive. Now we have to extend the ordering to $\overline{V}_Q$ and we are exactly in the same situation as at the beginning, but with dimension $n-d$ less.

Therefore this procedure ends after at most n steps and clearly gives a continuous ordering on $\mathbb{R}^n$, which extends the given one on $\mathbb{Q}^n$. It is also clear by the construction that every time we meet a situation where $\dim V_{\mathbb{Q}} < \dim V$, we loose the unicity of the extension.

<u>EXAMPLE 1</u>. Let us consider the Example 4 of the first section. There $v_1 = (\sqrt{2},1)$ and $V = \{(x,y)/\sqrt{2}x+y = 0\}$ hence $V_{\mathbb{Q}} = \{(0,0)\}$; if we take a vector v in $\mathbb{R}^2 - V$ then $v > 0$ iff $v \cdot v_1 > 0$, but if we take $v \in V$ then we have <u>two</u> choices. Namely, let $v_2 = (1,-\sqrt{2})$; then we can say that if $v \in V$, $v > 0$ iff $v \cdot v_2 > 0$ or $v > 0$ iff $v \cdot (-v_2) > 0$; both extend continuously the ordering on $\mathbb{Q}^2$, which is obtained by extending the given ordering on $\mathbb{Z}^2$.

<u>DEFINITION 3</u>. The ordering on $\mathbb{R}^n$ $(\mathbb{Q}^n, \mathbb{Z}^n)$ defined by the rule: $(a_1,\dots,a_n) > 0$ iff the first nonzero coordinate is positive is called lexicographic and denoted by lex.

<u>PROPOSITION 2.4</u>. <u>Let</u> $<$ <u>be a continuous ordering on</u> $\mathbb{R}^n$. <u>Then there exists an ordered isomorphism</u> $\alpha : (\mathbb{R}^n,<) \longrightarrow (\mathbb{R}^n,\mathrm{lex})$.

<u>PROOF</u>. Of course $<$ induces an ordering on $\mathbb{Q}^n$, hence we get a vector space V of dimension $n-1$ (lemma 2.2) hence a vector v_1 orthogonal to V and such that if $v \in \mathbb{R}^n - V$, $v > 0$ iff $v \cdot v_1 > 0$ (see the proof of Proposition 2.3.)

Let $\{w_1,\dots,w_{n-1}\}$ be a base of V and denote by $\tilde{V}$ the set $\mathbb{Q}w_1+\dots+\mathbb{Q}w_{n-1}$ which is a $\mathbb{Q}$-vectorspace of dimension $n-1$. Now we repeat the same argument for the couple $V,\tilde{V}$ as we did for $\mathbb{R}^n,\mathbb{Q}^n$. Eventually we get $\{v_1,\dots,v_n\}$ which is an orthogonal base of $\mathbb{R}^n$ with the property that $v>0$ iff the first nonzero coordinate of the vector $(v\cdot v_1,\dots,v\cdot v_n)$ is positive. To conclude it is sufficient to put $\alpha(v) = (v\cdot v_1,\dots,v\cdot v_n)$.

<u>THEOREM 2.5</u>. <u>Let</u> $<$ <u>be an ordering on</u> $\mathbb{Q}^n$. <u>Then there exists an integer</u> s <u>with</u> $1 \leq s \leq n$ <u>and an ordered injective homomorphism</u> $\alpha : (\mathbb{Q}^n,<) \longrightarrow (\mathbb{R}^s,\mathrm{lex})$.

<u>PROOF</u>. Let v_1 be chosen as we did in the proof of Proposition 2.3 ; we give it the new name u_1. Looking again at the proof of Proposition 2.3, we see that $v_2,\dots v_{n-d}$ are orthogonal to $V_{\mathbb{Q}}$, hence if $v\cdot u_1 = 0$ then $v\cdot v_i = 0$ $i = 2,\dots,n-d$. So the next vector which is relevant to the ordering of $\mathbb{Q}^n$ is v_{n-d+1} ; we give it the new name u_2. Going on in this way we eventually get a subarray $(u_1,\dots,u_s)$ of $(v_1,\dots,v_n)$ where $u_1 = v_1$. Now we consider the homomorphism $\alpha : (\mathbb{Q}^n,<) \longrightarrow (\mathbb{R}^s,\mathrm{lex})$ given by $\alpha(v) = (v\cdot u_1,\dots,v\cdot u_s)$ and we get that α is injective since $v\cdot u_i = 0$, $i = 1,\dots s$ implies $v\cdot v_j = 0$, $j = 1,\dots,n$ and it is ordered just because of the given description.

DEFINITION 4. Let $<$ be an ordering on $\mathbb{Z}^n(\mathbb{Q}^n)$. Then, the minimum number s such that there exists an injective homomorphism α like in Theorem 2.5 is denoted by $s(<)$. We say that $<$ is of lexicographic type if $s(<) = n$.

REMARK. Of course $<$ is archimedean iff $s(<) = 1$.

EXAMPLE 2. Let $A = k[x_1,\ldots,x_n]$ where $\deg(x_i) = 1$ for $i = 1,\ldots,n$ and let Γ be the free group generated by $\{x_1,\ldots,x_n\}$. In many problems arizing in computer algebra the following total ordering on Γ is considered: if M_1, M_2 are terms i.e. monomials with coefficient 1, then $M_1 < M_2$ if either $\deg M_1 < \deg M_2$ or $\deg M_1 = \deg M_2$ and $M_1 < M_2$ in the lexicographic ordering generated by $x_1 < \ldots < x_n$. If we identify Γ with $\mathbb{Z}^n$ we see that the above given ordering is of lexicographic type and the vectors $u_1,\ldots u_n$ are
$u_1 = (1,1,\ldots,1)$, $u_2 = (-n+1,1,\ldots,1)$, $u_3 = (0,-n+2,1,\ldots,1)\ldots$
$\ldots$ $u_{n-1} = (0,\ldots,0,-2,1,1)$, $u_n = (0,\ldots,0,-1,1)$.

REMARK. In computer algebra the most important orderings on the set T of the terms of $A = k[x_1,\ldots,x_n]$ are the so-called "term-orderings". A term-ordering is defined to be a total ordering on T such that a) For every $M \in T, M \neq 1$ then $1 < M$ and b) If $N < M$ and $M' \in T$ then $NM' < MM'$. But this simply means that the free group Γ generated by $\{x_1,\ldots,x_n\}$ is given a total ordering such that it becomes an ordered group and $T \geq 0$. So our description gives a <u>full classification of all the term-orderings on</u> A . (for some more details see Robbiano, L.: Term orderings on the polynomial ring. Preprint).

<u>COROLLARY 2.6</u> <u>Let</u> $<$ <u>be an ordering on</u> $\mathbb{Z}^n$, <u>let</u> $u_1,\dots,u_t$ <u>be elements of</u> $\mathbb{Z}^n$ <u>and</u> $\Gamma_1,\dots,\Gamma_t$ be <u>finitely generated non negative (i.e. contained in</u> $(\mathbb{Z}^n)^+ \cup \{0\}$) <u>sub-semigroups of</u> $\mathbb{Z}^n$. <u>Then</u> $E = \bigcup_{i=1}^{t} (u_i+\Gamma_i)$ <u>is well-ordered</u>.

<u>PROOF</u>. We may assume $t = 1$, hence that E itself is a finitely generated non-negative semigroup of $\mathbb{Z}^n$. We extend the ordering $<$ to $\mathbb{Q}^n$ and we consider the injective ordered homomorphism $\alpha : (\mathbb{Q}^n,<) \longrightarrow (\mathbb{R}^s,\text{lex})$ of Theorem 2.5, so that we may assume that E is a finitely generated non-negative subsemigroup of $(\mathbb{R}^s,\text{lex})$. Let us take a subset $F \subseteq E$, $F \neq \emptyset$. We may consider the integer t , $1 \leq t \leq s$ such that every element of F has the first $t-1$ coordinates zero, and there exists an element in F having the t^{th}-coordinate different from zero. If we denote by π the projection of $\mathbb{R}^s$ to the t^{th}-factor, since $\mathbb{R}^s$ is ordered lexicographically, we are reduced to prove that $\pi(F)$ has a first element. On the other hand $\pi(E)$ turns out to be a finitely generated non-negative semigroup of $\mathbb{R}$ with the usual ordering. If $r \in E$ it is therefore sufficient to prove that $\{r' \in E / r' \leq r\}$ is finite and this is clear since $\mathbb{R}$ with the usual ordering is archimedean.

REMARK. Let us consider again the Example 4 of the first section. The map α of Theorem 2.5 is given by

$$\alpha : (\mathbb{Z}^2,<) \longrightarrow (\mathbb{R},<) \qquad \alpha(x,y) = \sqrt{2}\,x+y$$

so that $<$ is archimedean. Again by using Theorem 36 p. 30 of Hardy-Wright "The teory of numbers", we can find a sequence (x_n,y_n) with $x_n > 0$ and $0 < \sqrt{2}\,x_n + y_n < \frac{1}{x_n}$. This implies that with respect to this ordering $(\mathbb{Z}^2)^+$ is not a finitely generated subsemigroup.

THEOREM 2.7. Let A be a graded noetherian structure, such that $\Gamma^\circ(A) \geq 0$. Then every finite A-module is a Krull-A-module; in particular A is a strong Krull-structure.

PROOF. Let $M = (M,\Gamma,w,T,\emptyset)$; then there is a surjective graded homomorphism $\oplus_1^t G(-\gamma_i) \longrightarrow T$, hence $\Gamma^\circ(M) \subseteq \bigcup_{i=1}^{t} (-\gamma_i+\Gamma^\circ(A))$ and $\Gamma^\circ(A)$ generates a finitely generated subsemigroup of Γ (see Proposition 1.3). Since $\Gamma^\circ(A) \geq 0$ we get from Corollary 2.6 that $\Gamma^\circ(M)$ is well-ordered so that in $\Gamma^\circ(M)$ there are no strictly decreasing sequences and we are done.

§3. DISTINGUISHED BASES.

We start this section with the following fundamental

DEFINITION 1. Let $A = (A,\Gamma,v,G,F)$ be a graded noetherian structure, let $M = (M,\Gamma,w,T,\emptyset)$ be a finite A-module and let $m_1,\ldots m_r$ be nonzero elements of M. We say that $\{m_1,\ldots m_r\}$ is a distinguished base (d-base) of M if every nonzero element m of M can be written in the following way:

$m = \sum_1^r a_i m_i$ where $a_i \in A$ and for every $i \in \{1,\ldots,r\}$ such that $a_i \neq 0$, $w(m) \geq v(a_i) + w(m_i)$.

REMARK 1. By using the axioms M_4, M_7 it is clear that the condition $w(m) \geq v(a_i) + w(m_i)$ can be replaced by: $w(m) = \mathrm{Max}(v(a_i) + w(m_i))$ where the maximum is taken over the set of indexes i such that $a_i \neq 0$.

REMARK 2. When we say that $\{m_1,\ldots,m_r\}$ is a d-base, we mean that it is a d-base of the induced structure on the submodule generated by $\{m_1,\ldots,m_r\}$.

THEOREM 3.1. Let $A, M, m_1,\ldots,m_r$ be as before and let us consider the following conditions

i) $\{m_1,\ldots,m_r\}$ is a d-base of M.

ii) $\{\emptyset(m_1),\ldots,\emptyset(m_r)\}$ is a base of T as a G-module.

Then i) implies ii) and if M is a Krull-module they are equivalent.

PROOF. i) $\Rightarrow$ ii) Let $h \in T$ be a nonzero homogeneous element of degree γ ; then $h = \phi(m)$ where $w(m) = \gamma$. We write $m = \Sigma a_i m_i$ according to definition 1 and we let I be the subset of $\{1,\ldots,r\}$ of the indexes j's such that $w(m) = v(a_j)+w(m_j)$. By using the axioms we get
$\phi(m) = \sum_{j\in I} F(a_j)\phi(m_j)$.

Now we assume M to be a Krull-module and we prove ii) $\Rightarrow$ i). We consider the following subgroups of M
$U_\gamma = F_{\gamma-w(m_1)}\cdot m_1+\ldots+F_{\gamma-w(m_r)}\cdot m_r$. Of course $U_\gamma \subseteq F_\gamma M$ and we are done if we prove that equality holds for every $\gamma \in \Gamma$. For, we take the finite free module $L(w(m_1),\ldots,w(m_r))$ (see Example 8 and Definition 10 of the first section) and we consider the homomorphism $\lambda : A^r \longrightarrow M$ defined by $\lambda(a_1,\ldots,a_r) = \Sigma a_i m_i$, which gives rise to a morphism $\lambda : L(w(m_1),\ldots,w(m_r)) \longrightarrow M$. Now $\lambda(F_\gamma A^r) = U_\gamma$ and since M is assumed to be a Krull-module, it will suffice to show that $F_\gamma M \subseteq \bigcap_n (U_\gamma + F_{\gamma_n} M)$ for a suitable decreasing sequence $\{\gamma_n\}_{n\in\mathbb{N}}, \gamma_n \in \Gamma^\circ(M)$. So let m be a nonzero element of M such that $w(m) \leq \gamma$; we know that
$\phi(m) = \sum_{i=1}^{r} R_i\ \phi(m_i)$ where either $R_i = 0$ or $\deg R_i = w(m) - w(m_i)$. The nonzero R_i's are of the form $F(a_i)$ with $v(a_i) = w(m) - w(m_i)$, therefore $\phi(m) = \phi(\Sigma a_i m_i)$ where the summation is taken over the

set of indexes i's such that $R_i \neq 0$. Therefore $\gamma_1 = w(m - \Sigma a_i m_i) < w(m)$, whence $m \in U_\gamma + F_{\gamma_1} M$. Now we apply our argument to $m - \Sigma a_i m_i$ and so on, so that we get the required decreasing sequence and we are done. As a straightforward consequence of this theorem we get

COROLLARY 3.2. *If A is a noetherian graded structure, then every finite Krull-A-module has d-bases.*

REMARK 3. If we look at Definition 3 of section 1 and at Proposition 1.4, it turns out that Theorem 3.1 yields a criterion for "computing" the quotients of the Krull modules. See also Proposition 4.6 of the last section.

LEMMA 3.3. *Let $M = (M,\Gamma,w,T,\emptyset)$, $M' = (M',\Gamma,w',T',\emptyset')$ be two modules over a noetherian graded structure A . Let $(\lambda,\Lambda) : M' \longrightarrow M$ be a morphism and let K be the induced structure on $\ker \lambda$. Consider the following conditions*

i) *For every $\gamma \in \Gamma$, $\lambda(F_\gamma M') = F_\gamma M$.*

ii) *M'/K is a v-filtered structure on $M'/\mathrm{Ker}(\lambda)$ and (λ,Λ) induces an isomorphism $(\overline{\lambda},\overline{\Lambda}) : M'/K \longrightarrow M$.*

iii) *Λ is surjective*

Then i) $\Leftrightarrow$ ii) $\Rightarrow$ iii) .

PROOF. Condition ii) is equivalent to the condition: λ induces isomorphisms $\lambda_\gamma : F_\gamma M'/(F_\gamma M' \cap \mathrm{Ker}(\lambda)) \longrightarrow F_\gamma M$

whence i),ii) are clearly equivalent, while the implication i) $\Rightarrow$ iii) follows directly from the axioms.

REMARK 4. Of course conditions i),ii) can be expressed in categorical language by saying that (λ,Λ) is an epimorphism.

EXAMPLE 1. $\mathcal{A} = (A, \mathbf{Z}, F_A)$ be the v-filtered structure on $A = k[x]$, where $F_A = \{F_{-n}A\}_{n\in\mathbf{Z}}$ $F_{-n}A = (x)^n$ and let $\mathcal{J}$ be the induced structure on (x) . The A-homomorphism $\lambda : A \longrightarrow A$ defined by $\lambda(1) = x-x^2$ extends to a morphism $(\lambda,\Lambda) : \mathcal{A}(+1) \longrightarrow \mathcal{J}$. Now the associated graded module in $\mathcal{A}(+1)$ is isomorphic to $A(+1)$ (where A is graded by $\deg(x) = -1$) and the associated graded module in $\mathcal{J}$ is isomorphic to (x) . Then it turns out that $\Lambda : A(+1) \longrightarrow (x)$ is defined by $\Lambda(1) = x$, hence Λ is surjective, but $F_0A = A$, $F_0(x) = (x)$ and $\lambda(F_0A) \subset F_0(x)$. So in general iii) does not imply i) in Lemma 3.3. However we have:

<u>LEMMA 3.4</u>. <u>Let</u> $\mathcal{A}$ <u>be a noetherian graded structure, let</u> $\mathcal{M}$ <u>be a finite Krull</u> $\mathcal{A}$-<u>module, let</u> $m_1,\ldots,m_r$ <u>be elements of</u> M <u>and put</u> $w_i = w(m_i)$ $i = 1,\ldots,r$. <u>Let</u> (λ,Λ) <u>be the canonical morphism from</u> $\mathcal{L}(w_1,\ldots,w_r)$ <u>to</u> $\mathcal{M}$ <u>associated to</u> $m_1,\ldots,m_r$ <u>(see Section 1 Ex. 8). Then</u> i),ii),iii) <u>of Lemma 3.3 are equivalent to the condition</u>

iv) $\{m_1,\ldots,m_r\}$ <u>is a</u> d-<u>base</u>.

PROOF. iii) $\Rightarrow$ i) Let $m \in M$, $m \neq 0$ and put $\gamma = w(m)$. Since $\emptyset(m) = \emptyset^{+}(l)$ for a suitable $l \in A^{r}$, we get $\Lambda(\emptyset^{+}(l)) = \emptyset(\lambda(l))$ and $w^{+}(l) = w(m) = \gamma$. Therefore, if m_1 denotes $m-\lambda(l)$, then $w(m_1) < \gamma$. Now either $m_1 = 0$ and we are done or we repeat the same argument for m_1 and we get $l_1 \in A^{r}$ such that $w^{+}(l_1) = w(m_1) < \gamma$. Therefore $w(m-\lambda(l+l_1)) < w^{+}(m_1) < \gamma$ and $w^{+}(l+l_1) = w(l) = \gamma$. Going on in this way we get a strictly decreasing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_n \in \Gamma^{\circ}(M)$ and

$$m \in \bigcap_{n}(\lambda(F_{\gamma}A^{r}) + F_{\gamma_n}M) = \lambda(F_{\gamma}A^{r})$$

where the last equality follows from the Krull-type assumption on M.

Now the equivalence between iii) and iv) follows from the definition of λ and from Theorem 3.1.

THEOREM 3.5. If A is a noetherian graded structure and M is a finite Krull A-module, then M has a free resolution.

PROOF. Since every submodule of a finite Krull A-module is also finite Krull, the proof is done if we can show that for every finite Krull A-module N there exists a finite free module L and an epimorphism $(\lambda,\Lambda) : L \longrightarrow N$, because then we replace N with K, the induced structure on $\operatorname{Ker}\lambda$, and so on.

Now N has a d-base by Corollary 3.2, hence we are done by using Lemma 3.4.

THEOREM 3.6. *Let* $\mathcal{M} = (M,\Gamma,w,T,\emptyset)$, $\mathcal{M}' = (M',\Gamma,w',T',\emptyset')$ *be two finite Krull-modules over a noetherian graded structure* A *and let* $(\lambda,\Lambda) : \mathcal{M}' \longrightarrow \mathcal{M}$ *be a morphism. Then the following conditions are equivalent*

1) (λ,Λ) *is an epimorphism.*
2) Λ *is surjective*
3) λ *is surjective and for every homogeneous nonzero element* $\sigma \in \mathrm{Ker}(\Lambda)$, *there exists an element* $s \in \mathrm{Ker}(\lambda)$ *such that* $\emptyset'(s) = \sigma$.
4) λ *is surjective and there exist a homogeneous base* $\{\sigma_1,\dots,\sigma_t\}$ *of* $\mathrm{Ker}(\Lambda)$ *and elements* $s_1,\dots,s_t$ *of* $\mathrm{Ker}(\lambda)$ *such that* $\emptyset'(s_i) = \sigma_i$, $i = 1,\dots,t$.

PROOF. 1) $\Rightarrow$ 3) It is clear that λ is surjective. Let now σ be a homogeneous nonzero element in $\mathrm{Ker}(\Lambda)$. Then $\sigma = \emptyset'(m')$ for a suitable m' , and $\Lambda\emptyset'(m') = 0$. So either $\lambda(m') = 0$ and we are done, or $w(\lambda(m')) < w'(m')$. By assumption $\lambda(m') = \lambda(m'')$ with $w'(m'') = w(\lambda(m'')) = w(\lambda(m')) < w'(m')$. Therefore $m'-m'' \in \mathrm{Ker}(\lambda)$ and $\emptyset'(m'-m'') = \emptyset'(m') = \sigma$.

3) $\Rightarrow$ 2) Let t be a nonzero homogeneous element in T of degree γ . Then $t = \emptyset(\lambda(m'))$ for some $m' \in M'$. If $w'(m') = \gamma$ then $\emptyset(\lambda(m')) = \Lambda\emptyset'(m')$ and we are done. If $w'(m') > \gamma$ then $\Lambda\emptyset'(m') = 0$, hence $\emptyset'(m') \in \mathrm{Ker}(\Lambda)$; let $m \in \mathrm{Ker}(\lambda)$ be such that $w'(m) = w'(m') > \gamma$ and $\emptyset'(m) = \emptyset'(m')$. We put $m_1 = m'-m$ so that we have

$t = \emptyset(\lambda(m_1))$ with $\gamma_1 = w'(m_1) < w'(m')$. After a finite number of steps like this we get $\gamma_{\overline{n}} = \gamma$, otherwise we could construct a strictly decreasing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$, $\gamma_n \in \Gamma^\circ(M')$ such that $m' \in \bigcap_n (\mathrm{Ker}(\lambda) + F_{\gamma_n} M') = \mathrm{Ker}(\lambda)$, since M' is a Krull-module; but then we get $t = \emptyset(\lambda(m')) = 0$ a contradiction.

2) $\Rightarrow$ 1) The proof of this implication would be exactly the same as the proof of iii) $\Rightarrow$ i) in Lemma 3.4 if we knew that

$$\bigcap_n (\lambda(F_\gamma M') + F_{\gamma_n} M) = \lambda(F_\gamma M')$$

for every $\gamma \in \Gamma \cup \{-\infty\}$ and every strictly decreasing sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ of elements of $\Gamma^\circ(M)$.

Now M' is a Krull-module, hence it has a d-base, say $\{m'_1,\ldots,m'_r\}$ by Corollary 3.2.

Let $w_i = w'(m'_i)$, $i = 1,\ldots,r$ and consider the canonical morphism $(\delta,\Delta) : L(w_1,\ldots,w_r) \longrightarrow M'$ associated to the elements $m'_1,\ldots,m'_r$. We get

$$\bigcap_n (\lambda(F_\gamma M') + F_{\gamma_n} M) \overset{(1)}{=} \bigcap_n (\lambda(\delta(F_\gamma A^r)) + F_{\gamma_n} M) \overset{(2)}{=} \lambda(\delta(F_\gamma A^r)) \overset{(3)}{=} \lambda(F_\gamma M')$$

where (1) and (3) follow from Lemma 3.4 applied to M' and (2) follows from the fact that M is a Krull-module.

3) $\Rightarrow$ 4) Obvious

4) $\Rightarrow$ 3) Let σ be a homogeneous nonzero element in $\mathrm{Ker}(\Lambda)$

of degree γ . Then we can write $\sigma = \Sigma g_i \sigma_i$ where $0 \neq g_i \in G$ are homogeneous, $\deg g_i + \deg \sigma_i = \gamma$ and the σ_i's are among the elements of the base $\{\sigma_1, \ldots, \sigma_t\}$ of $\mathrm{Ker}(\Lambda)$. But then it follows directly from the axioms that $\sigma = \emptyset'(\Sigma a_i s_i)$ where $g_i = F(a_i)$, $\sigma_i = \emptyset'(s_i)$.

COROLLARY 3.7. With the same assumption as in Theorem 3.6., let $m_1, \ldots, m_r$ be elements of M such that $\{\emptyset(m_1), \ldots, \emptyset(m_r)\}$ is a base of $\mathrm{Im}(\Lambda)$. Then all the conditions of Theorem 3.6 are equivalent to

5) $\{m_1, \ldots, m_r\}$ is a d-base of $\mathcal{M}$.

PROOF. Since $\{\emptyset(m_1), \ldots, \emptyset(m_r)\}$ generates $\mathrm{Im}(\Lambda)$, Theorem 3.1 tells us that $\{m_1, \ldots, m_r\}$ is a d-base of $\mathcal{M}$ if and only if $\mathrm{Im}(\Lambda) = T$ i.e. if and only if condition 2) of Theorem 3.6 holds.

COROLLARY 3.8. Let $\mathcal{A}$ be a strong Krull structure. Let $\mathcal{M} = (M, \Gamma, w, T, \emptyset)$ be a finite Krull $\mathcal{A}$-module and $m_1, \ldots, m_r$ be nonzero elements which generate M . Let $w_i = w(m_i)$, $i = 1, \ldots, r$ and let $(\lambda, \Lambda) : \mathcal{L}(w_1, \ldots, w_r) \longrightarrow \mathcal{M}$ be the canonical morphism associated to $m_1, \ldots, m_r$. Then the following conditions are equivalent

1) (λ, Λ) is an epimorphism

2) Λ is surjective

3) <u>For every homogeneous nonzero element</u> $\sigma \in \mathrm{Ker}(\Lambda)$, <u>there exists an element</u> $s \in \mathrm{Ker}(\lambda)$ <u>such that</u> $\emptyset^+(s) = \sigma$

4) <u>There exists a homogeneous base</u> $\{\sigma_1,\dots,\sigma_t\}$ <u>of</u> $\mathrm{Ker}(\Lambda)$ <u>and elements</u> $s_1,\dots,s_t \in \mathrm{Ker}(\lambda)$ <u>such that</u> $\emptyset^+(s_i) = \sigma_i$, $i = 1,\dots,t$.

5) $\{m_1,\dots,m_r\}$ <u>is a</u> d-<u>base</u>.

<u>PROOF</u>. It is a consequence of Corollary 3.7 and Theorem 3.6, since the conditions "$\{\emptyset(m_1),\dots,\emptyset(m_r)\}$ is a base of $\mathrm{Im}(\Lambda)$" and " λ is surjective" are fulfilled by the very definition of (λ,Λ) .

<u>EXAMPLE 2</u>. Let $A = k[x,y,z]$, $I = (f_1,f_2)$ where $f_1 = x-y^4$, $f_2 = z^2-xy^3$, $\Gamma = \mathbb{Z}$, $v(f) = \deg(f)$. We get a graded structure $\mathcal{A} = (A,\Gamma,v,G,F)$ on A , where $\Gamma^\circ(\mathcal{A}) = \mathbb{N}$ $G = A$, $F(f) = H(f)$, where $H(f)$ is the form of f of maximal degree (see Ex. 2 of section 1). Let $\mathcal{J}$ be the induced structure on I and let $(\lambda,\Lambda) : \mathcal{L}(4,4) \longrightarrow \mathcal{J}$ be the canonical map associated to f_1,f_2 . Let $f = z^4-x^3y^2 = -x^2y^2f_1 + (z^2+xy^3)f_2$ and $g = yf = yz^4-x^3y^3 = -x^2y^3f_1 + (yz^2+xy^4)f_2$. We have $F(g) = -x^3y^3 = x^2F(f_2) = \Lambda(0,x^2)$. Since f_1,f_2 is a regular sequence , g can be written as a combination of f_1,f_2, only in the following way $g = (rf_2-x^2y^3)f_1 + (yz^2+xy^4-rf_1)f_2$, $r \in A$. Therefore $g = \lambda(rf_2-x^2y^3,\ yz^2+xy^4-rf_1), r \in A$ . .
Claim: for every choice of $r \in A$ we have $w^+(rf_2-x^2y^3,yz^2+xy^4-rf_1) > v(g)$.

For, it is sufficient to show that for every choice of $r \in A$, $\deg(rf_2 - x^2y^3) > 2$; it is clear that the minimum is attained when $r = -x$ and it is 3.

Now the morphism (λ, Λ) certainly satisfies the hypotheses of Theorem 3.6 , hence the conditions "(λ,Λ) is an epimorphism" and "Λ is surjective" are equivalent. However $F(g) \in \mathrm{Im}(\Lambda)$, $v(g) = 6$ but $g \in \lambda(F_7A^2) - \lambda(F_6A^2)$.

Another remark is that a d-base in this example is $\{f_1,f_2,f_3,f_4,f_5\}$ where $f_3 = x^2-yz^2$, $f_4 = f = z^4-x^3y^2$, $f_5 = x^5y-z^6$. So we may consider the canonical map $(\delta,\Delta) : L(4,4,3,5,6) \longrightarrow J$ associated to f_1,f_2,f_3,f_4,f_5 . Then Δ is surjective by Corollary 3.8 and $g = \delta(0,0,0,y,0) \in \delta(F_6A^5)$.

<u>EXAMPLE 3</u>. Let $A = (A,\Gamma,v,G,F)$ where $A = k[x]$, $\Gamma = \mathbb{Z}$, $v(f) = \deg(f)$ and consider the induced morphism $A(-1) \longrightarrow A$ by the identity map $A \longrightarrow A$. Then $\Lambda = 0$ and $\{1\}$ is a d-base of A , but the conclusions of Corollary 3.7 cannot be applied to this situation, since $F(1) \notin \mathrm{Im}(\Lambda)$.

Let now A be a graded structure, $M = (M,\Gamma,w,T,\emptyset)$ a finite A-module.

<u>DEFINITION 2</u>. Let $m_1,\dots,m_r \in M$. We say that $(m_1,\dots,m_r)$ is a stepwise d-base if $\{m_1\}$, $\{m_1,m_2\}$, ..., $\{m_1,\dots,m_r\}$ are d-bases. We say that $\{m_1,\dots,m_r\}$ is a strong d-base if every subset of $\{m_1,\dots,m_r\}$ is a d-base.

In the following, if $\mathcal{A} = (A,\Gamma,v,G,F)$ is a graded structure and $x_1,\dots,x_r \in A$, we denote by $\underline{x}$ the sequence $x_1,\dots,x_r$, by $\{\underline{x}\}$ the set $\{x_1,\dots,x_r\}$, by $(\underline{x})$ the ideal generated by $\{\underline{x}\}$ and by $F(\underline{x})$ the sequence $F(x_1),\dots,F(x_r)$.

<u>THEOREM 3.9</u>. <u>Let</u> $\mathcal{A} = (A,\Gamma,v,G,F)$ <u>be a strong Krull structure and let</u> $x_1,\dots,x_r \in A$. <u>Then the following conditions are equivalent</u>

1) $\underline{x}$ <u>is a regular sequence and a stepwise</u> d-<u>base</u>.

2) $F(\underline{x})$ <u>is a regular sequence</u>.

<u>PROOF</u>. 1) $\Rightarrow$ 2) Let $g.F(x_r) = \sum_1^{r-1} {}_i g_i F(x_i)$ where $g,g_1,\dots,g_{r-1}$ are homogeneous, $g \neq 0$ and $\deg(g) + v(x_r) = \deg(g_i) + v(x_i)$ for those i's such that $g_i \neq 0$. This implies that $(g_1,\dots,g_{r-1},-g)$ is a homogeneous syzygy of $F(x_1),\dots,F(x_r)$, hence it can be lifted to $(a_1,\dots,a_r)$ such that $\Sigma a_i x_i = 0$ and $w^+(a_1,\dots,a_r) =$ $= \deg(g_1,\dots,g_{r-1},-g) = \deg(g) + v(x_r)$ by means of Corollary 3.8.

Now $a_r \neq 0$ and $a_r \in (x_1,\dots,x_{r-1})$ since $\underline{x}$ is a regular sequence; moreover $\{x_1,\dots,x_{r-1}\}$ is a d-base, hence $a_r = \sum_1^{r-1} {}_i b_i x_i$ with $v(a_r) = \operatorname{Max}_i(v(b_i) + v(x_i))$. Therefore $g = F(a_r) \in (F(x_1),\dots,F(x_{r-1}))$.

Of course the other steps in the proof that $F(\underline{x})$ is a regular sequence can be proved in the same way by the very nature of the assumptions.

2) $\Rightarrow$ 1) We prove that $\{\underline{x}\}$ is a d-base. The syzygies of $F(x_1),\dots,F(x_r)$ are generated by the trivial ones, which can be lifted to the corresponding trivial syzygies of $x_1,\dots,x_r$, hence we are done by Corollary 3.8. Of course the same remark as before shows that

$\{x_1\}$, $\{x_1,x_2\},\dots,\{x_1,\dots,x_{r-1}\}$ are also d-bases. Now we prove that $a_r x_r \in (x_1,\dots,x_{r-1})$ implies $a_r \in (x_1,\dots,x_r)$. We know already that $x_1,\dots,x_{r-1}$ is a d-base hence $a_r x_r = \sum_1^{r-1} {}_i a_i x_i$ with $v(a_r x_r) = \operatorname{Max}_i (v(a_i) + v(x_i))$.

If $v(a_r x_r) < v(a_r) + v(x_r)$ then $F(a_r)F(x_r) = 0$; if $v(a_r x_r) = v(a_r)+v(x_r)$ then $F(a_r)F(x_r) = \Sigma g_i F(x_i)$ where the sum is taken over the set of the i's such that $v(a_i)+v(x_i)$ is the maximum. In both cases we get that $F(a_r) = \Sigma h_i F(x_i) = \Sigma F(b_i)F(x_i)$, whence $v(a_r - \Sigma b_i x_i) < v(a_r)$. Going on in this way, we get a strictly decreasing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $a_r \in \bigcap_n ((x_1,\dots,x_{r-1}) + F_{\gamma_n} A) = (x_1,\dots,x_{r-1})$ by the Krull-type assumption. Of course again the other steps have the same proof.

<u>LEMMA 3.10</u>. <u>Let</u> A <u>and</u> $\underline{x}$ <u>be as in the Theorem 3.9</u>. <u>Assume that</u> $\Gamma^\circ(A) \geq 0$ (<u>or</u> $\Gamma^\circ(A) \leq 0$), <u>that</u> $\underline{x}$ <u>is a regular sequence and</u> $\{\underline{x}\}$ <u>is a</u> d-<u>base</u>. <u>Moreover assume that for every</u> x_i <u>such that</u> $v(x_i) = 0$, $F(x_i)$ <u>is in the Jacobson radical of</u> G_0. <u>Then</u> $\{\underline{x}\}$ <u>is a stepwise</u> d-<u>base</u>.

<u>PROOF</u>. Let $E = \{x \in (x_1,\dots,x_{r-1}) / F(x) \notin (F(x_1),\dots,F(x_{r-1}))\}$.

We want to show that E is empty. Suppose not, then for every $x \in E$, we have $x \neq 0$ and $x = \sum_1^{r-1} {}_i a_i x_i + a_r x_r$ with $v(x) \geq v(a_i) + v(x_i)$ and $v(x) = v(a_r) + v(x_r)$, so that $F(x) = \sum_1^{r-1} {}_i g_i F(x_i) + F(a_r) F(x_r)$.

Let us consider the ring $\overline{G} = G/(F(x_1), \ldots, F(x_{r-1}))$ and in it the principal ideal generated by $\overline{F(x_r)}$. Since $\overline{G}$ is noetherian, we get

$$\bigcap_t (\overline{F(x_r)})^t = \{\overline{g} \in (\overline{F(x_r)}) / \exists\, \overline{h} \text{ with } (1+\overline{h}\overline{F(x_r)})\overline{g} = 0\}.$$

Now, if $v(x_r) \neq 0$ we get $\bigcap_t (\overline{F(x_r)})^t = 0$ by using the assumption $\Gamma^\circ(A) \geq 0$ and the given description of the intersection; if $v(x_r) = 0$, then $1 + \overline{h}\,\overline{F(x_r)}$ is invertible by assumption, hence we get again $\bigcap_t (\overline{F(x_r)})^t = 0$. Therefore for every $x \in (\underline{x})$ it is well-defined the number $t(x) = \max\{n / F(x) \in (F(x_1), \ldots, F(x_{r-1})) + (F(x_r))^n\}$. Let $y \in E$ be such that $t(y)$ is minimum in E; then again $y = \sum_1^{r-1} {}_i b_i x_i + b_r x_r$ where $v(y) = v(b_r) + v(x_r)$ and $F(y) = \sum_1^{r-1} {}_i g_i F(x_i) + F(b_r) F(x_r)$. But $y \in (x_1, \ldots, x_{r-1})$ and $x_1, \ldots, x_{r-1}$ is a regular sequence, hence $b_r \in (x_1, \ldots, x_{r-1})$; since $t(y) > t(b_r)$ we deduce that $b_r \notin E$, therefore $F(b_r) \in (F(x_1), \ldots, F(x_{r-1}))$ whence $F(y) \in (F(x_1), \ldots, F(x_{r-1}))$ a contradiction. The other steps have the same proof.

THEOREM 3.11. *Let $\mathcal{A} = (A,\Gamma,v,G,F)$ be a strong Krull structure such that $\Gamma^\circ(\mathcal{A}) \geq 0$ (or $\Gamma^\circ(\mathcal{A}) \leq 0$) and let $\underline{x} = x_1,\ldots,x_r$ be a sequence of elements of A such that for every x_i with $v(x_i) = 0$ then $F(x_i)$ is in the Jacobson radical of G_0. Then the following conditions are equivalent*

1) *$\underline{x}$ is a regular sequence and $\{\underline{x}\}$ is a d-base*
2) *$F(\underline{x})$ is a regular sequence.*

PROOF. After Theorem 3.9 we need only proving 1) $\Rightarrow$ 2) . But this is an immediate consequence of Theorem 3.9 and Lemma 3.10.

COROLLARY 3.12. *With the same assumptions as in Theorem 3.11, the following conditions are equivalent*

1) *$\underline{x}$ is a permutable regular sequence and $\{\underline{x}\}$ is a d-base*
2) *$F(\underline{x})$ is a permutable regular sequence.*

EXAMPLE 4. Let $A = k[x,y]$, $\Gamma = \mathbb{Z}$ $v(f) = -v_{(x,y)}(f)$ (see Ex. 1 of section 1) . Let $x_1 = x(x-1)$ $x_2 = y(x-1)$. Then $F(x_1)$, $F(x_2)$ is a regular sequence and x_1,x_2 is not . This happens because $\mathcal{A}$ is noetherian but not Krull.

§4. DOUBLE STRUCTURES AND HILBERT FUNCTIONS.

Let $\mathcal{A} = (A,\Gamma,\mathcal{F}_A) \cong (A,\Gamma,v,G,F)$ and $\mathcal{B} = (B,\Delta,\mathcal{F}_B) \cong (B,\Delta,w,H,\emptyset)$ be two v-filtered (graded) structures.

<u>DEFINITION 1</u>. A morphism of $\mathcal{A}$ in $\mathcal{B}$ is a couple (α,λ) where $\alpha : \Gamma \longrightarrow \Delta$ is an ordered homomorphism (i.e. a homomorphism such that $0 \leq \gamma$ implies $0 \leq \alpha(\gamma)$) and $\lambda : A \longrightarrow B$ is a ring-homomorphism such that $\lambda(F_\gamma A) \subseteq F_{\alpha(\gamma)} B$ or, equivalently, $w(\lambda(a)) \leq \alpha(v(a))$ for every $a \in A - \mathrm{Ker}(\lambda)$.

<u>REMARK 1</u>. Let $\mathcal{A}$ be a graded Krull structure and I an ideal of A ; then there is an obvious canonical morphism $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{J}$ where $\mathcal{J}$ is the structure induced on I ; it is clear that in this case it may happen that $w(\lambda(a)) < \alpha(v(a))$.

<u>REMARK 2</u>. Let us consider the following example; on $\mathbb{Z}^2$ we put the ordering $<$ given by $u_1 = (1,1)$, $u_2 = (-1,1)$ (see section 2); we let $A = k[x,y]$ and we consider the function $v : A - \{0\} \longrightarrow \mathbb{Z}^2$ which to every polynomial associates the couple of exponents of the maximal monomial (with respect to $<$) ; this gives us a graded structure $\mathcal{A}$. Let now $w : A - \{0\} \longrightarrow \mathbb{Z}$ be the function which to every polynomial associates its degree; this yields another graded structure, say $\mathcal{B}$, on A .

Let now $\alpha : \mathbb{Z}^2 \longrightarrow \mathbb{Z}$ be defined by $\alpha(a,b) = a+b$ and $\lambda : A \longrightarrow A$ be the identity map. Then it is easy to check that (α,λ) is a morphism; moreover we have $w(\lambda(a)) = \alpha(v(a))$ for every $a \neq 0$. Since $(1,1) < (0,2)$, $xy \in F^{\circ}_{(0,2)}A$, but $\lambda(xy) = xy \in F_2A - F^{\circ}_2A$, where $2 = \alpha(0,2)$. This shows that there is no hope in general to deduce from a morphism (α,λ) a homomorphism between the two graded objects G,H.

<u>DEFINITION 2</u>. Given a morphism (α,λ) as before, we denote by G^{Δ} the Δ-graded ring defined by

$$G^{\Delta} = \bigoplus_{\delta\in\Delta} G^{\Delta}_{\delta} \quad \text{where} \quad G^{\Delta}_{\delta} = \lambda^{-1}(F_{\delta}B)/\lambda^{-1}(F^{\circ}_{\delta}B)$$

We denote by G_{Δ} the Δ-graded ring defined by

$$G_{\Delta} = \bigoplus_{\delta\in\Delta} G_{\Delta,\delta} \quad \text{where} \quad G_{\Delta,\delta} = \bigoplus_{\alpha(\gamma)=\delta} G_{\gamma}$$

<u>LEMMA 4.1</u>. G^{Δ} <u>is the graded ring associated to a</u> v-<u>filtered structure if and only if</u> λ <u>is injective</u>.

<u>PROOF</u>. If λ is not injective, then $\bigcap_{\delta} \lambda^{-1}(F_{\delta}B) \supseteq \mathrm{Ker}(\lambda)$ hence the elements of $\mathrm{Ker}(\lambda)$ have no valuation. Conversely assume that λ is injective and let $a \in A$, $a \neq 0$; let $\delta = w(\lambda(a))$. Then $a \in \lambda^{-1}(F_{\delta}B)$ and $a \notin \lambda^{-1}(F^{\circ}_{\delta}B)$, so we see that the filtration $\{\lambda^{-1}(F_{\delta}B)\}$ is valued.

<u>LEMMA 4.2.</u> <u>Given a morphism</u> (α,λ) <u>as before, there are two canonical homomorphisms</u>

$$i : G \longrightarrow G_\Delta \qquad \overline{\lambda} : G^\Delta \longrightarrow H$$

<u>where</u> $i(g) = g$ <u>for every</u> $g \in G$ <u>(but it changes the degrees) and</u> $\overline{\lambda}$ <u>is defined through</u> λ <u>and it is an injective</u> Δ-<u>homogeneous homomorphism</u>.

<u>PROOF</u>. Obvious.

<u>LEMMA 4.3.</u> <u>Given a morphism</u> (α,λ) , <u>if</u> α <u>is injective (equivalently strictly ordered) then there is a canonical map</u> $G_\Delta \longrightarrow G^\Delta$, <u>hence a canonical map</u> $\Lambda : G \longrightarrow H$.

<u>PROOF</u>. Obvious.

At this point we can say that, given a morphism $(\alpha,\lambda) : \mathcal{A} \longrightarrow \mathcal{B}$, there are circumstances where it induces a map $\Lambda : G \longrightarrow H$ (see Lemma 4.3), while in general this does not happen. So the question is to get informations about the relation between G_Δ and G^Δ . Henceforth we are going to consider a special situation, which nevertheless will be general enough for several applications. Essentially we are going to consider the case when $A = B$ and $\alpha(v_\Gamma(a)) = v_\Delta(a)$ for every $a \neq 0$.

<u>DEFINITION 3</u>. We denote by $\mathbb{A}_\alpha$ the triple $(A_\Gamma, A_\Delta, \alpha)$ where

$$A_\Gamma = (A,\Gamma,F_{\Gamma,A}) \cong (A,\Gamma,v_\Gamma,G(\Gamma),F_\Gamma)$$
$$A_\Delta = (A,\Delta,F_{\Delta,A}) \cong (A,\Delta,v_\Delta,G(\Delta),F_\Delta)$$

are graded structures over the ring A and $\alpha : \Gamma \longrightarrow \Delta$ is an ordered homomorphism such that $\alpha(v_\Gamma(a)) = v_\Delta(a)$ for every $a \neq 0$ (hence $F_{\Gamma,\gamma} \subseteq F_{\Delta,\alpha(\gamma)}$) . $\mathbb{A}_\alpha$ will be called a "double structure on A ".

We denote by $\mathfrak{M}$ the couple (M_Γ, M_Δ) where

$$M_\Gamma = (M,\Gamma,F_{\Gamma,M}) \cong (M,\Gamma,w_\Gamma,T(\Gamma),\emptyset_\Gamma)$$
$$M_\Delta = (M,\Delta,F_{\Delta,M}) \cong (M,\Delta,w_\Delta,T(\Delta),\emptyset_\Delta)$$

are modules over A_Γ, A_Δ respectively. If α has the property that $\alpha(w_\Gamma(m)) = w_\Delta(m)$ for every $m \neq 0$, we say that $\mathfrak{M}$ is an $\mathbb{A}_\alpha$-module (on M). An example of double structure is that one described in Remark 2.

<u>DEFINITION 4</u>. We say that $\mathbb{A}_\alpha$ is noetherian (Krull,...) if A_Γ and A_Δ are noetherian (Krull,...) . We say that $\mathfrak{M}$ is finite (Krull,...) if M_Γ is finite (Krull,...) over A_Γ and M_Δ is finite (Krull,...) over A_Δ .

<u>DEFINITION 5</u>. As in definition 2, we put

$$G(\Gamma)_{\Delta} = \bigoplus_{\delta\in\Delta} G(\Gamma)_{\Delta,\delta} \quad \text{where} \quad G(\Gamma)_{\Delta,\delta} = \bigoplus_{\alpha(\gamma)=\delta} G(\Gamma)_{\gamma}$$
$$T(\Gamma)_{\Delta} = \bigoplus_{\delta\in\Delta} T(\Gamma)_{\Delta,\delta} \quad \text{where} \quad T(\Gamma)_{\Delta,\delta} = \bigoplus_{\alpha(\gamma)=\delta} T(\Gamma)_{\gamma}$$

We observe that, being $\lambda = id$, the corresponding $G(\Gamma)^{\Delta}$ is nothing but $G(\Delta)$ and $T(\Gamma)^{\Delta} = T(\Delta)$.

Therefore, in this situation the question is to get informations about the relation between $G(\Gamma)_{\Delta}$ and $G(\Delta)$, more generally between $T(\Gamma)_{\Delta}$ and $T(\Delta)$.

<u>LEMMA 4.4</u>. <u>Let</u> $\mathbb{A}_{\alpha} = (A_{\Gamma}, A_{\Delta}, \alpha)$ <u>be a double structure and</u> $\mathfrak{M} = (M_{\Gamma}, M_{\Delta})$ <u>a module on it; then</u>

a) <u>The map</u> α <u>restricts to a map</u> $\Gamma^{\circ}(M_{\Gamma}) \longrightarrow \Delta^{\circ}(M_{\Delta})$ <u>which is surjective</u>.

b) <u>The map</u> α <u>is surjective, hence</u> $\mathrm{Ker}(\alpha)$ <u>is an isolated subgroup of</u> Γ.

<u>PROOF</u>. a) Let $\delta \in \Delta^{\circ}(M_{\Delta})$; then $\delta = w_{\Delta}(m) = \alpha(w_{\Gamma}(m))$.

b) It follows from a), since Γ and Δ are generated by $\Gamma^{\circ}(A_{\Gamma})$ and $\Delta^{\circ}(A_{\Delta})$ respectively (see section 1, after definition 11).

<u>PROPOSITION 4.5</u>. <u>Let</u> $\mathbb{A}_{\alpha} = (A_{\Gamma}, A_{\Delta}, \alpha)$ <u>be a double structure and assume that</u> α <u>is injective; then</u>

a) α <u>is an isomorphism</u>

b) <u>After identifying</u> Γ <u>with</u> Δ <u>via</u> α, <u>for every</u> $\mathbb{A}_{\alpha}$-<u>module</u> $\mathfrak{M} = (M_{\Gamma}, M_{\Delta})$ <u>there is a canonical identification of</u> M_{Γ} <u>with</u> M_{Δ}.

c) <u>The categories of</u> $\mathbb{A}_\alpha$<u>-modules and</u> A_Γ<u>-modules are equivalent</u>.

<u>PROOF</u>. a) Follows from Lemma 4.4.

b) We have $w_\Gamma(m) \le \gamma$ iff $\alpha(w_\Gamma(m)) \le \alpha(\gamma)$; but $\alpha(w_\Gamma(m)) = w_\Delta(m)$, whence $F_{\Gamma,\gamma}M = F_{\Delta,\alpha(\gamma)}M = F_{\Delta,\gamma}M$ where the last equality depends on the identification of Γ with Δ .

c) Follows from b).

Let now $\mathbb{A}_\alpha = (A_\Gamma, A_\Delta, \alpha)$ be a strong Krull double structure on A and $\mathbb{M} = (M_\Gamma, M_\Delta)$ a finite $\mathbb{A}_\alpha$-module . Let N be a submodule of M and $I = \mathrm{Ann}_A(M/N)$; let J_α denote the "ideal" of $\mathbb{A}_\alpha$ given by (J_Γ, J_Δ) , where J_Γ, J_Δ are the structures induced on I by A_Γ, A_Δ respectively. Let $\mathbb{N}$ denote the "submodule" of $\mathbb{M}$ given by (N_Γ, N_Δ) , where N_Γ, N_Δ are the structures induced on N by M_Γ, M_Δ respectively. Finally let us denote by $\mathbb{A}_\alpha/J_\alpha$ the triple $(A_\Gamma/J_\Gamma, A_\Delta/J_\Delta, \alpha)$ and by $\mathbb{M}/\mathbb{N}$ the couple $(M_\Gamma/N_\Gamma, M_\Delta/N_\Delta)$.

<u>PROPOSITION 4.6</u>. <u>With these assumptions and notations, we have</u>

a) $\mathbb{A}_\alpha/J_\alpha$ <u>is a strong Krull double structure</u>.

b) $\mathbb{M}/\mathbb{N}$ <u>is an</u> $\mathbb{A}_\alpha/J_\alpha$ - <u>module</u>

c) <u>If</u> $\{n_1,\dots,n_r\}$ <u>is a</u> d-<u>base of</u> N_Γ , <u>then it is also a</u> d-<u>base of</u> N_Δ .

<u>PROOF</u>. a) A_Γ/J_Γ and A_Δ/J_Δ are structures on A/I by Proposition 1.4. Let now $\bar{a} \in A/I$, $\bar{a} \neq 0$; then $v_\Gamma(\bar{a}) = \min_{x-a\in I} v_\Gamma(x)$; so what we have to show is that $\alpha(\min_{x-a\in I} v_\Gamma(x)) = \min_{x-a\in I} v_\Delta(x)$ and this is true, because $\alpha(v_\Gamma(x)) = v_\Delta(x)$ for every $x \neq 0$. So now we know that A_α/J_α is a double structure and to show that it is strong Krull, we may use again Proposition 1.4.

b) Same arguments as in a).

c) Every nonzero element n of N can be written as $n = \Sigma_1^r a_i n_i$ where $a_i \in A$ anf for every i such that $a_i \neq 0$, $w_\Gamma(n) \geq v_\Gamma(a_i) + w_\Gamma(m_i)$. By using the properties of α , we get also $w_\Delta(n) \geq v_\Delta(a_i) + w_\Delta(m_i)$.

<u>REMARK 3</u>. If $A_\alpha = (A_\Gamma, A_\Delta, \alpha)$ is a double structure on A and A_Δ is noetherian, then A_Γ need not being noetherian, as the following example shows.

Let us consider the structure A_Γ of the Example 9 of the first section, where $\Gamma = \mathbf{Z}^2$, $A = k[x,y,z]$ and $G(\Gamma)$ turns out to be non noetherian. Let us consider $\Delta = \mathbf{Z}$, $v_\Delta(f) = \deg(f)$ and $\alpha : \mathbf{Z}^2 \longrightarrow \mathbf{Z}$, $\alpha(a,b) = a+b$. Then $G(\Delta) = k[x,y,z]$; moreover $\Delta° = \mathbb{N}$, $\Gamma° = \mathbb{N}^2$ and $\alpha(v_\Gamma(f)) = \deg(f) = v_\Delta(f)$.

<u>QUESTION</u>. I do not know whether A_Γ noetherian implies A_Δ noetherian. However it should be noted that the answer is negative if we drop the assumption that

$\alpha(v_\Gamma(a)) = v_\Delta(a)$ (so we do not have a double structure). Namely we put $\Delta = \mathbb{Z}^2$ and as A_Δ the structure described in the Example 9 of the first section. Then we put $\Gamma = \mathbb{Z}$, $v_\Gamma(f) = \deg(f)$ and $\alpha : \mathbb{Z} \longrightarrow \mathbb{Z}^2$, $\alpha(n) = (n,0)$. Then $G(\Gamma) = k[x,y,z]$, but of course $\alpha(v_\Gamma(f)) \leq v_\Delta(f)$ and the strict inequality occurs.

<u>PROPOSITION 4.7</u>. <u>Let</u> $\mathbb{A}_\alpha = (A_\Gamma, A_\Delta, \alpha)$ <u>be a double structure on</u> A <u>and let</u> M_Γ <u>be an</u> A_Γ<u>-module</u> . <u>Then</u>

a) <u>If we put</u> $F_{\Delta,\delta}M = \bigcup_{\alpha(\gamma)=\delta} F_{\Gamma,\gamma}M$ <u>we get an</u> A_Δ<u>-module</u> , <u>which we denote by</u> $M_{\alpha(\Gamma)}$, <u>such that</u> $(M_\Gamma, M_{\alpha(\Gamma)})$ <u>is an</u> $\mathbb{A}_\alpha$<u>-module.</u>

b) <u>Conversely, if</u> $\mathfrak{M} = (M_\Gamma, M_\Delta)$ <u>is an</u> $\mathbb{A}_\alpha$<u>-module , then</u> $M_\Delta = M_{\alpha(\Gamma)}$.

c) <u>If moreover</u> α <u>is not injective, and</u> $\mathfrak{M} = (M_\Gamma, M_\Delta)$ <u>is an</u> $\mathbb{A}_\alpha$<u>-module , then</u> $F^\circ_{\Delta,\delta}M = \bigcap_{\alpha(\gamma)=\delta} F_{\Gamma,\gamma}M$ <u>for every</u> δ .

<u>PROOF</u>. a) By Lemma 4.4 we know that α is an ordered surjective homomorphism, so that it is easy to see that the given one is a valued filtration on M . Moreover if $m \in M$ is such that $w_\Gamma(m) = \gamma$ then $w_\Delta(m) = \alpha(\gamma)$ by definition.

b) If $\mathfrak{M} = (M_\Gamma, M_\Delta)$ is an $\mathbb{A}_\alpha$-module, then clearly $F_{\Delta,\delta}M \supseteq \bigcup_{\alpha(\gamma)=\delta} F_{\Gamma,\gamma}M$; on the other hand if $w_\Delta(m) = \delta$, then $\alpha(w_\Gamma(m)) = \delta$ and of course $m \in \Gamma_{w_\Gamma(m)}M$; if $w_\Delta(m) < \delta$ and $\alpha(\gamma) = \delta$, then $w_\Gamma(m) \leq \gamma$, hence $m \in \Gamma_\gamma M$.

c) Let m be such that $w_\Delta(m) < \delta$ and let γ be such that $\alpha(\gamma) = \delta$. We get $\alpha(w_\Gamma(m)) = w_\Delta(m) < \delta = \alpha(\gamma)$ hence

$w_\Gamma(m) < \gamma$ and the inclusion "$\subseteq$" is proved.

Let now $m \in \bigcap_{\alpha(\gamma)=\delta} F_{\Gamma,\gamma}M$, $m \neq 0$; since $w_\Gamma(m) \leq \gamma$ implies $\alpha(w_\Gamma(m)) \leq \alpha(\gamma) = \delta$ and we have to show that $\alpha(w_\Gamma(m)) < \delta$, we have only to exclude that $\alpha(w_\Gamma(m)) = \delta$. Suppose, for contradiction, that $\alpha(w_\Gamma(m)) = \delta$; then $\alpha^{-1}\{\delta\} = w_\Gamma(m) + \mathrm{Ker}(\alpha)$; let γ' $\mathrm{Ker}(\alpha)$ be such that $\gamma' < 0$ (such a γ' exists since $\mathrm{Ker}(\alpha)$ is nontrivial). Then $\gamma'' = w_\Gamma(m) + \gamma' < w_\Gamma(m)$ and $m \notin F_{\Gamma,\gamma''}M$, a contradiction.

<u>PROPOSITION 4.8</u>. <u>Let</u> $A_\alpha = (A_\Gamma, A_\Delta, \alpha)$ <u>be a noetherian double structure on</u> A ; <u>let</u> S <u>be the semigroup generated by</u> $\Gamma^\circ(A_\Gamma)$ <u>and assume that</u> $\mathrm{Ker}(\alpha) \cap S = \{0\}$. <u>Then</u>

a) <u>For every finite</u> A_α<u>-module</u> $\mathfrak{M} = (M_\Gamma, M_\Delta)$ <u>and every</u> $\delta \in \Delta$, <u>the set</u> $\alpha^{-1}\{\delta\} \cap \Gamma^\circ(M_\Gamma)$ <u>is finite. If we assume in addition that</u> $\Gamma^\circ(A_\Gamma)$ <u>is either positive or negative then</u>

b) <u>For every finite</u> A_α<u>-module</u> $\mathfrak{M} = (M_\Gamma, M_\Delta)$, <u>if</u> M_Δ <u>is a Krull module, also</u> M_Γ <u>is a Krull module</u>

c) <u>If</u> A_Δ <u>is a strong Krull structure, also</u> A_Γ <u>is a strong Krull structure</u>.

<u>PROOF</u>. a) Arguing as in the proof of Theorem 2.7, we know that $\Gamma^\circ(M_\Gamma)$ is contained in a finite union of subsets of the type $\gamma + \Gamma^\circ(A_\Gamma)$. Since $\alpha^{-1}\{\delta\} \cap (\gamma + \Gamma^\circ(A_\Gamma) \subseteq \gamma + (\alpha^{-1}\{\delta - \alpha(\gamma)\} \cap S)$, it is enough to show that $\alpha^{-1}\{\delta\} \cap S$ is finite for every δ . By Proposition 1.3 we know that S is a finitely generated semigroup; let

us identify Γ with $\mathbb{Z}^n$ and embed it into $\mathbb{R}^n$; moreover let P be the polyhedron (finite intersection of closed half-spaces) spanned by S . Being S finitely generated, we get $\mathrm{Ker}(\alpha) \cap P = \{0\}$, so that the linear space generated by $\alpha^{-1}\{\delta\}$ in $\mathbb{R}^n$ intersects P in a compact region and we are done.

Of course there is nothing to prove in b) and c) if $\Gamma^\circ(A_\Gamma) \geq 0$ because then every finite A_Γ-module is Krull by Theorem 2.7. Moreover if L_Γ is a finite free A_Γ-module then $L_\Gamma \simeq L(\gamma_1,\ldots,\gamma_r)$ and it is easy to see that $L_{\alpha(\Gamma)} \simeq L(\alpha(\gamma_1),\ldots,\alpha(\gamma_2))$. Now we know from Lemma 4.7 a) that $(L_\Gamma, L_{\alpha(\Gamma)})$ is an A_α-module, hence c) is a consequence of b).

So we have only to prove b) under the assumptions that $\Gamma^\circ(A_\Gamma) \leq 0$. If $L_\Gamma = L(\gamma_1,\ldots,\gamma_r)$ $\lambda : L_\Gamma \longrightarrow M_\Gamma$ is a morphism, then $\lambda(F_\gamma L) = F_{\gamma-\gamma_1}A\cdot m_1+\ldots+F_{\gamma-\gamma_r}A\cdot m_r$ where $m_1,\ldots,m_r \in M$; but $F_{\gamma-\gamma_i}A$ is an ideal of A by Lemma 1.6, hence $\lambda(F_\gamma L)$ is a submodule of M .

Therefore, to show that M_Γ is a Krull module, it is sufficient to show that, given a submodule N on M and a strictly decreasing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of elements of $\Gamma^\circ(M_\Gamma)$, then $\bigcap_n(N+F_{\gamma_n}M) = N$. On the other hand $\bigcap_n(N+F_{\gamma_n}M) \subseteq \bigcap_n(N+F_{\alpha(\gamma_n)}M)$ and we know that M_Δ is a Krull module.

To conclude it is sufficient to show that $\{\alpha(\gamma_n)\}_{n\in\mathbb{N}}$ has a strictly decreasing subsequence.

For, we know by Proposition 4.7 that $F_{\Delta,\alpha(\gamma)} = \bigcup_{\gamma'\in\alpha^{-1}\{\alpha(\gamma)\}} F_{\Gamma,\gamma',M}$ and we know from a) that $\alpha^{-1}\{\alpha(\gamma)\} \cap \Gamma^\circ(M_\Gamma)$ is finite, so that in the sequence $\{\alpha(\gamma_n)\}_{n\in\mathbb{N}}$, we have $\#\{n'/n' \geq n\ ,\ \alpha(n') = \alpha(n)\}$ is finite for every n.

QUESTION. I do not know if b) and c) are still valid if we drop the extra-condition that $\Gamma^\circ(A_\Gamma)$ is positive or negative.

Now we come to the main result of this section; we keep the notations introduced in Definition 3.

THEOREM 4.9. Let A_α be a noetherian double structure on A ; let S be the semigroup generated by $\Gamma^\circ(A_\Gamma)$ and assume that $\mathrm{Ker}(\alpha) \cap S = \{0\}$; let $\mathfrak{M}$ be a finite A_α-module. Then

1) $G(\Gamma)_0 = G(\Delta)_0$ and we shall denote it by G_0.
2) If $[\ldots]$ denotes the image in the Grothendieck group of finitely generated G_0-modules, then

$$[T(\Gamma)_{\Delta,\delta}] = [T(\Delta)_\delta] \qquad \text{for every} \quad \delta\in\Delta^\circ(M_\Delta)\ .$$

PROOF. If α is injective, we use Proposition 4.5 and there is nothing to prove. So let us assume that α is not injective; we first prove the following

CLAIM: If $\gamma > 0$, $\gamma \notin \mathrm{Ker}(\alpha)$ then $\gamma > \gamma'$ for every $\gamma' \in \mathrm{Ker}(\alpha)$. If $\gamma < 0$, $\gamma \notin \mathrm{Ker}(\alpha)$, then $\gamma < \gamma'$ for every $\gamma' \in \mathrm{Ker}(\alpha)$. Namely, suppose that $\gamma > 0$ and there exists $\gamma'\in\mathrm{Ker}(\alpha)$

such that $\gamma < \gamma'$; then $-\gamma' < -\gamma < \gamma < \gamma'$, whence $\gamma \in \operatorname{Ker}(\alpha)$ (being $\operatorname{Ker}(\alpha)$ an isolated subgroup), a contradiction; of course the same proof works for $\gamma < 0$.

Now we come to the proof of 1).

By the assumption and the fact that $0 \in \Gamma^\circ(A_\Gamma)$ (see Remark 1 of section 1) we get $\operatorname{Ker}(\alpha) \cap \Gamma^\circ(A_\Gamma) = \{0\}$; this means that $\{a \in A / v_\Gamma(a) \in \operatorname{Ker}(\alpha) - \{0\}\}$ is empty, hence $F_{\Gamma,\gamma}A = F_{\Gamma,0}A$ for every $\gamma \geq 0$, $\gamma \in \operatorname{Ker}(\alpha)$, and all the $F_{\Gamma,\gamma}A$ are equal for every $\gamma < 0$, γ $\operatorname{Ker}(\alpha)$, hence they are equal to $F^\circ_{\Gamma,0}A$ by the claim.

So we conclude by Proposition 4.7 applied to A_α considered as a module over itself.

2) We know that $T(\Gamma)_{\Delta,\delta} = \bigoplus_{\gamma\in\alpha^{-1}\{\delta\}} T(\Gamma)_\gamma$ and Lemma 4.7 tells us that $\{F_{\Gamma,\gamma}M\}_{\gamma\in\alpha^{-1}\{\delta\}}$ is a filtration such that $F^\circ_{\Delta,\delta} = \bigcap_{\gamma\in\alpha^{-1}\{\delta\}} F_{\Gamma,\gamma}M$ and $F_{\Delta,\delta}M = \bigcup_{\gamma\in\alpha^{-1}\{\delta\}} F_{\Gamma,\gamma}M$; moreover $T_{\Delta,\delta} = F_{\Delta,\delta}M / F^\circ_{\Delta,\delta}M$, hence to conclude it is sufficient to know that in the filtration $\{F_{\Gamma,\gamma}M\}_{\gamma\in\alpha^{-1}\{\delta\}}$ only a finite number of strict inequalities occur. And this follows from Proposition 4.8. a).

I want to conclude by showing two different applications of Theorem 4.9.

Let us consider $A = k[x_1,\ldots,x_n]$, $\Gamma = \mathbb{Z}^n$ and let $<$ be an ordering on $\mathbb{Z}^n$ such that $u = (q_1,\ldots,q_n)$ where $q_i \in \mathbb{N}^+$, $i = 1,\ldots,n$ (see Theorem 2.5) . Then we consider the graded structure on A described in Example 3 of section 1;

as a consequence of Theorem 2.7 we get that it is a strong Krull structure; we denote it by A_Γ .
Now if $\alpha : \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$ is an ordered nonzero homomorphism, $\mathrm{Ker}(\alpha)$ is an isolated subgroup, hence it has to be orthogonal to u_1 .
Let $m = 1$ and consider the usual ordering on $\mathbb{Z}$; then $\alpha : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ has to be defined by
$\alpha(a_1,\dots,a_n) = \sum_1^n{}_i a_i q_i$. Let us consider $q_1,\dots,q_n$ as weights of the variables $x_1,\dots x_n$ and then let us consider the graded structure on A described in Example 2 of section 1; also this structure, which we denote by $A_{\mathbb{Z}}$ is a strong Krull structure and it is clear that $A_\alpha = (A_\Gamma, A_{\mathbb{Z}}, \alpha)$ is a strong Krull double structure on A . Since $\Gamma^\circ(A_\Gamma) = \mathbb{N}^n$, the hypotheses of Theorem 4.9 are satisfied.
Let now I be an ideal of A and let $J_\alpha = (J_\Gamma, J_{\mathbb{Z}})$ be the induced double structure on I . By Proposition 4.6 we know that A_α/J_α is a strong Krull double structure on A/I and that if $\{f_1,\dots,f_r\}$ is a d-base of J_Γ , then it is also a d-base of J , and the hypotheses of Theorem 4.9 are satysfied. If we apply Theorem 4.9 to A_α/J_α we get the following

<u>COROLLARY 4.10</u> <u>Let</u> $A = k[x_1,\dots,x_n]$, I <u>an ideal of</u> A . <u>Let</u> $M(I)$ <u>denote the ideal generated by the maximal monomials of the elements of</u> I , <u>with respect to an ordering on</u> $\mathbb{Z}^n$ <u>with</u> $u_1 = (q_1,\dots,q_n)$, $q_i \in \mathbb{N}^+$ <u>and let</u> $F(I)$ <u>denote the ideal generated by the forms of maximum degree of the</u>

elements of I, where $\deg x_i = q_i$, $i = 1,\ldots,n$. Let $A/M(I)$ and $A/F(I)$ be considered as graded over $\mathbb{N}$ by the graduations induced by the graduation on A defined by the total degree, where $\deg x_i = q_i$, $r = 1,\ldots,n$. Finally let $H(A/M(I))$, $H(A/F(I))$ be the Hilbert functions. Then $H(A/M(I)) = H(A/F(I))$.

In particular we get

COROLLARY 4.11. (Macaulay, see [5]) With the same notations as before, let us assume that $u_1 = (1,1,\ldots,1)$ and that I is homogeneous with respect to the usual total degree. Then $H(A/M(I)) = H(A/I)$.

REMARK. Let us consider the ring $S = k[x_0,\ldots,x_n]$ graded by the total degree, where $\deg x_0 = 1$, $\deg x_i = q_i$, $i = 1,\ldots,n$; let us denote by ${}^hI$ the ideal generated by all the ${}^hf$, $f \in I$, where ${}^hf$ is the homogenization of f with respect to x_0. With the notations of Corollary 4.10, it is clear that $({}^hI, x_0) = (F(I), x_0)$. As before we denote by $H(\ldots)$ the Hilbert function and by $H^1(\ldots)$ the function defined by $H^1(\ldots,n) = \sum_{0}^{n}{}_i H(\ldots,i)$. Since x_0 is a nonzerodivisor modulo ${}^hI$, homogeneous of degree 1, we get $H^1(A/M(I)) = H^1(A/F(I)) = H^1(S/(F(I),x_0)) =$
$= H^1(S/({}^hI,x_0)) = H(S/{}^hI)$. Hence we deduce that $H(S/(M(I)\cdot S)) = H(S/{}^hI)$. On the other hand it is easily seen that if $\{f_1,\ldots,f_r\}$ is a d-base of J_Γ (this is usually called a Gröbner-base of I with respect to the given

ordering) then not only $M(I) = (M(f_1),\dots,M(f_r))$ but also ${}^hI = ({}^hf_1,\dots,{}^hf_r)$.

Therefore the knowledge of such a base allows to "compute" the equations and the Hilbert function of the compactification of $\operatorname{Spec}(A/I)$ in the weighted projective space $\mathbb{P}(1,q_1,\dots,q_r) = \operatorname{Proj}(S)$.

The second application is dealing

with the local ring $R = k[[x_1,\dots,x_n]]$ of power series. We consider on it the noetherian structure $A_{\mathbb{Z}}$ described in Example 1 of section 1. Let now $\Gamma = \mathbb{Z}^n$ and let $<$ be an ordering on it such that $u_1 = (-1,-1,\dots,-1)$. To every serie we associate its maximum monomial (it exists, because the maximum monomial with respect to the given ordering is among those of minimum degree) and we get again a noetherian graded structure A_Γ on R , since the associated graded ring is clearly isomorphic to $k[x_1,\dots,x_n]$.

Let now $\alpha : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ be defined by $\alpha(a_1,\dots,a_n) = -\sum_1^n i a_i$. It is an ordered homomorphism and $\operatorname{Ker}(\alpha) \cap \Gamma^\circ(A_\Gamma) = \{0\}$. In this case $\Gamma^\circ(A_\Gamma) = \mathbb{N}^n$, hence it is already a semigroup; moreover $\Gamma^\circ(A_\Gamma) \leq 0$ (remember that $u_1 = (-1,-1,\dots,-1)$) , $A_{\mathbb{Z}}$ is a strong Krull structure by Proposition 1.7 and $A_\alpha = (A_\Gamma, A_{\mathbb{Z}}, \alpha)$ is clearly a double structure on R . So we may use Lemma 4.8 and we get that A_Γ is strong Krull.

Let now I be an ideal of R and let $J_\alpha = (J_\Gamma, J_{\mathbb{Z}})$ be the induced double structure on I . By Proposition 4.6 we know that A_α/J_α is a strong Krull double structure on R/I and that if $\{f_1,\dots,f_r\}$ is a d-base of J_Γ (it exists, since A_Γ is strong Krull), then it is also a d-base of $J_{\mathbb{Z}}$ (which is usually called a standard base of I with respect to the maximal ideal of R (see [7] and [8])) . Moreover, by using Theorem 4.9 we see again that the computation of the Hilbert function of the tangent cone to $\mathrm{Spec}(R/I)$ at the origin can be computed by means of a monomial ideal. An algorithm which shows the effectiveness of such a computation is discussed in [6], while the study of graded structures on R (and on the ring of convergent power series) such as A_Γ was in some sense started by Hironaka and Grauert and developed by Galligo (see [2]) and others.

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