Embedding 3-manifolds in simply connected positive definite 4-manifolds and the smooth structures of open 4-manifolds

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MPI 96-19

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January 23, 1996

Abstract

Let M be a noncompact 4-manifold with at least two open ends. Suppose that one of these ends is homeomorphic to $N \times R$, where N is an oriented 3-manifold satisfying one of the following conditions:
(i). $\pi_1(N)$ is an extension of free group by a perfect normal subgroup.
(ii). $H_1(N) \cong Z/n_1 \oplus \cdots \oplus Z/n_k$, $k \leq 4$, and the link form of N is isomorphic to $(\frac{1}{n_1}) \oplus \cdots \oplus (\frac{1}{n_k})$ where each n_i, $1 \leq i \leq k$ is a product of primes which are $1(mod4)$.
(iii). $H_1(N) \cong Z/n_1 \oplus \cdots \oplus Z/n_k$, $k \leq 2$ and $n_i, 1 \leq i \leq k$, is a power of a prime which are $3(mod8)$.
Then there exist uncountably many different smooth 4-manifolds which are homeomorphic to M.

0. Introduction

The most striking fact of 4-manifolds different from all other dimensions is the existence of an exotic $\mathbf{R}^4$, a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ but not diffeomorphic to it. The first example [8] was pointed out by M.Freedman in 1982 which follows from Donaldson's Theorem[3] on the nonexistence of closed, smooth 4-manifold with nonstandard definite intersection form, together with Freedman's 4-dimensional topological surgery theory. Subsequently, R.Gompf[9] proved there are infinite many different smooth manifolds doubly indexed family $\{\mathbf{R}_{m,n}\}_{m,n=0}^{\infty}$ which are all homeomorphic to $\mathbf{R}^4$. Donaldson's

*Supported partially by Sonderforschungsbereich 343 and the Max-Planck-Institut fuer Mathematik. The author would like to thank Universität Bielefeld and MPI for their hospitality during the preparation of this paper.

Theorem was generalized by Taubes[16] to the case of open 4-manifolds of end-periodic, this leads to construct uncountably many different smooth structures on $\mathbf{R}^4$. Freedman and Taylor[6] proved that there exists a universal exotic $\mathbf{R}^4$ which includes all others as a smooth submanifold. It is conjectured that this smoothing can not be embedded into any smooth compact 4-manifold. More recently, DeMichelis and Freedman [1] proved that there exist uncountably many different smooth structures on $\mathbf{R}^4$ embeds in to $\mathbf{R}^4$ smoothly. On the enumeration problem of smooth structures on open 4-manifolds with nontrivial homotopy type, R.Gompf [10] proved that any punctured four manifold admits uncountably many smooth structures. Gompf mentioned in [10] that it is plausible to make a conjecture, namely, every open 4-manifold admits uncountably different smooth structures. In this paper, we shall address to verify this in some cases.

Theorem A. *Let M be a noncompact 4-manifold which possesses at least two open ends and one of them is homeomorphic to $N \times \mathbf{R}$ where N is an oriented closed 3-manifold. Suppose that M has an end collared by $N \times \mathbf{R}$ where N is an oriented closed 3-manifold. Then there are uncountably many different smooth structures on M if N satisfies one of the following:*
(i). $\pi_1(N)$ is an extension of a free group by a perfect normal subgroup.
(ii). $H_1(N) \cong Z/n_1 \oplus \cdots \oplus Z/n_k$, $k \leq 4$, and *the link form of N is isomorphic to $(\frac{1}{n_1}) \oplus \cdots \oplus (\frac{1}{n_k})$ where each n_i, $1 \leq i \leq k$ is a product of primes which are $1(mod 4)$.*
(iii). $H_1(N) \cong Z/n_1 \oplus \cdots \oplus Z/n_k$, $k \leq 2$ and $n_i, 1 \leq i \leq k$, is a power of a prime which are $3(mod 8)$.

In particular, $N \times \mathbf{R}$ admits uncountably many smooth structures if N satisfies the above assumption.

1. Embeddings of 3-dimensional rational homology spheres

In M.Freedman's fundamental paper [4], he proves that every homology 3-sphere embeds locally flat into the 4-sphere. In the smooth category, it is much different. In fact, it is known for some years [11] that there are infinite homology 3-spheres which can not be embedded smoothly into 4-space. As remarked in [11], the same method applies to get infinite many examples which can not be embedded into any simply connected smooth closed 4-manifold with positive definite intersection form. Generally, rational homology sphere can not be embedded into the 4-sphere locally flat even in the topological category(c.f [12]). In this section, we are addressed the study of when a 3-dimensional rational homology sphere embeds locally flat into a positive definite closed 4-manifold. In particular, we will handle with the problem of when a 3-dimensional rational homology sphere can be embedded into the connected sum $\#_1^m CP^2$ locally flat. This particular result may be used to prove Theorem A advertised in the introduction.

Theorem 1.1. *Let M be a 3-dimensional rational homology sphere. Then M embeds locally flat into a simply connected closed 4-manifold with positive definite intersection*

form if one of the following conditions holds true:
(i): $H_1(M) \otimes Z_2 = 0$.
(ii): $M = N\#(-N)$.

The idea to show the theorem is to construct two simply connected 4-dimensional topological manifolds with boundary M and $-M$ and with positive definite intersection forms respectively. Gluing them together along the boundary one can obtain a desire simply connected 4-manifold with positive definite intersection form and the proof can be done.

The Lens space $L_3(k)$ and $S^3/Q(8k)$ embeds in $\#_1^n CP^2$ smoothly as the circle bundles over S^2 and RP^2 respectively for n large enough. It is natural to ask when a 3-manifold embeds in $\#_1^n CP^2$ locally flat. Unfortunately it is very hard to control the intersection form of the 4-manifold constructed in the proof of Theorem 1.1. Thanks to J.Conway and N.Elkies for informing me some results on integral lattice relevant to this question. As an example, we give the following partial result which will be used to prove Theorem A. We remark that, by the proof one can get some more result along the same method.

Corollary 1.2. *Let M be a 3-dimensional rational homology sphere. Suppose that $H_1(M) \cong Z/n_1 \oplus \cdots \oplus Z/n_k$ where $k \leq 4$. Then M embeds into $\#_1^{3k} CP^2$ if one of the following conditions holds:*
(1) The link form of M is isomorphic to $(\frac{1}{n_1}) \oplus \cdots \oplus (\frac{1}{n_k})$ where each n_i, $1 \leq i \leq k$ is a product of primes which are $1 (mod 4)$.
(2) $k \leq 2$ and n_i, $1 \leq i \leq k$, *is a power of prime which are $3 (mod 8)$.*

Let M be a 3-dimensional closed oriented manifold. Recall that the linking form of M is a symmetric nonsingular bilinear form

$$\phi_M : tor H_1(M) \times tor H_1(M) \to Q/Z$$

defined by Poincare duality(c.f. [17]). By [13], any symmetric nonsingular bilinear form over a finite group can be realized as the linking form of a 3-manifolds. It is often convenient to identify a linking form (G, ϕ) with a matrix which represents ϕ relative to the generators of a cyclic splitting of G.

As in [17], write N and N_p for the monoid of isomorphism classes of finite groups and finite p-groups with linking forms, where addition is defined as orthogonal sum. Recall

Theorem (Wall 1962). *The monoid N is the direct sum of N_p. For p odd, N_p has generators A_p^k, $B_p^k (k \geq 1)$ and the sole relations $2A_p^k = 2B_p^k$. The generators of N_2 are A_2^k, $E_2^k (k \geq 1)$, $B_2^k (k \geq 2)$, $F_2^k (k \geq 2)$, C_2^k, $D_2^k (k \geq 3)$; where $A_p^k = (\frac{1}{p^k})$, $B_p^k = (\frac{a}{p^k})$ and a is a positive integer such that the Jacobi symbol $(\frac{a}{p}) = -1$ if p odd. $A_2^k = (\frac{1}{2^k})$, $B_2^k = (\frac{-1}{2^k})$ if $k \geq 2$;*

$C_2^k = (\frac{5}{2^k})$, $D_2^k = (-\frac{5}{2^k})$ *if* $k \geq 3$;

$$E_2^k = \begin{bmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{bmatrix}; \quad F_2^k = \begin{bmatrix} 2^{1-k} & 2^k \\ 2^k & 2^{1-k} \end{bmatrix}$$

Let H be a free abelian group of finite rank n with a basis $e_1, \cdots, e_n$. Write S for a nondegenerate $n \times n$ matrix over integer. S gives a monomorphism of H to H whose quotient group G is of order $detS$. From S one obtains a unique linking form ϕ_S over G such that $\phi_S(e_i, e_j) = a_{ij}$, where a_{ij} is the (i,j)-entry of $S^{-1}(modZ)$. We say that S is a presentation of the linking form ϕ. Recall that every linking form over a finite abelian group can be captured in this way.

We say symmetric matrices over integer S_1 and S_2 are *closed related* if there exists a unimodular integer matrix P such that

$$P(S_1 \oplus (\pm 1) \oplus \cdots \oplus (\pm 1))P' \cong S_2 \oplus (\pm 1) \oplus \cdots \oplus (\pm 1).$$

A fundamental theorem of Kneser-Puppe[14] says that S_1 and S_2 are *closed related* if and only if they present isomorphic linking forms.

Lemma 1.4. *Every link form over a cyclic p-group(where p is a prime) can be presented by a positive definite matrix.*

Before proving this lemma we first complete the proof of the Theorem 1.1.

Proof of Theorem1.1. By the Wall's Theorem, if the condition (i) holds, the linking form $(H_1(M), \phi_M)$ can be written as the direct sum $\phi_1 \oplus \cdots \oplus \phi_n$ where ϕ_i $(1 \leq i \leq n)$ are all linking pairings over a cyclic group of order $p_i^{n_i}$, here p_i denotes an odd prime. Note that M bounds a 4-dimensional oriented simply connected manifold V. The intersection form I_V of V is a presentation of ϕ_V. By lemma 1.4, ϕ_M has a presentation $S_1 \oplus \cdots \oplus S_n$ where $S_1, \cdots, S_n$ are positive definite matrices over integer. Generally, if M is a 3-dimensional rational homology sphere. Suppose that the 2 torsion subgroup of $H_1(M)$ is isomorphic to $Z_{2^{i_1}} \oplus \cdots \oplus Z_{2^{i_k}}$. We want to show that the link pairing of $\phi_M \oplus (\frac{1}{2^{i_1}}) \oplus \cdots \oplus (\frac{1}{2^{i_k}})$ can be presented by a positive definite matrix. By lemma 1.4 and Wall's Theorem above, we need only to consider the case that the 2 torsion part of ϕ_M consists of the direct sum of at most $[\frac{k}{2}]$ terms which are either $F_{2^{i_j}}$ and $E_{2^{i_j}}$ of N_2. When k is even, say $k = 2r$, the matrix

$$\begin{pmatrix} \frac{2^{2r-1}+1}{3} & b & 0 \\ b & 4b & 2^{2r+1} \\ 0 & 2^{2r+1} & 3b \end{pmatrix}$$

is a positive definite matrix over integer to present F_{2^k} where $b = \frac{2^{2r+1}+1}{3}$. By [13],

$E_{2^k}\oplus(\frac{5}{2^{k+1}})\cong F_{2^k}\oplus(\frac{1}{2^{k+1}})$. The identity

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2^k & 0 \\ 2^k & 0 & 0 \\ 0 & 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}=\begin{pmatrix} 2^k & 0 & 0 \\ 0 & -2^k & 0 \\ 0 & 0 & 2^k \end{pmatrix}.$$

shows that $E_{2^k}\oplus(\frac{1}{2^k})\cong(\frac{1}{2^k})\oplus(\frac{1}{2^k})\oplus(\frac{-1}{2^k})$. Thus $\phi_M\oplus(\frac{1}{2^{i_1}})\oplus\cdots\oplus(\frac{1}{2^{i_k}})$ is equivalent to direct sum of linking forms over cyclic groups. Applying lemma 1.4 again it can be presented by a positive matrix. Note the lens space L_{2^k} has the link form $(\frac{1}{2^k})$. Therefore the linking form of $M\#L_3(2^{i_1})\#\cdots\#L_3(2^{i_k})$ is isomorphic to $\phi_M\oplus(\frac{1}{2^{i_1}})\oplus\cdots\oplus(\frac{1}{2^{i_k}})$. If (i) does not hold, we use $M\#L_3(2^{i_1})\#\cdots\#L_3(2^{i_k})$ to instead of M and still use V to denote a simply connected manifold with boundary $M\#L_3(2^{i_1})\#\cdots\#L_3(2^{i_k})$.

The Kneser-Puppe's Theorem [14] says that I_V and $S_1\oplus\cdots\oplus S_n$ are closed related and so there are positive integers k, l, m and q such that

$$I_V\oplus k(+1)\oplus l(-1)\cong S_1\oplus\cdots\oplus S_n\oplus m(+1)\oplus q(-1).$$

We may assume $m\geq 1$. By Freedman [4] there is a simply conncted manifold V' such that $V'\#(m-1)CP^2\#q(-CP^2)$ is homeomorphic to $V\#kCP^2\#l(-CP^2)$. The intersection form of V' is $S_1\oplus\cdots\oplus S_n\oplus(1)$ which is positive definite.

For the same reasoning, we can obtain another oriented simply connected 4-manifold W' with boundary $-M/(-M)\#(-L_3(2^{i_1}))\#\cdots\#(-L_3(2^{i_k}))$ and possessing a positive intersection form. Let $X=V'\cup_M W'$. It is a simply connected closed 4-manifold with positive definite intersection form. $M/M\#L_3(2^{i_1})\#\cdots\#L_3(2^{i_k})$ is a locally flat embedded submanifold as the boundary of V'. In the second case, note that $M-intD^3$ embeds into X locally flat and so the boundary of this embedding, $M\#(-M)$ embeds in X locally flat too. This ends the proof. $\diamondsuit$

Proof of Lemma 1.4. Case (i). $p=-1(mod4)$; As $(\frac{-1}{p})=-1$.
The linking pairing $(\frac{1}{p^k})$ is not isomorphic to $(-\frac{1}{p^k})$. The former is presented by the 1×1 matrix (p^k). And $(-\frac{1}{p^k})$ is presented by the $(p^k-1)\times(p^k-1)$ positive definite matrix

$$A=\begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ & \cdots & & \cdots & \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}$$

whose determinant is p^k. Over rational numbers A is congruent to the diagonal matrix

$$\begin{pmatrix} 2 & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \ddots & \\ & & & & \frac{p^k}{p^k-1} \end{pmatrix}.$$

The matrix A presents the link form $(\frac{-1}{p^k})$ as the diagonal element of its inversion are all $\frac{p^k-1}{p^k}$.

Case (ii). $p = 1(mod 4)$;
Note now $(\frac{-1}{p^k})$ and $(\frac{1}{p^k})$ are equivalent. So we need only to consider the form $(\frac{a}{p^k})$ where a is not quadratic residue. Let n be the minimal positive prime not residue mod p. We can present $(\frac{n}{p^k})$ by a positive definite matrix over integer constructed from the Euclidean algorithm, using the fact

$$\begin{aligned} 1 &= nd_1 - p^k d_2 \\ d_1 &= a_1 d_2 - d_3 \\ &\vdots \\ d_{i-1} &= a_{i-1} d_i - 1 \\ d_i &= a_i \end{aligned}$$

The matrix B^{-1} is a positive definite matrix over integer presenting the link form $(\frac{n}{p^k})$ where

$$B = \begin{pmatrix} np^{-k} & 1 & 0 & \cdots 0 & 1 \\ 1 & a_1 & 1 & \cdots 0 & 0 \\ 0 & 1 & a_2 & \cdots 0 & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots 1 & a_i \end{pmatrix}.$$

Note the arguments in case (i) and (ii) apply identically to show that A_{2^k}, B_{2^k} and C_{2^k}, D_{2^k} can be presented by positive definite matrices. This completes the proof. $\diamondsuit$

Proof of Proposition 1.2. If (1) holds, the link form $(\frac{1}{n_1}) \oplus \cdots \oplus (\frac{1}{n_k})$ and $(\frac{-1}{n_1}) \oplus \cdots \oplus (\frac{-1}{n_k})$ are equivalent as the Jacobi-Legendre symbol $(\frac{-1}{p}) = 1$ for every prime $p = 1(mod 4)$. For a simply connected 4-manifold V with boundary $-M$, its intersection form is closed related to $(n_1) \oplus \cdots \oplus (n_k)$. This is of odd type. By [5] 10.3, V is homeomorphic to the connected sum of a closed simply connected manifold and a compact manifold with boundary $-M$ with intersection form isomorphic to $(n_1) \oplus \cdots \oplus (n_k)$. As in the proof of Theorem 1.1 we can obtain an embedding of M in a simply connected closed 4-manifold X whose intersection form is positive definite of rank $2k \leq 8$. Note that each positive definite unimodular form of odd type over integer and rank not excess 8 is canonical. Thus X is either $2kCP^2$ or $(2k-1)CP^2 \# *CP^2$, where $*CP^2$ is the manifold with the same homotopy type of CP^2 but nontrivial Kirby-Siebenmann obstruction. $*CP^2 \# * CP^2$ is homeomorphic to $2CP^2$. If X is $(2k-1)CP^2 \# * CP^2$, we may sum $*CP^2$ to X and get the canonical $(2k+1)CP^2$ and M embeds in $(2k+1)CP^2$ topological locally flat.

In the case (2), the symbol $(\frac{2}{p}) = -1$ if $p = 3(mod 8)$. Note that $(\frac{2}{p^r})$ has a presentation of odd type positive definite matrix $\begin{pmatrix} 2 & 1 \\ 1 & \frac{p^r+1}{2} \end{pmatrix}$ or $\begin{pmatrix} 4 & 1 \\ 1 & \frac{p^r+1}{4} \end{pmatrix}$ by r even or odd. Thus M can be embedded into a 4-manifold X with positive definite intersection form of rank $3k \leq 6$ and hence is standard. The rest of the proof is similarly. This ends the proof.

2. Embeddings of 3-manifolds with torsion free homology group

This section is devoted to discuss the problem of when a 3-manifold M with $H_1(M)$ torsion free can be embedded into a positive definite 4-manifold. This is relavant to the cobordism problem of links. The situation now is much complicated and seems hard to obtain a complete answer.

Let $K \subset S^3$ be a knot. Performing framing zero surgery on S^3 one get a manifold M_K with the homology of $S^1 \times S^2$. It is easy to show that M_K embeds into S^4 if K is a slice knot, and conversely if M_K embeds into S^4 the knot K is slice in a homology 4-ball, hence is algebraically slice. So M_K can not be embedded into S^4 topologically where K is the trefoil knot. The analogue relation holds true for link instead of knot. The following proposition shows that each smooth/topologcal link is smooth/topological slice in the connected sum $D^4 \#(\#_1^n CP^2)$ for n large. However, those embedded disks extending the link does not always give the zero framings. This can not produce an embedding of the manifold resulted from the zero framing surgeries on a link.

Proposition 2.1. *Let $L \subset S^3$ be a smooth/topological link. Then L is smooth/ topological slice in $D^4 \#(\#_1^n CP^2)$ for n large.*

Proof. Let L be a link with k components $L_1, \cdots, L_k$. It is easy to see that there are k generic immersed discs $\Delta_1, \cdots, \Delta_k$ in D^4 with boundary the link L. Note that there are two embedded 2-spheres S_1^2 and S_2^2 in CP^2 which intersects in one ponit. For each double point A of Δ_i, the connected sum $\Delta_i \# S_1^2 \subset D^4 \# CP^2$ is a generic immersed disc which intersects S_2^2 in one point. By the technique of [7] we can remove A and keep all other double points and intersection points fixed. Proceeding this program we can get a slice for the link L in $D^4 \#(\#_1^n CP^2)$ for n large. This ends the proof. $\diamondsuit$

Recall [5] showed that a knot $K \subset S^3$ is Z-slice(i.e. the complement of the embedded disc in D^4 has Z as the fundamental group) if and only if the Alexander polynomial $\Delta(K) = 1$. In this case the manifold M_K embeds locally flat in S^4. By using the CP^2-stable surgery we conclude that

Proposition 2.2. *Let M be a closed oriented 3-manifold. Assume that the fundamental group π of M is an extension of free group by a perfect normal subgroup. Then M embeds into nCP^2 locally flat for n large enough.*

Proof. By [5] 11.6C, there is a Poincare pair (X, Y) where $X \simeq \vee_1^k S^1$ and $Y \simeq M$ which is unique up to homotopy. Moreover there is a degree one normal map $f : (W, M) \rightarrow (X, Y)$, where (W, M) is a manifold with boundary. The CP^2-stable surgery theory [7] applies to show that $(W\#(\#_1^n CP^2), M)$ is normal cobordant to a pair (W', M) which is simple homotopy equivalent to $(X\#(\#_1^n CP^2), Y)$ for some n. Note that the intersection form of (W', M) is $n(1)$. By surgery it is easy to get a simply connected spin manifold V with boundary M such that $H_2(M) \rightarrow H_2(V)$ is an isomorphism. Gluing V and W' along the boundary we obtain a simply connected 4-manifold with intersection form $n(1)$. This gives an embedding of M in this manifold. By Freedman[4], this manifold either nCP^2 or $(n-1)CP^2 \# * CP^2$. If the latter occurs, we sum $*CP^2$ to it and obtain $(n+1)CP^2$. This completes the proof.

If $L \subset S^3$ is a good boundary link with n-components, i.e, there is a homomorphism of the fundamental group of its complemnet to the free group of n-letters with perfect kernel so that the image of the linking circles forms a set of generators. The framing zero surgeries on L gives a manifold M_L whose fundamental group is an extension of free group of n-letters by a perfect normal subgroup. By Proposition 2.2 M_L embeds into nCP^2 for n large enough.

3. Proof of Theorem A

In this section, we shall give the proof of the result advertised in the introduction. Our strategy to show this theorem is to construct certain open 4-manifold with periodic ends in the sense of Taubes. We refer to Taubes [16] for the detailed definition of end-periodic. Recall the following generalization of Donaldson's Theorem.

Theorem.(Taubes)*Let M be a smooth simply connected open end-periodic 4-manifold. If the intersection form of M is definite, then it can be diagonalized over integer.*

In [2], using the same method of Gompf [9] which is used to produce two parameters family of uncountably many exotic R^4, Ding claimed that, if M is a noncompact 4-manifold with at least two open ends and one of them is topological collared by $N \times R$, where N is a closed oriented 3-dimensional flat submanifold of nCP^2, then there are uncountably smooth structures on M. The proof of Theorem A follows from this claim and corollary 1.2, proposition 2.2. For reader's convenient, I include this details here.

Proof of Theorem A: By corollary 1.2 and Proposition 2.2, under the assumptions, N can be embedded into nCP^2 as a flat submanifold for n large. First we are going to show that there are uncountably many exotic structures on $N \times R$. Note that $N \times R$ is a submanifold of nCP^2. Let $j : N \times R \hookrightarrow nCP^2$ be the inclusion. Endow nCP^2 with a canonical smooth structure. $N \times R$ has an induced smooth structure from j. By Quinn[15], we may assume that the smooth structure on $N \times R$ is canonical near $x \times R$, where $x \in N$ is a fixed point. Let R_∞ be an exotic R^4 whose end is diffeomorphic to $|E_8 \oplus (1)| - pt$, where $|E_8 \oplus (1)|$ is a simply connected closed 4-manifold with the intersection form $E_8 \oplus (1)$, where E_8 is the positive definite unimodular form over integer with rank and signature are both 8. Notice that, by Quinn[15], $|E_8 \oplus (1)| - pt$ is smoothable. We should understand that it has an endowed smooth structure. Let $R_s = intB_s$, B_s is the topological ball of radius s. For s large, namely, $s \geq r_0$, R_s is a family of exotic R^4 which are pairwise different(c.f: Taubes[16] or [9] for the details of this construction). Let X_s denote the open 4 -dimensional submanifold of $|E_8 \oplus (1)| - pt$ with the same end as R_s by cutting a cyclinder along infinity.

Now we want to show that $N \times R\natural R_s$ and $N \times R\natural R_t$are not diffeomorphic if $s \neq t$, $s, t \geq r_0$. Obviously $N \times R\natural R_s \approx N \times R$ for all s. Thus we obtain uncountably many exotic structures on $N \times R\natural R_s$ and the claim is proved.

Suppose not, then there are reals, namely $s < t$, $N \times R\natural R_s$ and $N \times R\natural R_t$are diffeomorphic. We have therefore an embedding smoothly $g : R_t \hookrightarrow N \times R\natural R_t \hookrightarrow N \times R\natural R_s \hookrightarrow nCP^2 \# R_t$. One should note that here R_t is embedded into the target manifold with a compact support. Let ϵ be a small real so that $t - \epsilon > s - \epsilon \geq r_0$. Let $V = nCP^2 \# R_t - g(B_{t-\epsilon})$. Notice that V is a open manifold and two ends of V are both diffeomorphic to $intB_t - B_{t-\epsilon}$. By the same method of [9] or [16], one may construct an end periodic manifold, namely W through splicing X_s with infinite many copies of V along the ends. Notice that the intersection form of W is $E_8 \oplus Q_\infty$, here Q_∞ denotes the direct sum of infinite copies of (1). This contradicts with Taubes Theorem mentioned before. This proves that $N \times R$ admits uncountably many smooth structures.

Now let M be a 4-manifold with at least to open ends and one of them, namely ε is topological collared as $N \times R$, where N satisfies the assumption in the theorem. Put the induced smooth structure on $N \times R$ as above. By Quinn[15] this smooth structure can be extended to M. Now we may understand M as a smooth manifold. We can form the end sums $M\natural R_s$ for all $s \geq r_0$ along the end ε. Note that $M\natural R_s$ are all homeomorphic to M. Thus we get a family of smooth manifolds which are homeomorphic to M but the collection of the ends of the manifolds in this family is a uncountable set. Notice that there are at most countably many different ends for every smooth manifold with countably base. Thus the collection $\{M\natural R_s, r_0 \leq s\}$ are uncountably up to diffeomorphism. This completes the Theorem A. $\diamondsuit$

Acknoledgement: This work was accomplished during the author's visit at the Universität Bielefeld and the Max-Planck-Institut fuer Mathematik. The author would like to express his thanks to Prof. F.Waldhausen and Prof. F.Hirzebruch for their invitations and very warm hospitality. He also benefitted from R.Gompf, R.Stong and Z.Wang for discussion in correspondences.

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