ON HOMOTOPY CLASSIFICATION PROBLEMS OF J.H.C. WHITEHEAD

by

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A principal task of homotopy theory arises from the <u>homotopy classification problems</u>:

(1) Classify homotopy types of polyhedra $X, Y \ldots$ by computable algebraic data!

(2) Compute the set of homotopy classes of maps, $[X,Y]$, in terms of the classifying data for X and Y! Moreover, compute the group of homotopy equivalences, $\mathrm{Aut}(X)$!

The rich structure of homotopy groups of spheres $[S^m, S^n]$, however, shows that the difficulties for a solution of these problems increase rapidly when, for the spaces involved, the <u>range</u> = (dimension) - (degree of connectedness) is growing. Whitehead [21], [22], [23] examined examples of the homotopy classification of polyhedra in a small range. In particular, he classified simply connected 4-dimensional homotopy types. Moreover, he classified $(n-1)$-connected $(n+2)$-dimensional polyhedra which he calls A_n^2-polyhedra. The following related problems ever since remained unsolved though they are just first steps beyond Whitehead's results:

(3) Compute all homotopy classes of maps between simply connected 4-dimensional polyhedra in terms of Whitehead's classifying data!

(4) Compute the groups of homotopy classes of maps between A_n^2 - polyhedra, $n \geq 3$!

(5) Classify homotopy types of all simply connected 5-dimensional polyhedra and of all A_n^3 - polyhedra, $n \geq 3$.

We obtained solutions of these problems which will appear in [3], [4]. This paper (which is an addendum of my lecture in Göttingen) describes some of the results. I would like to acknowledge the support of the 'Sonderforschungsbereich 40, Theoretische Mathematik' and of the 'Max-Planck-Institut für Mathematik', in Bonn.

Part I. On the homotopy classification of simply connected polyhedra of finite dimension.

We describe some results on

Problem (*): Classify algebraically the homotopy types of n-dimensional polyhedra which are simply connected!

Our main tool is a new kind of invariant which we call the boundary invariant of a simply connected complex. Using these invariants one obtains a solution of problem (*) by the inductive computation of Γ-groups. This extends nicely the solution of J.H.C. Whitehead for $n = 4$.

§ 1 The Γ-groups

Let X be a simply connected CW-complex with basepoint and let $SP^\infty X$ be the infinite symmetric product of X. The canonical inclusion $j: X \subset SP^\infty X$ yields the fiber sequence

(1.1) $$\Gamma X \xrightarrow{i} X \xrightarrow{j} SP^\infty X \quad .$$

By the result of Dold-Thom [10] we have the natural equation $\pi_n SP^\infty X = H_n(X)$ where H_n denotes the singular homology with integral coefficients. Moreover, j induces the Hurewicz homomorphism h. Therefore we obtain by (1.1) the natural exact sequence of homotopy groups

(1.2) $$H_{n+1}X \xrightarrow[b_{n+1}]{} \Gamma_n X \xrightarrow[i_*]{} \pi_n X \xrightarrow[h]{} H_n X \xrightarrow[b_n]{} \Gamma_{n-1} X$$

where we set $\Gamma_n X = \pi_n \Gamma X$. In fact, this sequence is the 'certain exact sequence' of J.H.C. Whitehead [23] compare the results of Kan in [17]. The connecting homomorphism $b_{n+1} = b_{n+1}X$ is called the secondary boundary operator. Whitehead defines $\Gamma_n X$ by the cell structure of X, namely

(1.3) $$\Gamma_n X = \mathrm{im}(i_{n*}: \pi_n X^{n-1} \longrightarrow \pi_n X^n) \quad .$$

Since X is 1-connected the Hurewicz theorem shows that ΓX is 2-connected. Thus we have $\Gamma_2 X = 0$. The group $\Gamma_3 X$ was computed by Whitehead in [23]; he introduces the universal quadratic functor Γ from abelian groups to abelian groups and he obtaines the natural equation

(1.4) $$\Gamma_3 X = \Gamma(H_2 X)\text{, compare (III. 1.2) below.}$$

We also use homotopy groups with coefficients in an abelian group A:

(1.5) $$\pi_n(A;X) = [M(A,n),X] \quad , \quad n \geq 2 \quad ,$$

which are defined by the Moore-space $M(A,n)$ with $H_n M(A,n) = A$.

Here one has to be careful since $\pi_n(A;X)$ is not a contravariant functor in A . A homomorphism $\xi\colon A \longrightarrow B$ between abelian groups induces a subset

(1.6) $$\xi^{\#} \subset \mathrm{Hom}(\pi_n(B;X),\pi_n(A;X))$$

$$\xi^{\#} = \{\bar{\xi}^* \mid \bar{\xi} \in [M(A,n),M(B,n)] \ , \ H_n\bar{\xi} = \xi\} \quad .$$

The well known <u>universal coefficient theorem</u> [15] yields the binatural short exact sequence

(1.7) $$\mathrm{Ext}(A,\pi_{n+1}X) \overset{\Delta}{\rightarrowtail} \pi_n(A;X) \overset{\mu}{\twoheadrightarrow} \mathrm{Hom}(A,\pi_n X) \quad .$$

We introduce the <u>Γ-groups with coefficients</u>:

(1.8) $$\Gamma_n(A;X) = \pi_n(A;\Gamma X) \quad , \quad \text{see (1.1).}$$

Clearly, this group is embeded in the binatural short exact sequence

(1.9) $$\mathrm{Ext}(A,\Gamma_{n+1}X) \overset{\Delta}{\rightarrowtail} \Gamma_n(A;X) \twoheadrightarrow \mathrm{Hom}(A,\Gamma_n X) \quad .$$

Since X is simply connected we derive from (1.4)

(1.10) $$\Gamma_2(A;X) = \mathrm{Ext}(A,\Gamma(H_2X)) \quad .$$

All groups $\Gamma_n(A;X)$, $n \in \mathbb{Z}$, are abelian, we set $\Gamma_n = 0$ for $n \leq 1$.

§ 2 The classification of simply connected 4-dimensional polyhedra by J.H.C. Whitehead

A graded abelian group $H = \{H_n\}$ is r-connected if $H_0 = \mathbb{Z}$ and $H_i = 0$ for $i < 0$ and $0 < i \le r$. Moreover, H has dimension $\le N$ if H_N is free abelian and if $H_i = 0$ for $i > N$. For example the homology $H_*(X)$ of an r-connected N-dimensional polyhedron X is an r-connected N-dimensional graded abelian group and in fact each such group arises this way.

We now fix a simply connected graded abelian group H of dimension N and we slightly alter problem (*) above by:

Problem(**): Classify algebraically the simply connected homotopy types with homology H.

In this lecture graded abelian groups H and homotopy types $\{X\}$ always are simply connected. For $\dim H = 4$ problem (**) was solved by J.H.C. Whitehead as follows:

(2.1) Theorem [23]: *Let $\dim H = 4$. Then proper equivalence classes of pairs (b_4, β_4) with*

$$b_4 \in Hom(H_4, \Gamma(H_2)) \quad ,$$

$$\beta_4 \in Ext(H_3, cok\ b_4)$$

are 1-1 corresponded to the homotopy types $\{X\}$ with homology H.

Two pairs (b_4, β_4) and (b_4', β_4') are proper equivalent if there exists an automorphism $\varphi: H \cong H$ of degree 0 such that

(2.2) $$\Gamma(\varphi_2)_* b_4 = \varphi_4^* b_4' \quad , \quad \varphi_{4*}\beta_4 = \varphi_3^* \beta_4' \quad .$$

The correspondence in (2.1) is given by

(2.3) $$X \longmapsto (b_4X, \beta_4X) \quad \text{with}$$

$$b_4X = \text{secondary boundary in (1.2)},$$

$$\beta_4X = \{\pi_3X\} = \text{class of the extension}$$

$$\operatorname{cok}(b_4X) \rightarrowtail \pi_3X \twoheadrightarrow H_3X \quad \text{in (1.2)}.$$

These invariants are given by Whitehead's exact sequence . By naturality of this sequence the correspondence (2.3) between homotopy types and proper equivalence classes is welldefined , compare also (III. § 1).

A new proof of Whitehead's result was described by Chang in [4], I will give further proofs in [3], [4] , in fact, Whitehead's result is the special case n=4 of our general result in section § 4 below.

(2.4) Remark: The result of J.H.C. Whitehead above was generalized in the literature in various directions. Actually Whitehead first used a quite complicated 'extended cohomology ring' together with the Pontrjagin square for the classification, [21]. In his second approach [23] he points out that the classification by the Γ - sequence is much simpler. Whitehead also obtained a solution of problem (**) if H is (n-1) - connected and (n+2) - dimensional, this is the classifiaction of the A_n^2 - polyhedra, [22], [23]. Next Shiraiwa [19] considers A_n^3 - polyhedra, that is the case of problem (**) where H is (n-1) - connected and (n+3) - dimensional; he restricts to $n \geq 3$. However, his result is not correct as was pointed out by Chow [9]. A different approach on A_n^3 - polyhedra, $n \geq 3$, is due to Chang [8], but also his result is not correct. We have the following counterexample for Shiraiwa's result as well as for Chang's result:

Consider the homology H with $H_n = \mathbb{Z}_2$ and $H_{n+2} = \mathbb{Z}_2$ and $H_i = 0$ otherwise, $i > 0$, $n > 3$. Then each complex X with homology H has the homotopy type of a mapping cone C_f where

$$f \in [M(\mathbb{Z}_2,n+1), M(\mathbb{Z}_2,n)] = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad .$$

This shows that there are atmost 4 such homotopy types, in fact, there are exactly 4. However, the result of Shiraiwa [19] yields 6 and the result of Chang [9] yields 5 homotopy types with homology H . Since also Chow [9] did not clarify the realizability of his invariants the classification of A_n^3-polyhedra, $n \geq 3$, remained open. It can be achieved for $n \geq 2$ by our method below, for n=2 this is the classification of 1-connected 5-dimensional polyhedra.

Recently Henn [12] obtained a further solution of problem (**). He considers for an odd prime p and for large n (stable range) homology groups H which are p-local, $(n-1)$-connected and $n+4p-5$-dimensional. His result is exactly an analogue of Whitehead's original result on A_n^2-polyhedra, $n \geq 3$, since only primary (co)homology operations are needed. The classification of A_n^3 - polyhedra is made more difficult by the secondary Adem operation.

§ 3 The boundary invariants

For an abelian group A and for a simply connected complex X we define the group $\Gamma^{b}_{n-1}(A,X)$ by the push out diagram (see (1.9)):

(3.1)
$$\begin{tikzcd}
\mathrm{Ext}(A,\Gamma_n X) \arrow[r, "p_*"'] \arrow[d, tail, "\Delta"'] \arrow[dr, phantom, "\text{push}"] & \mathrm{Ext}(A,\mathrm{cok}\, b_{n+1}) \arrow[d, tail] \\
\Gamma_{n-1}(A;X) \arrow[r] \arrow[dr, "\mu"'] & \Gamma^{b}_{n-1}(A;X) \arrow[d, "\mu"] \\
 & \mathrm{Hom}(A,\Gamma_{n-1}X)
\end{tikzcd} .$$

Here $p\colon \Gamma_n X \longrightarrow \mathrm{cok}\, b_{n+1}$ is the projection for the secondary boundary $b_{n+1}\colon H_{n+1}X \longrightarrow \Gamma_n X$ in the exact sequence (1.2). By naturality of this sequence also p is natural and therefore diagram (3.1) is binatural in X and A, see (1.6).

The exact sequence (1.2) yields the short exact sequence of abelian groups

(3.2)
$$\mathrm{cok}\, b_{n+1} \rightarrowtail \pi_n X \twoheadrightarrow \ker b_n \qquad \text{with}$$
$$\{\pi_n X\} \in \mathrm{Ext}(\ker b_n, \mathrm{cok}\, b_{n+1}) \quad .$$

(3.3) <u>Theorem</u>: *To each 1-connected CW-complex there is canonically associated a sequence of elements* $\beta = (\beta_4, \beta_5, \dots)$ *with*

$$\beta_{n+1} = \beta_{n+1}X \in \Gamma^{b}_{n-1}(H_n X; X)$$

such that the following properties are satisfied:

(a) Naturality: For a map $F\colon X \longrightarrow Y$ *we have the equation*

$$F_*\beta_{n+1}X = (H_nF)^{\#}\beta_{n+1}Y \quad \text{in} \quad \Gamma^b_{n-1}(H_nX;Y) \quad .$$

(b) $$\mu(\beta_{n+1}X) = b_nX \in Hom(H_nX,\Gamma_{n-1}X)$$

(c) $$\{\pi_nX\} = \Delta^{-1}i^{\#}(\beta_{n+1}X)$$

where $i\colon ker(b_nX) \subset H_nX$ *denotes the inclusion.*

The equations in (a) and (b) also show that the subsets $(H_nF)^{\#}\beta_{n+1}Y$ and $i^{\#}\beta_{n+1}X$ consist of a single element, see (1.6). By (b) and by exactness in the column of (3.1) equation (c) is welldefined. For n=3 we have

(3.4) $$\beta_4X = \{\pi_3X\} \in \Gamma^b_2(H_3X;X) = \mathrm{Ext}(H_3X,\mathrm{cok}\ b_4X) \quad ,$$

compare (2.3).

(3.5) <u>Remark</u>: The boundary invariant $\beta_{n+1}X$ depends only on the (n+1) - skeleton of X or at the n - th section of a homology decomposition of X , see [15]. The homology decomposition was introduced by Eckmann and Hilton as a dual of the Postnikov decomposition. The homology decomposition, however, turned out to have a major disadvantage, namely it failed to be natural while the Postnikov decomposition is natural. In particular, the homotopy type of the n - th section of a homology decomposition is not an invariant of the homotopy type of X , see [5]. Therefore also the k' - invariants (= attaching maps of the homology decomposition) do not have the desired property of naturality. The boundary invariants in the theorem above can be thought of as being the 'natural part' of the k' - invariants. The proof of the theorem relies on the 'principal reduction' of a 1 - connected complex as described in [1] and on a study of 'twisted maps', see [3], [4].

§ 4 The classification of simply connected N - dimensional polyhedra

Let H be a simply connected graded abelian group of dimension N. Then there is a space X which realizes the homology H, we can take the one point union of Moore spaces:

$$X = \bigvee_{n=2}^{N} M(H_n, n) \quad . \tag{4.1}$$

On the other hand we get:

(4.2) Theorem: *Let X be a 1 - connected N - dimensional polyhedron with homology H. Then X has the homotopy type of a one point union of Moore spaces as in (4.1) if and only if all boundary invariants $\beta_n(X)$ are trivial, $n \geq 2$.*

One obtains a classification of all simply connected homotopy types with homology H by the following kind of a system of operators:

(4.3) Definition: A Γ - system associated to H is given by operators Γ_n and $\Gamma_{n-1}(A)$, $2 \leq n < N$, which are defined on the set D_n. This set consists of sequences

$$\text{(1)} \qquad \begin{aligned} &(b,\beta)^n = (b_4, \beta_4, \ldots, b_n, \beta_n) \quad \text{with} \\ &(b,\beta)^{n-1} = (b_4, \beta_4, \ldots, b_{n-1}, \beta_{n-1}) \in D_{n-1} \quad . \end{aligned}$$

in addition we set $b_i = 0$ and $\beta_i = 0$ for $i = 2,3$ so that D_2 and D_3 contain only the sequence $(0,\dots,0)$. The operator Γ_n associates with $(b,\beta)^n \in D_n$ the abelian group $\Gamma_n = \Gamma_n(b,\beta)^n$ and the operator $\Gamma_{n-1}(A)$ associates with an abelian group A and with $(b,\beta)^n \in D_n$ an abelian group $\Gamma_{n-1}(A) = \Gamma_{n-1}(A)(b,\beta)^n$ together with a short exact sequence

$$\text{(2)} \qquad \operatorname{Ext}(A,\Gamma_n(b,\beta)^n) \overset{\Delta}{\rightarrowtail} \Gamma_{n-1}(A)(b,\beta)^n \twoheadrightarrow \operatorname{Hom}(A,\Gamma_{n-1}(b,\beta)^{n-1}) \quad .$$

For n=2 we have $\Gamma_2 = 0$ and $\Gamma_1(A) = 0$ and for n=3 we have

$$\text{(3)} \qquad \Gamma_3 = \Gamma(H_2) \quad \text{and} \quad \Gamma_2(A) = \operatorname{Ext}(A,\Gamma(H_2)) \quad ,$$

compare (1.4) and (1.10). The domain D_n , $n \geq 3$, is defined inductively by the Γ-system, that is: $(b,\beta)^{n+1} \in D_{n+1}$ if and only if the following condition are satisfied:

$$\text{(4)} \qquad \begin{cases} (b,\beta)^n \in D_n & , \\ b_{n+1} \in \operatorname{Hom}(H_{n+1},\Gamma_n(b,\beta)^n) & , \\ \beta_{n+1} \in \Gamma^b_{n-1}(H_n)(b,\beta)^n & , \\ \mu\beta_{n+1} = b_n & . \end{cases}$$

Here the group $\Gamma^b_{n-1}(A)$ is defined by the push out diagram

$$\begin{tikzcd}
\operatorname{Ext}(A,\Gamma_n) \arrow[r, "p_*"'] \arrow[d, tail, "\Delta"'] \arrow[dr, phantom, "\text{push}"] & \operatorname{Ext}(A,\operatorname{cok} b_{n+1}) \arrow[d, tail] \\
\Gamma_{n-1}(A) \arrow[r] \arrow[dr, dashed, "\mu"'] & \Gamma^b_{n-1}(A) \arrow[d, "\mu"'] \\
 & \operatorname{Hom}(A,\Gamma_{n-1})
\end{tikzcd}$$

which is given by the exact sequence in (2) and by b_{n+1} in (4). We denote by $p\colon \Gamma_n = \Gamma_n(b,\beta)^n \longrightarrow \operatorname{cok} b_{n+1}$ the quotient map, compare (3.1).

(4.4) Theorem: *Let H be a 1-connected graded abelian group of dimension N. Then there exists a Γ-system of operators associated to H such that proper equivalence classes of sequences*

$$(b_4, \beta_4, \ldots, b_N, \beta_N) \in D_N$$

are 1-1 corresponded to the simply connected homotopy types with homology H.

Clearly, by (3) and (4) in (4.3) we know D_4. Therefore the theorem with $N=4$ is exactly the result of J.H.C. Whitehead in (2.1). The theorem raises two highly non trivial problems:

Problem (***): (a) Compute a Γ-system as in (4.4).

(b) Compute the relation of proper equivalence on D_N.

The solution of these problems yields by (4.4) a solution of problem (**). We do not say that we can solve problem (***), in fact, this would be a never ending task. Still various examples show that theorem (4.4) leads much further than the results previously obtained in the literature.

For the proof of (4.4) we use the following results. We show that for a simply connected complex X the Γ-groups $\Gamma_n(X)$ and $\Gamma_{n-1}(A;X)$ in § 2 are essentially determined by the homology $H_*(X)$ and by the sequences of boundary invariants $(b_4X, \beta_4X, \ldots, b_nX, \beta_nX)$. Here we use inductively the 'CW-tower of categories' which is introduced in [3], [4]. Moreover, the CW-tower shows that all boundary invariants b_{n+1} and β_{n+1} with the properties in (4) of (4.3) are realizable by an appropriate X. This, together with conditions on the realizability of homology homomorphisms, yields the result.

Actually, the result (4.4) above is an abbreviation of our result in [3], [4] since we describe many properties of a Γ-system associated to H .

(4.5) Remark: If the homology H is free abelian we have $\mathrm{Ext}(H_{n+1},\Gamma_n) = 0$ and therefore the isomorphism

$$\mu: \Gamma^b_{n-1}(H_n) \cong \mathrm{Hom}(H_n,\Gamma_{n-1}) \quad .$$

Thus β_{n+1} is determined by $\mu\beta_{n+1} = b_n$ in this case. Therefore we can omit the boundary invariant β_n in this case and a Γ-system associated to H consists of domains D_n with

$$(b_4,\ldots,b_n) \in D_n$$

and of operators Γ_n defined on D_n with $b_{n+1} \in \mathrm{Hom}(H_{n+1},\Gamma_n)$. This simplifies the computation of the Γ-system.//

(4.6) Remark: If H_n is a rational vector space, $n>0$, then also Γ_n is a rational vector space and in this case $b_{n+1} \in \mathrm{Hom}(H_{n+1},\Gamma_n)$ essentially is the boundary in the minimal model $(L(s^{-1}H),d)$ of the Quillen model, this minimal model was constructed in [2]. This shows that rationally a Γ-system can be computed by using differential Lie algebras.//

§ 5 The classification of simply connected 5 - dimensional polyhedra

For $\dim H = 4$ we have the example of J.H.C. Whitehead in § 2. Thus the next case is $\dim H = 5$.

The domain D_4 is the set of all pairs (b_4, β_4) with

(5.1)
$$b_4 \in \mathrm{Hom}(H_4, \Gamma H_2) \quad ,$$
$$\beta_4 \in \mathrm{Ext}(H_3, \mathrm{cok}\, b_4) \quad .$$

We have to compute the operators Γ_4 and $\Gamma_3(A)$ on D_4; this solves part (a) of problem (***) for N=5:

Our computation uses the natural short exact sequences (A = abelian group):

(5.2)
$$\Gamma_2^2(A) \xrightarrow{j} \pi_4(M(A,2)) \twoheadrightarrow \Gamma T(A) \quad .$$

Here Γ_2^2 and the ΓT are functors from abelian groups to abelian groups. We call ΓT the <u>Γ - torsion</u>. We give an explicit description of these functors in (III. 2.1) and (III. 2.3) below. We point out that in (5.2) the group $\pi_4(M(A,2))$ is not a functor in A, see (1.6). The extension problem for (5.2) is solved.

The group $\Gamma_2^2(A)$ is a natural quotient

(5.3) γ
$$\Gamma_2^2(A) = (\Gamma(A) \otimes \mathbb{Z}_2 \oplus \Gamma A \otimes A)/\sim \quad .$$

Here the equivalence relation $\sim$ is explicitly described by certain natural structure of the functor Γ.

We now obtain the group $\Gamma_4(b_4,\beta_4)$ as follows. For β_4 we choose an extension

$$\text{cok } b_4 \rightarrowtail \pi_3 \twoheadrightarrow H_3 \tag{5.4}$$

with $\beta_4=\{\pi_3\}$. By use of the composition

$$i\colon \Gamma(H_2) \xrightarrow{p} \text{cok } b_4 \rightarrowtail \pi_3$$

we obtain the push out diagram

$$\begin{tikzcd}
\Gamma(H_2)\otimes \mathbb{Z}_2 \oplus \Gamma(H_2)\otimes H_2 \arrow[r, two heads, "p"] \arrow[d, "i\otimes \mathbb{Z}_2 \oplus i\otimes H_2"'] & \Gamma_2^2 H_2 \arrow[r, tail, "j"] \arrow[d, phantom, "\text{push}"] & \pi_4(M(H_2,2)) \arrow[d, dashed, "q"] \\
\pi_3\otimes \mathbb{Z}_2 \oplus \pi_3\otimes H_2 \arrow[rr, dashed, "\bar{p}"] & & \Gamma_4(b_4,\beta_4)
\end{tikzcd}$$

This yields $\Gamma_4(b_4,\beta_4)$ as an abelian group. The map p in (5.4) is the quotient map, see (5.2), and j is the inclusion in (5.2).

Next we obtain the group $\Gamma_3(A)(b_4,\beta_4)$ by the push out

$$\begin{tikzcd}
\mathrm{Ext}(A,\pi_4 M(H_2,2)) \arrow[r, "q_*"] \arrow[d, tail, "\Delta"] \arrow[dr, phantom, "\text{push}"] & \mathrm{Ext}(A,\Gamma_4(b_4,\beta_4)) \arrow[d, tail, "\Delta"] \\
\pi_3(A;M(H_2,2)) \arrow[r] \arrow[d, dashed, two heads, "\mu"] & \Gamma_3(A)(b_4,\beta_4) \arrow[d, two heads, "\mu"] \\
\mathrm{Hom}(A,\pi_3 M(H_2,2)) \arrow[r, "\cong"] & \mathrm{Hom}(A,\Gamma(H_2))
\end{tikzcd} \tag{5.6}$$

Again the extension problem for the left hand column in (5.6) is solved.

Thus this diagram determines $\Gamma_3(A)(b_4,\beta_4)$ as an abelian group. By (5.5) and (5.6) the domain D_5 can be computed, see (4) in (4.3). The classification of 1 - connected 5 - dimensional polyhedra is complete if we describe the relation of proper equivalence on D_5, see (4.4). This relation, however, is fairly complicated. For simplicity we here consider only the case that the homology H is <u>free abelian</u>, see (4.5). In this case we have

$$(b_4,b_5) \in D_5$$

iff $b_4 \in \mathrm{Hom}(H_4,\Gamma H_2)$ and $b_5 \in \mathrm{Hom}(H_5,\Gamma_4)$. For $\Gamma_4 = \Gamma_4(b_4)$ the map

$$\pi_3 \otimes \mathbb{Z}_2 \oplus \pi_3 \otimes H_2 \xrightarrow{\bar{p}} \Gamma_4$$

is surjective since $\Gamma T(H_2) = 0$, see (5.5). Now (b_4,b_5) is properly equivalent to (b_4',b_5') if and only if there is an isomorphism $\varphi\colon H \cong H$ and a map $\bar{\varphi}$ such that

(5.7)
$$\begin{aligned} \Gamma(\varphi_2)_* b_4 &= \varphi_4^* b_4' \quad , \\ \Gamma_4(\bar{\varphi},\varphi_2)_* b_5 &= \varphi_5^* b_5' \quad . \end{aligned}$$

Here $\bar{\varphi}$ is any map for which the diagram

$$\begin{tikzcd} \operatorname{cok} b_4 \arrow[r, tail] \arrow[d, "\Gamma(\varphi_2)"] & \pi_3 \arrow[r, two heads] \arrow[d, dashed, "\bar{\varphi}"] & H_3 \arrow[d, dashed, "\varphi_3"] \\ \operatorname{cok} b_4 \arrow[r, tail] & \pi_3 \arrow[r, two heads] & H_3 \end{tikzcd}$$

commutes. Such a map induces a homomorphism $\Gamma(\bar{\varphi},\varphi_2)$ by the commutative diagram

$$\begin{array}{ccc} \pi_3 \otimes \mathbb{Z}_2 \oplus \pi_3 \otimes H_2 & \longrightarrow\!\!\!\!\!\rightarrow & \Gamma_4 \\ \Big\downarrow \bar{\varphi}\otimes\mathbb{Z}_2 \oplus \bar{\varphi}\otimes\varphi_2 & & \Big\downarrow \Gamma(\bar{\varphi},\varphi_2) \\ \pi_3 \otimes \mathbb{Z}_2 \oplus \pi_3 \otimes H_2 & \longrightarrow\!\!\!\!\!\rightarrow & \Gamma_4 \end{array} .$$

The subclass of all simply connected 5 - dimensional Poincaré complexes was recently classified by Stöcker [20]. These complexes have invariants (b_4, β_4, b_5) since $H_3 = 0$ and thus $\beta_3 = 0$, moreover $H_5 = \mathbb{Z}$.

§ 6 The example of Unsöld

My student H.M. Unsöld who is working on his dissertation solved problem (***) for the following special types of homology groups:

$$(6.1) \qquad \begin{cases} H \text{ is free abelian,} \\ H \text{ is } (n-1)\text{-connected and } (n+4)\text{-dimensional,} \\ n \geq 6 \quad \text{(stable range)} . \end{cases}$$

In this case D_{n+4} consists of sequences $(b_{n+2}, b_{n+3}, b_{n+4})$. D_{n+4} is computed explicitly by

$$\Gamma_{n+2} = H_n \otimes \mathbb{Z}_2 \quad ,$$

$$\Gamma_{n+3}(b_{n+2}) = \operatorname{cok}(b_{n+2}) \oplus H_{n+1} \otimes \mathbb{Z}_2 \quad ,$$

$$\Gamma_{n+3}(b_{n+2}, b_{n+3}) = \operatorname{cok}((i \oplus 1) b_{n+3}) \oplus \ker(\bar{b}_{n+2}) \quad .$$

Here $\bar{b}_{n+2}\colon H_{n+2} \otimes \mathbb{Z}_2 \longrightarrow H_n \otimes \mathbb{Z}_2$ is induced by b_{n+2} and the inclusion i is given by the commutative diagram

$$\begin{tikzcd}
H_n \otimes \mathbb{Z}_2 \arrow[r, two heads] \arrow[d, tail, "j"] & \operatorname{cok} b_{n+2} \arrow[d, tail, "i"] \\
H_n \otimes \mathbb{Z}_{24} \arrow[r, two heads] & \operatorname{cok}(jb_{n+2})
\end{tikzcd}$$

where j is given by the inclusion $\mathbb{Z}_2 \subset \mathbb{Z}_{24}$. Moreover, Unsöld obtained explicit formulas for proper equivalences in D_{n+4}. Thus by (4.4) all simply connected homotopy types with homology H as in (6.1) are classified.

For example there are exactly 89 simply connected homotopy types with homology

$$H_o = H_n = H_{n+1} = H_{n+2} = H_{n+3} = H_{n+4} = \mathbb{Z}$$

and $H_i = 0$ otherwise, $n \geq 6$. Moreover, there are exactly 27 simply connected homotopy types with

$$H_o = H_n = H_{n+2} = H_{n+4} = \mathbb{Z}$$

and $H_i = 0$ otherwise, $n \geq 6$, the stable complex projecitve 3-space $\Sigma^{n-2}\mathbb{C}P_3$ is one of these 27 types.

Part II. An example: The classification of A_n^3 - polyhedra, $n \geq 4$.

A_n^3 -polyhedra are (n-1)-connected CW-complexes which are (n+3) - dimensional. For $n \geq 4$ we define below A_n^3-systems for which we have:

(1) Theorem: *Proper isomorphism classes of A_n^3-systems are 1-1 corresponded to homotopy types of A_n^3-polyhedra ($n \geq 4$).*

This result is a special case of (I.4.4) above. We use the following notation: Let A be an abelian group. The extension $\mathbb{Z}/2 \rightarrowtail \mathbb{Z}/4 \twoheadrightarrow \mathbb{Z}/2$ yields the exact sequence

$$A * \mathbb{Z}/2 \rightarrowtail A * \mathbb{Z}/4 \longrightarrow A * \mathbb{Z}/2 \xrightarrow[\{G(A)\}]{} A \otimes \mathbb{Z}/2 \longrightarrow A \otimes \mathbb{Z}/4 \twoheadrightarrow A \otimes \mathbb{Z}/2$$

By $\mathrm{Hom}(A * \mathbb{Z}/2, A \otimes \mathbb{Z}/2) = \mathrm{Ext}(A * \mathbb{Z}/2, A \otimes \mathbb{Z}/2)$ we choose an extension $G(A)$ which represents the connecting homomorphism $\{G(A)\}$ above:

$$A \otimes \mathbb{Z}/2 \overset{\mu}{\rightarrowtail} G(A) \overset{\Delta}{\twoheadrightarrow} A * \mathbb{Z}/2 \qquad . \tag{2}$$

For each $\varphi \in \mathrm{Hom}(A,B)$ there is a homomorphism $\bar{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccccc} A \otimes \mathbb{Z}/2 & \overset{\mu}{\rightarrowtail} & G(A) & \overset{\Delta}{\twoheadrightarrow} & A * \mathbb{Z}/2 \\ \downarrow{\scriptstyle \varphi \otimes 1} & & \downarrow{\scriptstyle \bar{\varphi}} & & \downarrow{\scriptstyle \varphi * 1} \\ B \otimes \mathbb{Z}/2 & \rightarrowtail & G(B) & \twoheadrightarrow & B * \mathbb{Z}/2 \end{array} \qquad . \tag{3}$$

Moreover, we define $\bar{G}(A)$ by the commutative diagram

(4)
$$
\begin{tikzcd}
\mathrm{Ext}(A,\mathbb{Z}/2) \arrow[r, tail, "\bar{\Delta}"] \arrow[d, equal] & \bar{G}(A) \arrow[r, two heads, "\bar{\mu}"] \arrow[d, equal] & \mathrm{Hom}(A,\mathbb{Z}/2) \arrow[d, equal] \\
\mathrm{Hom}(A*\mathbb{Z}/2,\mathbb{Z}/4) \arrow[r, tail, "\Delta^*"] & \mathrm{Hom}(G(A),\mathbb{Z}/4) \arrow[r, two heads, "\mu^*"] & \mathrm{Hom}(A\otimes\mathbb{Z}/2,\mathbb{Z}/4)\ .
\end{tikzcd}
$$

Remark: For the Moore space of A in degree n we have isomorphisms $(n \geq 4)$

$$G(A) = \pi_{n+2}M(A,n) \quad ,$$
$$\bar{G}(A) = \pi^n M(A,n+1) = [M(A,n+1),S^n] \quad .$$

Thus these groups are Spanier - Whitehead duals of each other.//

(5) Definition: Consider the diagrams (a) and (b) below. An A_n^3 - system S with $n \geq 4$ is a diagram of unbroken arrows as in (a) together with an element β_{n+3} as in (b); (all arrows are homomorphisms between abelian groups, as usual $\rightarrowtail$ and $\twoheadrightarrow$ denote injective and surjective maps respectively).

(a)
$$
\begin{tikzcd}
\mathrm{cok}(b_{n+3}) & & & H_n\otimes\mathbb{Z}/2 \arrow[lll, dashed, "v"'] \arrow[dll, dashed, "i\otimes 1"'] \arrow[d, dashed, tail, "\mu"] & & & \\
& \pi_{n+1}\otimes\mathbb{Z}/2 \arrow[d, tail, "\mu"] \arrow[rr, phantom, "\text{push out}"] & & G(H_n) \arrow[dll, dashed] & & & \\
H_{n+3} \arrow[r, "b_{n+3}"] & \Gamma_{n+2}(i) \arrow[r, dashed] \arrow[uul, dashed, "q"] \arrow[d, two heads, "\Delta"] & \pi_{n+2} \arrow[r, dashed] & H_{n+2} \arrow[r, "b_{n+2}"] & H_n\otimes\mathbb{Z}/2 \arrow[r, "i"] & \pi_{n+1} \arrow[r, two heads] & H_{n+1} \\
& H_n*\mathbb{Z}/2 & & & & &
\end{tikzcd}
$$

$\underbrace{H_{n+2} \xrightarrow{b_{n+2}} H_n\otimes\mathbb{Z}/2 \xrightarrow{i} \pi_{n+1} \twoheadrightarrow H_{n+1}}_{\text{exact sequence}}$

The column is a short exact sequence, H_{n+3} is free abelian and $i\otimes 1$ is given by the homomorphism i in the row of the diagram. By (2) we obtain the push out $\Gamma_{n+2}(i)$ and the map Δ. The map v is the composition $q\mu(i\otimes 1)$ where q is the quotient map. We use v for the definition of the following push out, see (4):

(b)
$$\begin{tikzcd}
\mathrm{Ext}(H_{n+2},\mathbb{Z}/2)\otimes H_n \arrow[r, "\bar\Delta\otimes 1"] \arrow[d, equal] & \bar G(H_{n+2})\otimes H_n \arrow[r, two heads, "\bar\mu\otimes 1"] \arrow[dd] & \mathrm{Hom}(H_{n+2},\mathbb{Z}/2)\otimes H_n \arrow[dd, equal] \\
\mathrm{Ext}(H_{n+2},H_n\otimes\mathbb{Z}/2) \arrow[d, "v_*"] \arrow[dr, phantom, "\text{push out}"] & & \\
\mathrm{Ext}(H_{n+2},\mathrm{cok}\, b_{n+3}) \arrow[r, tail] & \Gamma^b_{n+1}(H_{n+2},v) \arrow[r, two heads, "\mu"] & \mathrm{Hom}(H_{n+2},H_n\otimes\mathbb{Z}/2) \\
 & \beta_{n+3} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & b_{n+2} \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

μ is induced by $\bar\mu\otimes 1$ and we have $\mu\beta_{n+3}=b_{n+2}$, see (a). The rows of the diagram are exact.

For a map $\psi\colon A\longrightarrow H_{n+2}$ let $\bar\psi\colon G(A)\longrightarrow G(H_{n+2})$ be a map as in (3) and let

(6)
$$\bar\psi^*\colon \Gamma^b_{n+1}(H_{n+2},v)\longrightarrow \Gamma^b_{n+1}(A,v)$$

be the map between push outs, see (5)(b), induced by $\bar\psi^*\otimes H_n\oplus \mathrm{Ext}(\psi,\mathrm{cok}\, b_{n+3})$ with $\bar\psi^* = \mathrm{Hom}(\bar\psi,\mathbb{Z}/4)$, see (4).

For the inclusion $j=\psi\colon \ker b_{n+2}\subset H_{n+2}$ we get $\mu\bar j^*\beta_{n+3}=j^*\mu\beta_{n+3} = j^*b_{n+2}=0$. Therefore the element

(7)
$$\{\pi_{n+2}\} = \Delta^{-1}\bar j^*\beta_{n+3}\in \mathrm{Ext}(\ker b_{n+2},\mathrm{cok}\, b_{n+3})$$

is welldefined. An extension π_{n+2}, which represents this element, fits into the row of (5)(a) such that this row is an exact sequence. Since $\Delta^{-1}\mathfrak{J}^*$ is surjective on $\mu^{-1}(b_{n+2})$ we see that each exact row as in (a) is obtained via (7) by an appropriate β_{n+3}.

Next we define proper maps.

(8) Definition: Let S and S' be A_n^3-systems as in (5). A proper map $\varphi: S \longrightarrow S'$ is a tuple of homomorphisms

(a)
$$\varphi = \begin{cases} \varphi_i: H_i \longrightarrow H_i', & i=n,n+1,n+2,n+3, \\ \varphi_\pi: \pi_{n+1} \longrightarrow \pi'_{n+1}, & \\ \varphi_\Gamma: \Gamma_{n+2}(i) \longrightarrow \Gamma_{n+2}(i'), & \end{cases}$$

such that φ is compatible with all unbroken arrows in (5)(a) and such that

(b)
$$(\varphi_\Gamma,\varphi_n)_*(\beta_{n+3}) = \bar{\varphi}^*_{n+2}(\beta'_{n+3})$$

in $\Gamma^b_{n+1}(H_{n+2},v')$. Here $\bar{\varphi}^*_{n+2}$ is defined as in (6) and $(\varphi_\Gamma,\varphi_n)_*$ is induced on the push out (5)(b) by $\mathrm{Ext}(H_{n+2},\varphi_\Gamma) \oplus \bar{G}(H_{n+2}) \otimes \varphi_n$ where φ_Γ: cok $b_{n+2} \longrightarrow$ cok b'_{n+2} is welldefined by φ_Γ. //

A proper isomorphism is a proper map φ for which all φ_i and thus also φ_π and φ_Γ are isomorphisms. We are now ready to formulate the following theorem for $n \geq 4$.

(9) Theorem: *Let* $n \geq 4$.

(A) Each A_n^3-polyhedron X induces an A_n^3-system $S=S(X)$ such that the row in (5)(a) is the certain exact sequence of J.H.C. Whitehead,

β_{n+3} *is the 'boundary invariant of* X' , *see (I.3.3).*

(B) *Each map* $F\colon X \longrightarrow X'$ *between* A_n^3*-polyhedra induces a proper map* $F_*\colon S(X) \longrightarrow S(X')$.

(C) *Each* A_n^3*-system* S *is realizable, that is, for* S *there exists an* A_n^3*-polyhedron* X *such that* $S \cong SX$ *are properly isomorphic.*

(D) *Each proper map* $\varphi\colon S(X) \longrightarrow S(X')$ *is realizable by a continous map* $X \longrightarrow X'$.

Theorem (1) is an easy corollary of this result and of (14). Actually A_n^3 - systems and proper maps form a category and S is a functor, $S\colon A_n^3 \longrightarrow A_n^3$ - systems , (here A_n^3 denotes the full homotopy category of A_n^3-polyhedra).

Part III. The classification of maps between 1-connected 4-dimensional polyhedra

Whitehead [23] and also Steenrod, in his review of Whitehead's paper [21], point out the problem of determing the homotopy classes of maps between 1-connected 4-dimensional polyhedra. We here describe a solution of this problem. In particular, we compute the 4-th homotopy group $\pi_4(X)$ and the group of homotopy equivalences $E(X)$ of an arbitrary simply connected 4-dimensional polyhedron X. The computation of π_4 solves a problem of P.J. Hilton in [13]. As an example we compute the group $E(X)$ for a simply connected 4-dimensional manifold.

§ 1 <u>The realizability of homology homophisms for 1-connected 4-dimensional polyhedra</u>

We consider the following part of Whitehead's certain exact sequence in (I. 1.2)

$$(1.1) \qquad H_5 \xrightarrow[b_5]{} \Gamma_4 \xrightarrow[i]{} \pi_4 \xrightarrow[h]{} H_4 \xrightarrow[b_4]{} \Gamma_3 \xrightarrow[i]{} \pi_3 \xrightarrow[h]{} H_3 \longrightarrow 0 \, .$$

which is a functor on simply connected polyhedra. For the group Γ_3 we have the natural equation

$$(1.2) \qquad \overline{\eta^*} : \Gamma(H_2X) = \Gamma_3X = \Gamma_3$$

where Γ is the <u>universal quadratic functor</u>.

(1.3) <u>Definition</u> of Γ : A map $f\colon A \longrightarrow B$ between abelian groups is <u>quadratic</u> if $f(a)=f(-a)$ and if $f(a+b) - f(a) - f(b)$ is bilinear, $a,b \in A$. There is a universal quadratic map

$$\gamma : A \longrightarrow \Gamma A$$

such that for each quadratic map f there is a unique homomorphism $\bar{f}: \Gamma A \longrightarrow B$ between abelian groups with $\bar{f}\gamma = f$. This defines the functor Γ on abelian groups. We define the 'Whitehead product'

$$[\ ,\ ]: A \otimes A \longrightarrow A \quad ,$$

$$[a,b] = \gamma(a+b) - \gamma(a) - \gamma(b) \ . \quad //$$

It is easy to compute the abelian group ΓA by the following formulas

(1.4)
$$\Gamma(\mathbb{Z}/n) = \begin{cases} \mathbb{Z}/2n & , \ n \ \text{even} \ , \\ \mathbb{Z}/n & , \ n \ \text{odd} \ , \end{cases}$$

$$\Gamma(A \oplus B) = (\Gamma A) \oplus (\Gamma B) \oplus (A \otimes B) \ .$$

The isomorphism in (1.2) is induced by the Hopf map $\eta: S^3 \longrightarrow S^2$ which induces the quadratic map

(1.5)
$$\eta^*: H_2X = \pi_2X \longrightarrow \Gamma_3X \ .$$

This yields $\overline{\eta^*}$ by the universal property. We also will use the homomorphisms

(1.6)
$$\sigma: \ A \longrightarrow A \otimes \mathbb{Z}/2 \ ,$$
$$\tau: \ A \longrightarrow A \otimes A \ ,$$

which are induced by the quadratic maps $a \longmapsto a \otimes 1$ and $a \longmapsto a \otimes a$ respectively. Via (1.3) the map σ is the suspension homomorphism. For Whitehead products we get

(1.7)
$$\sigma[a,b] = 0 \ ,$$
$$\tau[a,b] = a \otimes b + b \otimes a \ .$$

By naturality of (1.1) and (1.2) the secondary boundary $b_4: H_4 \longrightarrow \Gamma H_2$ is a primary homology opearation. The <u>Pontrjagin square</u> can be deduced from b_4 and from $\{\pi_3\} \in \mathrm{Ext}(H_3, \mathrm{cok}\ b_4)$. Moreover, we have the following natural commutative diagram

(1.8)
$$\begin{tikzcd}
 & & H_2 \otimes H_2 \\
H_4 \arrow[urr, dashed, "\tilde{\Delta}"] \arrow[rr, "b_4"'] \arrow[drr, "Sq_2"'] & & \Gamma(H_2) \arrow[u, "\tau"'] \arrow[d, "\sigma"] \\
 & & H_2 \otimes \mathbb{Z}/2
\end{tikzcd}$$

Here $\tilde{\Delta}$ is the reduced diagonal and Sq_2 is the integral Steenrod square. If H_* is free abelian and finitely generated then b_4 is determined by the cup product in $H^* = H^*(X, \mathbb{Z})$.

Whitehead determined the groups π_3 which possibly appear as a third homotopy group by the following result.

(1.9) <u>Theorem</u> (J.H.C. Whitehead [23]): *Each exact sequence*

$$H_4 \longrightarrow \Gamma H_2 \longrightarrow \pi_3 \longrightarrow H_3 \longrightarrow 0 \quad ,$$

where H_4 *is free abelian, is realizable by a 1-connected 4-dimensional space* X. *For two such spaces* X *and* X' *a homology homomorphism* $\varphi : H_*X = H \longrightarrow H_*X' = H'$ *is realizable if and only if there is a commutative diagram*

$$\begin{tikzcd}
H_4 \arrow[r, "b_4"'] \arrow[d, "\varphi_4"] & \Gamma H_2 \arrow[r, "i_3"'] \arrow[d, "\Gamma\varphi_2"] & \pi_3 \arrow[r] \arrow[d, dashed] & H_3 \arrow[r] \arrow[d, "\varphi_3"] & 0 \\
H'_4 \arrow[r, "b'_4"'] & \Gamma H'_2 \arrow[r, "i'_3"'] & \pi'_3 \arrow[r] & H'_3 \arrow[r] & 0
\end{tikzcd}$$

This result implies the classification of 1-connected 4-dimensional polyhedra in (I. 2.1).

By (1.9) we know exactly the image $H_*[X,X']$ of the homology functor

(1.10) $$H_* : [X,X'] \longrightarrow \mathrm{Hom}(H_*X, H_*X') \quad .$$

In § 3 below we compute the fibres of this map in terms of the classifying data $(b_4, \{\pi_3\})$ for X and X' respectively.

§ 2 The homotopy groups π_4 and Γ_4 and the realizability of Whitehead's exact sequence in dimension 5

In this section we determine the groups π_4 which possibly appear as a fourth homotopy group of a simply connected space with prescribed homology. We first compute the group Γ_4 in (1.1); this group clearly has much more structure than the group Γ_3 in (1.2). We show that Γ_4 depends only on the homology H_*X , on the secondary boundary $b_4 \in \mathrm{Hom}(H_4, \Gamma H_2)$ and on the extension class $\{\pi_3\} \in \mathrm{Ext}(H_3, \mathrm{cok}\, b_4)$.

For the computation of Γ_4 we have to introduce two new functors ΓT and Γ_2^2 which carry abelian groups to abelian groups and which are derived from Whitehead's functor Γ in (1.3).

(2.1) Definition: Let A be an abelian group. Then the Γ-torsion $\Gamma T(A)$ is defined as follows. Choose a short exact sequence

$$0 \longrightarrow C \xrightarrow{d} D \longrightarrow A \longrightarrow 0$$

where C and D are free abelian. Then we get the sequence

$$C \otimes C \xrightarrow{\partial_1} \Gamma C \oplus C \otimes D \xrightarrow{\partial_2} \Gamma D \twoheadrightarrow \Gamma A \quad ,$$

$$\partial_1 = ([1,1], -1 \otimes d) \quad \text{with } 1 = \text{identity on } C \quad ,$$

$$\partial_2 = (\Gamma(d)\, ,\, [d,1]) \quad \text{with } 1 = \text{identity on } D \quad .$$

We have $\operatorname{cok}(\partial_2) = \Gamma A$ and $\partial_2\partial_1 = 0$. Now we set

$$\Gamma T(A) = \ker \partial_2 / \operatorname{im} \partial_1 \quad . \quad //$$

It is easy to check that ΓT is a welldefined functor in A . The abelian group $\Gamma T(A)$ can be easily computed by the following formulas which are similar to (1.4):

(2.2)
$$\Gamma T(A) = A * \mathbb{Z}/2 \quad , \text{ if } A \text{ is cyclic,}$$
$$\Gamma T(A \oplus B) = \Gamma T(A) \oplus \Gamma T(B) \oplus A * B$$

Here $A * B$ denotes the torsion product of abelian groups over $\mathbb{Z}$. Next we define via the natural structure $[\ ,\ ]$ and γ in (1.3) the functor Γ_2^2 .

(2.3) Definition: Let

$$\Gamma_2^2(A) = (\Gamma(A) \otimes \mathbb{Z}/2 \oplus \Gamma(A) \otimes A)/\sim$$

be the abelian group given by the relations

(i) $$0 \sim [x,y] \otimes z + [z,x] \otimes y + [y,z] \otimes x \ ,$$

(ii) $$0 \sim (\gamma x) \otimes x \ ,$$

(iii) $$0 \sim [x,y] \otimes 1 + (\gamma x) \otimes y + [y,x] \otimes x \ ,$$

for $x,y,z \in A$. //

Here (i) corresponds to the Jacobi identity for Whitehead products, (ii) is forced by the triviality of the Whitehead product $[\eta,i] = 0$ in $\pi_4(S^2)$; $i \in \pi_2 S^2$ denotes a generator. Moreover, (iii) is the Barcus - Barratt formula for $[i_1\eta, i_2]$ where i_1 and i_2 are inclusions of S^2 in $S^2 \vee S^2$; in (iii) the element 1 is the generator in $\mathbb{Z}_2$.

For the Moore space $M(A,2)$ of the abelian group A in dimension 2 we have:

(2.4) Theorem: *There is a natural short exact sequence*

$$\Gamma_2^2(A) \rightarrowtail_{i} \pi_4 M(A,2) \twoheadrightarrow \Gamma T(A) \quad .$$

We identify $\pi_3 M(A,2) = \Gamma A$ by (1.2) and $\pi_2 M(A,2) = A$. Then i on $A \otimes \mathbb{Z}/2$ is given by $(\Sigma\eta)^*$ and i on $\Gamma A \otimes A$ is the Whitehead product. This way we obtain the geometric interpretation of the relations as described above in (2.3).

As an abelian group, $\Gamma_2^2 A$ is easily computable by the formula

$$\Gamma_2^2 A = A \otimes \mathbb{Z}/2 \oplus \Lambda^2(A \otimes \mathbb{Z}/2) \oplus L(A,1)_3 \quad . \tag{2.5}$$

This equation is not natural in A . The term Λ^2 denotes the second exterior power of a $\mathbb{Z}/2$-vector space and $L(A,1)_3$ is the group of Lie elements of degree 3 in the tensoralgebra $T(A)$ where A is concentrated in degree 1. In fact, $L(A,1)_3$ splits of naturally in (2.5).

Now let $i_3 : \Gamma(H_2) \longrightarrow \pi_3$ be the map in (1.1). We obtain the group Γ_4 by the push out diagram of abelian groups

$$\begin{tikzcd} H_2 \otimes \mathbb{Z}/2 \oplus \Gamma H_2 \otimes H_2 \arrow[r, two heads, "p"] \arrow[d, "\bar{i}_3"] \arrow[dr, phantom, "\text{push}"] & \Gamma_2^2 H_2 \arrow[r, tail, "i"] \arrow[d] \arrow[dr, phantom, "\text{push}"] & \pi_4 M(H_2,2) \arrow[d, "\alpha_*"] \\ \pi_3 \otimes \mathbb{Z}/2 \oplus \pi_3 \otimes H_2 \arrow[r, two heads, "\bar{p}"] & \Gamma_2^2(i_3) \arrow[r, tail] & \Gamma_4 \end{tikzcd} \tag{2.6}$$

with $\bar{i}_3 = i_3 \otimes \mathbb{Z}/2 \oplus i_3 \otimes H_2$. Geometrically α_* is induced by a map $\alpha: M(H_2,2) \longrightarrow X$ which induces an isomorphism on H_2 . The map p is the quotient map, see (2.3). Diagram (2.6) completes our computation of Γ_4 for which we thus have the natural short exact sequence

(2.7) $$\Gamma_2^2(i_3) \rightarrowtail \Gamma_4 \xrightarrow{t} \Gamma T(H_2) \quad .$$

Via the secondary boundary b_5 in (1.1) we obtain a new <u>primary homology operation</u>

(2.8) $$tb_5 : H_5 \longrightarrow \Gamma T(H_2) \quad .$$

The following diagram commutes

(2.9)
$$\begin{tikzcd} H_5 \arrow[r, "tb_5"'] \arrow[d, "Sq_2"'] & \Gamma T(H_2) \arrow[d, "\sigma"] \\ H_3(X,\mathbb{Z}/2) \arrow[r, two heads, "\Delta"'] & H_2 * \mathbb{Z}/2 \end{tikzcd} \quad .$$

Here Δ is the surjection in the universal coefficient theorem and Sq_2 is the integral Steenrod square. The map σ is given on $\Gamma(C) \oplus C \otimes D$ in (2.1) by σ on $\Gamma(C)$ and by the trivial map on $C \otimes D$; we use $H_2 * \mathbb{Z}/2 \subset C * \mathbb{Z}/2$.

We now can describe all realizable sequences in (1.1) which start with b_5 . This question of realizability also was asked by J.H.C. Whitehead [23].

(2.10) <u>Theorem</u>: *Let* H *be a graded abelian group with* H_5 *free abelian and* $H_i = 0$ *for* $i > 5$. *Then we can choose arbitrary elements*

$$b_4 \in Hom(H_4, \Gamma H_2) \quad , \quad b_5 \in Hom(H_5, \Gamma_4) \quad ,$$
$$\{\pi_3\} \in Ext(H_3, cok\ b_4) \quad , \quad \{\pi_4\} \in Ext(ker\ b_4, cok\ b_5) \quad ,$$

where Γ_4 *is given by* $(b_4, \{\pi_3\})$ *as in (2.6). These choices yield exactly the sequences in (1.1) which are realizable by a* 1*-connected* 5*-dimensional polyhedron.*

This result corresponds to (1.9), however, the sequence (1.1) does not classify 1-connected 5 - dimensional polyhedra. Thus the direct analogue of Whitehead's classification in (1.9) is not true in dimension 5.

We still obtain an extension of (1.9) to the 5 - dimensional case by introducing the boundary invariant, β_5 , which replaces the element $\{\pi_4\}$ in (2.10); see (I. § 5).

<u>Remark</u>: P.J. Hilton in [13] computed the homotopy group π_{n+2} of A_n^2 - polyhedron for $n \geq 3$. Our computation of π_4 solves this problem for $n = 2$. //

§ 3 <u>The classification of maps between simply connected 4 - dimensional polyhedra</u>

With the notation in section § 2 we can state our result on the set of homotopy classes $[X,X']$ where X and X' are simply connected 4 - dimensional polyhedra. Let H and H' be the homology of X and X' respectively and assume the homotopy types of X and X' are determined by exact sequences as in (1.9). Thus we have by (1.10) a good characterization of the subset $H_*[X,X'] \subset \mathrm{Hom}(H,H')$. For the full computation of the set $[X,X']$ we have the following result:

(3.1) <u>Theorem</u>: *There is a canonical decomposition of the function* H_* *on* $[X,X']$ *as in the following diagram:*

$$
\begin{tikzcd}
G_\varphi = \dfrac{H^4(X,\Gamma_2^2 i_3')}{\mathrm{im}(\Delta\beta) + \mathrm{im}\ d(\varphi_2)} \arrow[r, tail, "+"] & {[X,X']} \arrow[d, two heads, "\lambda"] \arrow[ddddrr, "H_*"] & & \\
G_1 = \mathrm{Ext}(H_2,\pi_3') \arrow[r, tail, "+"] & M_1(X,X') \arrow[d, two heads, "\lambda_1"] & & \\
G_2 = H^4(X,\Gamma T(H_2')) \arrow[r, tail, "+"] & M_2(X,X') \arrow[d, two heads, "\lambda_2"] & & \\
G_3 = \mathrm{Ext}(H_3,\ker b_4') \arrow[r, tail, "+"] & M_3(X,X') \arrow[d, two heads, "\lambda_3"] & & \\
G_4 = \mathrm{Hom}(H_3,\mathrm{cok}\ b_4') \arrow[r, tail, "+"] & M_4(X,X') \arrow[d, two heads, "\lambda_4"] & & \\
 & \varphi \in H_*[X,X'] & \subset & \mathrm{Hom}(H,H')
\end{tikzcd}
$$

All functions λ are surjective maps. The groups G_i act transitively and effectively on all fibers of λ_i $(i=1,2,3,4)$. Moreover, the group G_φ acts transitively and effectively on all fibers $\lambda^{-1}(f)$ for which $f \in M_1(X,X')$ induces φ in $H_*[X,X']$.

For the definition of G_φ recall the definitions of $\Gamma_2^2(i_3')$ in (2.6) and recall that we have the short exact sequence

$$(3.2) \qquad \Gamma_2^2(i_3') \rightarrowtail \Gamma_4' \twoheadrightarrow \Gamma T(H_2') \quad ,$$

see (2.7). This sequence induces the <u>connecting homomorphism</u> β in the following commutative diagram:

$$(3.3) \qquad \begin{tikzcd} \mathrm{Hom}(H_3,\Gamma TH_2') \arrow[d, "\beta"] & H^3(X,\Gamma TH_2') \arrow[l, two heads, "\mu"'] \arrow[d, "\partial"] & \\ \mathrm{Ext}(H_3,\Gamma_2^2 i_3') \arrow[r, tail, "\Delta"] & H^4(X,\Gamma_2^2 i_3') \arrow[r, two heads, "\mu"] & \mathrm{Hom}(H_4,\Gamma_2^2 i_3') \\ & H^2(X,\pi_3') \arrow[u, "d(\varphi_2)"] \arrow[r, equal] & \mathrm{Hom}(H_2,\pi_3') \arrow[u, "\bar d(\varphi_2)"'] \end{tikzcd}$$

Here ∂ is the Bockstein homomorphism for (3.2) and μ and Δ are defined by the universal coefficient theorem. Thus we see that in the definition of G_φ in (3.1) we have $\mathrm{im}(\Delta\beta) = \mathrm{im}\,\partial$. Next we define $d(\varphi_2)$ by

$$(3.4) \qquad d(\varphi_2)(\alpha) = ((\Sigma\eta)^*)_* Sq^2(\alpha) + [\ ,\ ]_*(\alpha \cup \varphi_2)$$

for $\alpha \in H^2(X,\pi_3')$. Here $Sq^2(\alpha) \in H^4(X,\pi_3' \otimes \mathbb{Z}/2)$ is given by the Steenrod squaring operation and $(\Sigma\eta)^*: \pi_3' \otimes \mathbb{Z}/2 \longrightarrow \Gamma_2^2(i_3')$ in (3.4) is the restriction of the quotient map $\bar p$ in (2.6). Moreover, for $\varphi_2 \in H^2(X,H_2')$ we have $\alpha \cup \varphi_2 \in H^4(X,\pi_3' \otimes H_2')$ and $[\ ,\ ]: \pi_3' \otimes H_2' \longrightarrow \Gamma_2^2(i_3')$ in (3.4) is the restriction of the quotient map $\bar{p}$ in (2.6). This completes the definition of $d(\varphi_2)$ in (3.4). The composition $\bar{d}(\varphi_2) = \mu d(\varphi_2)$ in (3.3) satisfies the formula

(3.5) $$\bar{d}(\varphi_2)(\alpha) = (\Sigma\eta)^*(\alpha \otimes \mathbb{Z}/2)\sigma b_4 + [\ ,\ ](\alpha\otimes\varphi_2)\tilde{\Delta}\ ,$$

see (1.8), $\alpha \in \mathrm{Hom}(H_2,\pi_3')$. Here the compositions are

$$H_4 \xrightarrow{b_4} \Gamma H_2 \xrightarrow{\sigma} H_2\otimes\mathbb{Z}/2 \xrightarrow{\alpha\otimes\mathbb{Z}/2} \pi_3'\otimes\mathbb{Z}/2 \xrightarrow{(\Sigma\eta)^*} \Gamma_2^2 i_3'\ ,$$

$$H_4 \xrightarrow{\tilde{\Delta}} H_2\otimes H_2 \xrightarrow{\alpha\otimes\varphi_2} \pi_3'\otimes\pi_2' \xrightarrow{[\ ,\ ]} \Gamma_2^2 i_3'\ .$$

An important feature of the decomposition of H_* in (3.1) is the following fact:

(3.6) <u>Addendum</u>: For $i=1,2,3,4$ the composition of maps, $\circ$, induces a commutative diagram

$$\begin{tikzcd}
{[X',X'']} \times {[X,X']} \arrow[r, "\circ"] \arrow[d, two heads] & {[X,X'']} \arrow[d, two heads] \\
M_i(X',X'') \times M_i(X,X') \arrow[r, dashed, "\circ"] & M_i(X,X'')
\end{tikzcd}\ .$$

Therefore the sets $M_i(X,X')$ are morphism sets of a category M_i such that λ and λ_i are quotient functors. Moreover, G_i yields a bifunctor on M_i such that the following distributivity law is satisfied

$$(f+\alpha)\ (g+\beta) = f\ g + g^*\alpha + f_*\beta\ \ ,$$

$f \in M_i(X',X'')$, $\alpha \in G_i(X',X'')$, $g \in M_i(X,X')$ and $\beta \in G_i(X,X')$. The same formula holds for $f \in [X',X'']$, $\alpha \in G_\varphi(X',X'')$, $g \in [X,X']$ and $\beta \in G_\psi(X,X')$ where $H_*f = \varphi$, $H_*g = \psi$. This shows that all functors $\lambda,\lambda_1,\dots,\lambda_4$ of the decomposition are '<u>linear extensions of categories</u>', see [3].

§ 4 The group of homotopy equivalences of a simply connected 4 - dimensional polyhedron

Let X be a simply connected 4 - dimensional polyhedron and let $E(X)$ be the group of homotopy equivalences of X . Thus $E(X)$ consists of all $f \in [X,X]$ which induce an isomorphism in homology. Composition of such maps yields the group structure in $E(X)$.

(4.1) <u>Theorem</u>: *The homomorphism $H_*\colon E(X) \longrightarrow Aut(H_*X)$ has the following canonical decomposition where $\varphi = id$ is the identity of H_*X .*

$$
\begin{tikzcd}
G_\varphi = \dfrac{H^4(X,\Gamma_2^2 i_3)}{\operatorname{im}(\Delta\beta) + \operatorname{im} d(\varphi)} \arrow[r, tail, "1^+"] & E(X) \arrow[d, two heads, "\lambda"] \arrow[dddd, bend left=60, dashed, two heads, "H_*"] \\
G_1 = \operatorname{Ext}(H_2,\pi_3) \arrow[r, tail, "1^+"] & E_1(X) \arrow[d, two heads, "\lambda_1"] \\
G_2 = H^4(X,\Gamma T(H_2)) \arrow[r, tail, "1^+"] & E_2(X) \arrow[d, two heads, "\lambda_2"] \\
G_3 = \operatorname{Ext}(H_3,\ker b_4) \arrow[r, tail, "1^+"] & E_3(X) \arrow[d, two heads, "\lambda_3"] \\
G_4 = \operatorname{Hom}(H_3,\operatorname{cok} b_4) \arrow[r, tail, "1^+"] & E_4(X) \arrow[d, two heads, "\lambda_4"] \\
 & \operatorname{Aut}(H) \cap H_*[X,X]
\end{tikzcd}
$$

Here λ and $\lambda_1,\dots,\lambda_4$ are surjective homomorphisms of groups with an abelian kernel. The kernel of λ_i is $1^+(G_i)$ and the kernel of λ is $1^+(G_\varphi)$. Here each homomorphism 1^+ is injective and is defined by $1^+(\alpha) = 1+\alpha$ where we use the action in (3.1) and where 1 denotes the identity of X. This result is actually an easy consequence of (3.1) and of (3.6). By the distributivity law in (3.6) we see that 1^+ is a homomorphism of groups since we have

$$\begin{aligned}(1^+(\alpha))\circ(1^+(\beta)) &= (1+\alpha)\circ(1+\beta)\\ &= 1\circ 1 + 1^*\alpha + 1_*\beta\\ &= 1 + (\alpha+\beta)\\ &= 1^+(\alpha+\beta)\end{aligned}$$

Clearly, 1^+ is injective by (3.1) and also kernel $\lambda_i = \lambda_i^{-1}(1) = \mathrm{im}(1^+)$ by (3.1). We have an algebraic characterization of all group extensions in (4.1) up to the group E_1X. The extension problem for λ is not solved.

§ 5 Symmetric bilinear forms and spaces

Symmetric bilinear forms appear naturally in topology as the intersection forms of manifolds, see for example [16]. We here are interested in the homotopy theory of 1-connected 4-dimensional manifolds; in fact, the homotopy type of such a manifold is determined by its intersection form.

(5.1) Definition: Let V be a finitely generated free abelian group. A symmetric bilinear form $(V,\cup)$ is a map $\cup: V\otimes V \longrightarrow \mathbb{Z}$ which is bilinear and which is symmetric, $v \cup w = w \cup v$. A map $f: (V,\cup) \longrightarrow (W,\cup)$ is a pair $(f: V \longrightarrow W\ ,\ f_o: \mathbb{Z} \longrightarrow \mathbb{Z})$ of homomorphisms which satisfies $f_o(v \cup w) = (fv)\cup(fw)$. These maps form a category which we denote by $\mathcal{SBF}$. We say that f is orientation preserving if $f_o = 1$. An orientation preserving automorphism in $\mathcal{SBF}$ is called an isometry. Let $\mathrm{Aut}(V,\cup)$ be the group of equivalences in $\mathcal{SBF}$. //

(5.2) Remark: Let $V^* = \mathrm{Hom}(V,\mathbb{Z})$. The symmetric bilinear forms $\cup$ on V are elements in $V^* \otimes V^* = \mathrm{Hom}(V \otimes V,\mathbb{Z})$, in fact, these elements are exactly those in the image of

$$\tau: \Gamma(V^*) \rightarrowtail V^* \otimes V^* \quad ,$$

see (1.6). Here τ is injective since V^* is free abelian. Thus we can identify the symmetric bilinear forms $\cup$ on V with the element

$$b = b_V = \tau^{-1}\cup \in \Gamma(V^*)$$

which we call the boundary element associated to $\cup$. //

The following homotopy category of spaces is highly related to the category $\mathcal{SBF}$ of symmetric bilinear forms.

(5.3) Definition: Let *SBF-spaces* be the full category consisting of simply connected CW-spaces X with cohomology groups

$$H^4(X,\mathbb{Z}) = \mathbb{Z} \quad ,$$

$$H^2(X,\mathbb{Z}) = \text{free abelian and finitely generated,}$$

$$\tilde{H}^i(X,\mathbb{Z}) = 0 \quad \text{otherwise} \quad . \quad //$$

The cup product $\cup\colon H^2\times H^2 \longrightarrow H^4=\mathbb{Z}$ of an SBF-space is a symmetric bilinear form, the intersection form. Moreover, the secondary boundary in Whitehead's exact sequence (1.1):

$$b_4\colon H_4=\mathbb{Z} \longrightarrow \Gamma(H_2) \quad ,$$

with $H_n=\mathrm{Hom}(H^n,\mathbb{Z})$, is given by the boundary element

(5.4) $$b_4(1) = b = \tau^{-1}(\cup) \quad , \quad \text{see (5.2).}$$

Now cohomology yields the contravariant functor

(5.5) $$\mathit{SBF}\text{-spaces} \xrightarrow{H^*} \mathit{SBF}$$

which by Whitehead's theorem (1.9) has the following properties:

(5.6) Theorem: *Each symmetric bilinear form* $(V,\cup)$ *is realizable by an SBF-space* X , *that is* $(H^*X,\cup)\cong(V,\cup)$. *Moreover, for SBF-spaces* X,Y *each map* $\varphi\colon (H^*Y,\cup) \longrightarrow (H^*X,\cup)$ *in* SBF *is realizable by a map* $F\colon X\longrightarrow Y$ *with* $H^*F=\varphi$.

We derive from this result that the equivalence classes of objects in *SBF* are 1-1 corresponded to the homotopy types of SBF-spaces. We write

(5.7) $$X = M(V,\cup)$$

if X is an SBF - space which realizes the symmetric bilinear form $(V,\cup)$. By (5.6) the homotopy type of $M(V,\cup)$ is welldefined.

(5.8) Remark: It is easy to see that each simply connected 4 - dimensional closed topological manifold has the weak homotopy type of an SBF - space. By Freedman's result Cor. 1.6 in[11] we do not know whether such a manifold is triangulable. On the other hand differentiable manifolds are well known to be triangulable. Since our results in this section are available for all SBF - spaces we do not restrict to manifolds.//

We want to compute the set of all maps between SBF - spaces which induce the same cohomology homomorphism φ. For this we need the abelian group $G(\varphi)$ which we define below in terms of the following natural structure of the Γ - functor, see § 1:

$$\begin{tikzcd} A \arrow[dr, "\gamma"] & & A \otimes A \\ & \Gamma A \arrow[ur, "\tau"] \arrow[dr, "\sigma"'] & \\ A \otimes A \arrow[ur, "{[\,,\,]}"'] & & A \otimes \mathbb{Z}/2 \end{tikzcd} .$$

(5.9) Definition: Let $\varphi\colon (V,\cup) \longrightarrow (W,\cup)$ be a map in $\mathcal{SBF}$ and let

$$\varphi^*\colon W^* \longrightarrow V^*$$

be the dual of φ. Then we set

$$G(\varphi) = (\Gamma(V^*) \otimes \mathbb{Z}/2 \oplus \Gamma(V^*) \otimes V^*) / U$$

where the subgroup U is generated by the following elements (where $1 \in \mathbb{Z}/2$ is a generator):

(i) $$[x,y]\otimes z + [z,x]\otimes y + [y,z]\otimes x \quad ,$$

(ii) $$\gamma x \otimes x \quad ,$$

(iii) $$[x,y]\otimes 1 + (\gamma x)\otimes y + [y,x]\otimes x \quad ,$$

(iv) $$b_V \otimes 1 \quad ,$$

(v) $$b_V \otimes y \quad ,$$

(vi) $$(\alpha \otimes 1_{\mathbb{Z}/2})\sigma b_W + (\alpha \otimes \varphi^*)\tau b_W \quad ,$$

where $x,y,z \in V^*$ and $\alpha \in \mathrm{Hom}(W^*,\Gamma(V^*))$. The elements $b_V \in \Gamma(V^*)$ and $b_W \in \Gamma(W^*)$ are the boundary elements in (5.2).//

For each symmetric bilinear form (V,U) we define the abelian group

(5.9)' $$G(V,U) = G(1) \quad \text{where } 1 = \text{identity of } (V,U) \quad .$$

This is an $\mathrm{Aut}(V,U)$ -module induced by the functor $\Gamma(V^*)\otimes\mathbb{Z}/2 \oplus \Gamma(V^*)\otimes V^*$ in V . From (3.1) and (4.1) we easily derive the following results:

(5.10) <u>Theorem</u>: *Let $Y = M(V,U)$ and $X = M(W,U)$ be SBF-spaces. Then the cohomology yields the surjective function*

$$H^*: [X,Y] \longrightarrow\!\!\!\!\!\rightarrow Hom((V,U),(W,U)) \quad .$$

For $\varphi: (V,U) \longrightarrow (W,U)$ the group $G(\varphi)$ acts transitively and effectively on the fiber $(H^*)^{-1}(\varphi)$.

In addition, H^* in (5.5) is a linear extension of categories, see (3.6).

(5.11) <u>Theorem</u>: *For the group of homotopy equivalences $E(Y)$ of the SBF-space $Y = M(V,U)$ we have the short exact sequence*

$$G(V,U) \rightarrowtail E(Y) \longrightarrow\!\!\!\!\!\rightarrow Aut(V,U)$$

the associated module of which is the one in (5.9).

Thus $E(X)$ represents the canonical element

(5.12) $$\{E(X)\} \in H^2(\mathrm{Aut}(V,U), G(V,U)) \quad ,$$

which, however, is not known.

The intersection form of a 1-connected 4-dimensional closed manifold is always unimodular. We therefore consider the following example of (5.12):

(5.13) Corollary: *Let V be a free $\mathbb{Z}$-module of dimension n and let $U: V \times V \longrightarrow \mathbb{Z}$ be an unimodular symmetric bilinear form which is realized by $Y = M(V,U)$. Then we get the short exact sequence of groups*

$$(\mathbb{Z}/2)^{n+\delta} \rightarrowtail E(Y) \twoheadrightarrow \mathrm{Aut}(V,U)$$

where $\delta = -1$ if the form U is odd and where $\delta = 0$ if the form U is even.

Proof: For an unimodular form (V,U) one shows

$$G(V,U) = (\mathbb{Z}/2)^{n+\delta}$$

by the definition in (5.9). Now the corollary is a consequence of (5.11). $\square$

(5.14) Remark: By the result of Freedman [11] we know that each unimodular symmetric bilinear form is realizable by a 1-connected 4-dimensional topological manifold. Thus the corollary is available for the weak homotopy types of all such manifolds, see (5.8). //

Part IV. The computation of maps between Chang's elementary A_n^2 - polyhedra, $n \geq 3$.

It is a well known fact that each A_n^2 - polyhedron (of finite type) is homotopy equivalent to the one point union of appropriate elementary A_n^2 - polyhedra, see [6], [14], [15]. Let $\eta : S^{n+1} \longrightarrow S^n$ be the Hopf element, let p be an odd prime and let r,t be natural numbers ≥ 1 . Then all elementary A_n^2 - polyhedra are given by the following 11 types:

$$X_1 := S^n$$

$$X_2 := S^{n+1}$$

$$X_3 := S^{n+2}$$

$$X_4(r) := S^n \cup_{2^r} e^{n+1}$$

$$X_5(p,r) := S^n \cup_{p^r} e^{n+1}$$

$$X_6 := S^n \cup_\eta e^{n+2}$$

$$X_7(t) := (S^n \vee S^{n+1}) \cup_{\eta+2^t} e^{n+2}$$

$$X_8(t) := S^{n+1} \cup_{2^t} e^{n+2}$$

$$X_9(p,r) := S^{n+1} \cup_{p^r} e^{n+2}$$

$$X_{10}(r) := S^n \cup_{(2^r,\eta)} (e^{n+1} \vee e^{n+2})$$

$$X_{11}(r,t) := (S^n \vee S^{n+1}) \cup_{(2^r,\eta+2^t)} (e^{n+1} \vee e^{n+2})$$

Using the formalism in [3] and [4] my student T. Schmidt [18] worked out the table below which describes the abelian groups $[X_i,X_j]$ where X_i and X_j are elementary A_n^2 - polyhedra. The homotopy groups $\pi_{n+2}(X_j)$, $\pi_{n+1}(X_j)$ were also obtained by Hilton [14]. Moreover, the groups $[X_i,X_j]$ were computed by Brown-Copeland [5] in case X_i and X_j are Moore spaces, that is $i,j \in \{4,5,9,10\}$. The more difficult parts of the table seem to be new results. We point out that we have Spanier Whitehead duality

$$[X_i,X_j] \cong [DX_j,DX_i]$$

where DD = identity and where

$DX_1 = X_3$

$DX_2 = X_2$

$DX_4(r) = X_8(r)$

$DX_5(p,r) = X_9(p,r)$

$DX_6 = X_6$

$DX_7(t) = X_{10}(t)$

$DX_{11}(r,t) = X_{11}(t,r)$

In the following table we use the notation:

$g := (p,p')$	$j := \max(t,r')$	$k := \min(t,r')$
$l := \min(r,r')$	$l' := \min(r+1,r')$	
$m := \min(r,t')$	$m' := \min(r+1,t')$	$m'' := \min(r,t'+1)$
$n := \min(t,t')$		$n'' := \min(t,t'+1)$

$Y \backslash X$	X_1	X_2	X_3	$X_4(r)$
X_1	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
X_2	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2^r$
X_3	0	0	$\mathbb{Z}$	0
$X_4(r')$	$\mathbb{Z}/2^{r'}$	$\mathbb{Z}/2$	$r'=1$: $\mathbb{Z}/4$ $r'>1$: $\mathbb{Z}/2\oplus\mathbb{Z}/2$	$r=r'=1$: $\mathbb{Z}/4$ sonst: $\mathbb{Z}/2^l\oplus\mathbb{Z}/2$
$X_5(p',r')$	$\mathbb{Z}/p'^{r'}$	0	0	0
X_6	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$X_7(t')$	$\mathbb{Z}$	$\mathbb{Z}/2^{t'+1}$	$\mathbb{Z}/2$	$\mathbb{Z}/2^{m''}$
$X_8(t')$	0	$\mathbb{Z}/2^{t'}$	$\mathbb{Z}/2$	$\mathbb{Z}/2^m$
$X_9(p',t')$	0	$\mathbb{Z}/p'^{t'}$	0	0
$X_{10}(r')$	$\mathbb{Z}/2^{r'}$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2^l$
$X_{11}(r',t')$	$\mathbb{Z}/2^{r'}$	$\mathbb{Z}/2^{t'+1}$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2^l\oplus\mathbb{Z}/2^{m''}$

X / Y	$X_5(p,r)$	X_6	$X_7(t)$	$X_8(t)$
1	0	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$t=1$: $\mathbb{Z}/4$ $t>1$: $\mathbb{Z}/2\oplus\mathbb{Z}/2$
2	$\mathbb{Z}/p^r$	0	0	$\mathbb{Z}/2$
3	0	$\mathbb{Z}$	$\mathbb{Z}/2^t$	$\mathbb{Z}/2^t$
4	0	$\mathbb{Z}/2^{r'}$	$\mathbb{Z}/2^{r'}\oplus\mathbb{Z}/2$	$t=1\wedge r'=1$: $\mathbb{Z}/2\oplus\mathbb{Z}/2$ $t>1\wedge r'=1$: $\mathbb{Z}/4\oplus\mathbb{Z}/2$ $t=1\wedge r'>1$: $\mathbb{Z}/2\oplus\mathbb{Z}/4$ $t>1\wedge r'>1$: $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$
5	$\mathbb{Z}/g^l$	$\mathbb{Z}/p^{,r'}$	$\mathbb{Z}/p^{,r'}$	0
6	0	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2^t$	$\mathbb{Z}/2^t$
7	0	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2^{n''}$	$\mathbb{Z}/2^{n''}\oplus\mathbb{Z}/2$
8	0	0	$\mathbb{Z}/2^n$	$t=t'=1$: $\mathbb{Z}/4$ sonst: $\mathbb{Z}/2^n\oplus\mathbb{Z}/2$
9	$\mathbb{Z}/g^m$	0	0	0
10	0	$\mathbb{Z}\oplus\mathbb{Z}/2^{r'}$	$\mathbb{Z}/2^{j+1}\oplus\mathbb{Z}/2^k$	$\mathbb{Z}/2^t\oplus\mathbb{Z}/2$
11	0	$\mathbb{Z}/2^{r'}$	$t'\geq t$: $\mathbb{Z}/2^{j+1}\oplus\mathbb{Z}/2^k$ $t'<t$: $\mathbb{Z}/2^{t'+1}\oplus\mathbb{Z}/2^{r'}$	$\mathbb{Z}/2^{n''}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$

X / Y	$X_9(p,t)$	$X_{10}(r)$	$X_{11}(r,t)$
1	0	$\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
2	0	$\mathbb{Z}/2^{r+1}$	$\mathbb{Z}/2^{r+1}$
3	$\mathbb{Z}/p^t$	$\mathbb{Z}$	$\mathbb{Z}/2^t$
4	0	$\mathbb{Z}/2^{l'}\oplus\mathbb{Z}/2$	$\mathbb{Z}/2^{l'}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$
5	0	0	0
6	$\mathbb{Z}/p^t$	$\mathbb{Z}$	$\mathbb{Z}/2^t$
7	0	$\mathbb{Z}/2^{m+1}$	$\mathbb{Z}/2^{n''}\oplus\mathbb{Z}/2^{m+1}$
8	0	$\mathbb{Z}/2^{m'}$	$\mathbb{Z}/2^{n}\oplus\mathbb{Z}/2^{m'}$
9	$\mathbb{Z}/g^n$	0	0
10	$\mathbb{Z}/p^t$	$\mathbb{Z}\oplus\mathbb{Z}/2^{l'}$	$r'\leqq r$: $\mathbb{Z}/2^{j+1}\oplus\mathbb{Z}/2^k$ $r'>r$: $\mathbb{Z}/2^{r+1}\oplus\mathbb{Z}/2^t$
11	0	$\mathbb{Z}/2^{l'}\oplus\mathbb{Z}/2^{m+1}$	$r'\leqq r\wedge t'\geqq t$: $\mathbb{Z}/2^{j+1}\oplus\mathbb{Z}/2^k\oplus\mathbb{Z}/2^{m+1}$ $r'>r\vee t'<t$: $\mathbb{Z}/2^{n''}\oplus\mathbb{Z}/2^{l'}\oplus\mathbb{Z}/2^{m+1}$

We have also results on the law of composition

$$[X_i, X_j] \times [X_j, X_k] \longrightarrow [X_i, X_k] \quad .$$

This in particular yields the following list of groups of homotopy equivalences:

X	Aut X		
X_1	Aut $\mathbb{Z}$	$= \mathbb{Z}/2$	
X_2	Aut $\mathbb{Z}$	$= \mathbb{Z}/2$	
X_3	Aut $\mathbb{Z}$	$= \mathbb{Z}/2$	
$X_4(r)$	Aut $\mathbb{Z}/2^r \oplus \mathbb{Z}/2$	$= \mathbb{Z}/2$	für $r=1$
		$\mathbb{Z}/2^{r-2} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	für $r \geqq 2$
$X_5(p,r)$	Aut $\mathbb{Z}/p^r$	$= \mathbb{Z}/(p-1) \oplus \mathbb{Z}/p^{r-1}$	
X_6	Aut $\mathbb{Z} \oplus$ Aut $\mathbb{Z}$	$= \mathbb{Z}/2 \oplus \mathbb{Z}/2$	
$X_7(t)$	Aut $\mathbb{Z}/2^{t+1} \oplus$ Aut $\mathbb{Z}$	$= \mathbb{Z}/2^{t-1} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	
$X_8(t)$	Aut $\mathbb{Z}/2^t \oplus \mathbb{Z}/2$	$= \mathbb{Z}/2$	für $t=1$
		$\mathbb{Z}/2^{t-2} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$	für $t \geqq 2$
$X_9(p,t)$	Aut $\mathbb{Z}/p^t$	$= \mathbb{Z}/(p-1) \oplus \mathbb{Z}/p^{t-1}$	
$X_{10}(r)$	Aut $\mathbb{Z} \oplus$ Aut $\mathbb{Z}/2^{r+1}$	$= \mathbb{Z}/2 \oplus \mathbb{Z}/2^{r-1} \oplus \mathbb{Z}/2$	

For $X = X_{11}(r,t)$ we have the short exact sequence

$$\mathbb{Z}/2^{\min(r,t)+1} \rightarrowtail \mathrm{Aut}\ X \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}/2^{t+1}) \oplus \mathrm{Aut}(\mathbb{Z}/2^{r+1})$$

which is split if $r \neq t$. The associated action is given by

$$\begin{aligned} \mathrm{Aut}(\mathbb{Z}/2^{t+1}) \oplus \mathrm{Aut}(\mathbb{Z}/2^{r+1}) &\longrightarrow \mathrm{Aut}(\mathbb{Z}/2^{\min(r,t)+1}) \\ (F_{n+2}, F_n) &\longmapsto p_* F_{n+2} q_* F_n^{-1} \end{aligned}$$

where $p : \mathbb{Z}/2^{t+1} \longrightarrow \mathbb{Z}/2^{\min(r,t)+1}$ and $q : \mathbb{Z}/2^{r+1} \longrightarrow \mathbb{Z}/2^{\min(r,t)+1}$ are the canonical projections. For $r = t$ we have the commutative diagram with exact rows and with exact columns:

$$\begin{tikzcd}
\mathbb{Z}/2 \arrow[r, tail] \arrow[d, equal] & \mathbb{Z}/2^{r+1} \arrow[rr, two heads] \arrow[d, tail] & & \mathbb{Z}/2^{r} \arrow[d, tail] \\
\mathbb{Z}/2 \arrow[r, tail] & \mathrm{Aut}\ X \arrow[rr, two heads] \arrow[dr, two heads] & & A \arrow[dl, two heads] \\
& & \mathrm{Aut}(\mathbb{Z}/2^{t+1}) \oplus \mathrm{Aut}(\mathbb{Z}/2^{r+1}) &
\end{tikzcd}$$

Here the extension A is split.

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