THE SEMI-CLASSICAL APPROXIMATION FOR MODULAR OPERADS

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The semi-classical approximation is an explicit formula of mathematical physics for the sum of Feynman diagrams with a single circuit. In this paper, we study the same problem in the setting of modular operads [5]; instead of being a number, the interaction at a vertex of valence n will be an $\mathbb{S}_n$-module.

The motivation for developing this theory was the desire to calculate the $\mathbb{S}_n$-equivariant Hodge polynomials of the Deligne-Mumford-Knudsen moduli spaces $\overline{\mathcal{M}}_{1,n}$ of stable curves of genus 1 with n marked smooth points. In performing these calculations, we use the formulas for the $\mathbb{S}_n$-equivariant Serre polynomials of $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{1,n}$ derived in [1] and [3] respectively.

A particular consequence of our calculations will be needed in [4] to find a relation among the codimension two cycles in $\overline{\mathcal{M}}_{1,4}$.

Theorem. *The $\mathbb{S}_4$-module $H^4(\overline{\mathcal{M}}_{1,4},\mathbb{Q})$ is isomorphic to*

$$\big(V_{(4)}\otimes\mathbb{Q}^7\big)\oplus\big(V_{(3,1)}\otimes\mathbb{Q}^4\big)\oplus\big(V_{(2,2)}\otimes\mathbb{Q}^2\big).$$

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1. Wick's theorem and the semi-classical approximation

Let $\Gamma_{g,n}$ be the small category whose objects are isomorphism classes of stable graphs G of genus $g(G)=g$ with n totally ordered legs [5], and whose morphisms are the automorphisms: if $G\in\Gamma_{g,n}$, its automorphism group $\operatorname{Aut}(G)$ is the subset of the permutations of the flags which preserve all the data defining the stable graph, including the total ordering of the legs. Because of the stability condition, $\Gamma_{g,n}$ is a finite category.

Define polynomials $\{\mathsf{M}v_{g,n}\mid 2(g-1)+n>0\}$ of a set of variables $\{v_{g,n}\mid 2(g-1)+n>0\}$ by the following formula:

$$\mathsf{M}v_{g,n}=\sum_{G\in\operatorname{Ob}\Gamma_{g,n}}\frac{1}{|\operatorname{Aut}(G)|}\prod_{v\in\operatorname{Vert}(G)}v_{g(v),n(v)}. \tag{1.1}$$

Introduce the sequences of generating functions

$$a_g(x)=\sum_{2(g-1)+n>0}v_{g,n}\frac{x^n}{n!},\quad\text{and}\quad b_g(x)=\sum_{2(g-1)+n>0}\mathsf{M}v_{g,n}\frac{x^n}{n!}.$$

Wick's theorem gives an integral formula for the generating functions $\{b_g\}$ in terms of $\{a_g\}$:

$$\sum_{g=0}^{\infty}b_g\hbar^{g-1}=\log\int_{-\infty}^{\infty}\exp\Big(\sum_{g=0}^{\infty}a_g\hbar^{g-1}-\frac{(x-\xi)^2}{2\hbar}\Big)\frac{dx}{\sqrt{2\pi\hbar}}.$$

As written, this is purely formal, since it involves the integration of a power series in x. It may be made rigourous by observing that the integral transform

$$f \longmapsto \int_{-\infty}^{\infty} f(\hbar, x) e^{-(x-\xi)^2/2\hbar} \frac{dx}{\sqrt{2\pi\hbar}}$$

induces a continuous linear map on the space of Laurent series $\mathbb{Q}((\hbar))[\![x]\!]$ topologized by the powers of the ideal $(\hbar, x)$.

The semi-classical expansion is a pair of formulas for b_0 and b_1 in terms of a_0 and a_1, which we now recall.

Definition (1.2). Let R be a ring of characteristic zero. The Legendre transform $\mathcal{L}$ is the involution of the set $x^2/2 + x^3 R[\![x]\!]$ characterized by the formula

$$(\mathcal{L}f) \circ f' + f = p_1 f'.$$

Theorem (1.3). *The series $x^2/2 + b_0$ is the Legendre transform of $x^2/2 - a_0$.*

The first few coefficients of b_0 may be calculated, either from the definition of $\mathsf{M}v_{0,n}$ or from Theorem (1.3):

n	$\mathsf{M}v_{0,n}$
3	$v_{0,3}$
4	$v_{0,4} + 3v_{0,3}^2$
5	$v_{0,5} + 10v_{0,4}v_{0,3} + 15v_{0,3}^3$
6	$v_{0,6} + 15v_{0,5}v_{0,3} + 10v_{0,4}^2 + 105v_{0,4}v_{0,3}^2 + 105v_{0,3}^4$

We now come to the formula for b_1, known as the semi-classical approximation.

Theorem (1.4). *The series b_1 and a_1 are related by the formula*

$$b_1 = \big(a_1 - \tfrac{1}{2}\log(1 - a_0'')\big) \circ (x + b_0').$$

By the definition of the Legendre transform, we see that $(\mathcal{L}f)' \circ f' = x$. It follows that Theorem (1.4) is equivalent to the formula

$$b_1 \circ (x - a_0') = a_1 - \tfrac{1}{2}\log(1 - a_0'').$$

This formula expresses the fact that the stable graphs contributing to b_1 are obtained by attaching a forest whose vertices have genus 0 to two types of graphs:

(i) those with a single vertex of genus 1 (corresponding to the term a_1);
(ii) stable graphs with a single circuit, and all of whose vertices have genus 0 — we call such a graph a *necklace*.

The presence of a logarithm in the term which contributes the necklaces is related to the fact that there are $(n-1)!$ cyclic orders of n objects.

The first few coefficients of b_1 are also easily calculated:

n	$\mathsf{M}v_{1,n}$
1	$v_{1,1} + \frac{1}{2}v_{0,3}$
2	$v_{1,2} + v_{1,1}v_{0,3} + \frac{1}{2}\big(v_{0,4} + v_{0,3}^2\big)$
3	$v_{1,3} + 3v_{1,2}v_{0,3} + v_{1,1}v_{0,4} + \frac{1}{2}\big(v_{0,5} + 3v_{0,4}v_{0,3} + 2v_{0,3}^3\big)$
4	$v_{1,4} + 6v_{1,3}v_{0,3} + 3v_{1,2}v_{0,4} + 15v_{1,2}v_{0,3}^2 + v_{1,1}v_{0,5}$ $+ \frac{1}{2}\big(v_{0,6} + 4v_{0,5}v_{0,3} + 3v_{0,4}^2 + 12v_{0,4}v_{0,3}^2 + 6v_{0,3}^4\big)$

2. The semi-classical approximation for modular operads

In the theory of modular operads, one replaces the sequence of coefficients $\{v_{g,n}\}$ considered above by a stable $\mathbb{S}$-module, that is, a sequence of $\mathbb{S}_n$-modules $\mathcal{V}((g,n))$. The analogue of (1.1) is the functor on stable $\mathbb{S}$-modules which sends $\mathcal{V}$ to

$$\mathbb{M}\mathcal{V}((g,n)) = \operatorname*{colim}_{G\in\Gamma_{g,n}} \bigotimes_{v\in\mathsf{Vert}(G)} \mathcal{V}((g(v),n(v))). \tag{2.1}$$

Thus, the coefficients in (1.1) are promoted to vector spaces, the product to a tensor product, the sum over stable graphs to a direct sum, and the weight $|\operatorname{Aut}(G)|^{-1}$ to $\operatorname{colim}_{\operatorname{Aut}(G)}$, that is, the coinvariants with respect to the finite group $\operatorname{Aut}(G)$. Note that this definition makes sense in any symmetric monoidal category $\mathcal{C}$ with finite colimits. We will need the Peter-Weyl theorem to hold for actions of the symmetric group $\mathbb{S}_n$ on $\mathcal{C}$; thus, we will suppose that $\mathcal{C}$ is additive over a ring of characteristic zero.

Definition (2.2). The characteristic $\operatorname{ch}_n(\mathcal{V})$ of an $\mathbb{S}_n$-module is defined by the formula

$$\operatorname{ch}_n(\mathcal{V}) = \frac{1}{n!}\sum_{\sigma\in\mathbb{S}_n} \operatorname{Tr}_\sigma(\mathcal{V})p_\sigma \in \Lambda_n \otimes K_0(\mathcal{C}),$$

where p_σ is the product of power sums $p_{|\mathcal{O}|}$ over the orbits $\mathcal{O}$ of σ.

Although this definition appears to require rational coefficients, this is an artifact of the use of the power sums p_n; it is shown in [2] that the characteristic is a symmetric function of degree n with values in the Grothendieck group of the additive category $\mathcal{C}$. If $\mathsf{rk} : \Lambda \to \mathbb{Q}[x]$ is the homomorphism defined by $h_n \mapsto x^n/n!$, we have

$$\operatorname{rk}(\operatorname{ch}_n(\mathcal{V})) = [\mathcal{V}]/n! \in K_0(\mathcal{C}) \otimes \mathbb{Q}.$$

Note that $\operatorname{rk}(f)$ is obtained from f by setting the powers sums p_n to 0 if $n > 1$, and to x if $n = 1$.

The place of the generating functions a_g and b_g is now taken by

$$\mathbf{a}_g = \sum_{2(g-1)+n>0} \operatorname{ch}_n(\mathcal{V}((g,n))) \in \Lambda\hat{\otimes}K_0(\mathcal{C}),$$
$$\mathbf{b}_g = \sum_{2(g-1)+n>0} \operatorname{ch}_n(\mathbb{M}\mathcal{V}((g,n))) \in \Lambda\hat{\otimes}K_0(\mathcal{C}).$$

Theorem (8.13) of [5], whose statement we now recall, calculates $\mathbf{b}_g$ in terms of $\mathbf{a}_h$, $h \le g$. Let Δ be the "Laplacian" on $\Lambda((\hbar))$ given by the formula

$$\Delta = \sum_{n=1}^{\infty} \hbar^n \left(\frac{n}{2}\frac{\partial^2}{\partial p_n^2} + \frac{\partial}{\partial p_{2n}} \right).$$

Theorem (2.3). *If $\mathcal{V}$ is a stable $\mathbb{S}$-module, then*

$$\sum_{g=0}^{\infty} \mathbf{b}_g \hbar^{g-1} = \operatorname{Log}\Big(\exp(\Delta) \operatorname{Exp}\Big(\sum_{g=0}^{\infty} \mathbf{a}_g \hbar^{g-1} \Big) \Big).$$

There is also a formula for $\mathbf{b}_0$ in terms of $\mathbf{a}_0$. To state it, we must recall the definition of the Legendre transform for symmetric functions. Let

$$\Lambda_*\hat{\otimes}K_0(\mathcal{C}) = \{ f \in \Lambda\hat{\otimes}K_0(\mathcal{C}) \mid \operatorname{rk}(f) = x^2/2 + O(x^3) \}.$$

If f is a symmetric function, let $f' = \partial f/\partial p_1$; this operation may be expressed more invariantly as $p_1^\perp$ (Ex. I.5.3, Macdonald [6]).

Definition (2.4). The Legendre transform $\mathcal{L}$ is the involution of $\Lambda_*\hat{\otimes}K_0(\mathcal{C})$ characterized by the formula $(\mathcal{L}f) \circ f' + f = p_1 f'$.

The Legendre transform $\mathcal{L}f$ of a function f is characterized by the formula $(\mathcal{L}f)'\circ f' = x$. For symmetric functions, although the analogue of this formula holds, in the form

$$(\mathcal{L}f)' \circ f' = h_1,$$

the situation is not as simple, since there is no single notion of integral for symmetric functions (the "constant" term may be any function of the power sums p_n, $n > 1$). Neverthless, there is a simple algorithm for calculating $\mathcal{L}f$ from f. Denote by f_n and g_n the coefficents of f and $g = \mathcal{L}f$ lying in $\Lambda_n \otimes K_0(\mathcal{C})$.

(i) The formula $f' \circ (\mathcal{L}f)' = h_1$ may be rewritten as

$$\sum_{n=3}^{N} g_n' + \sum_{n=3}^{N} f_n' \circ \Big(h_1 + \sum_{k=3}^{N-1} g_k' \Big) \cong 0 \mod \Lambda_N \otimes K_0(\mathcal{C}).$$

This gives a recursive procedure for calculating g_n'.

(ii) Having determined g', we obtain g from the formula $f = \mathcal{L}g$, or $g = p_1 g' - f \circ g'$.

We now recall Theorem (7.17) of [5], which is the generalization to modular operads of Theorem (1.3).

Theorem (2.5). *The symmetric function $h_2 + \mathbf{b}_0$ is the Legendre transform of $e_2 - \mathbf{a}_0$.*

The main result of this paper is a formula for $\mathbf{b}_1$ in terms of $\mathbf{a}_1$ and $\mathbf{a}_0$, generalizing Theorem (1.4). If f is a symmetric function, write $\dot{f} = \partial f/\partial p_2 = \frac{1}{2} p_2^{\perp} f$.

Theorem (2.6).

$$\mathbf{b}_1 = \left(\mathbf{a}_1 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(1 - \psi_n(\mathbf{a}_0'')) + \frac{\dot{\mathbf{a}}_0(\dot{\mathbf{a}}_0 + 1)}{1 - \psi_2(\mathbf{a}_0'')} \right) \circ (h_1 + \mathbf{b}_0')$$

Here, $\phi(n)$ is Euler's function, the number of prime residues modulo n.

Remark. The first two terms inside the parentheses on the right-hand side of Theorem (2.6) are analogues of the corresponding terms in the formula of Theorem (1.4). In particular, the second of these terms is closely related to the sum over necklaces in the definition of $\mathbb{M}\mathcal{V}((1,n))$, as is seem from the formula

$$\sum_{n=1}^{\infty} \mathrm{ch}_n \big(\mathrm{Ind}_{\mathbf{Z}_n}^{\mathbf{S}_n} \mathbb{1} \big) = - \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(1 - p_n).$$

The remaining term may be understood as a correction term, which takes into account the fact that necklaces of 1 or 2 vertices have non-trivial involutions (while those with more vertices do not). A proof of the theorem could no doubt be given using this observation; however, we prefer to derive it directly from Theorem (2.3).

If we take the plethysm on the right of the formula of Theorem (2.6) with the symmetric function $h_1 - \mathbf{a}_0'$, and apply the formula $(h_1 + \mathbf{b}_0') \circ (h_1 - \mathbf{a}_0') = h_1$, we obtain the equivalent formulation of this theorem:

$$\mathbf{b}_1 \circ (h_1 - \mathbf{a}_0') = \mathbf{a}_1 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log(1 - \psi_n(\mathbf{a}_0'')) + \frac{\dot{\mathbf{a}}_0(\dot{\mathbf{a}}_0 + 1)}{1 - \psi_2(\mathbf{a}_0'')}.$$

Proof of Theorem (2.6). The symmetric function $\mathbf{b}_1$ is a sum over graphs obtained by attaching forests whose vertices have genus 0 to either a vertex of genus 1, or to a necklace. In other words,

$$\mathbf{b}_1 = \big(\mathbf{a}_1 + \text{sum over necklaces} \big) \circ (h_1 + \mathbf{b}_0').$$

To prove the theorem, we must calculate the sum over necklaces.

To do this, observe that a necklace is a graph with flags coloured red or blue, such that each vertex has exactly two red flags, each edge is red, and all tails are blue. Let $\mathcal{W}((n))$, $n \geq 1$, be the sequence of representations of $\mathbb{S}_2 \times \mathbb{S}_n$

$$\mathcal{W}((n)) = \operatorname{Res}^{\mathbb{S}_{n+2}}_{\mathbb{S}_n \times \mathbb{S}_2} \mathcal{V}((0, n+2));$$

think of the first factor of the product $\mathbb{S}_n \times \mathbb{S}_2$ as acting on the blue flags at a vertex, and the second factor as acting on the red flags. Applying Theorem (2.3), we see that

$$\operatorname{Log}\big(\exp(1 \otimes \Delta) \operatorname{Exp}(\operatorname{Ch}(\mathcal{W}))\big) \in \Lambda \hat{\otimes} \Lambda \hat{\otimes} K_0(\mathcal{C})$$

is the sum over stable graphs all of whose edges are red. To impose the condition that all tails are blue, we set the variables q_n to zero before taking the Logarithm.

We now proceed to the explicit calculation. We set $\hbar = 1$, since it plays no rôle when all graphs have genus 1. In writing elements of $\Lambda \hat{\otimes} \Lambda$, we will denote power sums in the first factor of Λ by p_n, and in the second by q_n.

Lemma (2.7). *The characteristic* $\operatorname{Ch}(\mathcal{W})$ *of* $\mathcal{W}$ *is the "bisymmetric" function*

$$\operatorname{Ch}(\mathcal{W}) = \tfrac{1}{2}\mathbf{a}_0'' q_1^2 + \dot{\mathbf{a}}_0 q_2 \in \Lambda \hat{\otimes} \Lambda_2 \hat{\otimes} K_0(\mathcal{C}).$$

Proof. We have $\operatorname{Ch}(\mathcal{W}) = h_2^{\perp}\mathbf{a}_0 \otimes h_2 + e_2^{\perp}\mathbf{a}_0 \otimes e_2$. Expressing this in terms of power sums, we have

$$\begin{aligned} h_2^{\perp}\mathbf{a}_0 \otimes h_2 + e_2^{\perp}\mathbf{a}_0 \otimes e_2 &= \big(\tfrac{1}{2}(p_1^{\perp})^2 + p_2^{\perp}\big)\mathbf{a}_0 \otimes \tfrac{1}{2}(q_1^2 + q_2) + \big(\tfrac{1}{2}(p_1^{\perp})^2 - p_2^{\perp}\big)\mathbf{a}_0 \otimes \tfrac{1}{2}(q_1^2 - q_2) \\ &= \tfrac{1}{2}(p_1^{\perp})^2\mathbf{a}_0 \otimes q_1^2 + p_2^{\perp}\mathbf{a}_0 \otimes q_2. \quad \square \end{aligned}$$

From this lemma, it follows that

$$\operatorname{Exp}(\operatorname{Ch}(\mathcal{W})) = \prod_{n=1}^{\infty} \exp\Big(\psi_n(\mathbf{a}_0'')\frac{q_n^2}{2n}\Big) \prod_{n=1}^{\infty} \exp\Big(\psi_n(\dot{\mathbf{a}}_0)\frac{q_{2n}}{n}\Big) \in \Lambda \hat{\otimes} \Lambda \hat{\otimes} K_0(\mathcal{C}),$$

We now apply the heat kernel and separate variables:

$$\begin{aligned} \exp(1 \otimes \Delta) \operatorname{Exp}(\operatorname{Ch}(\mathcal{W}))\big|_{q_n=0} &= \prod_{n \text{ odd}} \exp\left(\frac{n}{2}\frac{\partial^2}{\partial q_n^2}\right) \exp\Big(\psi_n(\mathbf{a}_0'')\frac{q_n^2}{2n}\Big)\Big|_{q_n=0} \\ &\times \prod_{n \text{ even}} \exp\left(\frac{n}{2}\frac{\partial^2}{\partial q_n^2} + \frac{\partial}{\partial q_n}\right) \exp\Big(\psi_n(\mathbf{a}_0'')\frac{q_n^2}{2n} + \psi_{n/2}(\dot{\mathbf{a}}_0)\frac{2q_n}{n}\Big)\Big|_{q_n=0}. \end{aligned}$$

We now insert the explicit formulas for the heat kernel of the Laplacian, namely

$$\exp\left(\frac{n}{2}\frac{\partial^2}{\partial q_n^2}\right) f(q_n)\big|_{q_n=0} = \int_{-\infty}^{\infty} f(q_n) \exp\Big(-\frac{q^2}{2n}\Big)\frac{dq}{\sqrt{2\pi n}}.$$

For the odd variables, matters are quite straightforward:

$$\begin{aligned} \exp\left(\frac{n}{2}\frac{\partial^2}{\partial q_n^2}\right) \exp\Big(\frac{q_n^2}{2n}\psi_n(\mathbf{a}_0'')\Big)\Big|_{q_n=0} &= \int_{-\infty}^{\infty} \exp\left(\psi_n(\mathbf{a}_0'')\frac{q_n^2}{2n} - \frac{q_n^2}{2n}\right)\frac{dq_n}{\sqrt{2\pi n}} \\ &= \big(1 - \psi_n(\mathbf{a}_0'')\big)^{-1/2}. \end{aligned}$$

For the even variables, things become a little more involved:

$$\begin{aligned} \exp\left(\frac{n}{2}\frac{\partial^2}{\partial q_n^2} + \frac{\partial}{\partial q_n}\right) &\exp\Big(\psi_n(\mathbf{a}_0'')\frac{q_n^2}{2n} + \psi_{n/2}(\dot{\mathbf{a}}_0)\frac{2q_n}{n}\Big)\Big|_{q_n=0} \\ &= \exp\left(\frac{n}{2}\frac{\partial^2}{\partial q_n^2}\right) \exp\Big(\psi_n(\mathbf{a}_0'')\frac{q_n^2}{2n} + \psi_{n/2}(\dot{\mathbf{a}}_0)\frac{2q_n}{n}\Big)\Big|_{q_n=1} \\ &= \int_{-\infty}^{\infty} \exp\left(\psi_n(\mathbf{a}_0'')\frac{q_n^2}{2n} + \psi_{n/2}(\dot{\mathbf{a}}_0)\frac{2q_n}{n} - \frac{(q_n-1)^2}{2n}\right)\frac{dq_n}{\sqrt{2\pi n}}. \end{aligned}$$

To perform this gaussian integral, we complete the square in the exponent:

$$\begin{aligned}\psi_n(\mathbf{a}_0'')\frac{q_n^2}{2n}+\psi_{n/2}(\dot{\mathbf{a}}_0)\frac{2q_n}{n}-\frac{(q_n-1)^2}{2n}\\ &=-\big(1-\psi_n(\mathbf{a}_0'')\big)\frac{q_n^2}{2n}+\big(2\psi_{n/2}(\dot{\mathbf{a}}_0)+1\big)\frac{q_n}{n}-\frac{1}{2n}\\ &=-\frac{1-\psi_n(\mathbf{a}_0'')}{2n}\Big(q_n-\frac{2\psi_{n/2}(\dot{\mathbf{a}}_0)+1}{1-\psi_n(\mathbf{a}_0'')}\Big)^2+\frac{2}{n}\Big(\frac{\psi_{n/2}(\dot{\mathbf{a}}_0)\big(\psi_{n/2}(\dot{\mathbf{a}}_0)+1\big)}{1-\psi_n(\mathbf{a}_0'')}\Big).\end{aligned}$$

Thus, the gaussian integral equals

$$\big(1-\psi_n(\mathbf{a}_0'')\big)^{-1/2}\exp\frac{2}{n}\Big(\frac{\psi_{n/2}(\dot{\mathbf{a}}_0)\big(\psi_{n/2}(\dot{\mathbf{a}}_0)+1\big)}{1-\psi_n(\mathbf{a}_0'')}\Big).$$

Putting these calculations together, we see that

$$\begin{aligned}\exp(1\otimes\Delta)\operatorname{Exp}(\operatorname{Ch}(\mathcal{W}))|_{q_n=0}&=\prod_{n=1}^\infty\big(1-\psi_n(\mathbf{a}_0'')\big)^{-1/2}\exp\frac{1}{n}\Big(\frac{\psi_n(\dot{\mathbf{a}}_0)\big(\psi_n(\dot{\mathbf{a}}_0)+1\big)}{1-\psi_{2n}(\mathbf{a}_0'')}\Big)\\ &=\prod_{n=1}^\infty\big(1-\psi_n(\mathbf{a}_0'')\big)^{-1/2}\operatorname{Exp}\Big(\frac{\dot{\mathbf{a}}_0(\dot{\mathbf{a}}_0+1)}{1-\psi_2(\mathbf{a}_0'')}\Big),\end{aligned}$$

and, applying the operation Log, that

$$\operatorname{Log}\big(\exp(1\otimes\Delta)\operatorname{Exp}(\operatorname{Ch}(\mathcal{W}))|_{q_n=0}\big)=\operatorname{Log}\prod_{n=1}^\infty\big(1-\psi_n(\mathbf{a}_0'')\big)^{-1/2}+\frac{\dot{\mathbf{a}}_0(\dot{\mathbf{a}}_0+1)}{1-\psi_2(\mathbf{a}_0'')}.$$

The proof of the theorem is completed by the following lemma., applied to $f=1-\mathbf{a}_0''$.

Lemma (2.8). *Let $f\in\Lambda\hat{\otimes}K_0(\mathcal{C})$ have constant term equal to 1; that is, $\operatorname{rk}(f)=1+O(x)$. Then*

$$\operatorname{Log}\prod_{n=1}^\infty\psi_n(f)^{-1/2}=-\frac{1}{2}\sum_{n=1}^\infty\frac{\phi(n)}{n}\log(\psi_n(f)).$$

Proof. By definition,

$$\operatorname{Log}\prod_{n=1}^\infty\psi_n(f)^{-1/2}=\sum_{k=1}^\infty\frac{\mu(k)}{k}\log\prod_{n=1}^\infty\psi_{nk}(f)^{-1/2}=-\frac{1}{2}\prod_{n=1}^\infty\Big(\sum_{d|n}\frac{\mu(d)}{d}\Big)\log(\psi_n(f)).$$

The lemma follows from the formula

$$\sum_{d|n}\frac{\mu(d)}{d}=\frac{\phi(n)}{n},$$

which follows by Möbius inversion from $\sum_{d|n}\phi(d)=n$. □

Corollary (2.9). *Define $a_g=\operatorname{rk}(\mathbf{a}_g)$, $b_g=\operatorname{rk}(\mathbf{b}_g)$, and $\dot{a}_0=\operatorname{rk}(\dot{\mathbf{a}}_0)$. Then we have*

$$a_1\circ(x-a_0')=a_1-\tfrac{1}{2}\log(1-a_0'')+\dot{a}_0(\dot{a}_0+1).$$

Example (2.10). Suppose $\mathcal{V}((0,n))=\mathbb{1}$ is the trivial one-dimensional representation for all $n\geq 3$, while $\mathcal{V}((1,n))=0$. Then $\mathbb{M}\mathcal{V}((1,n))$ is an $\mathbb{S}_n$-module whose rank is the number of graphs in $\Gamma^0_{1,n}$, where $\Gamma^0_{1,n}\subset\Gamma_{1,n}$ is the subset of stable graphs all of whose vertices have genus 0. We have

$$\mathbf{a}_0=\sum_{n=3}^\infty h_n=\exp\Big(\sum_{n=1}^\infty\frac{p_n}{n}\Big)-1-h_1-h_2.$$

Theorem (2.6) leads to the following results; the calculations were performed using J. Stembridge's symmetric function package SF for maple [7].

n	$\mathrm{ch}_n(\mathbb{M}\mathcal{V}((1,n)))$	$\lvert\Gamma^0_{1,n}\rvert$
1	s_1	1
2	$3\,s_2$	3
3	$7\,s_3 + 4\,s_{21}$	15
4	$20\,s_4 + 17\,s_{31} + 14\,s_{2^2} + 4\,s_{21^2}$	111
5	$52\,s_5 + 78\,s_{41} + 71\,s_{32} + 33\,s_{31^2} + 34\,s_{2^21} + 4\,s_{21^3} + s_{1^5}$	1104

An explicit formula for the generating function of the numbers $\lvert\Gamma^0_{1,n}\rvert$ may be obtained from Corollary (2.9), using the formulas $a_0' = e^x - 1 - x$, $a_0'' = e^x - 1$ and $\dot a_0 = \frac{1}{2}(e^x - 1)$.

Proposition (2.11).

$$\sum_{n=1}^{\infty} \lvert\Gamma^0_{1,n}\rvert \frac{x^n}{n!} = \Big(-\frac{1}{2}\log(2-e^x) + \frac{1}{4}(e^{2x}-1)\Big) \circ (1+2x-e^x)^{-1}.$$

3. The $\mathbb{S}_n$-equivariant Hodge polynomial of $\overline{\mathcal{M}}_{1,n}$

A more interesting application of Theorem (2.6) is to the stable $\mathbb{S}$-module in the category of $\mathbb{Z}$-graded mixed Hodge structures

$$\mathcal{V}((g,n)) = H^\bullet_c(\mathcal{M}_{g,n}, \mathbb{C}).$$

Let KHM be the Grothendieck group of mixed Hodge structures. The $\mathbb{S}_n$-equivariant Serre polynomial $\mathsf{e}^{\mathbb{S}_n}(\mathcal{M}_{g,n})$ is by definition the characteristic $\mathrm{ch}_n(\mathcal{V}((g,n))) \in \Lambda_n \otimes \mathsf{KHM}$. It follows from the usual properties of Serre polynomials (see [2] or Proposition (6.11) of [5]) that $\mathrm{ch}_n(\mathbb{M}\mathcal{V}((g,n)))$ is the $\mathbb{S}_n$-equivariant Serre polynomial $\mathsf{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{g,n})$ of the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves. Since the moduli space $\overline{\mathcal{M}}_{g,n}$ is a complete smooth Deligne-Mumford stack, its kth cohomology group carries a pure Hodge structure of weight k; thus, the Hodge polynomial of $\overline{\mathcal{M}}_{g,n}$ may be extracted from $\mathsf{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{g,n})$. Using Theorem (2.6), we will calculate the Serre polynomials $\mathsf{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{1,n})$.

It is shown in [1] (see also [2]) that

$$\mathbf{a}_0 = \sum_{n=3}^{\infty} \mathsf{e}^{\mathbb{S}_n}(\mathcal{M}_{0,n}) = \frac{\Big\{\prod_{n=1}^{\infty}(1+p_n)^{\frac{1}{n}\sum_{d|n}\mu(n/d)(1+\mathsf{L}^d)}\Big\} - 1}{\mathsf{L}^3 - \mathsf{L}} - \frac{h_1}{\mathsf{L}^2 - \mathsf{L}} - \frac{h_2}{\mathsf{L}+1},$$

where L is the pure Hodge structure $\mathbb{C}(-1)$ of weight 2. Theorem (2.5) implies that

$$h_2 + \mathbf{b}_0 = h_2 + \sum_{n=3}^{\infty} \mathsf{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{0,n})$$

is the Legendre transform of $e_2 - \mathbf{a}_0$; this was used in [1] to calculate $\mathsf{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{0,n})$.

Let S_{2k+2} be the pure Hodge structure $\mathrm{gr}^W_{2k+1} H^1_c(\mathcal{M}_{1,1}, \mathrm{Sym}^{2k}\mathsf{H})$, where H is the local system $R^1\pi_*\mathbb{Q}$ of rank 2 over the moduli stack of elliptic curves. (Here, $\pi : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,1}$ is the universal elliptic curve.) This Hodge structure has the following properties:

(i) $\mathsf{S}_{2k+2} = F^0\mathsf{S}_{2k+2} \oplus \overline{F^0\mathsf{S}_{2k+2}}$;

(ii) there is a natural isomorphism between $F^0\mathsf{S}_{2k+2}$ and the space of cusp forms S_{2k+2} for the full modular group $\mathrm{SL}(2,\mathbb{Z})$. (In particular, $\mathsf{S}_{2k+2} = 0$ for $k \le 4$.)

It is shown in [3] that

$$\mathbf{a}_1=\sum_{n=1}^{\infty}\mathrm{e}^{\mathbb{S}_n}(\mathcal{M}_{1,n})=\operatorname{res}_0\Bigg[\Bigg(\frac{\prod_{n=1}^{\infty}(1+p_n)^{\frac{1}{n}\sum_{d|n}\mu(n/d)(1-\omega^d-\mathsf{L}^d/\omega^d+\mathsf{L}^d)}-1}{1-\omega-\mathsf{L}/\omega+\mathsf{L}}\Bigg) \times\Bigg(\sum_{k=1}^{\infty}\Big(\frac{\mathsf{S}_{2k+2}+1}{\mathsf{L}^{2k+1}}\Big)\omega^{2k}-1\Bigg)(\omega-\mathsf{L}/\omega)\,d\omega\Bigg],$$

where $\operatorname{res}_0[\alpha]$ is the residue of the one-form α at the origin.

We may now apply Theorem (2.6) to calculate the generating function of the $\mathbb{S}_n$-equivariant Serre polynomials $\mathrm{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{1,n})$. We do not give the details, since they are quite straightforward, though the resulting formulas are tremendously complicated when written out in full. However, we do present some sample calculations, performed with the package SF.

n	$\mathrm{e}(\overline{\mathcal{M}}_{1,n})$	$\chi(\overline{\mathcal{M}}_{1,n})$
1	$(\mathsf{L}+1)s_1$	2
2	$(\mathsf{L}^2+2\mathsf{L}+1)s_2$	4
3	$(\mathsf{L}^3+3\mathsf{L}^2+3\mathsf{L}+1)s_3+(\mathsf{L}^2+\mathsf{L})s_{21}$	12
4	$(\mathsf{L}^4+4\mathsf{L}^3+7\mathsf{L}^2+4\mathsf{L}+1)s_4+(2\mathsf{L}^3+4\mathsf{L}^2+2\mathsf{L})s_{31}+(\mathsf{L}^3+2\mathsf{L}^2+\mathsf{L})s_{2^2}$	49
5	$(\mathsf{L}^5+5\mathsf{L}^4+12\mathsf{L}^3+12\mathsf{L}^2+5\mathsf{L}+1)s_5+(3\mathsf{L}^4+11\mathsf{L}^3+11\mathsf{L}^2+3\mathsf{L})s_{41}$ $+(2\mathsf{L}^4+7\mathsf{L}^3+7\mathsf{L}^2+2\mathsf{L})s_{32}+(\mathsf{L}^3+\mathsf{L}^2)(s_{32}+s_{2^21})$	260

In a table at the end of the paper, we give a table of non-equivariant Serre polynomials of $\overline{\mathcal{M}}_{1,n}$ for $n\le 15$; these give an idea of the way in which the Hodge structures S_{2k+2} typically enter into the cohomology. In particular, we see that the even-dimensional cohomology of the moduli spaces $\overline{\mathcal{M}}_{1,n}$ is spanned by Hodge structures of the form $\mathbb{Q}(\ell)$, while the odd dimensional cohomology is spanned by Hodge structures of the form $\mathsf{S}_{2k+2}(\ell)$.

The rational cohomology groups of $\overline{\mathcal{M}}_{1,n}$ satisfy Poincaré duality: there is a non-degenerate $\mathbb{S}_n$-equivariant pairing of Hodge structures

$$H^k(\overline{\mathcal{M}}_{1,n},\mathbb{Q})\otimes H^{2n-k}(\overline{\mathcal{M}}_{1,n},\mathbb{Q})\longrightarrow\mathbb{Q}(-n).$$

Unfortunately, our formula for $\mathrm{e}^{\mathbb{S}_n}(\overline{\mathcal{M}}_{1,n})$ does not render this duality manifest.

4. The Euler characteristic of $\overline{\mathcal{M}}_{1,n}$

As an application of Corollary (2.9), we give an explicit formula for the generating function of the Euler characteristics $\chi(\overline{\mathcal{M}}_{1,n})$.

Theorem (4.1). *Let $g(x)\in x+x^2\mathbb{Q}[\![x]\!]$ be the solution of the equation*

$$2g(x)-(1+g(x))\log(1+g(x))=x.$$

Then

$$\sum_{n=1}^{\infty}\chi(\overline{\mathcal{M}}_{1,n})\frac{x^n}{n!}=-\frac{1}{12}\log\big(1+g(x)\big)-\frac{1}{2}\log\big(1-\log(1+g(x))\big)+\epsilon(g(x)),$$

where

$$\epsilon(x)=\frac{1}{12}\big(19\,x+23\,x^2/2+10\,x^3/3+x^4/2\big).$$

Proof. We apply Corollary (2.9) with the data

$$\begin{aligned}
a_0' &= \sum_{n=2}^{\infty} \chi(\mathcal{M}_{0,n+1}) \frac{x^n}{n!} = \sum_{n=2}^{\infty} (-1)^n (n-2)! \frac{x^n}{n!} = (1+x)\log(1+x) - x, \\
a_0'' &= \log(1+x), \quad \dot{a}_0 = \frac{1}{4} x(x+2), \\
a_1 &= \chi(\mathcal{M}_{1,1}) x + \chi(\mathcal{M}_{1,2}) \frac{x^2}{2} + \chi(\mathcal{M}_{1,3}) \frac{x^3}{6} + \chi(\mathcal{M}_{1,4}) \frac{x^4}{24} + \frac{1}{12} \sum_{n=5}^{\infty} (-1)^n (n-1)! \frac{x^n}{n!} \\
&= x + \frac{x^2}{2} - \frac{1}{12} \log(1+x) + \frac{1}{12} \Big(x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \Big),
\end{aligned}$$

where we have used that $\chi(\mathcal{M}_{1,1}) = \chi(\mathcal{M}_{1,2}) = 1$ and $\chi(\mathcal{M}_{1,3}) = \chi(\mathcal{M}_{1,4}) = 0$. The function $g(x)$ of the statement of the theorem is $x + \mathbf{b}_0'(x)$. □

The following corollary was shown us by D. Zagier.

Corollary (4.2).

$$\chi(\overline{\mathcal{M}}_{1,n}) \sim \frac{(n-1)!}{4(e-2)^n} \Big(1 + C n^{-1/2} + O(n^{-3/2}) \Big),$$

where

$$C = \sqrt{\frac{e-2}{18\pi e}} (1 + 4e + 9e^2 + 4e^3 + 2e^4) \approx 18.31398807.$$

Proof. To show this, we analytically continue $g(x)$ to the domain $\mathbb{C} \setminus [e-2, \infty)$. The resulting function has an asymptotic expansion of the form

$$g(x) \sim e - 1 - \sqrt{2e(e-2-x)} + \sum_{k=3}^{\infty} a_k (e-2-x)^{k/2}.$$

The asymptotics ((4.2)) follow by applying Cauchy's integral formula to the right-hand side of Theorem (4.1), with contour the circle $|x| = e-2$. □

The peculiar polynomial $\epsilon(x)$ of Theorem (4.1) combines the error terms in the formula for $\chi(\mathcal{M}_{1,n})$ with the correction terms involving $\dot{a}_0$ in Corollary (2.9). Omitting the term $\epsilon(g(x))$ in Theorem (4.1), we obtain the generating function not of the Euler characteristics $\chi(\overline{\mathcal{M}}_{1,n})$, but rather of the virtual Euler characteristics $\chi_v(\overline{\mathcal{M}}_{1,n})$ of the underlying smooth moduli stack (orbifold). The asymptotic behaviour of the virtual Euler characteristics is the same as that of the Euler characteristics, with C replaced by $\widetilde{C} = \big(\frac{e-2}{18\pi e}\big)^{1/2} \approx 0.06835794$. The ratio between these Euler characteristics has the asymptotic behaviour

$$\frac{\chi(\overline{\mathcal{M}}_{1,n})}{\chi_v(\overline{\mathcal{M}}_{1,n})} \sim (C - \widetilde{C}) n^{-1/2} + O(n^{-1}),$$

giving a statistical measure of the ramification of $\overline{\mathcal{M}}_{1,n}$ for large n.

n	$\mathsf{e}(\overline{\mathcal{M}}_{1,n})$
1	$\mathsf{L}+1$
2	$\mathsf{L}^2+2\,\mathsf{L}+1$
3	$\mathsf{L}^3+5\,\mathsf{L}^2+5\,\mathsf{L}+1$
4	$\mathsf{L}^4+12\,\mathsf{L}^3+23\,\mathsf{L}^2+12\,\mathsf{L}+1$
5	$\mathsf{L}^5+27\,\mathsf{L}^4+102\,\mathsf{L}^3+102\,\mathsf{L}^2+27\,\mathsf{L}+1$
6	$\mathsf{L}^6+58\,\mathsf{L}^5+421\,\mathsf{L}^4+756\,\mathsf{L}^3+421\,\mathsf{L}^2+58\,\mathsf{L}+1$
7	$\mathsf{L}^7+121\,\mathsf{L}^6+1612\,\mathsf{L}^5+5077\,\mathsf{L}^4+5077\,\mathsf{L}^3+1612\,\mathsf{L}^2+12\,\mathsf{L}+1$
8	$\mathsf{L}^8+248\,\mathsf{L}^7+5802\,\mathsf{L}^6+31072\,\mathsf{L}^5+52402\,\mathsf{L}^4+31072\,\mathsf{L}^3+5802\,\mathsf{L}^2+248\,\mathsf{L}+1$
9	$\mathsf{L}^9+503\,\mathsf{L}^8+19925\,\mathsf{L}^7+175036\,\mathsf{L}^6+480097\,\mathsf{L}^5+480097\,\mathsf{L}^4+175036\,\mathsf{L}^3+19925\,\mathsf{L}^2+503\,\mathsf{L}+1$
10	$\mathsf{L}^{10}+1014\,\mathsf{L}^9+66090\,\mathsf{L}^8+920263\,\mathsf{L}^7+3975949\,\mathsf{L}^6+6349238\,\mathsf{L}^5+3975949\,\mathsf{L}^4+920263\,\mathsf{L}^3+66090\,\mathsf{L}^2+1014\,\mathsf{L}+1$
11	$\mathsf{L}^{11}+2037\,\mathsf{L}^{10}+213677\,\mathsf{L}^9+4577630\,\mathsf{L}^8+30215924\,\mathsf{L}^7+74269967\,\mathsf{L}^6+30215924\,\mathsf{L}^5+\dots+1-\mathsf{S}_{12}$
12	$\mathsf{L}^{12}+4084\,\mathsf{L}^{11}+677881\,\mathsf{L}^{10}+21793602\,\mathsf{L}^9+213725387\,\mathsf{L}^8+784457251\,\mathsf{L}^7+1196288936\,\mathsf{L}^6+\dots+4084\,\mathsf{L}+1-11(\mathsf{L}+1)\mathsf{S}_{12}$
13	$\mathsf{L}^{13}+8179\,\mathsf{L}^{12}+2120432\,\mathsf{L}^{11}+100226258\,\mathsf{L}^{10}+1424858788\,\mathsf{L}^9+7603002045\,\mathsf{L}^8+17095248952\,\mathsf{L}^7+\dots$
	$-(66\,\mathsf{L}^2+429\,\mathsf{L}+66)\mathsf{S}_{12}$
14	$\mathsf{L}^{14}+16370\,\mathsf{L}^{13}+6563147\,\mathsf{L}^{12}+448463866\,\mathsf{L}^{11}+9049174765\,\mathsf{L}^{10}+68547770726\,\mathsf{L}^9+221071720149\,\mathsf{L}^8+324314241400\,\mathsf{L}^7+\dots$
	$-(286\,\mathsf{L}^3+6006\,\mathsf{L}^2+286\,\mathsf{L})\mathsf{S}_{12}$
15	$\mathsf{L}^{15}+32753\,\mathsf{L}^{14}+20153930\,\mathsf{L}^{13}+1963368663\,\mathsf{L}^{12}+55228789080\,\mathsf{L}^{11}+581636563570\,\mathsf{L}^{10}+2627427327522\,\mathsf{L}^9+5488190927216\,\mathsf{L}^8+\dots$
	$-(1001\,\mathsf{L}^4+53053\,\mathsf{L}^3+186263\,\mathsf{L}^2+53053\mathsf{L}+1001)\mathsf{S}_{12}-\mathsf{S}_{16}$

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