Notes on Affine Hecke Algebras I.

(Degenerated Affine Hecke Algebras and Yangians in Mathematical Physics)

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1 Introduction

The main aim of these notes, based on my lectures at Bonn University (May-June, 1990), is to demonstrate that degenerated affine Hecke algebras (and Yangians) are quite natural from mathematical and physical points of view, and are necessary to understand the very classic object — the symmetric group S_n. There is no need to discuss much its important role in mathematics and natural sciences, the representation theory and the theory of combinations. But S_n is a key thing not only for science.

The great philosopher Immanuel Kant found that our consciousness is based on two a priori notions of space and time. I like his books very much and consider him as the best philosopher in Europe. Nevertheless, I think he made a mistake. One should add a third notion to these two. I mean the *notion of combination.*

I do not attempt to discuss this point here. One can find easily many arguments in favor of the thesis that calculus of permutations (transpositions) interchanges is the main work our brain can do and likes to do. Gambles give good models. They are among the most exciting things for human beings. Moreover, the primitive ones are more exciting, because here the fundamental concept of combination is as pure as possible. However, gambling is nothing else but the applied representation theory of S_n.

It is strange enough for me that the symmetric group is not a necessary thing to study for students in physical departments. The space-time is, but not S_n. I can understand to a certain extent why it is so for mathematicians (they do not try to understand the universe and for them consistent notions are on equal grounds). As for physics (old and new) its most significant parts (e.g. the diagrammatic method, the string theory, the two-dimensional conformal theory) are very closely connected with combinatorics. Moreover, the modern representation theory was created as a base for quantization and is now such a base. But its classic ground is undoubtedly in Young's works on S_n.

I shall begin now the detailed exposition of the simplest example of a physical theory based on S_n only. This one is the most natural way to quantum groups. We will discuss A. Zamolodchikov's world of *two-dimensional elementary factorized particles.* Let the space be $\mathbb{R}$ with the only coordinate x,and let t be the usual time. The life of a free particle can be represented as a line in $\mathbb{R}^2$. This line (the graph of the movement of the particle) can be determined (see fig. 1) by some (initial) point $(t_0, x(t_0))$ and by the *angle* θ from the t-axis to it $(-\pi/2 < \theta < \pi/2)$. The momentum of the particle is $p = mtg(\theta)$ (in the proper units), where m is its mass.

Let us suppose that the masses of all the particles are the same and

(a) *there is no mechanical interaction.*

It means that individual momenta of two or more particles are conserved at the points of intersections of corresponding lines. In some evident sense the particles are transparent to each other (see fig. 2). It is not a billiard.

This world is too boring. We should add some quantum scattering to make the life of these particles more interesting. Let us consider "coloured" particles of types (colours) $1, 2, ..., N$. The particles are permitted to

(b) *change the colours (types) only at the points of intersections.*

It is the second postulate. Moreover,

(c) *the S-matrix of any collision (the set of all the amplitudes) does not depend on concrete initial positions of the particles.*

I should comment on axiom (c). One can always slightly deform the initial positions (the x-coordiantes $x_1, ..., x_n$ at the time $t = t_0$) to have only two-particle intersections in the future and in the past like in fig. 2. By the standard principles of quantum mechanics any element of the total S-matrix can be obtained as a certain sum of products of two-particle amplitudes over the intersection points. Thus, it results from (c) that if we know two-particle S-matrices we can calculate any amplitude, depending in fact only on the angles and on the order of the points of the in-state and out-state.

The last things we have to declare to obtain Zamolodchikov's world are some locality (or causality) properties and some invariances:

(d) *every two-particle S-matrix depends only on the colours of these two particles and on the difference of their angles.*

Other particles do not affect it.

A few words about the origin of all these properties. It was conjectured in Zamolodchikov brothers' paper [1] that the quantum scattering processes should be of this type for O(N)-symmetric chiral field models in two-dimensions like Pohlmeyer's σ-model. Besides the asymptotic freedom and the existence of isovector N-plets of massive particles ("coloured particles") the following property is of great importance. This quantum model possesses an infinite set of conservation laws extending their classical counterparts. It gives us the conservation of individual momenta.

Zamolodchikov's conjecture was proved (at some physical level of strictness). I am not able to discuss the details here. It is worth mentioning that particles in [1] and in other papers are relativistic. Therefore, the authors use rapidities θ ($p^0 = mch(\theta), p^1 = msh(\theta)$ for the relativistic momentum $p = (p^0, p^1)$) instead of angles. The dependence of S-matrices on differences of angles (see (d)) is nothing else but the relativistic invariance. Some unitary conditions and crossing-symmetry relations are of physical importance as well. We will omit the latter here and pay little attention to the first.

2 Yang-Baxter identities

Let us describe a sequence of n particles at the moment $t = t_0$ with the x-coordinates $x_1 < x_2 < \dots < x_n$ (see fig. 2) by the *symbol* $A_J(\Theta) = A_{j_1}(\theta_1)\cdots A_{j_n}(\theta_n)$, where $\Theta = (\theta_1, \cdots, \theta_n)$ and $J = (j_1, \cdots, j_n)$ are the corresponding sets of the angles and the colours ($-\pi/2 < \theta_k < \pi/2, 1 \le j_k \le N$). Owing to property (c) the previous or subsequent changes of a set of particles depend only on its symbol $A_J(\Theta)$ (on the colours, angles and on the order of x-coordinates only). Given $A_I(\Theta')$ one has

$$A_I(\Theta')_{in} = \sum_J S_I^J(\Theta, \Theta') A_J(\Theta)_{out} \tag{1}$$

for some $I = (i_1, \cdots, i_n), \Theta' = (\theta'_1, \cdots, \theta'_n)$ describing the set of particles at $t = t'_0 < t_0$. We have expressed $A_I(\Theta')$ considered as an *in-state* in terms *out-states*. Every (scalar) coefficient S_I^J is the *S-matrix element* (the amplitude) from $A_I(\Theta')$ to $A_J(\Theta)$ (by definition). Here I and J can be arbitrary ($1 \le i_k, j_k \le N$) but not the set Θ'. The S-matrix is nontrivial only if

$$\Theta' = w(\Theta) = \left(\theta_{w^{-1}(1)}, \theta_{w^{-1}(2)}, \cdots \theta_{w^{-1}(n)}\right) \tag{2}$$

for an appropriate permutation w from the symmeric group S_n.

In this formula some misunderstanding is possible. I will comment on it. Any permutation $w : (1, 2, ..., n) \to (1', 2', \cdots, n')$ acts on an ordered set $s = (x, y, z, \cdots,)$ of some elements (e.g. coordinates) by the substitution the element at place No. i for the element at place No. i'. For example, the transposition $w = (12) : (1, 2, 3, \cdots, n) \to 21$ interchanges the content of the first and second places. This definition results in the natural formula $v(w(s)) = (v \cdot w)(s), w, v \in S_n$. We see that w^{-1} is necessary in the second equality of (2). In fig. 4 the corresponding w is equal to

(4,5,1,3,2) in the one-line notations or $\begin{smallmatrix}1&2&3&4&5\\4&5&1&3&2\end{smallmatrix}$ in the two-line notation, i.e. takes (1 2 3 4 5) to (3 5 4 1 2) ; $w^{-1} = \binom{1\,2\,3\,4\,5}{3\,5\,4\,1\,2} = (3,5,4,1,2)$

Let us discuss examples. We will omit the indications "in" and "out". One has for $n = 2$ and $\theta_1 < \theta_2, \theta_1' = \theta_2, \theta_2' = \theta_1$:

$$A_{i_1}(\theta_2)A_{i_2}(\theta_1) = \sum_{i_1,i_2} S_{i_1 i_2}^{j_1 j_2}(\theta_{12})A_{j_1}(\theta_1)A_{j_2}(\theta_2), \tag{3}$$

where $\theta_{ij} = \theta_i - \theta_j$ by definition. Given $I = (i_1, i_2, i_3), J = (j_1, j_2, j_3)$ in the case of fig. 3a we obtain the following relations:

$$\begin{aligned} A_{i_1}(\theta_3)A_{i_2}(\theta_2)A_{i_3}(\theta_1) &= \sum S_{i_2 i_3}^{k_2 k_3}(\theta_{12})A_{i_1}(\theta_3)A_{k_2}(\theta_1)A_{k_3}(\theta_2) \\ &= \sum S_{i_2 i_3}^{k_2 k_3}(\theta_{12})S_{i_1 k_2}^{j_1 l_2}(\theta_{13})A_{j_1}(\theta_1)A_{l_2}(\theta_3)A_{k_3}(\theta_2) \\ &= \sum S_{i_2 i_3}^{k_2 k_3}(\theta_{12})S_{i_1 k_2}^{j_1 l_2}(\theta_{13})S_{l_2 k_3}^{j_2 j_3}(\theta_{23})A_J(\Theta), \end{aligned}$$

where the sum is over all free indices $(j_1, j_2, j_3, k_2, k_3, l_2)$. The analogical calculation for fig. 3b should give the same result. We arrive at the identity:

$$\sum_{k_2,k_3,l_2} S_{i_2 i_3}^{k_2 k_3}(\theta_{12})S_{i_1 k_2}^{j_1 l_2}(\theta_{13})S_{l_2 k_3}^{j_2 j_3}(\theta_{23}) = \sum_{k_1,k_2,l_2} S_{i_1 i_2}^{k_1 k_2}(\theta_{23})S_{k_2 i_3}^{l_2 j_3}(\theta_{13})S_{k_1 l_2}^{j_1 j_2}(\theta_{12}) \tag{4}$$

Let us rewrite (4) in a tensor form. We will keep the following notations. Let us consider "multi-matrices" $T = \left(T_{i_1\ i_2\ \cdots\ i_n}^{j_1\ j_2\ \cdots\ j_n}\right)$ with the multi-indices $I = (i_1, \cdots, i_n), J = (j_1, \cdots, j_n)$, respectively, of rows and columns ($1 \le i_k, j_k \le N$ for $1 \le k \le n$). These T act on "multi-vectors" $x = (x_{i_1 i_2 \cdots i_n})$ by the natural formula $Tx = (\sum_J T_I^J x_J)$. If multi-indices are assumed to be (lexicographically) ordered then x and T are usual vectors and matrices for $\mathbf{C}^{N^n}$ in place of $\mathbf{C}^N$. Given two N x N-matrices $X = (X_i^j), Y = (Y_i^j)$ (from the matrix algebra M_N) one can define the tensor product $T = X \otimes Y : T = \left(T_{i_1\ i_2}^{j_1\ j_2}\right), T_{i_1\ i_2}^{j_1\ j_2} = X_{i_1}^{j_1} Y_{i_2}^{j_2}$. The definition of $X \otimes Y \otimes Z \otimes \cdots$ is quite analogous.

Later on, δ_i^j will be the Kronecker symbol. Put

$$\begin{aligned} {}^k X &= \left(\prod_{m \ne k} \delta_{i_m}^{j_m}\right) X_{i_k}^{j_k}, \\ {}^{kl} T &= \left(\prod_{m \ne k,l} \delta_{i_m}^{j_m}\right) T_{i_k i_l}^{j_k j_l} \end{aligned} \tag{5}$$

for $X = (X_i^j), T = (T_{i_1 i_2}^{j_1 j_2}), 1 \le k \ne l \le n, 1 \le m \le n$. These matrices are the natural images of X, T in the $M_N^{\otimes n} = M_{N^n}$ with respect to the indices k and (k,l). Note that ${}^{kl}(X \otimes Y) = {}^k X {}^l Y$ and ${}^k X$ commutes with ${}^l Y$ for any $X, Y \in M_N, k \ne l$.

Let us introduce $S(\theta)$ as the following matrix (depending on $\theta = \theta_{12}$) from $M_N^{\otimes 2}$: $S = \left(S_{i_1\ j_2}^{j_1\ j_2}(\theta)\right)$. Now one can represent (4) in the elegant form:

$${}^{23}S(\theta_{12})^{12}S(\theta_{13})^{23}S(\theta_{23}) = {}^{12} S(\theta_{23})^{23}S(\theta_{13})^{12}S(\theta_{12}). \tag{6}$$

If we put $S(\theta) = PR(\theta)$ for

$$P = \left(P_{i_1\ i_2}^{j_1\ j_2}\right), P_{i_1\ i_2}^{j_1\ j_2} = \delta_{i_1}^{j_2}\delta_{i_2}^{j_1} \tag{7}$$

we get the *Yang-Baxter equation*

$${}^{12}R(\theta_{12})^{13}R(\theta_{13})^{23}R(\theta_{23}) = {}^{23}R(\theta_{23})^{13}R(\theta_{13})^{12}R(\theta_{12}). \tag{8}$$

To check it (to deduce it from (6)) one should carry all the ${}^{12}P, {}^{23}P$ across the other terms in (6) after the substitution $S = PR$. Here we have to use the following properties of P:

$$ {}^{23}P\,{}^{12}P\,{}^{23}P = {}^{12}P\,{}^{23}P\,{}^{12}P, \tag{9a} $$

$$ {}^{12}P\,{}^{1}X = {}^{2}X\,{}^{12}P \quad \text{for} \quad X \in M_N. \tag{9b} $$

It is easy either to prove (9) directly or verify them without matrix calculations using the following natural interpretation of ${}^{ij}P$.

3 Factorization

By definition

$$ P(x \otimes y) = \left(\sum_{j_1, j_2} \delta^{j_2}_{i_1} \delta^{j_1}_{i_2} x_{j_1} y_{j_2} \right) = y \otimes x, x, y \in \mathbf{C}^n $$

This relation proves (9b) and give us that both sides of (9a) induce the same permutations of components under the above action of $M_N \otimes M_N \otimes M_N$ on $\mathbf{C}^N \otimes \mathbf{C}^N \otimes \mathbf{C}^N$ by left multiplications. Hence, (9a) is true. Moreover, we have the following more general property.

Let us denote by id the identical permutation and by $s_1, s_2, \cdots, s_{n-1}$, the *adjacent transpositions* (12), (23) and so on. Then given $w = (1', 2', \cdots, n') \neq id$ one has

$$ w = s_{i_l} \cdots s_{i_1} \tag{10} $$

for some indices $1 \leq i_1, \cdots, i_l \leq n-1$ and some l. If l is of minimal possible length then it is called *the length of* w (written $l(w)$)). We see that the product

$$ P_w = P_{s_{i_l}} \cdots P_{s_{i_1}} \quad \text{for} \quad P_{s_i} = {}^{i\,i+1}P \tag{10'} $$

acts on $(\mathbf{C}^N)^{\otimes n}$ as the permutation of components:

$$ P_w(x^1 \otimes x^2 \otimes \cdots \otimes x^n) = x^{w^{-1}(1)} \otimes x^{w^{-1}(2)} \otimes \cdots \otimes x^{w^{-1}(n)}, x^1, \cdots, x^n \in \mathbf{C}^N \quad (cf.(2)) $$

Hence, the matrix $Z = (Z^J_I)$, where $Z^J_I = \delta^{j_1}_{i'_1}, \delta^{j_2}_{i'_2} \cdots \delta^{j_n}_{i'_n}$, coincides with P_w. In particular, P_w is independent of the choice of decomposition (10).

The latter can be proven in a more abstract way without the above interpretation of P_w as some interchange of components. Which properties of $P_{s_1}, \cdots, P_{s_{n-1}}$ should one check to prove that the right-hand-side product of (10′) does not depend on the choice of decomposition? If we consider in (10) only products of minimal length (*reduced decompositions*), then they are as follows

$$ P_{s_i} P_{s_{i+1}} P_{s_i} = P_{s_{i+1}} P_{s_i} P_{s_{i+1}} \quad (1 \leq i < n) \tag{11a} $$

$$ P_{s_i} P_{s_j} = P_{s_j} P_{s_i} \qquad | i - j | > 1. \tag{11b} $$

The reason is that *relations (11) together with $P^2_{s_i} = 1$ are the defining ones for* S_n *as an abstract group* . In fact, it will be proven below by means of some pictures. Thus, we have two ways to prove the correctness of (10′). For S one has a priori the second way only.

Formula (11a) (being equivalent to (9a)) is very close to (6). There is the analog of (11b) for $S_i = {}^{i\,i+1}S$:

$$ S_i S_j = S_j S_i \text{ for } \quad | i - j | > 1. \tag{12} $$

Indeed, here S_i and S_j live in different components of the tensor product. For brevity, arguments have been omitted. Only two things are different. Firstly, we have the arguments $\theta_{12}, \theta_{13}, \theta_{23}$ in (6). Secondly, we do not know any interpretation of $\{S\}$ as some permutations. In the next sections we will find such an interpretation for our basic example of Yang's S. But now let us go back to (10).

All possible decompositions of $w \in S_n$ of minimal length $l = l(w)$ are in one-to-one correspondence with collisions for $A(\Theta) = A(\theta_1, \cdots, \theta_n)$ as an out-state and $A(w(\Theta)) = A(\theta_{w^{-1}(1)}, \cdots, \theta_{w^{-1}(n)})$ as an in-state (see (1)).

Indices I, J can be omitted here (nothing depends on them). Given a collision one can get a sequence $s_{i_l}, \cdots, s_{i_1}$ by writing s_{i_k} one after the other. The element s_{i_k} should stay at $t = t_k$, being the moment of the k-th intersection (of the i_k-th and (i_k+1)-th lines, where we number lines according to the position of their x-coordinates at $t = t_k + \epsilon(\epsilon > 0)$ from the bottom to the top. Then the consecutive product $s_{i_l} \cdots s_{i_1}$ is equal to w and $l = l(w)$. Look at fig. 4. It is clear. The converse (from (10) to some picture) can be proven by induction on l. It results from the above statement that (11) and $\{P_{s_i}^2 = 1\}$ are defining relations for S_n. By the way, it is evident from the pictures (like fig. 4) that there is only one element of maximal length $w_0 = (n, n-1, \cdots 2, 1)$ and $l(w_0) = n(n-1)/2$.

Now we are in a position to put down the formula for the set $S_I^J(\Theta, \Theta')$ from (1) considered as a multi-matrix function. Let $S(\Theta, \Theta') = (S_I^J(\Theta, \Theta'))$, where $\Theta' = w(\Theta)$ for some $w \in S_n$ (see (2)), $w = s_{i_l} \cdots s_{i_1}$ being a reduced decomposition of the *length* $= l = l(w)$ (see (10)), corresponding to some collision. Then we can use the same approach as we obtained (6) from (4). One gets the formula

$$S = S_{i_l}(s_{i_1} \cdots s_{i_1}(\Theta)) \cdots S_{i_2}(s_{i_1}\Theta) S_{i_1}(\Theta), \tag{13a}$$

$$S_i(\Theta) =^{i\,i+1} S(\theta_i - \theta_{i+1}), \qquad 1 \le i \le n. \tag{13b}$$

The best way to prove the independence of S of the choice of decomposition (10) is to pass from a given product for w to any other by some continious deformations of initial points. The only transformations in the formulas will be of type (6) for some indices in the place of 1,2,3, or like in formula (12). This reasoning is, in fact, due to R. Baxter.

4 An algebraic interpretation

Let us interpret formulas (6), (12), and (13) algebraically. We will now consider symbols $A_{i_1}(\theta) \cdots A_{i_n}(\theta_n)$ as products of generators $A_{i_1}(\theta), \cdots, A_{i_n}(\theta_n)$ in the free algebra TA with generators $A_i(\theta)$ (they have two indices i, θ, where $1 \le i \le N$, θ can be any number). The quotient algebra $\mathcal{A}$ of TA by relations (3) is called *Zamolodchikov's algebra*. We suppose here and further that S is analytical for Θ close to $0 = (0, \cdots 0)$ and $S(0) = 1$. The latter is quite natural physically (there should be no scattering, when two particles are "parallel" one to another). Here we identify 1 with $1 \otimes 1$ for 1 being the unit matrix.

Let us impose on S relation (6) and the so-called *unitaty condition*

$$S(\theta)\, S(-\theta) =^{12} 1 = 1 \otimes 1 = 1. \tag{14}$$

Given $\Theta = (\theta_1, \cdots, \theta_n)$ in some neighbourhood of 0 we wil denote by $TA\{\Theta\}$ the vector subspace in TA generated by $A_I(w(\Theta))$ for any $I = (i_1, \cdots, i_n), w \in S_n$. Let $\mathcal{A}\{\Theta\}$ be its image in $\mathcal{A}$.

Relations (6), (14) are in fact equivalent to a certain Wick (or Poincaré-Birkhoff-Witt) theorem for every $\mathcal{A}\{\Theta\}$. Namely (see e.g. [1,2]),

(a) *every $A_I(w(\Theta))$ is a linear combination of some $A_J(\Theta)$ in $\mathcal{A}\{\Theta\}$;*

(b) *all the $A_J(\Theta)$ are linearly independent in $\mathcal{A}\{\Theta\}$ for any multi-indices J.*

Here $\theta_1, \cdots \theta_n$ can be arbitrary complex numbers (possibly not distinct different). The appearance of (14) is quite evident. If one applies (3) twice, he should get the initial binomial because of (b). Physically (14) corresponds to the change of time t from t_0' to $t_0 > t_0'$ then back.

Our other problem is to find the best algebraic language for relations (6), (12), and formula (13). We will start with S_i (see (13b)). Let us denote by $S_w(\Theta)$ the product (13a) for w from (10). Then the set of functions $\{S_w(\Theta)\}$ satisfies the following conditions (cf. [3])

$$S_x(y(\Theta))S_y(\Theta) = S_{xy}(\Theta), \tag{15a}$$

if

$$l(xy) = l(x) + l(y), \quad x, y \in \mathrm{S}_n. \tag{15b}$$

(Deduce it from (13)).

Conversally, let $S_i = S_{s_i}$ be some undetermined functions of θ with values anywhere and let relation (12) together with

$$S_{i+1}(u)S_i(u+v)S_{i+1}(v) = S_i(v)S_{i+1}(u+v)S_i(v) \tag{16}$$

be valid for arbitrary $u = \theta_i - \theta_{i+1}, v = \theta_{i+1} - \theta_{i+2}$.

Then we claim that one can uniquely define the set of functions $\{S_w(\Theta)\}$ by (15). Moreover, it is possible to omit the condition (15b) in the case of unitary $S(S(\theta)S(-\theta) = 1)$.

Summarizing, we see that the tensor mode of rewriting (4) in the form of (6) is very convenient but not the best. The most natural way is to use S_n as the index set (see (15) and (16)).

5 Yang's S-matrix

Let us discuss the basic example of a *factorized S-matrix* (a solution of (6)). The following one is the so-called Yang's S-matrix:

$$S(\theta) = 1 + P\theta, \theta = \theta_{12} = \theta_1 - \theta_2. \tag{17}$$

Here and further we will denote by 1 the unit matrix in $M_N, 1 \otimes 1$ and so on. Setting $S(\theta)_\eta = 1\eta(\theta+\eta)^{-1} + P\theta(\theta+\eta)^{-1}$ for any η we get an unitary S-matrix, satisfying (6) and (14). In particular, $S(\theta)_0 = P$ corresponds to a world without any scattering. To verify (6) for S (or S_η) is an easy exercise. Nevertheless, one can wish to prove (6) without any calculations like it was made for P (see above). The best way is to find an interpretation of S as a transposition of something (P interchanges the tensor components). I know four ways to do this. Two of them are based, respectively, on some algebraic geometry (see [2]) and on the theory of the so-called Knizhnik-Zamolodchikov equation from the two-dimensional conformal field theory (see [4,5]). I will explain here and below only other two making use of (degenerated) affine Hecke algebras and the so-called Yangians [6].

Mathematically, the idea is simple enough. Let us substitute some operators Y_i for $\theta_i (1 \le i \le n)$ in $S_i(\Theta) =^{i\, i+1} S(\theta_i - \theta_{i+1})$, where S is from (17). We assume $\{Y_i\}$ to be pairwise commutative and impose the following conditions

$$S_i(Y_i - Y_{i+1})Y_i = Y_{i+1}S_i(Y_i - Y_{i+1}), \tag{18a}$$

$$S(Y_i - Y_{i+1})Y_{i+1} = Y_iS(Y_i - Y_{i+1}), \tag{18b}$$

$$S(Y_i - Y_{i+1})Y_j = Y_jS(Y_i - Y_{i+1}), \quad \text{for} \quad j \ne i, i+1 \tag{18c}$$

Formulas (18) are equivalent to the relations

$$Y_{i+1}s_i - s_iY_i = 1 = s_iY_{i+1} - Y_is_i \tag{19a}$$

$$Y_js_i = s_iY_j \quad \text{for} \quad j \neq i, i+1 \tag{19b}$$

Here and further we will identify $P_{s_i} =^{i\,i+1} P$ with s_i and 1 with 1. Let us check e.g. the first of these formulas. It results from (18a) that

$$Y_{i\,i+1} = (Y_{i+1}s_i - s_iY_i)Y_{i\,i+1}, Y_{ij} = Y_i - Y_j. \tag{20}$$

Then one can divide (20) by $Y_{i\,i+1}$. We see that (19) $\Rightarrow$ (18), but the converse holds true only for $\{Y_i\}$ in a "general position".

We have arrived at the following object. Let $\mathbf{C}[\mathrm{S}_n] =_w \oplus \mathbf{C}w$ be the *group algebra* of $\mathrm{S}_n \ni w$ with the natural multiplication law: $w \cdot w' = ww' \in \mathrm{S}_n$ (e.g. $(a + (12)) \cdot (b + (23)) = ab + a(23) + b(12) + (3,1,2), a, b \in \mathbf{C}$). It is nothing else but the algebra of formal linear combinations of permutations. Its extension by pairwise commutative symbols $Y_1, \cdots, Y_n$ with relations (19) is called *the degenerated affine Hecke algebra* (written $\mathcal{H}'_n$). It is due to Murphy and Drinfeld (see [6]). Starting with S_η instead of S we get $\mathcal{H}_n(\eta)$, where η should stay for 1 in (19a). The algebra $\mathcal{H}'_n(\eta)$ is isomorphic to $\mathcal{H}'_n$ for $\eta \neq 0$ (use the substitution $\{Y_i \to \eta Y_i\}$). But this η is important to understand $\mathcal{H}'_n$ as some quantum object.

To differentiate $S_i(\Theta)$ from ${}^{i\,i+1}S(Y_{i\,i+1})$ (see (13b)) let us denote the latter by :

$$\Sigma_i = 1 + s_i(Y_i - Y_{i+1}), \qquad 1 \leq i < n$$

The main point is that (16) is equivalent to the following identity in $\mathcal{H}'_n$:

$$\Sigma_i\Sigma_{i+1}\Sigma_i = \Sigma_{i+1}\Sigma_i\Sigma_{i+1} \qquad (1 \leq 1 < n). \tag{21}$$

To prove the equivalence we need some kind of Wick (or Poincaré-Birckhof-Witt) theorem for $\mathcal{H}'_n$. One can deduce directly from (19) that

> *each element $A \in \mathcal{H}'_n$ has the unique representation of the following type: $A = \sum_w wy_w$, where $w \in \mathrm{S}_n, y_w$ are some polynomials in $Y_1, \cdots, Y_n$.*

Let us denote this sum for A after the converse substitution $Y_i \to \theta_i$ by $< A >$. The only thing we need is to show that $< \Sigma_i\Sigma_{i+1}\Sigma_i >$ and $< \Sigma_{i+1}\Sigma_i\Sigma_{i+1} >$ coincide, respectively, with the l.h.s and r.h.s of (16). Let us carry all the Y_i, Y_{i+1}, Y_{i+2} in (21) over Σ_i, Σ_{i+1} by means of (18) from the left to the right. Then one obtains $Y_i - Y_{i+1}, Y_i - Y_{i+2}$ and $Y_{i+1} - Y_{i+2}$ instead of $Y_{i\,i+1}, Y_{i+1\,i+2}, Y_{i\,i+1}$ in the l.h.s of (21) and the same elements but in the opposite order in the r.h.s. These differences are exactly what we need. By the way, Σ_η in the natural notations is involutive in $\mathcal{H}'_n(\eta)$, i.e. $(\Sigma_\eta)(\Sigma_\eta) = 1$ (cf. (14)). The next theorem (see [7], proposition 3.1 and [8]) results directly from (21) and (18).

> Theorem 1. *The collection of elements $\{\Sigma_i\}$ from $\mathcal{H}'_n$ extends uniquely to the set $\{\Sigma_w, w \in \mathrm{S}_n\} \subset \mathcal{H}'_n$ with the following properties:*
>
> (a) $\quad \Sigma_x\Sigma_y = \Sigma_{xy}$ if $\ell(xy) = \ell(x) + \ell(y)$, $\Sigma_{id} = 1$
>
> (b) $\quad \Sigma_wY_i\Sigma_w^{-1} = Y_{w(i)}$, $w \in \mathrm{S}_n$, $1 \leq i \leq n$.

Here (a) is in fact (15). This property can be deduced from (b) (or (18)). Formulas (21), (6) are particular cases of this property. Therefore, the Yang-Baxter relation for Yang's S is a direct consequence of the definition of $\mathcal{H}'_n$. Let us discuss this point.

We see that the l.h.s and r.h.s of (a) induce (operating by conjugations) the same permutation of $\{Y_i\}$. Hence, the product $\Sigma_x \Sigma_y$ should be equal to Σ_{xy} modulo multiplications by some elements from the centralizer (commutant) of $\{Y_1, \cdots, Y_n\}$ in $\mathcal{H}'_n$. It is not difficult to prove that this centralizer coincides with the algebra $\mathbb{C}[Y_1, \cdots, Y_n]$ of polynomials of $\{Y_i\}$ (see theorem 3). In particular, (6) for Yang's S has to be true up to a multiplication by a scalar function in $\theta_1, \theta_2, \theta_3$ (use the Wick theorem for $\mathcal{H}'_n$). Then it is easy to get (6) from this weaker statement. Thus, we have verified, in principle, the Yang-Baxter identity without any calculations. Only by means of formula (b), which is the definition of $\mathcal{H}'_n$.

6 Quantization of angles

I shall try to interpret this mathematical trick as some quantization procedure. We will look for observables which correspond to $\theta_1, \cdots, \theta_n$. (I shall remind that $mtg\theta_i$ is the momentum of the i-th particle. Therefore, a quantization of angles is, in fact, a quantization of impulses).

Yang's S-matrix (17) is of a very symmeric type. Any in-state $A_{i_1}(\theta'_1) \cdots A_{i_n}(\theta'_n)$ for $\Theta' = w(\Theta)$ (see (1,2)) can be expressed in terms of $A_J(\Theta)$, where $J = x(I)$ for permutations $x \in \mathrm{S}_n$. Hence, it is natural to diminish the space of states. We fix Θ and some initial set of indices I. Let

$$A_x^w = A_{x(I)}(\Theta'),\ \Theta' = w(\Theta), x, w \in \mathrm{S}_n. \tag{22}$$

Here w and x play different roles. The orderings of $\theta_1, \cdots, \theta_n$ are indexed by w. They are in one-one correspondence with the "sectors" — the connected components of $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n, x_i \neq x_j$ for $1 \leq i \neq j \leq n\}$. Therefore, we will use the visiual name "sector" in place of "ordering". The states (for every given sector) are numbered by x. By definition, $P_x A_y^w = A_{xy}^w, x, y \in \mathrm{S}_n$.

The natural (but wrong) idea is to introduce the quantum angles $Y_1, \cdots, Y_n$ by relations $Y_i(A_x^w) = \theta_{w^{-1}(i)} A_x^w$. If $\{A_x^w\}$ were independent it might be possible. But they are linearly dependent. However, one can try the following:

$$Y_i(A_{id}^w) = \theta_{w^{-1}(i)} A_{id}^w. \tag{23}$$

Given I we define the action of $\{Y_i\}$ only on the "vacuum" states $A_{id}^w = A_{i_1 \ldots i_n}(\Theta')$ for each sector. Let us assume that $N \geq n$ and all $i_1, \cdots, i_n$ are pairwise distinct (for example $I = (1, 2, \cdots, n)$). Then the number of sectors (i.e. orderings of $\{\theta_i\}$) is equal to the number of states for each of them. Therefore, the definition (23) is, in principle, consistent. If the set $I = (i_1, \cdots, i_n)$ is not "generic" we should be more precise (we will not consider this case here).

All the sectors are glued together by the S-matrices $\{S_w(\Theta)\}$ (see (15)). In particular, $A_{id}^w = S_w(\Theta) A_{id}^{id}$. Identifying A_x^{id} with $x, \oplus_{x \in \mathrm{S}_n} \mathbb{C} A_x^{id}$ with $\mathbb{C}[\mathrm{S}_n]$ and P_x with x we obtain the basis $\{S_w(\Theta)\}$ of eigenvectors for $\{Y_1, \cdots, Y_n\}$ in the group algebra $\mathbb{C}[\mathrm{S}_n]$:

$$Y_i(S_w(\Theta)) = \theta_{w^{-1}(i)} S_w(\Theta),\ w \in \mathrm{S}_n,\ 1 \leq i \leq n. \tag{23'}$$

All these are true for $\theta_1, \cdots, \theta_n$ being in a general position only. Simple calculations show that $\{Y_i\}$ and $\{s_j\}$ satisfy the relations (18-19), where S_n acts on $\mathbb{C}[\mathrm{S}_n]$ by left multiplications (cf. [9]). Deduce this statement from (23').

We have collected the vacuum states $\{A_{id}^w\}$ (they linearly generate all the states) together in the space of states $\oplus_{x \in \mathrm{S}_n} \mathbb{C} A_x^{id}$ for the initial sector $\Theta' = \Theta$. Starting with other sectors we will obtain some isomorphic representations of $\mathcal{H}'_n$. The S-matrices will be *interwiners* between these "sector" representations. In fact, this interpretation of S is very close to the ideology of superselection sectors (see [10,11]). We will not discuss here the latter, but formulate the corresponding mathematical theorem. As a matter of fact it has been partially proven.

Theorem 2 (see [8,9]). *Given $\Theta = (\theta_1, \cdots, \theta_n)$ let us denote by $M(\Theta)$ the space $\mathbf{C}[\mathrm{S}_n]$ with the natural left (regular) action of S_n and the action of $Y_1, \cdots, Y_n \in \mathcal{H}'_n$, which can be uniquely determined by means of the following relations*

$$Y_i(1) = \theta_i \cdot 1\,,\ 1 = id \in \mathrm{S}_n\,,\ 1 \leq i \leq n. \tag{24}$$

Then $M(\Theta)$ is an irreducible $\mathcal{H}'_n$-module for Θ being in a general position ($\theta_i - \theta_j \neq 1$ for any i, j). The operator $\mathbf{C}[\mathrm{S}_n] \ni z \rightarrow z \cdot S_w(\Theta) \in \mathbf{C}[\mathrm{S}_n]$ gives an isomorphism $M(w(\Theta)) \rightarrow M(\Theta)$, which appears to be a $\mathcal{H}'_n$-isomorphism.

This theorem is in fact equivalent to theorem 1. Indeed, any $S_w(\Theta)$ considered as a function of $\theta_1, \cdots, \theta_n$ with its values in $\mathbf{C}[\mathrm{S}_n]$ is equal to $< \Sigma_w >$ (coincides with Σ_w after the substitution $\{Y_i \rightarrow \theta_i, 1 \leq i \leq n\}$, where all the $\{Y\}$ should be collected on the right). Therefore, the isomorphism above is a direct corollary of statement (b) of theorem 1. The irreducibility of $M(\Theta)$ is clear, because $\{S_w(\Theta), w \in \mathrm{S}_n\}$ form a basis in $\mathbf{C}[\mathrm{S}_n]$ of eigenvectors with respect to $\{Y_i\}$ with pairwise distinct eigenvalues (for Θ in a general position).

It is worth mentioning that $\mathcal{H}'_n$ is more natural than $\mathbf{C}[\mathrm{S}_n]$ from some other physical point of view. Let us summarize its "quantum" properties.

Theorem 3
(a) *The subalgebras $\mathcal{Y} = \mathbf{C}[Y_1, \cdots, Y_n]$ is a maximal commutative in $\mathcal{H}'_n$ i.e. the commutant (centralizer) of $\mathcal{Y}$ coincides with $\mathcal{Y}$.*
(b) *The centre C (commutant) of $\mathcal{H}'_n$ consists of all symmetric polinomials in $Y_1, \cdots, Y_n$ (due to I. Bernstein).*
c) *Haag-duality. The commutant of $\mathcal{H}'_m \subset \mathcal{H}'_n$, where $\mathcal{H}'_m$ is generated by $s_1, \cdots, s_{m-1}$ and $Y_1, \cdots, Y_m$, is equal to $C\mathcal{H}'_{n-m}$ generated by $C, s_{m+1}, \cdots, s_{n-1}, Y_{m+1}, \cdots, Y_n$ (see e.g. [7]).*

Compare (c) with the corresponding axiom from [10]. As for S_n the commutant of $\mathbf{C}[\mathrm{S}_m] \subset \mathbf{C}[\mathrm{S}_n]$ modulo the centre is more than complimentary $\mathbf{C}[\mathrm{S}_{n-m}]$.

Let us consider $\mathcal{H}'_n(\eta)$ (see above) with the relations $Y_{i+1}s_i - s_iY_i = \eta = s_iYi + 1 - Y_is_i$ in place of (19a). Here η plays the role of the Planck constant $\hbar$. For $\eta = 0$ we get the algebra $\mathcal{H}'_n(0)$, which is the "quasi-classical" limit of $\mathcal{H}'_n(\eta)$ and especially simple. For example, it is evident that any element $x \in \mathcal{H}'_n(0)$ can be represented in the form $x = \sum wy_w, w \in \mathrm{S}_n$ for appropriate polinomials $y_w = y_w(Y_1, \cdots, Y_n)$. Moreover, if $y_w \neq 0$ for some $w \neq id$ then $xY_k \neq Y_kx$ for any k such that $w(k) \neq k$. Indeed, $xY_k = Y_kx \Rightarrow y_w(Y_k - Y_{w^{-1}(k)} \equiv 0$. The latter is impossible. In particular, the subalgebra $\mathcal{Y} = \mathbf{C}[Y_1, \cdots, Y_n]$ coincides with its commutant in $\mathcal{H}'_n(0)$.

There is a nice mathematical trick to extend the above statement to any η sufficiently close to 0. To calculate this commutant for any η one should solve *linear* equations for coefficients of the polinomials $\{y_w, w \in \mathrm{C}_n\}$ in the decomposition $x = \Sigma wy_w$. If the commutant contains $x(\eta)$ with $y_w \neq 0$ for some $w \neq id$ then certain determinants of minors are to be equal to zero (and vice versa). To be more precise, given $k \in \mathbf{Z}_+$ the rank of the above system for polinomials y_w of degree $\leq k$ for such η is less than the corresponding rank for $\eta = 0$. The determinants are scalar polynomial functions in η. Some of them do not equal zero at $\eta = 0$ ($x = y_{id}$ for $\eta = 0$). Hence, they have no common zeroes not only at 0 but in a neighbourhood of $\eta = 0$. Therefore, rank $(\eta) =$ rank(0) and $x(\eta) \in \mathcal{Y}$ in this neighbourhood. We have proved the required statement for small $|\,\eta\,|$. But any $\mathcal{H}'_n(\eta)$ for $\eta \neq 0$ is isomorphic to $\mathcal{H}'_n(1) = \mathcal{H}'_n$ (see above). Hence, the coincidence of $\mathcal{Y}$ and its commutant holds true for arbitrary η as well.

The best way to prove (a, b, c) is to use the following statement.

each element $A \in \mathcal{H}'_n$ has the unique representation: $A = \Sigma_w \Sigma_w y_w$, where $w \in \mathrm{S}_n$, Y_w are some rational function in $y_1, \cdots, Y_n, \{\Sigma_w\}$ are from theorem 1.

7 Yangians

These algebras are the basic example of quantum groups. I think that they (and their q-analogs) should be more important for mathematics and physics than q-analogs of universal enveloping algebras being now in common use. You can find some mathematical arguments in favor of Yangians in [12]. Here I will try to demonstrate only that they are physically natural and give us another interpretation of Yang's S-matrix as an interwiner.

To introduce (explain) quantum groups one can follow Faddeev's ideology (the quantum inverse scattering method - [13]) or its particular case - Drinfeld's way [14]. Faddeev's point of view (as far as I understand it) is that a quantum group is more or less equivalent to the corresponding Bethe-ansatz (R-matrix's or not). To be more precises, it should be some hidden composition law of the latter. For Drinfeld the main prolem was to extend a given classical r-matrix to the quantum one. I'll try to explain here that it is quite possible to come to quantum groups without the concept of R-matrices and the inverse scattering technique.

The very first step for any scheme of quantization of a given Lie group G (or its Lie algebra g) is to place at each point z of some space-time the generators $\{g_\alpha\}$ of g with the natural commutation relations

$$[g_\alpha(z), g_\beta(z')] = \sum_\gamma c^\gamma_{\alpha\beta} g_\gamma(z)\delta(z-z') \tag{25a}$$

where $[g_\alpha, g_\beta] = \sum_\gamma c^\gamma_{\alpha\beta} g_\gamma$ in g. The r.h.s of (25a) can, in principle, have Schwinger and other terms. Let $g = g\ell_N$ (i.e. g is M_N considered as a Lie algebra) $\{g_\alpha\} = \{e_{k\ell}, 1 \le l, \ell \le N\}$, where $e_{k\ell} = 1^{k\ell} = (\delta^k_i \delta^\ell_j)$ has the only unit at place (i,j). The natural way to introduce states and observables is based on some initial representations V of g. Let V be $\mathbb{C}^N$ with the standard action of $g\ell_N$.

The first problem is to define the tensor product $\mathcal{V} = \otimes_z V(z)$ over all points of the space-time, where $V(z)$ is $\mathbb{C}^N$ at z:

$$e_{k\ell}(z')v(z) = (e_{k\ell}v)(z)\delta(z-z') \ \text{ for } \ v(z) \in V(z). \tag{25b}$$

To solve it one should choose some vacuum state and consider only such states, that are "close" to the vacuum (see works on von Neumann factors). The second problem is to introduce an algebra of observables $\mathcal{A}$ operating in $\mathcal{V}$ (see e.g. [10,11]). The pair $\{\mathcal{V}, \mathcal{A}\}$ is a *quantum group* by definition.

Elements of $\mathcal{A}$ can be expressed in terms of $\{e_{k\ell}(z)\}$. But one should avoid to include $\{e_{k\ell}(z)$ in $\mathcal{A}\}$. The latter is to be *the least* to make $\mathcal{V}$ irreducible with respect to the action of $\mathcal{A}$. The last (obscure enough) property and other similar principles give one some intuition. But, in fact, it is impossible to differ good and bad $\mathcal{A}$ without dealing with concrete physical problems.

Assume that the space-time is finite (written $z = 1, \cdots, n$). First of all, it is natural to include in $\mathcal{A}$ the elements $\sum_{z=1}^n e_{k\ell}(z)$ for any k, ℓ. One can add $\sum_{z=1}^{n-1}\sum_{k,\ell=1}^N e_{k\ell}(z)e_{\ell k}(z+1)$ to them (the hamiltonian for the Heisenberg ferromagnet or the so-called XXX-model). In the Bardeen, Cooper, Schrieffer (BCS) theory of superconductivity the hamiltonian of the following type (for $N = 2, k = \ell$) is important:

$$u\sum_{z=1}^n e_{k\ell}(z) + \sum_{z,z'=1}^n \sum_{m=1}^N e_{km}(z)e_{m\ell}(z').$$

Summarizing, we see that linear combinations of operators

$$\sum_z c_1(z)e_{k\ell}(z), \sum_{z,z'} c_2(z,z')\sum_m e_{km}(z)e_{m\ell}(z'), \ \sum_{z,z',z''} c_3(z,z',z'')\sum_{m,r} e_{km}(z)e_{mr}(z')e_{r\ell}(z''), \cdots$$

for some scalar functions $c_1, c_2, c_3, \cdots$ and every $1 \le k, \ell \le N$ are natural candidates to incorporate.

Of course, it is possible to consider analogous elements with two or more matrix free indices in place of (k,ℓ). But, generally speaking, $\mathcal{A}$ is already big enough without them. We will show below that for the simplest c the only above elements form an algebra acting irreducibly on $\mathcal{V}$. Although such more complicated combinations can be significant for another choice of c.

In our definition below the points $z, z', z'', \cdots$ will be orderded ($c \neq 0$ only for $z < z' < z'' \cdots$). If one changes the order he well get another algebra of observables and some other representation isomorphic to the initial pair. The corresponding interwiner will be precisely Yang's S.

In fact, this interpretation of S is dual to the above one (by means of $\mathcal{H}'_n$). We note that some points are in Yang's paper (Phys. Rev. 168 (1968)), which are close to our approach to Yangians.

Case $n = 2$. Formulas (25) show that we can use the tensor notations from sec. 2:

$$\mathcal{V} = V \otimes V,\ e_{k\ell}(z) =^z e_{k\ell},\ V = \mathbf{C}^N,\ z = 1,2.$$

Let us consider $\mathcal{V}$ as a module under the action of the algebra generated by $e^0_{k\ell} = e_{k\ell}(1) + e_{k\ell}(2) = e_{k\ell} \otimes 1 + 1 \otimes e_{k\ell}$ and $e^1_{k\ell} = u_1 e_{k\ell}(1) + u_2 e_{k\ell}(2) + \sum_{m=1} e_{mk} \otimes e_{\ell m}$ for all $1 \leq k,\ell \leq N$. Simple calculations give that V is irreducible for $u = u_2 - u_1 \neq \pm 1$.

Thus, $\mathcal{A}$ is big enough to make $\mathcal{V}$ irreducible (for a generic u). However, $\mathcal{A}$ is not very big. Namely, it is not far from $g\ell_N$ operating on $\mathcal{V}$ by $\{e^0_{k\ell}\}$, since for special $u = 1$ (respectively $u = -1$) the symmetric S^2V (external $\Lambda^2 V$) square of V is the only $\mathcal{A}$-submodule of $\mathcal{V}$. The idea is to define quantum groups (Yangians) like this $\mathcal{A}$ but for any initial representations and n.

The aim of the next general definition is to make $V^{\otimes n}$ irreducible (for some generic parameters) but not to loose the classic theory of decomposing of $V^{\otimes n}$ under the diagonal action of $g\ell_N$. For some special values of parameters we should reproduce in terms of $\mathcal{A}$ the classic results like the decomposition $V^{\otimes 2} = S^2V \oplus \Lambda^2 V$ above.

Let us use the rational function in $\lambda \in \mathbf{C}$

$$L(\lambda) = 1 + \sum_{r,k,\ell} \lambda^{-r} E^{r-1}_{k\ell} 1^{\ell k}, \tag{26}$$

where $1 \leq r \leq n, 1 \leq k,\ell \leq N, 1^{\ell k}$ is $e_{\ell k}$ considered as N x N-matrix (see above). Letters $E^{r-1}_{k\ell}$ are assumed to be pairwise non-commuting. E.g. for $n = 1, N = 2$

$$L(\lambda = \begin{pmatrix} 10 \\ 01 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} E^0_{11} E^0_{21} \\ E^0_{12} E^0_{22} \end{pmatrix}.$$

Although L is a matrix with non-commutative matrix elements we can use multi-index notations from sec. 2. In particular, ${}^1L = L \otimes 1, {}^2L = 1 \otimes L$. Let us impose on $\{E^{r-1}_{k\ell}\}$ the Yang-Baxter-Faddeev relation

$$R(\lambda_1 - \lambda_2)^1 L(\lambda_1)^2 L(\lambda_2) =^2 L(\lambda_2)^1 L(\lambda_1) R(\lambda_1 - \lambda_2) \tag{27}$$

for any $\lambda_1, \lambda_2 \in \mathbf{C}$, where (see (8))

$$R(\lambda) = PS(\lambda) = \lambda + P \tag{28}$$

One can show directly that (27) for Yang's R is equivalent to the system of the following relations:

$$\begin{aligned} [E^r_{ij}, E^s_{k\ell}] &= E^{r+s}_{i} \delta^k_j - E^{r+s}_{kj} \delta^\ell_i \\ &+ \sum_{a+b=r+s-1}^{a<r\leq b} \left(E^a_{kj} E^b_{i\ell} - E^b_{kj} E^a_{i\ell} \right). \end{aligned} \tag{29}$$

The quotient-algebra of the algebra of non-commutatitve polynomials in $\{E\}$ by relations (29) is called the *yangian of level n for $g\ell_N$* (written $\mathcal{Y}^n_N$). See [6,15].

Given a set $u = (u_1, \cdots, u_n)$ consider $e_{k\ell}(r) = {}^r e_{k\ell}$ acting on the corresponding components of $\mathcal{V} = V^{\otimes n}(V = \mathbf{C}^N)$ and put

$$\begin{aligned}\tilde{L}_u(\lambda) &= \left(1 + \frac{1}{\lambda - u_n}\sum_{k,\ell} e_{k\ell}(n)1^{\ell k}\right)\left(1 + \frac{1}{\lambda - u_{n-1}}\sum_{k,\ell} e_{k\ell}(n-1)1^{\ell k}\right)\cdots \\ &\cdots \left(1 + \frac{1}{\lambda - u_1}\sum_{k,\ell} e_{k\ell}(1)1^{\ell k}\right)\prod_{r=1}^{n}\left(\frac{\lambda - u_r}{\lambda}\right).\end{aligned}$$

Here $\{1^{\ell k}\}$ commute with $\{e\}$ and determine "the position" of $e_{\ell k}(r)$ in the corresponding N x N-matrix. This L is a function in λ having its values in N x N-matrices with the matrix elements from the algebra $M_N^{\otimes n}$ generated by $\{e(r), 1 \le r \le n\}$. It is convenient for the sake of more invariant writings to denote $1^{\ell k}$ by $e^{\ell k}(0)$ or ${}^0e_{\ell k}$. Then

$$\tilde{L}_u(\lambda) = {}^{0n}R(\lambda - u_n){}^{0n-1}R(\lambda - u_{n-1})\cdots{}^{01}R(\lambda - u_1)\lambda^{-n},$$

where R is from (28): $R(\lambda) = \lambda 1 + \sum_{k,\ell} e_{k\ell} \otimes e_{\ell k}$. It results directly from (8) that $\tilde{L}_u(\lambda)$ is a solution of equation (27). Hence, the corresponding $\tilde{E}_{k\ell}^{r-1}(1 \le r \le n)$ from the decomposition of $\tilde{L}_u(\lambda)$ (see (26)) give us the representation $\mathcal{Y}_N^n \ni E_{k\ell}^{r-1} \to \tilde{E}_{k\ell}^{r-1} \in M_N^{\otimes n} = End(\mathcal{V})$ of $\mathcal{Y}_N^w$ in $\mathcal{V}$ (written $\mathcal{V}(u)$). Two simple examples:

a) $\tilde{E}_{k\ell}^0 = e_{k\ell}(1) + \cdots e_{k\ell}(n) - (u_1 + \cdots + u_n)1$,

b) the operators $e_{k\ell}^0, e_{k\ell}^1$ for $n = 2$ (see above) are some linear combinations of $\tilde{E}_{k\ell}^0, \tilde{E}_{k\ell}^1$ modulo 1.

Theorem 4 a) *The space $\mathcal{V}(u)$ is an irreducible $\mathcal{Y}_N^n$-module if and only if $u_i - u_j \ne 1$ for every $1 \le i, j \le n$. For $w \in S_n$ one has*

$$R_w(u)L_u(\lambda) = L_{w(u)}(\lambda)R_w(u), \tag{30}$$

where $R_w(u) = P_w S_w(u), S_w$ is from (15). In particular, if $\mathcal{V}(u)$ is irreducible then the mapping

$$\mathcal{V}(u) \ni x \to R_w(u)x,\ \tilde{E}_{k\ell}^{r-1} \to R_w(u)\tilde{E}_{k\ell}^{r-1}R_w^{-1}(u)$$

is an isomoprhism from $\mathcal{V}(u)$ onto $\mathcal{V}(w(u))$.

Let us prove identity (30). Consider fig. 5. Let us calculate the corresponding S-matrices (see (13) and fig. 4). One has

$$\begin{aligned}&{}^{12}S(u_{13})\,{}^{01}S(u_{12})\,{}^{23}S(\lambda - u_3)\,{}^{12}S(\lambda - u_2)\,{}^{01}S(\lambda - u_1) = \\ = \ &{}^{23}S(\lambda - u_1)\,{}^{12}S(\lambda - u_3)\,{}^{01}S(\lambda - u_2){}^{23}S(u_{13})\,{}^{12}S(u_{12}).\end{aligned}$$

The simple rule of turning the latter into its R-matrix version is as follows. The upper left indices should be changed to coincide with the indices of the arguments. We obtain the identity

$${}^{13}R\,{}^{12}R\,({}^{03}R\,{}^{02}R\,{}^{01}R) = ({}^{01}R\,{}^{03}R\,{}^{02}R){}^{13}R\,{}^{12}R,$$

where the arguments are omitted. Here ${}^{13}R(u_{13})\,{}^{12}R(u_{12}) = {}^{13}P\,{}^{13}S(u_{13})\,{}^{12}P\ {}^{12}S(u_{12}) = P_w\ {}^{23}S(u_{13})\,{}^{12}S(u_{12}) = R_w(u)$ for $w = (3,1,2) = s_2 s_1$. The products in brackets are $L_u(\lambda)$ and $L_{w(u)}(\lambda)$.

Let us compare the corresponding mappings of theorem 4 and theorem 2. The latter is the right multiplication by $S_w(\Theta)$. The first is the conjugation by $R_w(u) = P_w S_w(u)$. The identification of

Θ and u makes it evident that these two should be very closely connected. In particular, they are degenerated (i.e. $S_w(\Theta), R_w(u)$ are non-invertible) for the same values of $\Theta = u$. Moreover, $\mathcal{V}(u)$ and $M(\Theta)$ are simultaneously irreducible.

We will not discuss here the precise mathematical statements (see [6,7]). Roughly speaking, the $\mathcal{Y}_N^n$-submodules of $\mathcal{V}(u)$ are in one-to-one correspondence with $\mathcal{H}'_n$-submodules of $M(\Theta)$ for $N > n$. The degeneration of $S_w(\Theta)$ and $R_w(u)$ for the same parameters is the particular case of this correspondence. Practically, if one can describe the submodules of $M(\Theta)$, he can construct all the submodules of $\mathcal{V}(u)$. Of course, the first problem is more convenient to settle.

8 Mirrors and polarizations

Let us complicate our space. The idea is to consider the half-line $\mathrm{R}_+ = \{x \geq 0\}$ instead of R with the reflection and its end, i.e. place a mirror at $x = 0$. Some typical picture of interactions is in fig. 6. As before N is the number of colours, axioms (a), (b), (c), (d) (see sec. 1) are valid. But now we have the refelction. We connect with it the scattering matrix $\Pi(\Theta) = (\Pi_{i_1}^{j_1}(-\theta_1))$, where $\Theta = (\theta_1, \cdots, \theta_n)$ are the angles in the out-state according to the conventions adopted. Each element of this matrix depends only on the angle θ_1 of the first particle (after the reflection at $x = 0$) and on its colours i_1 (before) and j_1 (after) the reflection.

Particles have two phases ($\mp$). The first is before ($\theta < 0$) and the second ($\theta > 0$) is after the reflection. Respectively, one should consider 3 types of two-particle amplitudes, when the phases (the signs of the angles) of particles in the out-state are $(-,-),(+,+),(-,+)$, (the combination $(+,-)$ is impossible). For the sake of simplicity we will identify the first two (written S). Let us denote the S-matrix of the third type $(-,+)$by $\hat{S}$ (cf. [3]). Look at fig. 7. Here the out-state is the state after the intersection.

We omit here the symbolic and multi-index language of sec. 1, 2 and we will use at once the notations $S_i(\theta)$ or $\hat{S}_i(\theta)$ (see (13b)) for scattering at the intersection point of the i-th and $(i+1)$-th particles (numbers are from the bottom to the top). We remind that S_i and $\hat{S}_i$ depend only on $\theta_i - \theta_{i+1}$ and on the corresponding colours of the i-th and $(i-1)$-th particles.

We should add to (16) its direct analogs $\hat{S}\,S\,\hat{S} = \hat{S}\,S\,\hat{S}, S\hat{S}\,\hat{S} = \hat{S}\,\hat{S}S$ (with the same indices and arguments), and the new one:

$$\Pi(u)\hat{S}_1(2u+v)\Pi(u+v)S_1(v) = S_1(v)\Pi(u+v)\hat{S}_1(2u+v)\Pi(u). \qquad (31)$$

Here (see fig. 8) $u = -\theta_1, v = \theta_1 - \theta_2, 2u + v = -\theta_1 - \theta_2, u + v = -\theta_2$.

We claim that the identities (with the indices and arguments from (16), (31))

$$S\;S\;S\;=S\;S\;S\,,\hat{S}\;S\;\hat{S}=\hat{S}\;S\;\hat{S}\,,S\;\hat{S}\;\hat{S}=\hat{S}\;\hat{S}\;S\,,\Pi\;\hat{S}\;\Pi\;S=S\;\Pi\;\hat{S}\;\Pi \qquad (32a)$$

together with the evident relations (see (12))

$$[S_i, S_j] = [\hat{S}_i, S_j] = [\hat{S}_i, \hat{S}_j] = [\Pi, S_j] = [\Pi, \hat{S}_j] = 0 \qquad (32b)$$

for $j \geq 2, \mid i - j \mid \geq 2$ provide the independence of any scattering matrix of the internal picture of intersections. In a word (32) is equivalent to axiom (c) from sec. 1.

Let us describe the corresponding group of symmetries. Now the transformation of a given out-state with the angles $\Theta = (\theta_1, \cdots, \theta_n)$ to some set of angles Θ' of the in-state can be represented as the sequence $\tilde{w} = (\varepsilon_1 1', \varepsilon_2 2', \cdots, \varepsilon_n n')$ for $\varepsilon_k = \pm 1, (1', 2', \cdots, n') = w \in \mathrm{S}_n$. It means that the angle $\varepsilon_k\theta_k$ in the set Θ' is situated at place k' (from the bottom to the top at moment $t = t_k$), where $1 \leq k \leq n$. One has $\tilde{w} = (3, -2, -1)$ for fig. 6. The composition of two elements $\tilde{w}, \tilde{w}'$ of this type is quite natural: if $k \to (\tilde{w})\varepsilon_k k', k' \to (\tilde{w}')\varepsilon'_{k'} k''$ then $k \to (\tilde{w}'\tilde{w})\,(\varepsilon_k \varepsilon'_{k'})\,k''$.

The group of all $\tilde{w}$ can be represented by permutations (acting on $\Theta \in \mathbf{R}^n$ in the usual manner) multiplied by any number of the reflections

$$\pi_k(\Theta) = (\theta_1, \cdots, \theta_{k-1}, -\theta_k, \theta_{k+1}, \cdots, \theta_n) \quad (1 \le k \le n).$$

Mathematically, it is the semi-direct product of S_n and $(\mathbf{Z}_2)^n = \{\pi_1^{\delta_1} \cdots \pi_n^{\delta_n}, \delta_k = 0, 1,\}$ with the natural action of S_n on the latter: $w\pi_k w^{-1} = \pi_{w(k)}$ for any $k, w \in \mathrm{S}_n$. As an abstract one this group is generated by $s_1, \cdots, s_{n-1} \in \mathrm{S}_n$ and $\pi = \pi_1$ with the following new relations: $s_1\pi s_1\pi = \pi s_1 \pi s_1, \pi s_j = s_j \pi, j \ge 2$. Each element $\tilde{w}$ of it can be represented in the form $\tilde{w} = (\Pi_{k=1}^n \pi_k^{\delta_k})w, w \in \mathrm{S}_n$. We will denote this group by $\tilde{\mathrm{S}}_n$. It is called *the Weyl group of type B_n (or C_n).*

Now we are in a position to calculate the S-matrix of any picture. To do it one should know Θ (in the out-state) and the corresponding transformation $\tilde{w} \in \tilde{\mathrm{S}}_n$ from Θ to Θ' for the in-state (see above). Let $\tilde{w} = s_{i_\ell} \cdots s_{i_1}$ be of minimal possible length (written $\ell = \ell(\tilde{w})$), where $0 \le i \le n$ and we denote π by s_o for the sake of uniformity. Then (cf. (13))

$$S_{\tilde{w}}(\Theta) = \tilde{S}_{i_\ell}(s_{i_{\ell-1}} \cdots s_{i_1}(\Theta)) \cdots \tilde{S}_{i_2}(s_{i_1}(\Theta))\tilde{S}_{i_2}(\Theta), \tag{33}$$

where

$$\tilde{S}_i(\Theta) = \Pi(-\theta_1) \text{ for } i = 0, \quad \tilde{S}_{i_k} = S_{ik} \text{ or } \hat{S}_{ik},$$

if $i_k \neq 0$ and pair $(k, k+1)$ of the angles from the set $s_{i_{k-1}} \cdots s_{i_1}(\Theta)$ has the coinciding signs or not. Elements from $\tilde{\mathrm{S}}_n$ act on Θ as have been explained. It follows from (32) that $S_{\tilde{w}}(\Theta)$ does not depend on decomposing of $\tilde{w}$.

The simplest example of such a theory is as follows. Let

$$S_i(\Theta) = \hat{S}_i(\Theta) = 1 + s_i(\theta_i - \theta_{i+1}),\ \Pi(\Theta) = 1 - \beta\pi\theta_1,\ \beta \in \mathbf{C}. \tag{34}$$

Then the only equation we need to verify is (31). To obtain some matrix interpretation of (34) one can use some tensor representation of S_n (see [3]). Our aim is to quantize the angles. We should substitute some pairwise commuting letters Y_i for θ_i in (34) and (according to the procedure of sec. 5) postulate relations (18) and the natural relations.

$$(1 - \beta\pi Y_1)Y_1 = -Y_1(1 - \beta\pi Y_1),\ [Y_j, \Pi] = 0 \text{ for } j > 1$$

Here $\Pi(Y_1)$ corresponds to π and therefore should act on $(Y_1 \cdots Y_n)$ as $Y_1 \to -Y_1$. One obtains the algebra $\tilde{\mathcal{H}}'_n$ generated by $\mathbf{C}[\tilde{\mathrm{S}}_n]$ and $Y_1, \cdots, Y_n$ with the relations (19) and some new ones

$$\pi Y_1 + Y_1 \pi = 2/\beta,\ [Y_j, \pi] = 0 \text{ for } j > 1. \tag{35}$$

This algebra is a certain *degeneration of the affine Hecke algebra of type B_n or C_n* (see e.g. [8]). To be more precise it is connected with B_n, C_n, D_n for $\beta = 1, 2, 0$ (see [3]).

We can use the group $\tilde{\mathrm{S}}_n$ for another problem. Let us consider the usual $\mathbf{R}$ as a space (without any reflections). However, assume that there are two different non-changing types of particles (two "polarizations"). We assume that the scattering process is described by Yang's two-particle S whenever they are of the same type (polarization). Otherwise the scattering is trivial. The simplest algebra of observables for collections of n polarised particles is $\mathbf{C}[\tilde{\mathrm{S}}_n]$. The operator $\pi_k (1 \le k \le n)$ corresponds to the polarization of k-th particle in a collection; $\pi_k\pi_{k+1}$ describes the change of polarization from the k-th to $(k+1)$-th particles. That means that $\pi_k(A_J(\Theta) = sgn(\theta_k)A_J(\Theta), \pi_k\pi_{k+1}(A_J(\Theta)) = sgn(\theta_k\theta_{k+1})A_J(\Theta)$.

Let us demonstrate that

$$S_i(\Theta) = (\pi_i\pi_{i+1} + 1)(\theta_i - \theta_{i+1})^{-1}/2 + s_i \tag{36}$$

is exactly what we need (see [5]). Fristly, it is a solution of the Yang-Baxter equation (in the form of (16)). Then $S_i = s_i$ for $\pi_i\pi_{i+1} = -1$ (i.e. there is no scattering in this case). At last, $S_i\pi_i = \pi_{i+1}S_i$ and $S_i\pi_{i+1} = \pi_i S_i$ (the polarizations are conserved). The quantization of angles give us the following relations (from [5]):

$$Y_{i+1}s_i - s_iY_i = (\pi_i\pi_{i+1} + 1)/2 = s_iY_{i+1} - s_iY_i$$

$$[s_i, Y_j] = [\pi_k, Y_j] = [Y_k, Y_j] = 0 \text{ for } j \neq i, i+1, 1 \leq k, j \leq n.$$

in place of (19).

9 Towards the CFT

There are several possibilities to generalize and extend the above constructions. I shall try to outline only some of them connected with the two-dimensional conformal field theory.

First of all, one can substitute everywhere the q-analog of Yang's S. It is written as

$$S_i^q(\theta) = T_i + (q - q^{-1})/(q^{2\theta} - 1), \quad 1 \leq i \leq n \tag{37}$$

where $q \in \mathbf{C}$, $\{T_i\}$ are the generators of the Hecke algebra H_n^q. They satisfy the following defining relations

$$(T_i - q)(T_i + q^{-1}) = 0, T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, [T_i, T_j] = 0 \tag{38}$$

for $|\, i - j \,| \geq 2, 1 \leq i, j \leq n$. For $q = 1$ we arrive at $\mathbf{C}[\mathrm{S}_n]$.

The function S^q was found independently in (one-dimensional) mathematical physics as some solution of (16) and in the theory of representations of p-adic affine Hecke algebras as an interwiner (see [7] for some details). The latter are defined as $\mathcal{H}'_n$ but with the term $(q^2-1)Y_{i+1}$ in place of 1 in formula (19a), where one should substitute T_i for s_i. In p-adic papers $q = p^m$ for a prime $p, m \in \mathbf{N}$. This way of definition is due to Bernstein, Zelevinsky. In many works $\{T_i\}$ are considered in some natural representation of H_n^q in $(\mathbf{C}^N)^{\otimes n}$ (Wenzl, Baxter).

Since $T^2 \neq 1$ we have two elements T, T^{-1} being on equal grounds. We omit the arguments, but it results in two possible pictures for two-particle S-matrices instead of the only one above. We can consider intersecting as a passage of a particle over or under the other.

The S^q from (36) is unitary after a proper normalization. But in other non-unitary theories this note can be important.

We have assumed the two-particle intersections to be the only elementary processes. But one can disagree with this assumption. Look at fig. 3a. There is a certain process between the intersections (θ_1, θ_2) and (θ_1, θ_2) of the corresponding particles. The particle with the angle θ_1 should move away from the particle with θ_2 after the first intersection and approach particle θ_3. This transference may be quantum as well. (In fact, any movement can be quantum in some general theory).

Let us consider the *arranged symbols* $\hat{A}_J(\Theta)$, which are $A_J(\Theta)$ from sec. 1 with some *complete* set of brackets between some $A_{j_k}(\theta_{j_k})$. For example, $(A_{j_1}(\theta_1)A_{j_2}(\theta_2))A_{j_3}(\theta_3), (A_{j_1}(\theta_1)A_{j_2}(\theta_2))$ $(A_{j_3}(\theta_3)A_{j_4}(\theta_4))$ are complete but $(A_1A_2)(A_3A_4)A_5$ is not. The correct arranged symbol should be either $((A_1A_2)(A_3A_4))A_5$ or $(A_1A_2)((A_3A_4)A_5)$. Here we have omitted j, θ. Physically, the last symbol can be interpreted as follows: the particles A_3, A_4 are very close one to another, A_5 is close to A_3 or A_4 (it is all the same, since A_3 and A_4 are very close, more close than A_5 to each of them), A_1 is close to A_2, the pair A_1, A_2 is not close to the triple A_3, A_4, A_5. In fact, we have the ordered sequence of relations "not close, close, very close, very very close and so on" on the set of $A_J(\Theta)$.

Formally speaking, a system of brackets is not complete if $\hat{A}$ contains a segment of type $(\hat{A}^1)(\hat{A}^2)(\hat{A}^3)$ for some arranged symbols $\hat{A}^1, \hat{A}^2, \hat{A}^3$. In this case $\hat{A}^1$ and $\hat{A}^3$ are at the same level (close, very close, ...) with respect to $\hat{A}^2$. We forbid it, that resembles very much the Pauli principle in quantum mechanics.

Given some arranged $\hat{A}_J(\Theta)$ for any $\Theta = (\theta_1, \cdots, \theta_n)$, we can define to following two *elementary operations*. One can interchange two adjacent terms $A_{j_k}(\theta_k)$ and $A_{j_{k+1}}(\theta_{k+1})$, but only if they are in brackets. The corresponding quantum S_k will be introduced like in sec. 1-3. Another operation (written Φ_k) is the passage

$$\hat{A}_J = \cdots \left(\hat{A}^1 \left(A_{j_k}(\theta_k)\hat{A}^2\right)\right) \cdots \to \cdots \left(\left(\hat{A}^1 A_{j_k}(\theta_k)\right) \hat{A}^2\right) \cdots \tag{39}$$

for some arranged $\hat{A}^1, \hat{A}^2$ or the analogous transformation from $(\hat{A}^1 A_k)\hat{A}^2$ to $\hat{A}^1(A_k\hat{A}^2)$. Given k the corresponding $\hat{A}^1, \hat{A}^2$ (if any) can be found uniquely. For example,

$$A_{i_1}(\theta_1)\left(A_{i_2}(\theta_2)A_{i_3}(\theta_3)\right) = \Sigma_J \Phi_I^J(\theta_1, \theta_2, \theta_3)\left(A_{j_1}(\theta_1)A_{j_2}(\theta_2)\right)A_{j_3}(\theta_3),$$

where $\Phi_{i_1 i_2 i_3}^{j_1 j_2 j_3}(\Theta)$ are the amplitudes from the in-state, where A_2 is more close to A_3 than to A_1, to the out-state, where A_2 is more close to A_1; $\Phi = (\Phi_I^J)$.

In a contrast with sec. 1 these two operations (processes) exist only for some k. E. g. let us consider

$$\hat{A} = ((A_1A_2)(A_3A_4))(A_5(A_6(A_7A_8))).$$

One can apply only the operations $S_1, S_3, S_7, \Phi_2, \Phi_3, \Phi_5, \Phi_6, \Phi_7$ to this $\hat{A}$. It is quite natural to postulate the identities

$$[\Phi_i, \Phi_j] = [S_i, S_j] = [S_i, \Phi_k] = 0, \mid i-j \mid > 1, k \neq i, i+1.$$

The reason is that $\Phi_i\Phi_j$ and $\Phi_j\Phi_i$ induce for $\mid i-j \mid > 1$ the same changes of brackets. It holds true for the permutations and changes of brackets in the case $[S, S]$ or $[S, \Phi]$ as well. The other relations are of the following type (see fig. 3). We begin with $\hat{A}_I = A_1(A_2A_3)$ and use here the abbreviations $\theta_{12} = (\theta_1, \theta_2), \theta_{123} = (\theta_1, \theta_2, \theta_3)$ and so on. One has

$$\begin{aligned} & S_2(\theta_{12})\Phi_2(\theta_{312})S_1(\theta_{13})\Phi_2(\theta_{132})S_2(\theta_{23})\Phi_2(\theta_{123}) \\ = \; & \Phi_2(\theta_{321})S_1(\theta_{23})\Phi_2(\theta_{231})S_2(\theta_{13})\Phi_2(\theta_{213})S_1(\theta_{12}). \end{aligned}$$

This equality is quite analogous to identity (2.6) from [17] (see also [16]). It is small wonder since our symbolic language and appropriate pictures are very close to these of [17,16].

Here we assume that S, Φ depend on the corresponding parameters in the natural order. In general, Φ can depend on many indices and parameters. E.g. Φ_6 for (39) may have 4 matrix indices and be a function of $\theta_{5\,6\,7\,8} = (\theta_5, \theta_6, \theta_7, \theta_8)$. In some sense the order of A_7 and A_8 is not important for Φ_6 since they both are at the same level with respect to A_6. In particular, the dependence of Φ_6 on the indices of A_7, A_8 should be symmetric. The development of this point can give some version of the axiom system from [17], where any Φ are defined by means of the comultiplication in terms of the least possible Φ (with 3 matrix indices). The penthagone relation arises in this way. We note that our angles are, in fact, parallel to the conformal dimensions (see e.g. [16]).

I'd like to give another example of connections between the two-dimensional conformal theory and the affine Hecke algebras. The so-called Knizhnik-Zamolodchikov equation for the n-point function of the Wess-Zumino-Witten model can be written in terms of $\mathrm{C}[\mathrm{S}_n]$ only. It has the following natural "affine" generalization

$$\kappa dG/dz_i = \left(\sum_j (ij)(z_i - z_j)^{-1} + x_i z_i^{-1}\right) G, \tag{40}$$

where $1 \leq i \neq j \leq n$, $G(z_1, \cdots, z_n)$ takes its values in the algebra $\mathcal{A}$ generated by $\mathrm{C}[\mathrm{S}_n]$ and some operators $\{x_i\}$ with the relations $w\kappa_i w^{-1} = x_{w(i)}, w \in \mathrm{S}_n$. Here (ij) are the usual transposition,

$\kappa \in \mathbf{C}$. It is easy to show (see [5]), that the cross-derivative integrability conditions for (40) are equivalent to the relations

$$[x_i, x_j + (ij)] = 0 = [(ij), x_i + x_j]. \tag{41}$$

The latter (together with the conditions $wx_iw^{-1} = x_{w(i)}$) coincide with the defining relations for Y_i (sec. 5) if

$$Y_i = -x_i - \sum_{n \geq j > i} (ij), 1 \leq i \leq n.$$

Hence, this $\mathcal{A}$ should be some quotient of the degenerated affine Hecke algebra $\mathcal{H}'_n$.

To get the usual Knizhnik-Zamolodchikov equation one should put $x_i = 0$ for any i. It gives us the so-called Murphy surjection $Y_i \to -\sum_{n \geq j > i}(ij)$ of $\mathcal{H}'_n$ onto $\mathbf{C}[\mathrm{S}_n]$ (see [6]). This homorphism of algebras is important in the theory of S_n. For example, the centre of $\mathbf{C}[\mathrm{S}_n]$ is generated by symmetric polynomials in the images of $\{Y_i\}$ (cf. theorem 3 and [7]).

To finish this part of my notes I will describe without going into detail some quantum counterpart of (41).

Let us consider the space $\mathbf{R}$ with the *glass* at the point $x = 0$. It is transparent for particles from sec. 1, but passing through this glass is assumed to be quantum. One can connect with this process two one-index matrices $X(+\theta), \hat{X}(-\theta)$ respectively for $\theta < 0$ and $\theta > 0$ (see fig. 9). E. g. for $\theta < 0$

$$A_i(\theta)_{in} \overset{glass}{=} \sum_j X_i^j(\theta) A_j(\theta)_{out}$$

The factorization relations are close to (6). We will write them down in term s of $R = PS$:

$$\begin{aligned}
{}^1X(\theta_1)^{12}R(\theta_{12})^2\hat{X}(-\theta_2) &= {}^2\hat{X}(-\theta_2)^{12}R(\theta_{12})^1X(\theta_1), \\
{}^{12}R(\theta_{12})^1X(\theta_1)^2X(\theta_2) &= {}^2X(\theta_2)^1X(\theta_1)^{12}R(\theta_{12}), \\
{}^{12}R(\theta_{12})^2\hat{X}(-\theta_2)^1\hat{X}(-\theta_1) &= {}^1\hat{X}(-\theta_1)^2\hat{X}(-\theta_2)^{12}R(\theta_{12}).
\end{aligned} \tag{42}$$

Of course, R should be a solution of (8) as well.

These equalities hold true (follow from (8)) if one formally substitute $X = {}^{10}R(\theta_{10}), \hat{X} = {}^{01}R(\theta_{01}), \theta_0 = 0$, where 0 is some other tensor index. Really, the transmission through the glass can be interpreted as intersecting with the particle of angle $\theta_0 = 0$ and colour $= 0$, where the latter does not change its colour in any quantum interactions.

The natural problem is to combain S, Φ , mirrors (not more than 2), polarizations (any number) and glasses (any number) in one picture. Then to consider more complicated spaces (circumferences, elliptic curves) and find interesting examples. Only some fragments of this heavy construction are clear (see [3,5,17]).

Let $R = R_\eta = \theta(\theta + \eta)^{-1} + \eta(\theta + \eta)^{-1}P$ (see sec. 5), $X = \hat{X}$. We can consider (42) over $\mathbf{C}[\mathrm{S}_n]$ in a natural manner. One identifies permutations with the corresponding matrices and supposes ${}^iX$ to be some undeterminate functions with the following action of S_n: $w\,{}^iXw^{-1} = {}^{w(i)}X$. We have $R_i =^{i\;i+1} R = 1 + \eta s_i/\theta_i + 0(\eta)$ as $\eta \to 0$. Let us impose the analogical restrictions $X_i = 1 + \eta x_i/\theta + o(\eta)$ on $X_i =^i X$ (in particular, $x_{w(i)} = wx_iw^{-1}$). Then (42) results in (41).

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List of Figure Captions

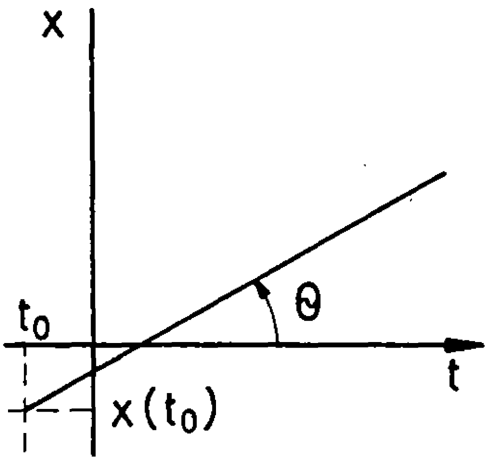

Fig.1

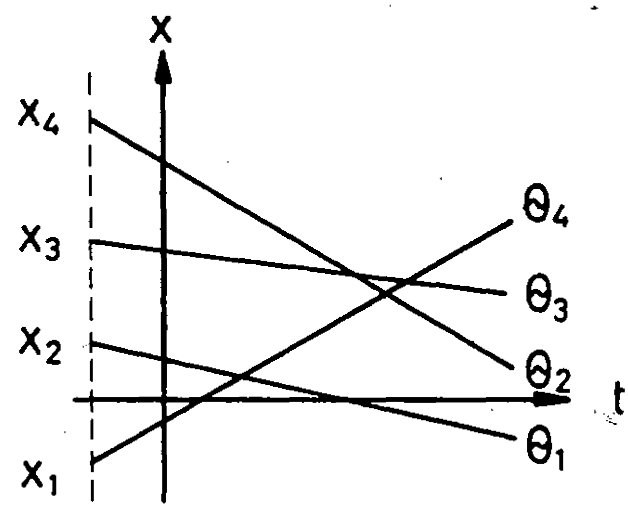

Fig. 2

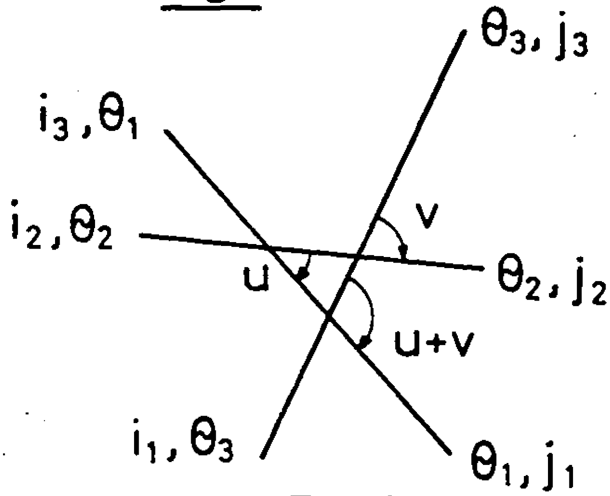

Fig. 3a

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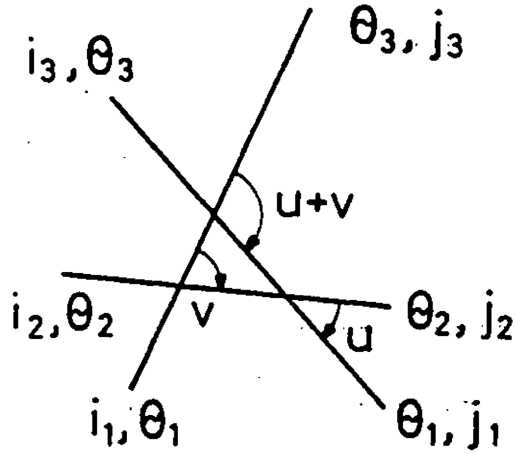
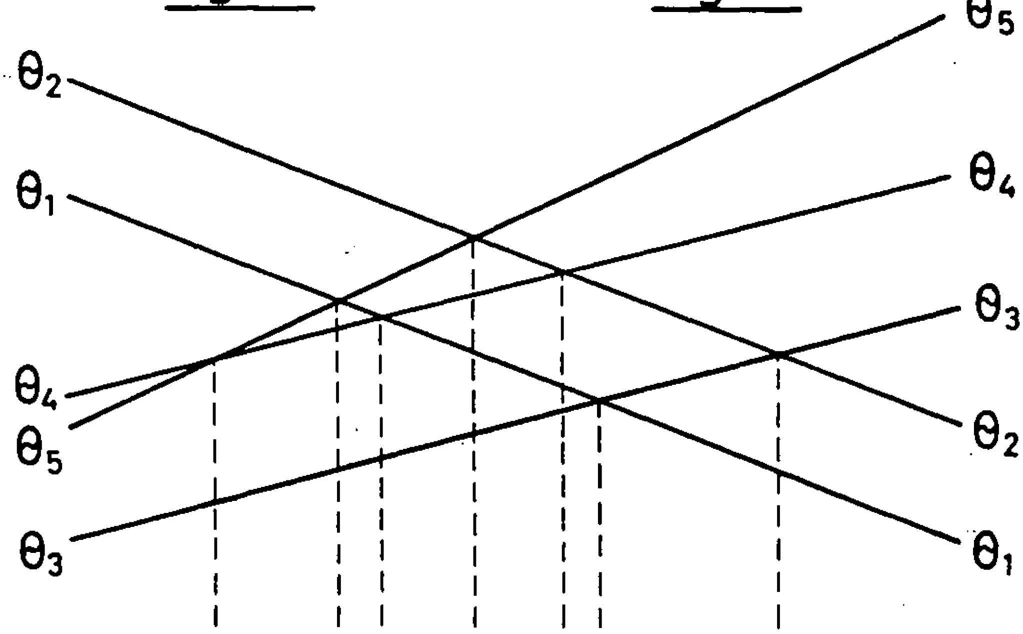


Fig. 3b



$$w = s_2 \quad s_3 \, s_2 \quad s_4 \quad s_3 \, s_1 \quad s_2 =$$

$$= (4,5,1,3,2) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 3 & 2 \end{pmatrix}$$

Fig. 4

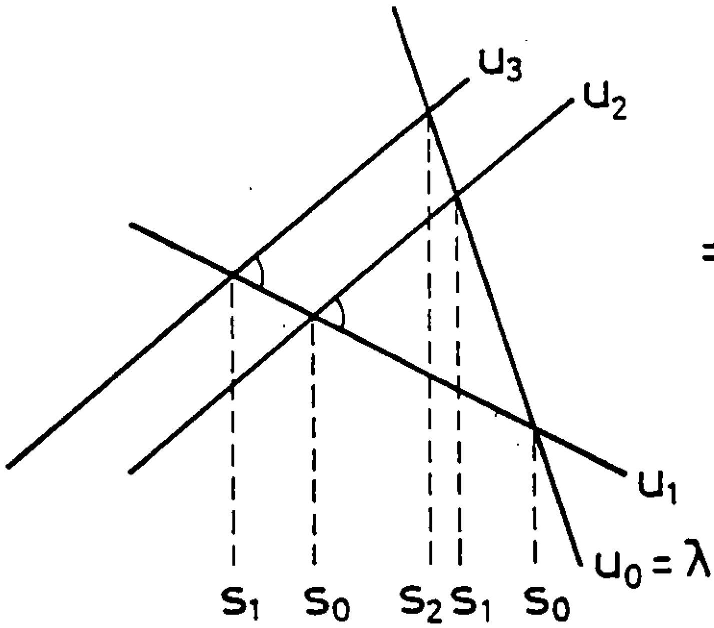


Fig. 5a

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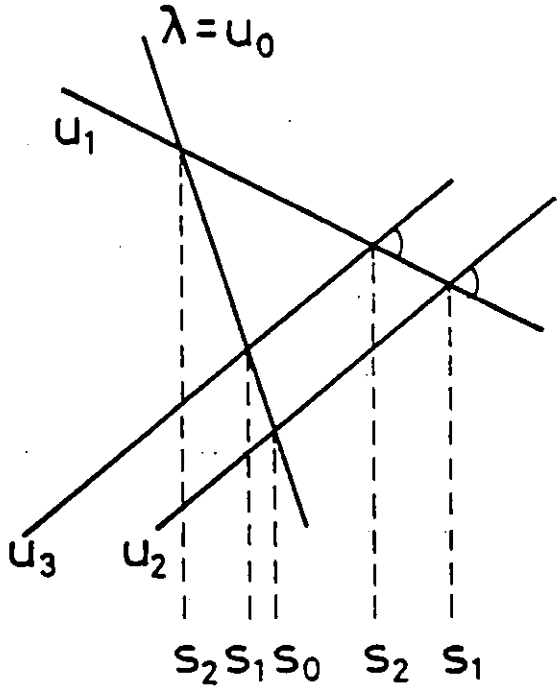
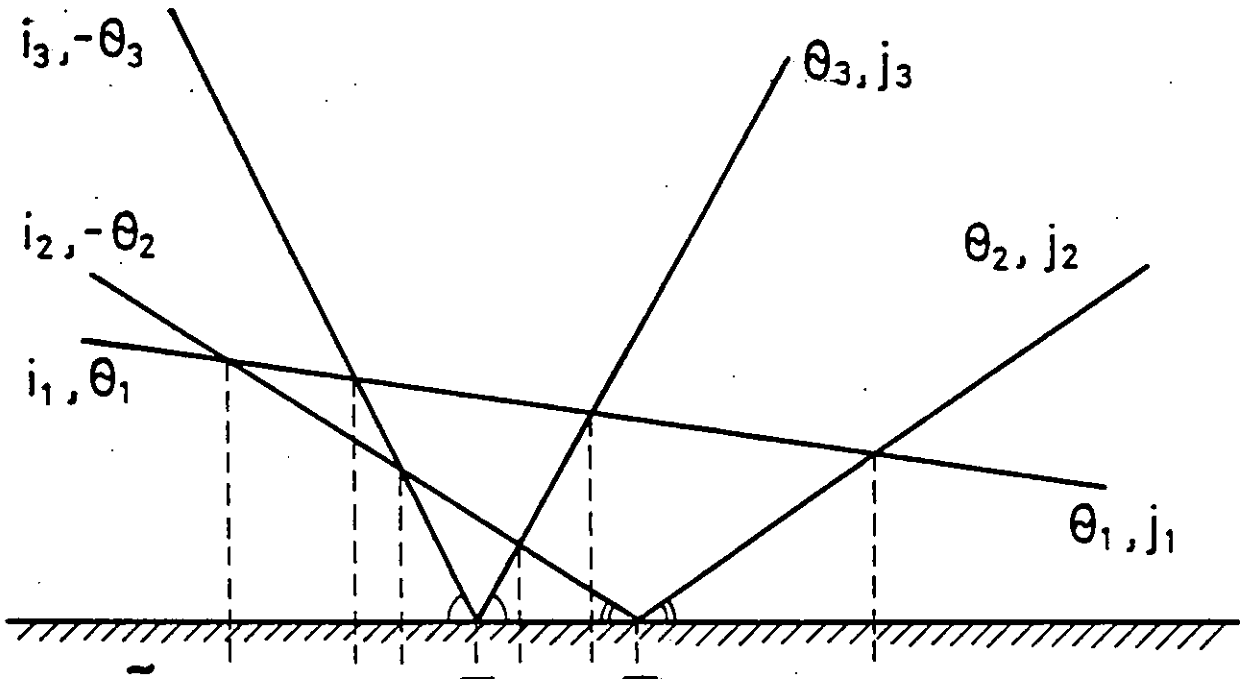


Fig. 5b



$$\tilde{w} = s_1 \; s_2 \, s_1 \, \pi \, s_1 \, s_2 \, \pi \; s_1 =$$
$$= (1, -2, -3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & -3 \end{pmatrix}$$

Fig. 6

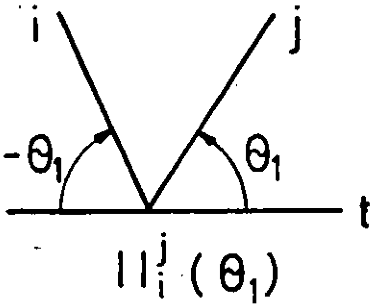


Fig. 7a

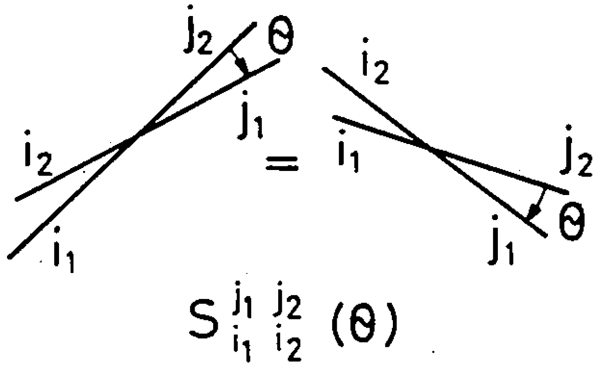


Fig. 7b

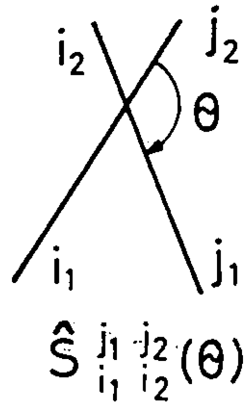


Fig. 7c

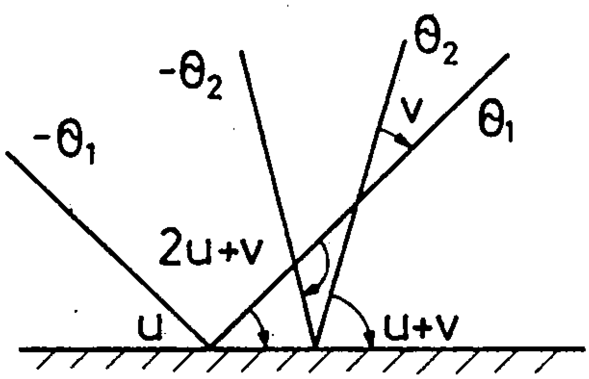


Fig. 8a

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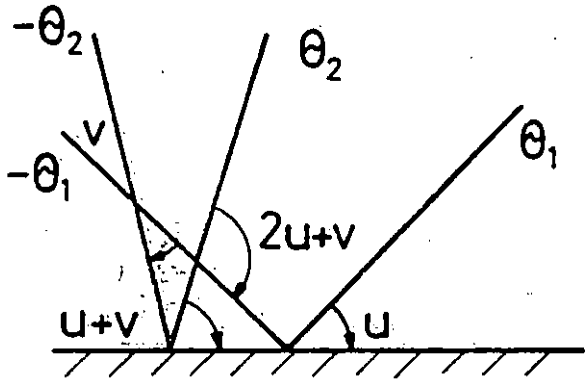

Fig. 8b

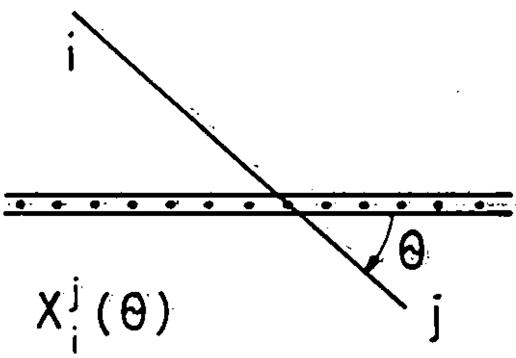

Fig. 9a

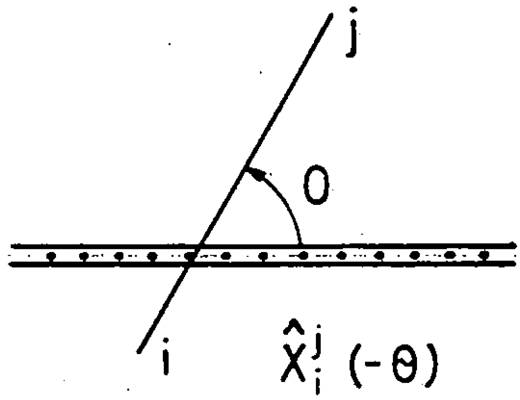

Fig. 9b