ON THE MODULE OF ZARISKI DIFFERENTIALS AND INFINITESIMAL DEFORMATIONS OF CUSP SINGULARITIES

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0. Introduction

In [1] we started the computation of the module of infinitesimal deformations T^1_X of a twodimensional cusp singularity X. We were able to construct a certain number of infinitesimal deformations, giving a lower bound for the dimension of T^1_X. The aim of this article is to prove that this bound does already give the correct dimension of T^1_X, as conjectured.

Theorem 3.4.: Let X be a twodimensional cusp singularity of embedding dimension $d \geq 5$, defined by a lattice M in a real quadratic number field K and a group V of totally positive units of K acting on M. Let $(a_0,\ldots,a_{d-1})$ be the cycle of selfintersection numbers associated with the dual lattice M^*. Then the dimension of T^1_X is equal to $\sum_{i=0}^{d-1} (a_i-1)$.

The idea of proof is to use local duality, which for the Gorenstein surface singularity X gives a perfect pairing of T^1_X and the first local cohomology $H^1_{\{\infty\}}(\Omega^1_X)$ of the Kähler differentials Ω^1_X. If (R,m) is the local ring of X at ∞, and if $\hat{R}$ is the m-adic completion of R, we investigate the natural map $\phi:\Omega^1_R \longrightarrow D_{\hat{R}}$ from the module of Kähler differentials to its double dual, the module of Zariski differentials of $\hat{R}$, which has $H^1_{\{\infty\}}(\Omega^1_R)$ as its cokernel. We give a criterion, in terms of Fourier series, for an element of $D_{\hat{R}}$ to be in the image of ϕ. In more geometric terms this is a criterion for a differential form ω, defined on a punctured neighbourhood of the singular point to have an extension across the singular point. This applies immediately to give the desired upper bound for the length of $H^1_{\{\infty\}}(\Omega^1_R)$.

As a by-product we get a precise description of the module of Zariski differnetials $D_{\hat{R}}$. For example one can easily see that it has a minimal set of generators of length 2d.

In the first section we recall the results from [1]. Then we study the module of Zariski differentials of the complete local ring $\hat{R}$ of the cusp singularity (X,∞), and in the third part we give the proofs of our results.

After finishing this manuscript we obtained the preprint 6 from I. Nakamura, which also contains a proof of our conjecture form [1] by different methods.

1. Cusp Singularities

In this section we recall briefly the results of [1] on infinitesimal deformations of cusp singularities. Let K be a real quadratic number field. For an element $\alpha \in K$ denote by α' the conjugate, by $N(\alpha) = \alpha\alpha'$ the norm, and by $Tr(\alpha) = \alpha+\alpha'$ the trace of α. The element α is called totally positive if both α and α' are greater than zero. Let $M \subset K$ be a complete lattice, let U_M^+ be the infinite cyclic group of totally positive units of K which act on M, and let $V \subset U_M^+$ be a subgroup of finite index.

The group $G = G(M,V)$ of all 2x2 matrices $\gamma = \begin{bmatrix} \varepsilon & \mu \\ 0 & 1 \end{bmatrix}$ with $\varepsilon \in V$ and $\mu \in M$ acts freely and properly discontinuously on the product HxH of upper half planes: $(z_1,z_2) \longmapsto \gamma(z_1,z_2) = (\varepsilon z_1+\mu, \varepsilon' z_2+\mu')$, and an ideal point ∞ can be added to the orbit space X' to obtain a normal complex space X. The singularity of X at ∞ is called a <u>twodimensional cusp</u>.

Let $M^* \subset K$ be the dual lattice of M, and assume that M has been chosen in its strict equivalence class so that $M^* = \mathbb{Z}.1 \oplus \mathbb{Z} w_*$ with $0 < w_*' < 1 < w_*$. Then w_* has a purely periodic expansion as a continued fraction $w_* = [[a_0, a_1, a_2, \ldots]] := a_0 - 1 \rceil a_1 - 1 \rceil \ldots$, with $a_i \geq 2$, and at least one $a_i \geq 3$. Let $(a_0, \ldots, a_{s-1})$ be a primitive period. If $(-b_0, \ldots, -b_{r-1})$ is the cycle of selfintersection numbers on the minimal resolution, in cyclic order, then after cyclic permutation for $d = s. [U_M^+ : V]$ the cycles $(a_0, \ldots, a_{d-1})$ and $(b_0, \ldots, b_{r-1})$ for $r \geq 2$, and (b_0+2) for $r = 1$ are dual.

Consider the sector M_+^* of totally positive elements of M^*. If M^* is mapped to $\mathbb{R}^2$ by sending $y-xw_*$ to $(x,y) \in \mathbb{R}^2$, the elements of M_+^* correspond to the integral lattice points of $\mathbb{R}^2$ with $y > xw_*$ and $y > xw_*'$. Denote by $A_k, k \in \mathbb{Z}$, the lattice points on the boundary of the convex hull of M_+^* in $\mathbb{R}^2$, numbered in consecutive order, such that $A_{-1} = w_*$, $A_o = 1$. The A_k are called <u>support points</u> of M_+^*. For proofs of the following properties of the support points see [3], sections 2.2 and 2.3:

1. For all k we have relations $A_{k-1} + A_{k+1} = a_k A_k$, where the a_k are the denominators from the continued fraction expansion of w_*, and all the relations among the A_k are generated by these.

2. Each element $\mu \in M_+^*$ can be written in a unique way as $\mu = nA_k + mA_{K+1}$ for integers k,n,m with $n > 0$, $m \geq 0$.

3. If the A_k are considered as elements of $K \subset \mathbb{R}$, we have chains of estimates $\cdots > A_{K-1} > A_k > A_{k+1} > \cdots$ and $\cdots < A_{k-1}' < A_k < A_{k+1}' < \ldots$.

4. A_d is the unique generator ε_1 of V with $0 < \varepsilon_1 < 1$.

5. A_d acts on $\{A_k\}$ by $A_d \cdot A_k = A_{k+d}$.

In [1], section 3, we constructed a fundamental domain F_+ for the action of V on M_+^*. The points of $F_+ = \{\mu \in M_+^* \,|\, \varepsilon_1 < |\mu/\mu'| \leq \varepsilon_1'\}$ correspond to the integral lattice points (x,y) which are elements of the convex cone defined by

$$y \geq ((\varepsilon_1' w_*' - w_*)/(\varepsilon_1' - 1))\ x\ ,\ y \geq ((w_* - \varepsilon_1 w_*')/(1-\varepsilon_1))\ x\ .$$

For all $\mu \in F_+$ and for all $\varepsilon \in V$ there we have $T_r(\varepsilon\mu) \geq T_r(\mu)$. Equality holds if and only if μ is on the boundary of F_+ and ε is the generator of V which maps μ to the other boundary component.

For $d = 2k+1$ the support points contained in F_+ are A_{-k}, A_k, and for $d = 2k+2$, $A_{-k-1},\dots,A_{k+1}$ are the support points contained in F_+. Observe that in the latter case A_{-k-1} and A_{k+1} are equivalent and are on the boundary of F_+.
The vectorspace of infinitesimal deformations of the cusp singularity X can be identified in a canonical way with a subspace of the cohomology group $H^1(X', \Theta_{X'})$, where $\Theta_{X'}$ is the tangent sheaf of X'.

Let Y be the Stein space H^2/M. Then V acts on Y, and we showed in Proposition 2.2. of [1], that $H^1(X', \Theta_{X'}) = \operatorname{coker}((\varepsilon_1 - \mathrm{id}): H^0(Y,\Theta_Y) \longrightarrow H^0(Y,\Theta_Y))$. A derivation δ $H^0(Y,\Theta_Y)$ is given by a Fourier series

$$\delta = \sum_{\mu\in M^*} a_\mu^{(1)} e(\mu z)\partial/\partial z_1 + \sum_{\mu\in M^*} a_\mu^{(2)} e(\mu z)\partial/\partial z_2 ,$$

where $e(\mu z)$ is used as an abbreviation for $\exp(2\pi i(\mu z_1+\mu' z_2))$.

We have from [1]:

<u>Theorem 1.1.: The restriction of the canonical projection $H^0(Y,\Theta_Y) \longrightarrow H^1(X',\Theta_{X'})$ to the subspace of derivations with $a_\mu^{(1)} = a_\mu^{(2)} = 0$ for μ not in $-F_+$ is an isomorphism.</u>

Moreover we know already a certain number of elements of T_X^1, namely:

<u>Proposition 1.2.: Assume $d \geq 3$, and for every point A_i contained in F_+ consider the derivations $\delta_{i,a} = e(-aA_i z)(A_i'\partial/\partial z_1 - A_i\partial/\partial z_2)$ for $1 \leq a \leq a_i - 1$. Their images in $H^1(X',\Theta_{X'})$ are linearly independent elements of T_X^1. In particular the dimension of T_X^1 is at least $\sum_{i=0}^{d-1} (a_i - 1)$.</u>

2. The module of Zariski differentials of a twodimensional cusp singularity.

Let R be the local ring of (X,∞), and let $\underline{m}$ be the maximal ideal of R. As usual denote by Ω^1_R the module of Kähler differentials, and let D_R be the double dual of Ω^1_R, the module of Zariski differentials. If $j: X' \hookrightarrow X$ denotes the embedding of the regular locus of X then D_R is the stalk of the direct image sheaf $j_*\Omega^1_{X'}$ at the singular point. Choosing any Stein neighbourhood of the singular point one has the exact sequence of local cohomology

$$0 \longrightarrow H^0_{\{\infty\}}(\Omega^1_X) \longrightarrow H^0(X, \Omega^1_X) \longrightarrow H^0(X', \Omega^1_{X'}) \longrightarrow H^1_{\{\infty\}}(\Omega^1_X) \longrightarrow 0.$$

Since kernel and cokernel are concentrated in the singular point, we can pass to the direct limit and obtain

$$0 \longrightarrow H^0_{\{\infty\}}(\Omega^1_X) \longrightarrow \Omega^1_R \longrightarrow D_R \longrightarrow H^1_{\{\infty\}}(\Omega^1_X) \longrightarrow 0 .$$

Kernel and cokernel are R-modules of finite length, so $\underline{m}$-adic completion will not change them. We want to give an intrinsic description of the $\underline{m}$-adic completions of R and D_R.

As in the first section let $e(\mu z) = \exp(2\pi i(\mu z_1+\mu' z_2))$. The local ring R consists of all V-invariant Fourier series $\sum_{\mu\in M^*} a_\mu e(\mu z)$, which converge for $Im(z_1)Im(z_2) >> 0$. The invariance is expressed by $a_\mu = a_{\mu\varepsilon}$ for all $\varepsilon\in V$, and the series converges for $Im(z_1)Im(z_2) \geq c$ if and only if $|a_\mu| \leq$ const. $\exp(2\pi(\alpha\mu+\alpha'\mu'))$ holds for all pairs of real numbers (α,α') with $\alpha\alpha' \geq c$. As an easy consequence all Fourier coefficients a_μ with $\mu\notin M^*_+ \cup \{0\}$ vanish.

An element of the module of Zariski differentials is represented by a V-invariant differential form $\underline{\omega} = f_1(z)dz_1 + f_2(z)dz_2$ on $U_c := \{Im(z_1)Im(z_2) \geq c\} \subset H \times H$, where again $f_i(z) = \sum_{\mu \in M^*} a_\mu^{(i)} e(\mu z)$, $i = 1,2$. The invariance of $\underline{\omega}$ under V is expressed by $a_{\mu\varepsilon}^{(1)} = \varepsilon a_\mu^{(1)}$ and $a_{\mu\varepsilon}^{(2)} = \varepsilon^{-1} a_\mu^{(2)}$ for all elements $\mu \in M^*$ and $\varepsilon \in V$. Together with the convergence of f_1 and f_2 this implies that $a_\mu^{(1)} = a_\mu^{(2)} = 0$ for all μ not in M_+^*.

In their article [2] Freitag and Kiehl defined a very natural filtration of the local ring R. Consider the ring $\hat{R}$ of all formal V- invariant Fourier series $\sum_{\mu \in M^* \cup \{0\}} a_\mu e(\mu z)$. Let r_0 be the positive generator of the infinite cyclic subgroup $Tr(M^*) \subset \mathbb{Q}$. For all natural numbers r define the ideals $\hat{\mathfrak{m}}_r := \{f \in \hat{R} \mid a_\mu = 0 \text{ for } Tr(\mu) < rr_0\}$, and let m_r be the intersection of $\hat{\mathfrak{m}}_r$ and R.

Then $m_1 \supset m_2 \supset \ldots$, $m_r m_s \subset m_{r+s}$ and $\bigcap_{r \geq 1} m_r = (0)$. Since m_1 is the maximal ideal of R, one has $m^r \subset m_r$, so the m_r - filtration is courser than the m - adic one. Freitag and Kiehl proved nevertheless that the formal ring $\hat{R}$ is the m - adic completion of R.

Now let $D_{\hat{R}}$ be the $\hat{R}$ - module of formal V invariant differential forms $\omega = f_1(z)dz_1 + f_2(z)dz_2$ with $f_j(z) = \sum_{\mu \in M_+^*} a_\mu^{(j)} e(\mu z)$ $j = 1,2$. Again there is an obvious filtration $D_{\hat{R}} = \hat{D}_1 \supset \hat{D}_2 \supset \ldots$ with $\hat{D}_r := \{\omega \in D_{\hat{R}} \mid a_\mu^{(1)} = a_\mu^{(2)} = 0 \text{ for } Tr(\mu) < rr_0\}$. Let $D_r = \hat{D}_r \cap D_R$ Then $m_r D_s \subset D_{r+s}$, and $\{D_r\}$ is an $\{m_r\}$ filtration of D_R. The following is an immediate consequence of the result of Freitag and Kiehl.

Proposition 2.1.: $D_{\hat{R}}$ is the m - adic completion of D_R. ∎

From now on we will work with the complete modules. Observe that the m - adic completion of Ω^1_R is just $\Omega^1_{\hat{R}}$.

3. Proof of the main result

In this section we want to show that for cusp singularities of degree d ≥5 the dimension of T^1_X is $\sum_{i=0}^{d-1}(a_i-1)$. Since cusp singularities are Gorenstein (see [4], or observe that $dz_1 \wedge dz_2$ descends to a holomorphic section of the canonical bundle over X' without zeros), one has the nice local duality pairing $Ext^1_{\hat{R}}(\Omega^1_{\hat{R}},\hat{R}) \times H^1_{\{\infty\}}(\Omega^1_{\hat{R}}) \longrightarrow \mathbb{C}$. In particular the module $H^1_{\{\infty\}}(\Omega^1_{\hat{R}})$ has the same length as T^1_X. From the preceeding section we can see that an upper estimate for the length of $H^1_{\{\infty\}}(\Omega^1_{\hat{R}})$ can be obtained by looking at the map $\phi : \Omega^1_{\hat{R}} \longrightarrow D_{\hat{R}}$. The image of $\Omega^1_{\hat{R}}$ is generated by the elements fdg, where f and g are formal series from $\hat{R}$.

Theorem 3.1.: Let $d \geq 5$ and let $\omega = \sum_{\mu \in M^*_+} a^{(1)}_\mu e(\mu z) dz_1 + \sum_{\mu \in M^*_+} a^{(2)}_\mu e(\mu z) dz_2$ be an element of $D_{\hat{R}}$ such that $a^{(1)}_\mu = a^{(2)}_\mu = 0$ for all lattice points $\mu = aA_i$ with $i \in \mathbb{Z}$ and $1 \leq a \leq a_i-1$. Then ω is in the image of ϕ .

Remark: The Fourier coefficients are completely determined by those for the points of the fundamental domain F_+. So the theorem gives only a finite number of conditions for a differential form to be in the image of ϕ.

Proof: We use the trace filtration $\{\hat{D}_r\}$ of $D_{\hat{R}}$. Let

$$\omega = \sum_{\mu \in M_+^*} e(\mu z) \ (a_\mu^{(1)} dz_1 + a_\mu^{(2)} dz_2)$$

be an element of $D_{\hat{R}}$ with $a_\mu^{(j)} = 0$, $j = 1,2$ for $\mu = aA_i$, $i \in \mathbb{Z}$, $1 \le a \le a_i - 1$.

Assume that $\omega \in \hat{D}_r$ for some $r > 0$. By induction it is sufficient to find an element $\eta \in \hat{D}_r \cap \operatorname{im} \phi$ such that

(1) The Fourier coefficients of η at $\mu = aA_i$, $1 \le a \le a_i - 1$ vanish, and

(2) $\omega - \eta \in \hat{D}_{r+1}$.

Let $B(\omega) = \{\mu \in M_+^* \mid T_r(\mu) = r$ and $(a_\mu^{(1)} \neq 0$ or $a_\mu^{(2)} \neq 0)\}$.

If $B(\omega) = \phi$, then ω is already in $\hat{D}_{r+1}$ and we are done. So let $\mu \in B(\omega)$. Clearly $B(\omega)$ is finite, so again by induction it is sufficient to construct $\eta \in \hat{D}_r \cap \operatorname{im} \phi$ with

(1) and

(2') $B(\omega - \eta) = B(\omega) - \{\mu\}$.

By construction of the fundamental domain F_+ (see Section 1) we can assume that μ is an element of F_+: since ω is invariant under V we have $a_{\mu\varepsilon}^{(1)} = \varepsilon a_\mu^{(1)} \neq 0$ or $a_{\mu\varepsilon}^{(2)} = \varepsilon^{-1} a_\mu^{(2)} \neq 0$ for all $\varepsilon \in V$. Hence $T_r(\varepsilon\mu) \ge T_r(\mu)$ for all $\varepsilon \in V$.

For an element $\nu \in M_+^*$ let F_ν be the series $\sum_{\varepsilon \in V} e(\varepsilon \nu z) \in \hat{R}$.

<u>Lemma 3.2</u> <u>Assume that μ can be written as a sum $\mu = \mu_1 + \mu_2$ with elements $\mu_1, \mu_2 \in F_+$, such that (μ_1, μ_1') and (μ_2, μ_2') are linearly independent over $\mathbb{R}$. Then there exists a complex linear combination η of dF_μ and $F_{\mu_1} dF_{\mu_2}$ with $B(\omega - \eta) = B(\omega) - \{\mu\}$.</u>

Remark:

(i) Since by assumption μ is not of the form aA_k, $1 \le a \le a_k - 1$, dF_μ has zero Fourier coefficient at these points. The Fourier coefficients of $F_{\mu_1} dF_{\mu_2}$ are different from zero at $\varepsilon^{(1)}\mu_1 + \varepsilon^{(2)}\mu_2$ with $\varepsilon^{(1)}, \varepsilon^{(2)} \in V$. But it follows from the first two properties of the support points from the first section that the multiples aA_i, $1 \le a \le a_i - 1$ can be written as a sum of two elements of M_+^* at most in a trivial way, that is $aA_k = bA_k + cA_k$, $b, c > 0$. So property (1) holds for η as in the Lemma.

(ii) The two embeddings of M_+^* in $\mathbb{R}^2$ by mapping $\nu = xw_*$ to (x,y) or to (ν,ν') are related by the nonsingular matrix $\begin{pmatrix} -w_* & 1 \\ -w_*' & 1 \end{pmatrix}$. Hence it is equivalent to say (μ_1, μ_2) and (μ_1', μ_2') are linearly independent or to say that (x_1, y_1) and (x_2, y_2) are linearly independent, where $\mu_i = y_i - x_i w_*$, $i = 1,2$.

Proof of the Lemma:

We can assume that μ_1, μ_2 are elements of F_+. So the estimate $Tr(\mu_i) \le Tr(\varepsilon \mu_i)$, $i = 1,2$, $\varepsilon \in V$ holds, with equality if and only if μ_i is on the boundary of F_+, and ε is the generator of V which maps μ_i to the other boundary component. Hence $Tr(\varepsilon^{(1)}\mu_1 + \varepsilon^{(2)}\mu_2) \ge Tr(\mu_1 + \mu_2) = r$ with equality if and only if $\varepsilon^{(1)} = \varepsilon^{(2)} = 1$. The differential form

$$F_{\mu_1} dF_{\mu_2} = 2\pi i \sum_{\varepsilon^{(1)}, \varepsilon^{(2)} \in V} e((\varepsilon^{(1)}\mu_1 + \varepsilon^{(2)}\mu_2)(\varepsilon^{(1)}\mu_2 dz_1 + (\varepsilon^{(2)}\mu_2)' dz_2)$$

can be written as a sum over infinitely many V-orbits. But by what we said before among these orbits there is exactly one, namely $V\mu$, which contains a point of trace less or equal to r.

So modulo $\hat{D}_{r+1}$

$F_{\mu_1} dF_{\mu_2} = \tilde{\eta} := 2\pi i \sum_{\varepsilon \in V} e(\varepsilon\mu\, z)(\varepsilon\mu_2 dz_1 + (\varepsilon\mu_2)' dz_2)$.

The complex vectorspace of differential forms

$\sum_{\varepsilon \in V} e(\varepsilon\mu\, z)(b^{(1)}_{\varepsilon\mu} dz_1 + b^{(2)}_{\varepsilon\mu} dz_2)$ which are invariant under V is of dimension 2. Since (μ,μ') and (μ_2,μ_2') are linearly independent $\tilde{\eta}$ and dF_μ are a basis. Hence we can find complex numbers α and β such that $\eta = \alpha\, dF_\mu + \beta\, F_{\mu_1} dF_{\mu_2}$ has properties (1) and (2').

Returning to the proof of our Theorem we write μ as the unique nonnegative linear combination $\mu = nA_i + mA_{i+1}$, $n > 0$, $m \geq 0$. If both A_i and A_{i+1} are elements of F_+ we can apply Lemma 3.2.. Let us call <u>bad points</u> of F_+ those, which are different from aA_i, $1 \leq a \leq a_{i-1}$, and for which the Lemma does not apply. If $d = 2k + 1$, then all bad points are contained in the sectors spanned by A_{-k-1}, A_{-k} and by A_k, A_{k+1} respectively (see figure below). For $d = 2k + 2$ the only bad points are aA_{-k-1}, aA_{k+1}, $a \geq a_{-k-1} = a_{k+1}$.

We will only treat the case $d = 2k+1$, $k \geq 2$, $\mu = nA_k + mA_{k+1}$, $n,m > 0$. The others are done more or less the same way.

Let $\mu_1 = nA_k$, $\mu_2 = mA_{-k} = \varepsilon_1^{-1}(mA_{k+1})$, and consider $F_{\mu_1} dF_{\mu_2}$. This differential form has nonzero Fourier coefficients at

$\varepsilon^{(1)}\mu_1 + \varepsilon^{(2)}\mu_2$, $\varepsilon^{(1)}$, $\varepsilon^{(2)} \in V$, so we have to find out which of these have trace less or equal than r. Since μ_1 and μ_2 are in F_+, and are not on the boundary, the traces of the μ_i can be estimated as follows:

$$\ldots > Tr(nA_{-k-1}) > Tr(nA_k) < Tr(nA_{3k+1}) < \ldots$$

$$\ldots > Tr(mA_{-3k-1}) > Tr(mA_{-k}) < Tr(mA_{k+1}) < \ldots \quad .$$

Clearly $\mu_1+\mu_2 = nA_k + mA_{-k}$ has the smallest trace among the points we have to study. We want to show that the point with the next largest trace is $\mu = nA_k + mA_{k+1}$. Observe that $nA_{-k-1}+mA_{-k} = \varepsilon_1^{-1}\mu$. By our assumption μ is an element of F_+, hence $\varepsilon_1^{-1}\mu$ is not. So we have $Tr(\mu) \leq Tr(\varepsilon_1^{-1}\mu)$, and by the V-invariance of our form ω we don't have to care about the case where equality holds.

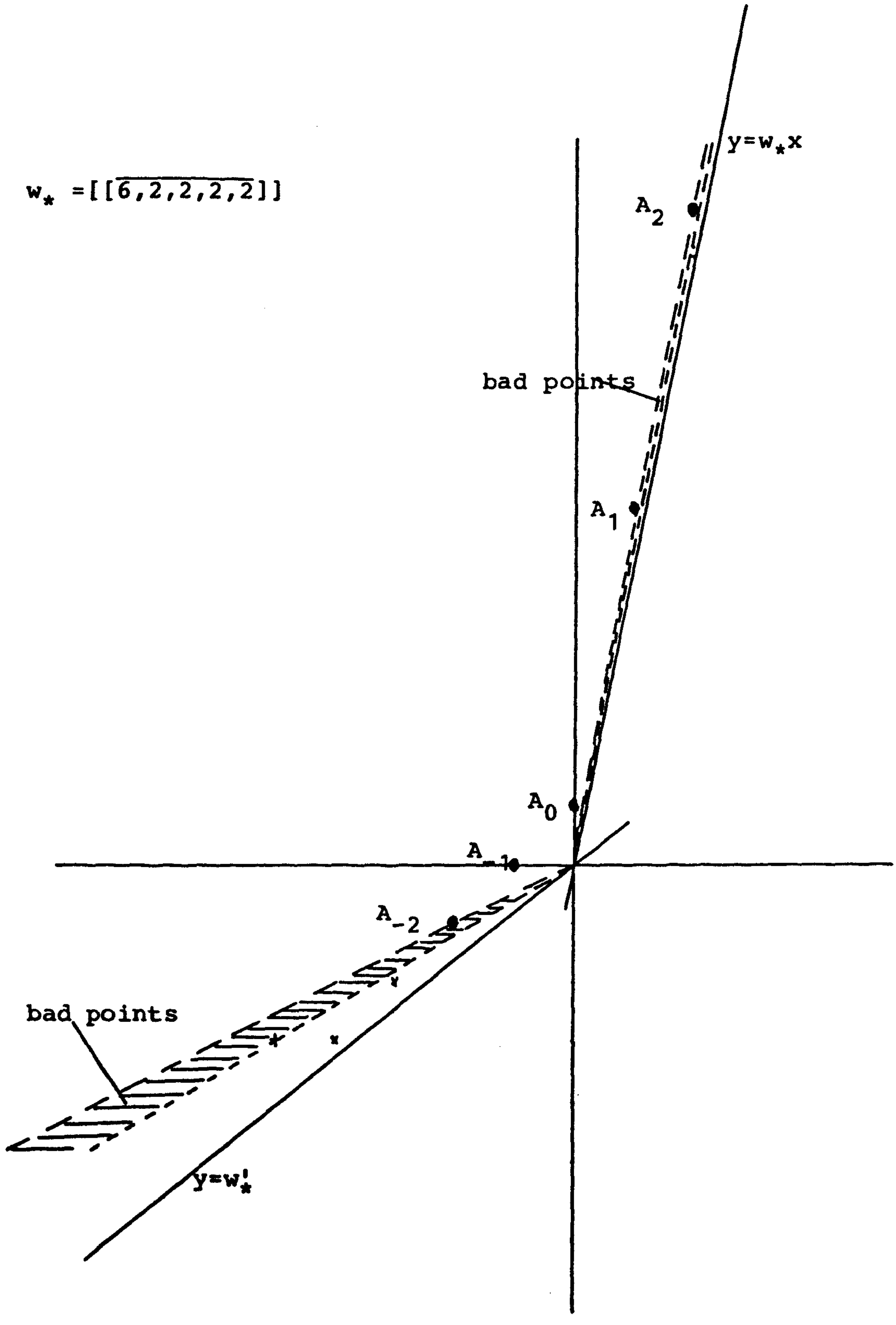
We claim that $Tr(A_{3k+1}) > Tr(A-k-1)$ and that $Tr(A_{-3k-1}) > Tr(A_{k+1})$. For example for the first inequality one computes easily that $Tr(A_{3k+1}-A_{-k-1}) = (\varepsilon_1 - \varepsilon_1^{-1})(A_k - A_k')$. But $\varepsilon_1 < 1$, and for positive indices i $0 < A_i < 1 < A_i'$.

Hence modulo $\hat{D}_{r+1}$:

$$1/2\pi i \, F_{\mu_1} dF_{\mu_2} = \sum_{\varepsilon \in V} e(\varepsilon(\mu_1+\mu_2)z)((\varepsilon\mu_2)dz_1 + (\varepsilon\mu_2)'dz_2) +$$

$$+ \sum_{\varepsilon \in V} e(\varepsilon\mu z)(\varepsilon\varepsilon_1\mu_2 dz_1 + (\varepsilon\varepsilon_1\mu_2)'dz_2).$$

By the same argument as before we can find a complex linear combination $\tilde{\eta} = \alpha\, dF_\mu + \beta\, F_{\mu_1} dF_{\mu_2}$ such that $\omega - \tilde{\eta}$ has Fourier coefficient zero at μ. Moreover all the other Fourier coefficients $a_\nu^{(i)}$, i=1,2, with $Tr(\nu) \leq r$ remain unchanged, except at $\nu = \mu_1+\mu_2$, where we possibly created a new one. But fortunately the point $\mu_1+\mu_2$ is no longer a bad point.



Figure

To complete the proof we have to show that we can remove this coefficient without destroying what we obtained so far. For this we need two differential forms η_1, η_2 with the following properties:

$$\eta_j = \sum_{\nu \in M_+^*} e(\nu z)\ (a_{\nu,j}^{(1)}\ dz_1 + a_{\nu,j}^{(2)}\ dz_2) \in \operatorname{im} \phi, \quad j = 1,2$$

such that

$$(a_{\mu_1+\mu_2,1}^{(1)},\ a_{\mu_1+\mu_2,1}^{(2)}) \quad \text{and} \quad (a_{\mu_1+\mu_2,2}^{(1)}\ a_{\mu_1+\mu_2,2}^{(2)})$$

are linearly independent over $\mathbb{C}$,and

$$a_{\nu,j}^{(i)} = 0 \quad \text{for} \quad i,j = 1,2 \quad \text{and} \quad \operatorname{Tr}(\nu) \leq r, \quad \nu \neq \mu_1+\mu_2 \ .$$

We can take $\eta_1 = dF_{\mu_1+\mu_2}$ which by the estimate we gave above has the second property. The following Lemma shows that η_2 exists, if $d \geq 5$.

<u>Lemma 3.3: If $d = 2k+1 \geq 5$, the point</u> $nA_k + mA_{-k}$, <u>$n,m > 0$, can always be written as a sum of elements B and C of F_+, linearly independent such that for any pair of units</u> $\varepsilon^{(1)}, \varepsilon^{(2)} \in V$, <u>not both equal to one, to estimate</u>

$$\operatorname{Tr}(\varepsilon^{(1)}B + \varepsilon^{(2)}C) > \operatorname{Tr}(mA_k + mA_{k+1})$$

<u>holds.</u>

<u>Proof.</u> Without loss of generality we can assume $m \leq n$. Consider the equation.

$nA_k + mA_{-k} = (n-m)A_k + m(a_{k+1}-1)A_{k-1} + (a_{k+2}-2)A_{k-2} + \dots + (a_{-k+1}-1)A_{-k+1}$

which is an easy consequence of the relations between the A_i mentioned in the first section.

Since all a_i are at least 2, the last summand on the right hand side is not zero. Let $C = mA_{-k+1}$, $B = nA_k + mA_{-k} - mA_{-k+1}$. Since $k \geq 2$, B and C are linearly independent elements of F_+. Next to $Tr(B+C)$ the smallest possible values for the trace of $\varepsilon^{(1)}B + \varepsilon^{(2}$ are the traces of $\varepsilon_1 B + C$, $B + \varepsilon_1^{-1}C$, $\varepsilon_1^{-1}B + C$, $B + \varepsilon_1 C$. The first and the third point are V-equivalent to the second and the last one respectively. Since these are in F_+ it is sufficient to estimate their traces.

One find easily that $B + \varepsilon_1 C = \mu + m(1-\varepsilon_1)(A_{-k} - A_{-k+1})$, and by $0 < \varepsilon_1 < 1 < \varepsilon_1'$, $A_{-k} > A_{-k+1}$ and $A'_{-k} < A'_{-k+1}$ the second summand on the right hand side has positive trace. This shows that $Tr(B + \varepsilon_1 C) > Tr(\mu)$. Finally $B + \varepsilon_1^{-1}C = \mu + m(\varepsilon_1^{-1}-1)(A_{-k+1} + A_{k+1})$. But $A_{-k+1} + A_{k+1}$ is totally positive, and since ε_1 is a totally postive unit of K $(\varepsilon_1 - 1)$ has positive trace. This ends the proof of the Lemma and of the Theorem.

Putting together the lower estimate of Proposition 1.2. and the preceeding result, we obtain immediately

<u>Theorem 3.3.</u>: <u>Let</u> X <u>be a cusp singularity of degree</u> <u>$d \geq 5$,</u> <u>defined by a lattice</u> M <u>and a subgroup</u> $V \subset U_M^+$ <u>of finite index. Let</u> $(a_0, \dots, a_{d-1})$ <u>be the cycle of selfintersection numbers of the dual lattice</u> M^*. <u>Then the dimension of</u> T_X^1 <u>is</u> $\sum_{i=0}^{d-1}(a_i - 1)$, <u>and a basis for</u> T_X^1 <u>is given by the derivations</u> $\delta_{j,a} \in H^1(X', \Theta_{X'})$, <u>for</u> $-k \leq j \leq k$, $1 \leq a \leq a_j - 1$, <u>in case</u> $d = 2k+1$, <u>and for</u> $-k-1 \leq j \leq k$, $1 \leq a \leq a_j - 1$, <u>if</u> $d = 2k+2$.

Proof: We need to show that $H^1_{\{\infty\}}(\Omega^1_{\hat{R}})$ can be generated by $\tau = \sum_{i=0}^{d-1}(a_i-1)$ elements as a complex vectorspace. By Theorem 3.1. $H^1_{\{\infty\}}(\Omega^1_{\hat{R}})$ is generated by the images of the $2\cdot\tau$ forms

$$\sum_{\varepsilon\in V} e(\varepsilon\mu z)dz_i \quad , \quad j = 1,2 \quad , \quad \mu = aA_i \in F_+ \ , \ 1\le a\le a_i-1 \quad .$$

For each A_i we choose a pair of complex numbers (c_i,c_i') which is not a multiple of (A_i,A_i'). Since dF_μ maps to zero in $H^1_{\{\infty\}}(\Omega^1_{\hat{R}})$ we are left with the τ generators

$$\omega(aA_i) := \sum_{\varepsilon\in V} e(\varepsilon aA_i)\ (c_i\varepsilon dz_1 + c_i'\varepsilon' dz_2) \ .$$

Remark: J. Wahl [8] and E. Looijenga [5] have computed the dimension of smoothing components of twodimensional cusp singularities. If a cusp is smoothable then possibly there are several components in the base space of the semiuniversal deformation, where smoothings occur, but they all have the same dimension $\sum_{i=0}^{d-1}(a_i-1)-2\cdot(d-5)$. For $d = 5$ this gives the same value as our formular for $\dim T^1_X$, which is no surprise, since the base space of the semiuniversal deformation of a Gorenstein surface singularity of embedding dimension 5 is smooth and the general fibre is nonsingular. For $6\le d\le 9$ all cusps are smoothable by [4] and [7]. Since the smoothing components have strictly smaller dimension that T^1_X the base space of the semiuniversal deformation has to be singular in this range. Just to indicate that our results give a precise description of the Zariski-differentials we mention

Corollary 3.5.: For a cusp singularity of degree $d \ge 5$ the minimal number of generators of $D_{\hat{R}}$ is 2d.

Proof: It is well known that in this case the embedding dimension of the singularity, which equals the corank of $\Omega^1_{\hat{R}}$, is also d. A minimal set of generators for $\Omega^1_{\hat{R}}$ is given by dF_{A_i}, $A_i \in F_+$. We consider the set of generators obtained at the end of the proof of Theorem 3.4. From the proof of Theorem 3.1. it follows that modulo the image of ϕ

$$F_{A_j} \cdot \omega(aA_i) = \delta_{ij} \cdot \omega((a+1)A_i)$$

if $a < a_i - 1$, and $F_{A_j} \cdot \omega((a_i-1)A_i) = 0$.

This shows that the corank of $H^1_{\{\infty\}}(\Omega^1_R)$ is also d so we need at most $2d$ elements to generate $D_{\hat{R}}$.

On the other hand it is easy to see that the elements

$\sum_{\varepsilon \in v} e(\varepsilon A_i z) dz_j$, $j=1,2$, $A_i \in F_+$, are mapped to linearly independent elements of $D_{\hat{R}} \otimes_{\hat{R}} \mathbb{C}$. A product of an element of $D_{\hat{R}}$ and of a formal series in $\hat{R}$ with zero constant term must have zero Fourier coefficient at support points. Otherwise one of the A_i would have a representation as a sum of two elements of M^*_+, which is impossible.

Remark: For $d \leq 4$ the structure of $H^1_{\{\infty\}}(\Omega^1_R)$ is more complicated, so that one cannot apply these arguments. In this range it is possibly simpler to work with the equations which here are explicitly known (see [4]).

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