# SPECTRAL DEFORMATIONS AND EISENSTEIN SERIES ASSOCIATED WITH MODULAR SYMBOLS

YIANNIS N. PETRIDIS

Abstract. We give a new proof of the analytic continuation of the Eisenstein series associated with modular symbols. We study its singularities, analytic properties, and functional equation using perturbation theory of the Laplace operator on character varieties.

## 1. Introduction

Let $f(z)$ be a holomorphic cusp form of weight 2 for the cofinite discrete subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbf{R})$. In [5], [6] Goldfeld introduced Eisenstein series associated with modular symbols. It is defined as

$$E^*(z,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\langle\gamma,f\rangle\Im(\gamma z)^s, \tag{1.1}$$

where for $\gamma\in\Gamma$ the modular symbol is given by

$$\langle\gamma,f\rangle=-2\pi i\int_{z_0}^{\gamma z_0}f(\tau)\,d\tau. \tag{1.2}$$

Here $z_0$ is an arbitrary point in $\mathbb{H}$. The aim is to study the distribution of the modular symbols. Goldfeld conjectured in [5] that

$$\sum_{c^2+d^2\le X}\langle\gamma,f\rangle\sim R(i)X, \tag{1.3}$$

where $R(z)$ is the residue at $s=1$ of $E^*(z,s)$, and we sum over the elements in $\Gamma$ with lower row $(c,d)$. In fact, he conjectured corresponding statements for the more general Eisenstein series associated with modular symbols $E^{m,n}(z,s)$, see (1.12). If we take $f(z)$ to be a Hecke eigenform for $\Gamma_0(N)$ and $E_f$ is the elliptic curve over $\mathbf{Q}$ corresponding to it by the Eichler-Shimura theory, then

$$\langle\gamma,f\rangle=n_1(f,\gamma)\Omega_1(f)+n_2(f,\gamma)\Omega_2(f),$$

where $n_i\in\mathbf{Z}$ and $\Omega_i$ are the periods of $E_f$. The conjecture $n_i\ll N^k$ for $|c|\le N^2$ and some fixed $k$ (Goldfeld's conjecture) is equivalent to Szpiro's conjecture $D\ll N^C$ for some $C$, where $D$ is the discriminant of $E_f$. This has been the motivation to look at the distribution of modular symbols.

In [14] the analytic continuation of the Eisenstein series has been proved and in [15] it is proved that the analytic continuation of them on the line $\Re(s)=1/2$ has poles at $s_j$, where $s_j(1-s_j)$ are the eigenvalues of the Laplace operator on $\Gamma\backslash\mathbb{H}$. A functional equation was also found. The action of Hecke operators on the Eisenstein series was studied in [4].

1991 *Mathematics Subject Classification.* Primary 11F67; Secondary 11F72, 11M36.

One of the problems is that the Eisenstein series is not a modular form in the classical setting, i.e., it is not invariant or transform nicely under the action of $\Gamma$. In fact, it transforms as

$$E^*(\gamma z, s) = E^*(z,s) - \langle \gamma, f\rangle E(z,s), \tag{1.4}$$

where $E(z,s)$ is the standard nonholomorphic Eisenstein series for $\Gamma$.

We study in this paper a new approach to this Eisenstein series. We consider Eisenstein series with characters depending on a parameter $\epsilon$ and we notice that the Eisenstein series with modular symbols is their derivative when $\epsilon = 0$. We define

$$E_\epsilon(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \chi_\epsilon(\gamma)\Im(\gamma z)^s, \tag{1.5}$$

where $\chi_\epsilon$ is a one-parameter family of characters of the group defined by

$$\chi_\epsilon(\gamma) = \exp\left(-2\pi i\epsilon\int_{z_0}^{\gamma z_0} f(\tau)\,d\tau\right). \tag{1.6}$$

This series is defined formally, because the character $\chi$ is not unitary. In practice one substitutes with a unitary character, by considering the real and imaginary part of the holomorphic differential $f(\tau)d\tau$. In this case convergence is guaranteed for $\Re(s) > 1$ by comparison with the standard Eisenstein series. The Eisenstein series with character transform as

$$E_\epsilon(\gamma z, s) = \bar\chi_\epsilon(\gamma)E_\epsilon(z,s) \tag{1.7}$$

They satisfy functional equations

$$E_\epsilon(z,s) = \phi_\epsilon(s)E_\epsilon(z,1-s). \tag{1.8}$$

In the domain of absolute convergence we see that

$$\frac{d}{d\epsilon}_{|\epsilon=0}E_\epsilon(z,s) = E^*(z,s), \tag{1.9}$$

by termwise differentiation. Also we differentiate (1.8) to get the functional equation

$$E^*(z,s) = \frac{d\phi(s)}{d\epsilon}_{|\epsilon=0}E(z,1-s) + \phi(s)E^*(z,1-s). \tag{1.10}$$

Here $\phi(s)$ is the (standard) scattering function for $\epsilon = 0$.

Our first theorem describes the analytic proporties of $E^*(z,s)$. It gives a new proof of the main result in [14] and another result in [15].

**Theorem 1.** *(a) The Eisenstein series associated with modular symbols $E^*(z,s)$ has a meromorphic continuation in the whole complex plane and satisfies the functional equation (1.10).*

*(b) For $\Re s \geq 1/2$ the poles of $E^*(z,s)$ are simple and contained in the set*

$$\{1/2\} \cup \bigcup_j \{s_j, \bar{s_j}\}.$$

*(c) At a cuspidal eigenvalue $s_j(1-s_j)$ of $\Delta$ corresponding to the cusp forms $\phi_l(z)$, $l = 1,\dots N$, the residue of $E^*(z,s)$ is equal to*

$$\sum_{l=1}^N \frac{c}{\pi^{s_j}}L(f\otimes\phi_l, s_j+1/2)\Gamma(s_j-1/2)\phi_l(z).$$

*Here $c$ is a certain constant, $\Gamma(s)$ is the Gamma function and $L(f \otimes \phi_l, s)$ is the Rankin-Selberg convolution of $f(z)$ with $\phi_l(z)$.*

We prove Theorem 1(a) in Section 2, (b) in Section (4), and (c) in Section 5.

It follows that the scattering function $\phi^*(s)$ identified in [14, 0.3] using Kloosterman sums is given simply by

$$\phi^*(s) = d\phi_\epsilon(s)/d\epsilon \tag{1.11}$$

at $\epsilon = 0$ and its functional equation [14, Th. 0.2] follows by the standard functional equation for the scattering matrix $\phi(s)\phi(1-s) = 1$ by differentiation. This gives the following theorem, see also [2, Theorem 1].

**Theorem 2.** *Let $i$ and $j$ be cusps. The entries of $d\phi_\epsilon(s)/d\epsilon$ at $\epsilon = 0$ are given by*

$$\phi^*_{ij}(s) = -2\pi i \int_j^i f(\tau)\, d\tau \cdot \phi_{ij}(s).$$

Since we are interested in the analytic continuation of Eisenstein series, we follow the method of Colin de Verdière [3], which is the shortest known method. We notice that at every step we can differentiate in the parameter $\epsilon$ and that everything remains meromorphic in $s \in \mathbb{C}$.

*Remark* 3. Our method also allows to prove the meromorphic continuation of more general Eisenstein series of the form

$$E^{m,n}(z,s) = \sum_{\Gamma_\infty \backslash \Gamma} \langle \gamma, f \rangle^m \overline{\langle \gamma, g \rangle}^n \Im(\gamma \cdot z)^s \tag{1.12}$$

for two cusp forms $f, g$ of weight 2, which are relevant to the distribution of modular symbols, as explained in [14, p. 165]. See (2.3).

A corollary of our method is to show the following theorem:

**Theorem 4.** *Assume that $s_j(1-s_j)$ has multiplicity one and the corresponding Maaß cusp form is $\phi_j(z)$. If the value of the L-series $L(f \otimes \phi_j, s_j + 1/2)$ is nonzero, then the perturbed Eisenstein series $E_\epsilon(z,s)$ has a pole close to $s_j$.*

The hypothesis

$$L(f \otimes \phi_l, s_j + 1/2) \neq 0 \tag{1.13}$$

is the Phillips-Sarnak condition and appeared in [20], [17]. See Remark 12.

We also study the behavior of $E^*(z,s)$ on vertical lines. We get the following theorem.

**Theorem 5.** *The Eisenstein series associated with modular symbols $E^*(z,s)$ is bounded on vertical lines with $\sigma > 1/2$. More precisely, for $z \in K$, a compact set, and for $s$ bounded away from the poles of $\phi(s)$ on $(1/2, 1]$ we have*

$$E^*(z,s) \ll_{K,\sigma} 1. \tag{1.14}$$

*Remark* 6. In fact Theorem 5 allows to improve the asymptotic formula (1.3), i.e., gives an estimate for the remainder of the form $O(X^a)$, with $a < 1$, using standards techniques in analytic number theory (contour integration). See the forthcoming article of Goldfeld and O'Sullivan [7].

*Remark* 7. We introduce sums over closed geodesics $\gamma$ with length $l(\gamma)$ as follows:

$$\pi_\epsilon(x) = \sum_{l(\gamma)\leq x} \chi_\epsilon(\gamma).$$

Then

$$\frac{d}{d\epsilon}_{|\epsilon=0}\pi_\epsilon(x) = -2\pi i \sum_{l(\gamma)\leq x} \int_\gamma f.$$

The asymptotic behavior of the sums $\pi_\epsilon(x)$ can be understood using the Selberg trace formula. To estimate their derivative one should differentiate the trace formula in $\epsilon$. On the other hand, to understand geodesics in homology classes as in [21], one integrates the trace formula over the character variety.

The study of $E^*(z,s)$ using perturbed Eisenstein series is a new application of the spectral deformations used in [17], [18] and [16]. Our contribution is to put the Eisenstein series with modular symbols into this framework. We avoid completely the Kloosterman sums with modular symbols introduced and used in [5], [14].

## 2. Proof of the analytic continuation of $E^*(z,s)$

We first notice that $E^*(z,s)$ is linear in the differential $f(\tau)d\tau$. So we can consider separately the real and imaginary part of $f(\tau)d\tau$. Let $w_i$ be either of the two. We let

$$\chi^i_\epsilon(\gamma) = \exp\left(-2\pi i\epsilon\int_{z_0}^{\gamma z_0} w_i\right), \tag{2.1}$$

which is now a unitary character of $\Gamma$. We define Eisenstein series

$$E_\epsilon(z,s,w_i) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \chi^i_\epsilon(\gamma)\Im(\gamma\cdot z)^s \tag{2.2}$$

for $\Re(s)>1$. More generally one can define Eisenstein series depending on a vector of parameters $\vec{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_m)$ and a vector of real-valued harmonic 1-forms $(w_1, w_2, \dots, w_m)$ as

$$E_{\vec{\epsilon}}(z,s,\vec{w}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \prod_{i=1}^m \chi^i_{\epsilon_i}(\gamma)\Im(\gamma\cdot z)^s. \tag{2.3}$$

The Eisenstein series (1.12) are linear combinations of their derivatives in $\epsilon_i$, when we set $\vec{\epsilon}=\vec{0}$. For simplicity we will restrict our attention to one parameter and one cusp. We drop the subscript $i$. The generalization to many cusps can proceed as in [12]. Using the identification of harmonic cuspical cohomology with cohomology with compact support, see [1], one can assume that $w$ is a compactly supported form. We consider the space $L^2(\Gamma\setminus\mathbb{H}, \bar{\chi}_\epsilon)$ of $L^2$ functions which transform as

$$h(\gamma\cdot z) = \bar{\chi}_\epsilon(\gamma)h(z), \quad \gamma\in\Gamma$$

under the action of the group. We introduce unitary operators

$$U_\epsilon : L^2(\Gamma\setminus\mathbb{H}) \to L^2(\Gamma\setminus\mathbb{H}, \bar{\chi}_\epsilon)$$

given by

$$(U_\epsilon h)(z) = \exp\left(2\pi i\epsilon\int_{z_0}^{z} w\right)h(z).$$

We set

$$L_\epsilon = U_\epsilon^{-1}\Delta U_\epsilon.$$

The operators $L_\epsilon$ on $L^2(\Gamma \setminus \mathbb{H})$ and $\Delta$ on $L^2(\Gamma \setminus \mathbb{H}, \chi_\epsilon)$ are unitarily equivalent. Also $L_\epsilon = \Delta$ outside the support of $w$. The cusp $C$ is isometric to $[b, \infty) \times \mathbb{R}/\mathbb{Z}$ with the metric $(dx^2 + dy^2)/y^2$, where $b$ is sufficiently large. We can assume that $\mathrm{supp}(w) \cap C = \emptyset$. We let $h(y) \in C^\infty(\mathbb{R}^+)$ be a function which is 0 for $y \leq b+1$ and 1 for $y \geq b+2$. Let

$$\Omega_\epsilon = \{s \in \mathbb{C}, \Re(s) > 1/2, s(1-s) \notin \mathrm{Spec}(L_\epsilon)\}.$$

**Lemma 8.** *For $s \in \Omega_\epsilon$ we can find a unique $D_\epsilon(z,s)$ such that*

$$(L_\epsilon + s(1-s))D_\epsilon(z,s) = 0$$

*and*

$$D_\epsilon(z,s) - h(y)y^s \in L^2(\Gamma \setminus \mathbb{H}).$$

*Moreover the functions*

$$s \to D_\epsilon(z,s), \quad s \to \frac{d}{d\epsilon} D_\epsilon(z,s)$$

*are holomorphic in $s \in \Omega_\epsilon$ and the function*

$$\epsilon \to D_\epsilon(z,s)$$

*is real analytic.*

*Remark* 9. The functions $D_\epsilon(z,s)$ are not the Eisenstein series themselves. These are $E_\epsilon(z,s) = U_\epsilon D_\epsilon(z,s)$.

*Proof.* We write $D_\epsilon(z,s) = h(y)y^s + g_\epsilon(z,s)$ with $g_\epsilon \in L^2(\Gamma \setminus \mathbb{H})$. We set

$$H_\epsilon(z,s) = -(L_\epsilon + s(1-s))(h(y)y^s) \tag{2.4}$$

and we see that $H_\epsilon$ has compact support and depends holomorphically on $s \in \mathbb{C}$ and real analytically in $\epsilon$. The same is true for $\dot{H}_\epsilon(z,s) = -\dot{L}_\epsilon(h(y)y^s)$. For notational convenience we put a dot to denote differentiation with respect to the parameter $\epsilon$. As long as $s(1-s)$ in not in the spectrum of $L_\epsilon$, the equation $(L_\epsilon + s(1-s))g_\epsilon(z,s) = H_\epsilon(z,s)$ can be inverted to give

$$g_\epsilon(z,s) = (L_\epsilon + s(1-s))^{-1} H_\epsilon(z,s)$$

and $g_\epsilon \in H^2(\Gamma \setminus \mathbb{H})$, the second Sobolev space. The resolvent is holomorphic outside the spectrum of $L_\epsilon$ and depends real analytically on the parameter $\epsilon$, see [10, p. 66-67]. □

We define pseudo-Laplacian operators associated with $L_\epsilon$ exactly as in [11] and [3]. We set

$$\mathcal{H}_a = \left\{f \in H^1(\Gamma \setminus \mathbb{H}), f_0|_{(a,\infty)} = 0\right\}$$

where $f_0$ is the zero Fourier coefficient at the cusp. We take $a \geq b+2$. The operator $L_{\epsilon,a}$ is the Friedrichs extension of the restriction to $H^1(\Gamma \setminus \mathbb{H})$ of the quadratic form $q(f) = \int ||\nabla U_\epsilon f||^2$ to $\mathcal{H}_a$. Intuitively we map $f$ to $L^2(\Gamma \setminus \mathbb{H}, \bar{\chi}_\epsilon)$ and we know that $L_\epsilon$ is unitarily equivalent to $\Delta$ on this space. As in [19], [11] the operators $L_{\epsilon,a}$ have compact resolvents and depend real analytically on $\epsilon$. Consequently this is true for their resolvents $R_{a,\epsilon}(s) = (L_{\epsilon,a} + s(1-s))^{-1}$ by standard perturbation theory, [10, p. 66–67]. We define

$$F_\epsilon(z,s) = h(y)y^s + (L_{\epsilon,a} + s(1-s))^{-1}(H_\epsilon(z,s)) \tag{2.5}$$

and we see that $F_\epsilon$ is meromorphic in $s$ and the same applies to

$$\dot{F}_\epsilon(z,s) = (L_{\epsilon,a} + s(1-s))^{-1}(\dot{H}_\epsilon(z,s)) + \dot{R}_{a,\epsilon}(s)H_\epsilon(z,s). \tag{2.6}$$

We notice that $L_{\epsilon,a}$ does not change the nonzero Fourier coefficients and it removes the zero Fourier coefficient at height $y = a$. For $b < y < a$ we see that $(L_\epsilon + s(1 - s))F_\epsilon(z,s) = 0$. Consequently the zero Fourier coefficient $F_{0,\epsilon}(z,s)$ of $F_\epsilon(z,s)$ is of the form

$$F_{0,\epsilon}(z,s) = A_\epsilon(s)y^s + B_\epsilon(s)y^{1-s}$$

for some holomorphic functions $A_\epsilon(s)$, $B_\epsilon(s)$, $s \neq 1/2$. The real analyticity of the expansion in $\epsilon$ is also obvious, as follows from the definition of the Fourier coefficients. By looking at height $y = a$ in (2.5) we get

$$A_\epsilon(s)a^s + B_\epsilon(s)a^{1-s} = a^s, \tag{2.7}$$

from which it follows that $A_\epsilon(s)$ and $B_\epsilon(s)$ are not identically 0 in $s$. We modify the functions $F_\epsilon(z,s)$ to relate them to the functions $D_\epsilon(z,s)$ as follows. We define

$$\tilde{F}_\epsilon(z,s) = F_\epsilon(z,s) + \chi_{[a,\infty)}(y)(A_\epsilon(s)y^s + B_\epsilon(s)y^{1-s} - y^s).$$

We notice that $(L_\epsilon + s(1-s))\tilde{F}_\epsilon(z,s) = 0$ for $y \geq a$. If $\Re(s) > 1/2$ and $s(1-s) \notin \operatorname{Spec}(L_\epsilon)$, all terms are in $L^2(\Gamma \setminus \mathbb{H})$ with the exception of $h(y)y^s + \chi_{[a,\infty)}(y)(A_\epsilon(s)y^s - y^s)$, so

$$\tilde{F}_\epsilon(z,s) - A_\epsilon(s)h(y)y^s \in L^2(\Gamma \setminus \mathbb{H})$$

and, therefore,

$$\tilde{F}_\epsilon(z,s) = A_\epsilon(s)D_\epsilon(z,s) \tag{2.8}$$

by Lemma 8. Similarly we see that $\tilde{F}_\epsilon(z,s) - \chi_{[a,\infty)}(y)B_\epsilon(s)y^{1-s} \in L^2(\Gamma \setminus \mathbb{H})$ for $\Re(s) < 1/2$ and $s(1-s) \notin \operatorname{Spec}(L_\epsilon)$, so

$$\tilde{F}_\epsilon(z,s) = B_\epsilon(s)D_\epsilon(z,1-s). \tag{2.9}$$

From (2.8), (2.9) we get the analytic continuation of $D_\epsilon(z,s)$ and its functional equation. As in [3] we see that $D_\epsilon(z,s)$ does not have poles on $\Re(s) = 1/2$ (using the Maaß-Selberg relations). The scattering matrix is

$$\phi_\epsilon(s) = \frac{B_\epsilon(s)}{A_\epsilon(s)}. \tag{2.10}$$

We mention the various formulas for the derivatives:

$$\frac{d\tilde{F}_\epsilon(z,s)}{d\epsilon} = \dot{F}_\epsilon(z,s) + \chi_{[a,\infty)}(y)(\dot{A}_\epsilon(s)y^s + \dot{B}_\epsilon(s)y^{1-s}),$$

$$\dot{D}_\epsilon(z,s) = \frac{dA_\epsilon^{-1}(s)}{d\epsilon}\tilde{F}_\epsilon(z,s) + A_\epsilon^{-1}(s)\frac{d\tilde{F}_\epsilon(z,s)}{d\epsilon}, \tag{2.11}$$

$$\dot{E}_\epsilon(z,s) = \dot{U}_\epsilon D_\epsilon(z,s) + U_\epsilon \dot{D}_\epsilon(z,s). \tag{2.12}$$

## 3. Proof of Theorem 2

We discuss the case of one cusp first. By Remark 61 in [9, p. 218] we know that $\phi_\epsilon(s) = \phi_{-\epsilon}(s)$. As a result $\dot{\phi}_0(s) = 0$, being the derivative of an even function at $\epsilon = 0$. Consequently, by (1.11) we have $\phi^*(s) = 0$.

We include a detailed proof of $\phi_\epsilon(s) = \phi_{-\epsilon}(s)$ to facilitate the understanding of the multiple-cusp case. Using the Bruhat decomposition $\Gamma_\infty \setminus \Gamma / \Gamma_\infty$ we can write the zero Fourier coefficient as

$$y^s + \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} \sum_{m \in \mathbb{Z}} \int_0^1 \chi_\epsilon(\gamma) \frac{y^s}{|cz + cm + d|^{2s}} \, dx$$

since $\chi_\epsilon(\gamma S^m) = \chi_\epsilon(\gamma)$ as $\chi_\epsilon$ is a character with $\chi_\epsilon(S) = 1$. Here $S$ is the standard parabolic generator. So

$$\phi_\epsilon(s) y^{1-s} = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} \frac{\chi_\epsilon(\gamma)}{|c|^{2s}} \int_{-\infty}^{\infty} \frac{y^s}{|x^2 + y^2|^s} \, dx.$$

The integral can be evaluted in terms of the Gamma function to be $y^{1-s}\sqrt{\pi}\Gamma(s - 1/2)/\Gamma(s)$, see [8, 8.380.3]. To show that $\phi_\epsilon(s) = \phi(s, \chi) = \phi(s, \bar{\chi}) = \phi_{-\epsilon}(s)$ it suffices to notice that we can take as coset representatives in the Bruhat decomposition $\gamma^{-1}$, where $\gamma \in \Gamma_\infty \setminus \Gamma / \Gamma_\infty$ and that $\gamma^{-1}$ has lower left entry $-c$. The same calculation works for the case of many cusps and the diagonal entries $\phi_{ii}(s, \chi)$ of the scattering matrix $\Phi_\epsilon(s)$. We get $\dot{\phi}_{ii}(s) = 0$.

*Remark* 10. In general, for a group $\Gamma$ with many cusps, $\Phi_\epsilon(s) = \Phi_{-\epsilon}(s)^T$. By differentiation we get that $\Phi^*(s)$ is skew-symmetric, which already gives Proposition 4.2 in [14].

In the case of many cusps the $ij$-entry of the scattering matrix is given by

$$\phi_{ij,\epsilon}(s) = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty} \frac{\chi_\epsilon(\sigma_i^{-1} \sigma_j \gamma)}{|c|^{2s}}. \tag{3.1}$$

Here $\sigma_j$ maps the $j$ cusp to $i\infty$ and $\sigma_i$ maps the $i$ cusp to $i\infty$. Since $\chi_\epsilon$ is a character we can write $\chi_\epsilon(\sigma_i^{-1}\sigma_j)\chi_\epsilon(\gamma)$ and differentiate to get

$$\phi_{ij}^*(s) = \left( -2\pi i \int_{z_0}^{\sigma_i^{-1} \sigma_j z_0} f(\tau) \, d\tau \right) \phi_{ij}(s) + \dot{\phi}_{ii}(s).$$

But we have shown that $\dot{\phi}_{ii}(s) = 0$. Since $\phi^*(s)$ does not depend on $z_0$, we take $z_0$ to be the $j$ cusp.

## 4. Poles of $E^*(z, s)$.

It is clear from (2.6) and (2.8) that we get potential poles for $E^*(z, s)$ at the eigenvalues of $L_{\epsilon,a}$ or at the poles of the scattering matrix. The eigenvalues of $L_{\epsilon,a}$ come in two types, [3]. Any cusp form for $\epsilon = 0$ is an eigenfunction of $L_{0,a}$ for all $a$, as its zero Fourier coefficient is identically zero. These are the eigenvalues of Type (I). The eigenvalues of Type (II) correspond to noncuspidal eigenfunctions: For $\Re(s_j) \geq 1/2$, $s_j \neq 1/2$ we must have

$$a^{s_j} + \phi_0(s_j) a^{1-s_j} = 0. \tag{4.1}$$

For $s_j = 1/2$ the condition is $\phi_0(1/2) = -1$ and $\phi_0'(1/2) = -2 \log a$. For details see [3, p. 93]. Recall that $\phi_0(s)$ is the (standard) scattering function $\phi(s)$. It is inconvenient to work with $L_{\epsilon,a}$, so we try to characterize the poles of $E^*(z, s)$ in terms of the eigenvalues of $\Delta$:

**Lemma 11.** *If $s_j$ does not correspond to a cuspidal eigenvalue of $\Delta$ on $L^2(\Gamma\setminus\mathbb{H})$ and $\phi_0(s)$ does not have a pole at $s_j$, then $E^*(z,s)$ is regular at $s_j$.*

*Proof.* If $A_0(s_j)=0$, then (2.7) implies that $B_0(s_j)\neq 0$. But then $\phi_0(s)$ has a pole at $s_j$. Consequently $A_0(s_j)\neq 0$ for all $a$ large, so we do not get a pole from the contribution of $A_\epsilon^{-1}$ and $dA_\epsilon^{-1}/d\epsilon$ in (2.11). We also need to arrange that we do not get a pole from the resolvent of of $L_{0,a}$ for some $a$, see (2.6). We need to exclude eigenvalues of type (II). Then all the formulas become regular at $s_j$. Notice that, by the second Neumann series for the resolvent [10, p. 66–67],

$$\dot{R}_\epsilon(z)=-R_\epsilon(z)L_\epsilon R_\epsilon(z), \tag{4.2}$$

so the derivative of the resolvent in (2.6) is regular away from the eigenvalues of $L_\epsilon$. However, the conditions (4.1) are satisfied for a discrete set of values of positive $a$. Once an $a$ is chosen (sufficiently large) not satisfying (4.1), for small enough $\epsilon$, $L_\epsilon$ do not have eigenvalue close to $s_j$. ☐

This lemma proves part (b) in Theorem 1.

## 5. Residues of $E^*(z,s)$ at cuspidal eigenvalues

Let $s_j$ be such that $s_j(1-s_j)$ is a cuspidal eigenvalue of $\Delta$. The formula for $L_\epsilon$ has been used in [16], [18] and is given by

$$L_\epsilon u=\Delta u-4\pi i\epsilon\langle du,w\rangle-4\pi^2\epsilon^2|w|_H^2u+2\pi i\epsilon(\delta w)u. \tag{5.1}$$

Here $\delta(p\,dx+q\,dy)=-y^2(p_x+q_y)$, $\langle p\,dx+q\,dy,f\,dx+g\,dy\rangle=y^2(p\bar{f}+q\bar{g})$ and $|p\,dx+q\,dy|_H^2=y^2(|p|^2+|q|^2)$. The difference in the signs is due to the fact that we use $-w$ in the formula in [16]. We have

$$(L_\epsilon+s(1-s))D_\epsilon(z,s)=0$$

away from the poles of $D_\epsilon(z,s)$. We differentiate it and evaluate at $\epsilon=0$ to get

$$(L_0+s(1-s))\dot{D}_0(z,s)=-\dot{L}_0D_0(z,s).$$

It follows from (2.8), (2.10) that the zero Fourier coefficient of $D_\epsilon(z,s)$ is

$$y^s+\phi_\epsilon(s)y^{1-s}.$$

Since $\dot{\phi}_0(s)=0$, by Theorem 2, $\dot{D}_0(z,s)$ is in $\mathcal{H}_a$. We can substitute $L_0$ with $L_{0,a}$ to get

$$(L_{0,a}+s(1-s))\dot{D}_0(z,s)=-\dot{L}_0D_0(z,s).$$

As in the previous section we can assume that we chose $a$ in such a way that $L_{0,a}$ does not have a Type (II) eigenvalue at $s_j(1-s_j)$. For $s(1-s)\notin\operatorname{Spec}(L_0)$, $\Re(s)>1/2$ we can introduce the resolvent to get

$$\dot{D}_0(z,s)=-R_0(s)\dot{L}_0D_0(z,s), \tag{5.2}$$

where $R_0(s)=(L_0+s(1-s))^{-1}$. The residue $A(z)$ of $\dot{D}_0(z,s)$ at $s_j$, which is the same as the residue of $\dot{E}_0(z,s)$ by (1.9) and (2.12) is the residue of $R_{0,a}(s)$ at $s_j$ applied to $-\dot{L}_0D_0(z,s)$. We recall that $D_0(z,s)$ is regular at $s_j$. The resolvent kernel for $L_{0,a}$ has an expansion at $s_j$ of the form

$$r_{L_{0,a}}(z,z',s)=\frac{1}{s(1-s)-s_j(1-s_j)}\sum_{l=1}^N\phi_l(z)\phi_l(z')+\text{analytic at } s_j,$$

where $\phi_l(z)$, $l=1,\dots,N$ are an orthonormal basis of cusp forms at $s_j$.

As a result the residue $A(z)$ is given by

$$\begin{aligned} A(z) &= \frac{4\pi i}{2s_j-1}\sum_{l=1}^{N}\phi_l(z)\int_{\Gamma\backslash\mathbb{H}}\phi_l(z')\langle dD_0(z',s_j),w\rangle d\mu(z') \\ &\quad -\frac{2\pi i}{2s_j-1}\sum_{l=1}^{N}\phi_l(z)\int_{\Gamma\backslash\mathbb{H}}\phi_l(z')(\delta w)D_0(z,s_j)d\mu(z'), \end{aligned} \tag{5.3}$$

where $d\mu(z)$ is the invariant hyperbolic measure $dx\,dy/y^2$. This residue is independent of $a$. We can change $w$ in its cohomology class, even though $U_\epsilon$ depends on it. Consequently we can approximate $\alpha$, the real or the imaginary part of the differential $f(z)\,dz$ by a family $w_l$, supported in a compact set $K_l$ with $\Gamma\backslash\mathbb{H}=\cup K_l$ and with convergence in the Sobolev space $H^1$. Then

$$\lim_l\int_{\Gamma\backslash\mathbb{H}}\phi_j(z)\langle dD_0(z,s_j),w_l\rangle d\mu(z)=\int_{\Gamma\backslash\mathbb{H}}\phi_j(z)\langle dD_0(z,s_j),\alpha\rangle d\mu(z),$$

and

$$\lim_l\int_{\Gamma\backslash\mathbb{H}}\phi_j(z)(\delta w_l)D_0(z,s_j)d\mu(z)=\int_{\Gamma\backslash\mathbb{H}}\phi_j(z)(\delta\alpha)D_0(z,s_j)d\mu(z).$$

Since $\alpha$ is harmonic, $\delta\alpha=0$. Since the modular symbols is linear, while $\langle\cdot,\cdot\rangle$ is antilinear in the second variable, we take $\alpha=\overline{f(z)dz}$. By linearity we are left to compute

$$\int_{\Gamma\backslash\mathbb{H}}\phi_j(z)\langle dD_0(z,s_j),\overline{f(z)dz}\rangle d\mu(z)=\int_{\Gamma\backslash\mathbb{H}}\phi_j(z)y^2f(z)E_{\bar z}(z,s_j)d\mu(z),$$

since $dD_0(z,s)=\partial_zD_0(z,s)dz+\partial_{\bar z}D_0(z,s)d\bar z$ and $\langle f_1dz+f_2d\bar z,g_1dz+g_2d\bar z\rangle=2y^2(f_1\overline{g_1}+f_2\overline{g_2})$.

5.1. **Relation with Rankin-Selberg convolutions.** We analyze the integral

$$I(s)=\int_{\Gamma\backslash\mathbb{H}}\phi_j(z)y^2f(z)E_{\bar z}(z,s)d\mu(z) \tag{5.4}$$

where $E(z,s)=D_0(z,s)$. For $\Re(s)$ sufficiently large we can differentiate the series

$$E(z,s)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\Im(\gamma z)^s$$

to get

$$E_{\bar z}(z,s)=\frac{is}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\Im(\gamma z)^{s-1}\overline{(cz+d)}^{-2}.$$

We can unfold as in the Rankin-Selberg method for modular forms of different weight to get

$$\frac{is}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\int_{\Gamma\backslash\mathbb{H}}\phi_j(z)\frac{y^2}{|cz+d|^4}f(z)(cz+d)^2\frac{y^{s-1}}{|cz+d|^{2s-2}}\,d\mu(z),$$

since $E_{\bar z}(z,s)$ has weight $(0,2)$. We then get

$$I(s)=\frac{is}{2}\int_{\Gamma_\infty\backslash\mathbb{H}}\phi_j(z)y^2f(z)y^{s-1}d\mu(z).$$

We write Fourier expansions for $\phi_j(z)$ and $f(z)$:

$$\phi_j(z) = \sum_{n\neq 0} a_n y^{1/2} K_{s_j-1/2}(2\pi|n|y)e^{2\pi inx}, \tag{5.5}$$

where $K_\nu(y)$ is the MacDonald Bessel function, and $f(z) = \sum_{n>0} b_n e^{2\pi inz}$. Using [8, 6.621.3, p. 733] we get

$$I(s) = \frac{is}{2}\sum_{n>0}\frac{a_{-n}b_n}{(2\pi n)^{s+1/2}}\frac{\sqrt{\pi}2^{-1/2}}{2^s}\frac{\Gamma(s+s_j)\Gamma(s-s_j+1)}{\Gamma(s+1)}.$$

We denote the Rankin-Selberg convolution of $f$ and $\phi_j$ as $L(f\otimes\phi_j, s)$. We use the duplication formula for the Gamma function, plug in $s = s_j$ and multiply by $4\pi i/(2s_j-1)$ to get

$$\frac{-\sqrt{\pi}}{2\pi^{s_j}(2s_j-1)}L(f\otimes\phi_j, s_j+1/2)\Gamma(s_j+1/2)$$

which gives Th. 1(c) and agrees up to a constant with the result in [15].

## 6. Proof of Theorem 4

If the value $L(f\otimes\phi_j, s_j+1/2)\neq 0$, then $\dot{D}_0(z,s)$ has definitely a pole at $s_j$. Since $D_0(z,s)$ is regular at $s_j$, the functions $D_\epsilon(z,s)$ should have poles $s_j(\epsilon)$ converging to $s_j$, as $\epsilon\to 0$.

*Remark* 12. According to [20] a pole of the perturbed Eisenstein series can occur if a cusp form eigenvalue becomes a scattering pole. This is so because for small $\epsilon$ the total multiplicity of the singular set in a small disc around $s_j$ remains constant. Our result creates the scattering pole out of the Phillips-Sarnak condition (1.13) without the use of the singular set. If we can show that a pole of $D_\epsilon(z,s)$ close to the unitary axis forces a Type (II) eigenvalue for $L_{\epsilon,a}$ for some $a$, then we would have a new proof of the destruction of cusp forms under (1.13).

## 7. Proof of Theorem 5

For simplicity we assume we have only one cusp and we fix $\sigma = \Re s > 1/2$. It follows from the Maaß-Selberg relations, as in [13, Lemma 8.8], that the scattering function $\phi_\epsilon(s)$ is bounded for $\Re s\geq 1/2$ and away from the finite number of poles in the interval $(1/2, 1]$. For the McDonald-Bessel function $K_s(x)$ the integral representation

$$K_s(x) = \int_0^\infty e^{-x\cosh t}\cosh(st)\,dt,$$

see [8, 8.432.1, p. 968], gives

$$|K_s(x)| \leq e^{-x/2}K_\sigma(2) \tag{7.1}$$

for $x\geq K$. This together with the polynomial bound on the Fourier coefficients of Eisenstein series gives

$$E_\epsilon(z,s) \ll_z y^\sigma + |\phi(s)|y^{1-\sigma} + O_\sigma(1) = O_{z,\sigma}(1). \tag{7.2}$$

Similarly $\partial_x E_\epsilon(z,s)\ll_z O(1)$. The estimates can clearly be made uniform in $z$ on compact sets. For $\partial_y E_\epsilon(z,s)$ we study $K'_s(x)$. Differentiating the integral in (7.1) we get

$$|K'_s(x)| \leq e^{-x/2}|K'_\sigma(2)|.$$

This implies that

$$\partial_y E_\epsilon(z,s) \ll_z |s|y^{\sigma-1} + |\phi(s)||1-s|y^{-\sigma} + O_z(1). \tag{7.3}$$

The estimates (7.2) and (7.3) show that both $E_\epsilon(z,s)$ and $(1/|t|)dE_\epsilon(z,s)$ are bounded on vertical lines for $\sigma > 1/2$. By (5.1) and (5.2) we get

$$\dot{D}_0(z,s) = -R_0(s)(-4\pi i\langle dD_0(z,s), w\rangle + 2\pi i(\delta w)D_0(z,s)).$$

The bounds for $D_0(z,s) = E_0(z,s)$ and its differential together with the fact that $\dot{L}$ has compact support give a polynomial bound for the $L^2$-norm of $\dot{L}D_0(z,s)$ in the $t$ aspect. Since

$$||R(z)|| \leq \frac{1}{\operatorname{dist}(z, \operatorname{spec} A)}$$

for the resolvent of a general self-adjoint operator $A$ on a Hilbert space and $\operatorname{dist}(s(1-s), \operatorname{spec} L_{0,a}) \geq |t|(2\sigma-1)$ we get

$$\dot{D}_0(z,s) \ll_\sigma \frac{|s|}{|t||2\sigma-1|}.$$

We last notice that the above estimate, together with (7.2) and (2.12) finish the proof of the theorem.

**Acknowledgments** The author would like to acknowledge the financial support of the Max-Planck-Institut für Mathematik during the completion of this project. The author would like to thank G. Chinta for helpful discussions, C. O'Sullivan for motivating Theorem 5 and P. Sarnak for a critical reading of the manuscript.

## References

[1] A. Borel; Stable real cohomology of arithmetic groups. II. Manifolds and Lie groups (Notre Dame, Ind., 1980), pp. 21–55, Progr. Math., 14, Birkhäuser, Boston, Mass., 1981.

[2] G. Chinta, D. Goldfeld; Grössencharakter $L$-functions of real quadratic fields twisted by modular symbols, Invent. Math **144**, 3 (2001) 435–449.

[3] Y. Colin de Verdière; Pseudo-laplaciens. II. Ann. Inst. Fourier (Grenoble) **33** (1983), no. 2, 87–113.

[4] N. Diamantis; C. O'Sullivan; Hecke theory of series formed with modular symbols and relations among convolution $L$-functions. Math. Ann. **318** (2000), no. 1, 85–105.

[5] D. Goldfeld; Zeta functions formed with modular symbols. Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), 111–121, Proc. Sympos. Pure Math., 66, Part 1, Amer. Math. Soc., Providence, RI, 1999.

[6] D. Goldfeld; The distribution of modular symbols. Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), 849–865, de Gruyter, Berlin, 1999.

[7] D. Goldfeld; C. O'Sullivan; in preparation.

[8] I. D. Gradshteyn, I. M. Ryzhik; Tables of Integrals, Series, and Products, 5th ed., ed. Alan Jeffrey, Academic Press, Boston, 1994.

[9] D. Hejhal; The Selberg trace formula for PSL(2, $R$). Vol. 2. Lecture Notes in Mathematics, 1001. Springer-Verlag, Berlin, 1983. viii+806pp.

[10] T. Kato; Perturbation theory for linear operators. Second edition. Grundlehren der Mathematischen Wissenschaften, Band 132. Springer-Verlag, Berlin-New York, 1976. xxi+619 pp.

[11] P. Lax, R. Phillips; Scattering theory for automorphic functions. Annals of Mathematics Studies, No. 87. Princeton Univ. Press, Princeton, N.J., 1976. x+300 pp.

[12] W. Müller; Spectral geometry and scattering theory for certain complete surfaces of finite volume. Invent. Math. **109** (1992), no. 2, 265–305.

[13] W. Müller, Spectral theory for Riemannian manifolds with cusps and a related trace formula. Math. Nachr. **111** (1983), 197–288.

[14] C. O'Sullivan; Properties of Eisenstein series formed with modular symbols. J. Reine Angew. Math. **518** (2000), 163–186.

[15] C. O'Sullivan; Ph. D. Thesis, Columbia University, 1998.

[16] Y. N. Petridis; Perturbation of scattering poles for hyperbolic surfaces and central values of $L$-series. Duke Math. J. **103** (2000), no. 1, 101–130.

[17] R. Phillips, P. Sarnak; Cusp forms for character varieties. Geom. Funct. Anal. 4 (1994), no. 1, 93–118.

[18] R. Phillips, P. Sarnak; The spectrum of Fermat curves. Geom. Funct. Anal. 1 (1991), no. 1, 80–146.

[19] R. Phillips, P. Sarnak; On cusp forms for co-finite subgroups of $\mathrm{PSL}(2, R)$. Invent. Math. **80** (1985), no. 2, 339–364.

[20] R. Phillips, P. Sarnak; Perturbation theory for the Laplacian on automorphic functions. J. Amer. Math. Soc. **5** (1992), no. 1, 1–32.

[21] R. Phillips, P. Sarnak; Geodesics in homology classes. Duke Math. J. **55** (1987), no. 2, 287–297.



Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, QC, Canada H3A 2K6

*E-mail address*: petridis@math.mcgill.ca