The First Eigenvalue of the Dirac Operator on Quaternionic Kähler Manifolds

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Abstract

In [KSW97a] we proved a lower bound for the spectrum of the Dirac operator on quaternionic Kähler manifolds. In the present article we show that the only manifolds in the limit case, i. e. the only manifolds where the lower bound is attained as an eigenvalue, are the quaternionic projective spaces. We use the equivalent formulation in terms of the quaternionic Killing equation introduced in [KSW97b] and show that a nontrivial solution defines a parallel spinor on the associated hyperkähler manifold.

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Contents

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1 Introduction

The square of the first eigenvalue of the Dirac operator on the sphere S^n of scalar curvature κ is $\frac{\kappa}{4}\frac{n}{n-1}$. In [Fri80] Friedrich showed that this eigenvalue is a universal lower bound for all eigenvalues on an arbitrary compact Riemannian spin manifold (M^n, g) with positive scalar curvature κ in the following sense: all eigenvalues of the Dirac operator satisfy

$$\lambda^2 \geq \frac{\min_M \kappa}{4}\frac{n}{n-1}\,.$$

An eigenspinor realizing this lower bound is characterized by a special differential equation called the Killing equation. Conversely, on manifolds admitting Killing spinors, i. e. nontrivial solutions of this Killing equation, the lower bound is realized as an eigenvalue. These manifolds have been characterized by C. Bär [Bär93] translating the Killing equation on M into the equation of a parallel spinor on the cone $\widehat{M} = \mathbb{R}^+ \times_{t^2} M$. The existence problem of solutions of the Killing equation was thus reduced to the description of manifolds with parallel spinors by M. Wang [Wan89].

Despite the fact that Friedrich's estimate is sharp, it is not optimal if M is assumed to have additional geometric structure, namely special holonomy. Due to a result of O. Hijazi and A. Lichnerowicz ([Hij84], [Lic87]), there is no solution of the Killing equation if M possesses a non–trivial parallel k–form, $k \neq 0, n$. There are two canonical classes of such manifolds which in addition have positive scalar curvature: Kähler manifolds and quaternionic Kähler manifolds.

The eigenvalue estimate for Kähler manifolds has been improved by K.–D. Kirchberg in [Kir86] and [Kir90] (see also [Lic90] and [Hij94]). Again, this estimate is sharp: the lower bound is attained as first eigenvalue on the complex projective space $\mathbb{C}P^m$ resp. on its product with a flat 2–torus in odd resp. even complex dimensions. On Kähler manifolds of odd complex dimension, a spinor with smallest possible eigenvalue is characterized by a suitable modification of the Killing equation, the Kählerian Killing equation. A. Moroianu showed in [Mor95] that a Kählerian Killing spinor defines an ordinary Killing spinor on the canonical S^1–bundle over M. Hence, the holonomy argument of Bär's work can be used to study the limit case. In even complex dimensions, the problem of characterizing the limit case is still open. Nevertheless, there are partial results by A. Moroianu [Mor97] and A. Lichnerowicz [Lic90].

A quaternionic Kähler manifold is an oriented $4n$–dimensional Riemannian manifold with $n \geq 2$ whose holonomy group is contained in the subgroup $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n) \subset \mathbf{SO}(4n)$. Equivalently they are characterized by the existence of a certain parallel 4–form Ω, the so–called fundamental or Kraines form (cf. [Bon67], [Kra66]). All quaternionic Kähler manifolds are Einstein with constant scalar curvature (cf. [Ale68b] or [Ish74]) and possess a unique spin structure if n is even, whereas for odd n only the quaternionic projective spaces are spin (cf. [Sal82]). In [KSW97a] we proved that on a compact quaternionic Kähler spin manifold (M^{4n}, g) of positive scalar curvature κ the eigenvalues λ of the Dirac operator satisfy

$$\lambda^2 \geq \frac{\kappa}{4}\frac{n+3}{n+2}\,.$$

As in the Riemannian or Kähler case, this estimate is sharp, and the lower bound is attained as first eigenvalue on the quaternionic projective space (cf. [Mil92]).

The natural task is to study the limit case and to find all manifolds which have $\frac{\kappa}{4}\frac{n+3}{n+2}$ in the spectrum of D^2. A first step in this direction was taken in [KSW97b], where we introduced the equation characterizing an eigenspinor with this particular eigenvalue. A new feature of this quaternionic Killing equation is that it involves not only the eigenspinor but also an auxiliary section of an additional bundle, which is not itself a spinor. We used it to show that no compact symmetric quaternionic Kähler manifolds besides the quaternionic projective spaces carry quaternionic Killing spinors. In the present article we prove the following more general

Theorem 1 *Let M be a compact quaternionic Kähler manifold of quaternionic dimension n and positive scalar curvature κ. If there is an eigenspinor for the Dirac operator with eigenvalue λ satisfying*

$$\lambda^2 = \frac{\kappa}{4}\frac{n+3}{n+2}\,,$$

then M is isometric to the quaternionic projective space.

For the proof we follow the approach of C. Bär and A. Moroianu. We consider the canonical $\mathbf{SO}(3)$–bundle S associated with any quaternionic Kähler manifold. Introducing an appropriate metric on the total space S the warped product $\widehat{M} := \mathbb{R}^+ \times_{t^2} S$ has a natural hyperkähler metric. Reformulating the Killing equation in terms of equivariant functions on the $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$–frame bundle of the quaternionic Kähler manifold we show that a quaternionic Killing spinor induces a Killing spinor on S and a parallel spinor on $\widehat{M}$. The result of M. Wang then implies that the hyperkähler manifold $\widehat{M}$ has to be locally isometric to $\mathbb{H}^{n+1}$ forcing M to be isometric to the quaternionic projective space.

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2 Semiquaternionic Vector Spaces and Representations

The tangent space of a quaternionic Kähler manifold is not *a priori* a quaternionic left vector space because in general the three local complex structures are not defined globally. This ambivalence gives rise to the weaker notion of a semiquaternionic structure on a real vector space:

Definition. A semiquaternionic structure on a real vector space T is a subalgebra $Q \subset \operatorname{End} T$ with $\mathrm{id}_T \in Q$ and $Q \cong \mathbb{H}$ as $\mathbb{R}$–algebras. It is said to be adapted to a euclidean scalar product $\langle , \rangle$ on T if

$$\langle\, qt_1,\, t_2\,\rangle = \langle\, t_1,\, \overline{q}t_2\,\rangle$$

for all $t_1, t_2 \in T$ and $q \in Q$, where $\overline{q}$ denotes conjugation defined by $Q = \mathbb{R} \oplus \operatorname{Im} Q := \mathbb{R}\,\mathrm{id}_T \oplus [Q, Q]$.

Thus, choosing an isomorphism from $\mathbb{H}$ to Q makes T a quaternionic left vector space, however no particular isomorphism is preferred. Accordingly, the notion of quaternionic linear map has to be refined:

Definition. An $\mathbb{R}$–linear map $f : T \to T'$ between vector spaces T, T' with semiquaternionic structures Q, Q' is semilinear, if there exists an isomorphism of $\mathbb{R}$–algebras $f^Q : Q \to Q'$ such that $f(qt) = f^Q(q)f(t)$ for all $t \in T$ and $q \in Q$. If f is semilinear and not identically zero f^Q is uniquely defined, because $\mathrm{id}_{T'} \in Q'$ and $Q' \cong \mathbb{H}$ implies that every non–zero endomorphism in Q' is invertible.

If T is a euclidean vector space with an adapted semiquaternionic structure, then the group of all semilinear isometries of T is isomorphic to $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n) := \mathbf{Sp}(1) \times \mathbf{Sp}(n)/\mathbb{Z}_2$. Choosing a particular isomorphism makes T a true representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ and any two representations T, T' defined this way are intertwined by a semilinear isometry, which is unique up to sign. For this reason we will call any such T the defining representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ with a choice of isomorphism tacitly understood. It turns out that the defining representation T comes along with a preferred isomorphism $\mathbb{H} \to Q$ given by the infinitesimal action of $i,\, j,\, k \in \mathfrak{sp}(1) \cong \operatorname{Im} \mathbb{H}$ on T.

Similarly, one may construct the defining representation of the group $\mathbf{Sp}(n)$ of unitary quaternionic $n \times n$–matrices. If E is a complex vector space of dimension $2n$ endowed with a symplectic form σ_E and an adapted positive quaternionic structure J, i. e. $\sigma_E(\,Je_1,\, Je_2\,) = \overline{\sigma_E(\,e_1,\, e_2\,)}$ for all $e_1, e_2 \in E$ and $\sigma_E(\,e,\, Je\,) > 0$ for all $0 \neq e \in E$, then the group of all $\mathbb{C}$–linear symplectic transformations of E commuting with J is isomorphic to $\mathbf{Sp}(n)$. Choosing a particular isomorphism makes E a true $\mathbf{Sp}(n)$–representation and any two representations E, E' defined this way are intertwined by a $\mathbb{C}$–linear symplectic map preserving the quaternionic structure, which is unique up to sign. For this reason we will call any such E the defining representation of $\mathbf{Sp}(n)$. Note that the Lie algebra of all infinitesimal symplectic transformations of E is canonically isomorphic to $\operatorname{Sym}^2 E$ with $e_1 e_2$ acting as the endomorphism $\sigma_E(e_1, \cdot)e_2 + \sigma_E(e_2, \cdot)e_1$, and elements of $\operatorname{Sym}^2 E$ commute with J if and only if they are real with respect to the real structure $\operatorname{Sym}^2 J$. Hence, the defining representation comes along with a canonical real $\mathbf{Sp}(n)$–equivariant isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{sp}(n) \to \operatorname{Sym}^2 E$ of Lie algebras. We will denote the defining representation of $\mathbf{Sp}(1)$ by H.

There are several possibilities to give explicit realizations of the representations introduced above. In calculations and proofs below we will use the following standard picture, differing somewhat from Salamon's conventions (cf. [Sal82]). Consider the space of row vectors $\mathbb{H}^n$ over the quaternions with complex and quaternionic structure given by multiplication with i and j from the left. The group of unitary quaternionic matrices $\mathbf{Sp}(n) := \{A \in M_{n,n}\mathbb{H} : \; A^H A = 1\}$ acts on $\mathbb{H}^n$ from the left by multiplying with A^H from the right. Thus, it commutes with the complex and quaternionic structure and preserves the linear form

$$\sigma_{\mathbb{H}^n}(\, v_1 \,,\, v_2 \,) := [\, v_1 v_2^H j \,]_{\mathbb{C}} \,,$$

where $[q]_{\mathbb{C}} := \frac{1}{2}(q - iqi) \in \mathbb{C}$ is the $\mathbb{C}$–part of $q \in \mathbb{H}$. The $\mathbb{C}$–part is obviously $\mathbb{C}$–bilinear, i. e. $[xqy]_{\mathbb{C}} = xy[q]_{\mathbb{C}}$ for all $x, y \in \mathbb{C} \subset \mathbb{H}$, and satisfies

$$[\overline{q}]_{\mathbb{C}} = \overline{[q]_{\mathbb{C}}} = [-jqj]_{\mathbb{C}} \,.$$

Using these properties it is easily checked that $\sigma_{\mathbb{H}^n}$ is indeed $\mathbb{C}$–bilinear symplectic and that the quaternionic structure is adapted and positive. In this way $\mathbb{H}^n$ becomes the defining representation of $\mathbf{Sp}(n)$.

With a slight modification of the construction above we can make $\mathbb{H}^n$ the defining representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$, too. The scalar multiplication with $q \in \mathbb{H}$ from the left on the row vectors in $\mathbb{H}^n$ determines a subspace $Q \subset \mathrm{End}\; \mathbb{H}^n$, which obviously is a semiquaternionic structure adapted to the standard scalar product on $\mathbb{H}^n$ given by

$$\langle\, v_1 \,,\, v_2 \,\rangle := \mathrm{Re}\; v_1 v_2^H = \mathrm{Re}\; \sigma_{\mathbb{H}^n}(\, v_1 \,,\, j v_2 \,) \,.$$

The group $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n) = \{z \cdot A : \; z \in \mathbf{Sp}(1) \text{ and } A \in \mathbf{Sp}(n)\}$ acts on $\mathbb{H}^n$ from the left through semilinear isometries by $(z \cdot A)v := zvA^H$.

An important point is particularly obvious in this standard picture and in consequence true for every defining representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$. The infinitesimal action of $i, j, k \in \mathfrak{sp}(1) \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ on T defines a canonical $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$–equivariant isomorphism of algebras $\mathbb{H} \to Q$ making T a quaternionic left vector space. Thus, the natural homomorphism $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n) \to \mathrm{Aut}\; Q$, $z \cdot A \mapsto (z \cdot A)^Q$ is trivial on the subgroup $\mathbf{Sp}(n)$ and descends to an isomorphism $\mathbf{SO}(3) := \mathbf{Sp}(1)/\mathbb{Z}_2 \cong \mathrm{Aut}\; Q$ on the complementary subgroup $\mathbf{Sp}(1)$ sending $z \cdot 1 \in \mathbf{Sp}(1)$ to $z^Q := (z \cdot 1)^Q$.

This observation allows a construction which becomes fundamental for quaternionic Kähler geometry once it is "gauged". Let T be the defining representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ and T' an arbitrary euclidean vector space with an adapted semiquaternionic structure Q'. The representations of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ on T and $\mathbf{SO}(3)$ on Q define simply transitive right group actions on

$$P = \{f : \; T \to T' \;\; \text{semilinear isometry}\}$$

and

$$S = \{f^Q : \; Q \to Q' \;\; \text{isomorphism of algebras}\}$$

such that the natural projection $f \mapsto f^Q$ is $\mathbf{Sp}(1)$–equivariant. Fixing a base point in P to identify P with $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$ we get the following diagram:

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ S \arrow[rr] & & \{T'\} \end{tikzcd} \qquad \begin{tikzcd} & z \cdot A \arrow[dl] \arrow[dr] & \\ z^Q \arrow[rr] & & 1 \end{tikzcd}$$

The quaternionic structure on T can be used to construct two important relations between the defining representations H, E and T of $\mathbf{Sp}(1)$, $\mathbf{Sp}(n)$ and $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$:

Lemma 2.1 *Let T be the defining representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$. Define the complex and quaternionic structure on T by the infinitesimal action of i, j (and k) in $\mathfrak{sp}(1) \cong \mathrm{Im}\; \mathbb{H}$. With the $\mathbb{C}$–bilinear symplectic form $\sigma_T(\, t_1 \,,\, t_2 \,) = \langle\, jt_1 \,,\, t_2 \,\rangle + i\langle\, kt_1 \,,\, t_2 \,\rangle$ the vector space T becomes the defining representation of $\mathbf{Sp}(n)$. Conversely, the complex and quaternionic structure of the defining representation E of $\mathbf{Sp}(n)$ generate a subalgebra Q in the $\mathbb{R}$–linear endomorphisms of E, which is a semiquaternionic structure adapted to the scalar product $\langle\, \cdot \,,\, \cdot \,\rangle := \mathrm{Re}\; \sigma_E(\, \cdot \,,\, J \cdot \,)$ making E the defining representation of $\mathbf{Sp}(1) \cdot \mathbf{Sp}(n)$. In particular, there is up to sign a unique $\mathbf{Sp}(n)$–equivariant, $\mathbb{C}$–linear symplectic isomorphism $\Psi : \; T \to E$ preserving the quaternionic structure. In the standard picture this isomorphism is simply the identity.*

This isomorphism has the disadvantage of spoiling the $\mathbf{Sp}(1)$–action on T. Consequently, it is impossible to use it directly on quaternionic Kähler manifolds. Nevertheless, we may use it to define a family of $\mathbf{Sp}(n)$–equivariant isomorphisms $\mathbb{C}\otimes_{\mathbb{R}} T \to H\otimes_{\mathbb{C}} E$ depending on the choice of a canonical base p, $q := Jp$ of H satisfying $\sigma_H(p,q) = 1$ by

$$\Phi : x\otimes_{\mathbb{R}} t \longmapsto \tfrac{1}{\sqrt{2}}\big(xp\otimes_{\mathbb{C}} \Psi(t) + xq\otimes_{\mathbb{C}} J\Psi(t)\big)\,. \tag{2.2}$$

All these isomorphisms are real with respect to the real structure $J\otimes_{\mathbb{C}} J$ on $H\otimes_{\mathbb{C}} E$ and isometries from $\langle\,,\,\rangle$ to $\sigma_H\otimes_{\mathbb{C}}\sigma_E$:

$$\begin{aligned}\sigma_H\otimes\sigma_E(\Phi(1\otimes_{\mathbb{R}} t_1),\Phi(1\otimes_{\mathbb{R}} t_2)) &= \tfrac{1}{2}\big(\sigma_E(\Psi(t_1),J\Psi(t_2)) - \sigma_E(J\Psi(t_1),\Psi(t_2))\big)\\ &= \tfrac{1}{2}\big(\langle\, jt_1,\, jt_2\,\rangle + i\langle\, kt_1,\, jt_2\,\rangle + \langle\, t_1,\, t_2\,\rangle - i\langle\, it_1,\, t_2\,\rangle\big) \;=\; \langle\, t_1,\, t_2\,\rangle\,,\end{aligned}$$

since $J\Psi(t) = \Psi(jt)$ and $\sigma_E(\Psi(t_1),\Psi(t_2)) = \langle\, jt_1,\, t_2\,\rangle + i\langle\, kt_1,\, t_2\,\rangle$ by construction. It turns out that there are exactly two canonical bases p, q for which the isomorphism above is not only $\mathbf{Sp}(n)$– but already $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$–equivariant, thus defining a more fundamental isomorphism better suited for globalization:

Lemma 2.3 *Up to sign there is a unique real* $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$*–equivariant isomorphism of complex vector spaces*

$$\Phi :\quad \mathbb{C}\otimes_{\mathbb{R}} T\cong H\otimes_{\mathbb{C}} E\,,$$

which is an isometry from the the $\mathbb{C}$*–bilinear extension of* $\langle\,,\,\rangle$ *to* $\sigma_H\otimes\sigma_E$.

Proof. It is sufficient to prove this lemma in the standard picture as it translates immediately to arbitrary realizations of the defining representations. In this picture the canonical base to choose is $p = j$ and $q = -1$ (or $p = -j$ and $q = 1$) leading to:

$$\begin{array}{rcccrcc}\Phi: & \mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}^n & \longrightarrow & \mathbb{H}\otimes_{\mathbb{C}}\mathbb{H}^n & \Phi^{-1}: \;\mathbb{H}\otimes_{\mathbb{C}}\mathbb{H}^n & \longrightarrow & \mathbb{C}\otimes_{\mathbb{R}}\mathbb{H}^n\\ & x\otimes_{\mathbb{R}} v & \longmapsto & \tfrac{1}{\sqrt{2}}(xj\otimes_{\mathbb{C}} v\,-\,x\otimes_{\mathbb{C}} jv) & q\otimes_{\mathbb{C}} v & \longmapsto & \tfrac{1}{\sqrt{2}}(1\otimes_{\mathbb{R}}\overline{q}jv\,+\,i\otimes_{\mathbb{R}}\overline{q}kv)\,.\end{array} \tag{2.4}$$

Note that $\Phi^{-1}(iq\otimes_{\mathbb{C}} v) = \Phi^{-1}(q\otimes_{\mathbb{C}} iv) = i\Phi^{-1}(q\otimes_{\mathbb{C}} v)$. Thus, Φ^{-1} is well defined and $\mathbb{C}$–linear. As $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$–equivariance of Φ^{-1} is obvious and Φ is an isometry it remains to show that Φ and Φ^{-1} are mutual inverses:

$$\begin{aligned}(\Phi^{-1}\circ\Phi)(x\otimes_{\mathbb{R}} v) &= \tfrac{1}{2}\big(1\otimes_{\mathbb{R}}(-j\overline{x}jv+\overline{x}v) + i\otimes_{\mathbb{R}}(-j\overline{x}ijv+\overline{x}iv)\big)\\ &= \tfrac{1}{2}\big(1\otimes_{\mathbb{R}} 2\,\mathrm{Re}\,(x)v + i\otimes_{\mathbb{R}} 2\,\mathrm{Re}\,(-xi)v\big)\\ &= x\otimes_{\mathbb{R}} v\,.\end{aligned}$$

A direct calculation of $\Phi\circ\Phi^{-1} = \mathrm{id}$ is more tedious, but can be done. □

3 Principal Bundles on Quaternionic Kähler Manifolds

"Gauging" the pointwise constructions of the previous section leads to the definitions of the basic objects of quaternionic Kähler geometry. However, in strict analogy with Kähler geometry one has to impose an additional integrability condition:

Definition. A quaternionic Kähler manifold is a Riemannian manifold M of dimension $4n$, $n\geq 2$ with an adapted semiquaternionic structure $Q_xM\subset \mathrm{End}\; T_xM$ on every tangent space which is respected by the Levi–Civitá connection of M:

$$\nabla\Gamma(Q)\subset\Gamma(T^*M\otimes Q)\,.$$

Thus, parallel transport of endomorphisms along arbitrary curves γ induces isomorphisms of $\mathbb{R}$–algebras $Q_{\gamma(0)}M\;\to\;Q_{\gamma(\tau)}M$, and *a fortiori* parallel transport of tangent vectors defines semilinear isometries

$T_{\gamma(0)}M \to T_{\gamma(\tau)}M$. In particular, the Levi–Civitá connection is tangent to the reduction of the frame bundle to the principal $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$–bundle of semilinear orthogonal frames

$$P := \{ f : T \to T_xM \;\; \text{semilinear isometry} \} \tag{3.5}$$

with projection $\pi_M : P \to M,\ f \mapsto x$. Additionally, P projects $\mathbf{Sp}(1)$–equivariantly to

$$S := \{ f^Q : Q \to Q_xM \;\; \text{isomorphism of algebras} \} \tag{3.6}$$

via $\pi_S : P \to S,\ f \mapsto f^Q$, which is in turn a principal $\mathbf{SO}(3)$–bundle over M with projection $\pi : S \to M,\ f^Q \mapsto x$. In this way P may be considered as a principal $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$–bundle over M or as a principal $\mathbf{Sp}(n)$–bundle over S:

$$\begin{tikzcd} & P \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_M"] & \\ S \arrow[rr, "\pi"] & & M \end{tikzcd} \qquad \begin{tikzcd} & f: T \to T_xM \arrow[dl, "\pi_S"'] \arrow[dr, "\pi_M"] & \\ f^Q : Q \to Q_xM \arrow[rr, "\pi"] & & x \end{tikzcd}$$

The tangent bundle TM of M is canonically isomorphic to the bundle associated to P by the defining representation T sending the class $[f,t] \in P \times_{\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)} T$ to $f(t)$. Alternatively, this isomorphism can be expressed by the soldering form $\theta_M := f^{-1} \circ (\pi_M)_* \in \Gamma(T^*P \otimes T)$ of M. Considering the fundamental isomorphism $\Phi : \mathbb{C} \otimes_{\mathbb{R}} T \to H \otimes E$ one might try to define vector bundles $\mathbf{H}$ and $\mathbf{E}$ from the defining representations H and E of $\mathbf{Sp}(1)$ and $\mathbf{Sp}(n)$. In this generality, this is only possible on the quaternionic projective spaces, because only for these manifolds the bundle P can be covered by a principal $\mathbf{Sp}(1)\times\mathbf{Sp}(n)$–bundle. Nevertheless, representations of $\mathbf{Sp}(1)\times\mathbf{Sp}(n)$ contained in some $H^{\otimes p}\otimes E^{\otimes q}$ with $p+q$ even descend to $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$ defining vector bundles on M associated to P. In this way M carries a multitude of naturally defined vector bundles besides bundles constructed out of the tangent bundle. The usefulness of these vector bundles has been shown by [Sal82] (see also [KSW97a]). In particular, the bundle $\mathbf{H}\otimes\mathbf{E}$ is globally defined and canonically isomorphic to $TM^{\mathbb{C}}$ as expressed e. g. by the soldering form $\theta_M^{H\otimes E} := \Phi\circ\theta_M \in \Gamma(T^*P\otimes(H\otimes E))$.

By definition, the Levi–Civitá connection is tangent to P and determined by a connection 1–form ω_M with values in the direct sum $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$; accordingly, ω_M splits into $\omega_M^{\mathfrak{sp}(1)} \oplus \omega_M^{\mathfrak{sp}(n)}$. Additionally, the Levi–Civitá connection defines a connection on the principal bundle S. Recall that horizontal lifts $X^h \in T_{f_0}P$ of tangent vectors $X = \frac{d}{d\tau}\big|_0 x_\tau \in T_{x_0}M$ can be represented by curves of semilinear orthogonal frames $f_\tau : T \to T_{x_\tau}M$ over x_τ satisfying $\frac{\nabla}{d\tau}\big|_0 f_\tau = 0$. Likewise the connection on S is defined by representing horizontal lifts $X^h \in T_{f_0^Q}S$ by curves $f_\tau^Q : Q \to Q_{x_\tau}M$ of algebra isomorphisms over x_τ satisfying $\frac{\nabla}{d\tau}\big|_0 f_\tau^Q = 0$. More succinctly, the connection 1–form ω on S for arbitrary tangent vectors $\frac{d}{d\tau}\big|_0 f_\tau^Q$ is given by

$$\omega\Big(\frac{d}{d\tau}\Big|_0 f_\tau^Q\Big) \;:=\; \big(f_0^Q\big)^{-1}\frac{\nabla}{d\tau}\Big|_0 f_\tau^Q \,.$$

Using the Leibniz rule for the covariant derivative along curves, it is immediately seen that π_S projects horizontal vectors $\frac{d}{d\tau}\big|_0 f_\tau$ on P to horizontal vectors $\frac{d}{d\tau}\big|_0 f_\tau^Q$ on S, because for arbitrary $q \in Q,\ t \in T$

$$\Big(\frac{\nabla}{d\tau}\Big|_0 f_\tau\Big)(qt) \;=\; \Big(\frac{\nabla}{d\tau}\Big|_0 f_\tau^Q\Big)(q)\, f_0(t) \;+\; f_0^Q(q)\,\Big(\frac{\nabla}{d\tau}\Big|_0 f_\tau\Big)(t)\,,$$

and consequently $\frac{\nabla}{d\tau}\big|_0 f_\tau = 0$ implies $\frac{\nabla}{d\tau}\big|_0 f_\tau^Q = 0$. Since the projection $\pi_S : P \to S$ is $\mathbf{Sp}(1)$–equivariant, and vectors tangent to the $\mathbf{Sp}(n)$–action on P are surely vertical with respect to the projection π_S we conclude:

Lemma 3.7 $\qquad \pi_S^*\omega \;=\; \omega_M^{\mathfrak{sp}(1)}\,.$

Remarkably, the curvature of this connection on S depends only on the scalar curvature κ of M. In fact, according to the classification of $\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$–curvature tensors due to Alekseevskii (cf. [Ale68a], [Sal82]) the curvature tensor of a quaternionic Kähler manifold can be expressed in terms of the scalar curvature κ and a section $\mathfrak{R}$ of $\mathrm{Sym}^4\mathbf{E}^*$. We have

$$R = -\frac{\kappa}{8n(n+2)}\Big(R^H + R^E\Big) + R^{hyper}, \tag{3.8}$$

where R^H, R^E and R^{hyper} are $\mathrm{Sym}^2\mathbf{H}$– or $\mathrm{Sym}^2\mathbf{E}$–valued 2–forms defined on sections of $\mathbf{H}\otimes\mathbf{E}\cong TM^{\mathbb{C}}$:

$$\begin{aligned} R^H_{h_1\otimes e_1, h_2\otimes e_2} &= \sigma_E(e_1,e_2)h_1\cdot h_2 \in \mathrm{Sym}^2\mathbf{H} \\ R^E_{h_1\otimes e_1, h_2\otimes e_2} &= \sigma_H(h_1,h_2)e_1\cdot e_2 \in \mathrm{Sym}^2\mathbf{E} \\ R^{hyper}_{h_1\otimes e_1, h_2\otimes e_2} &= \sigma_H(h_1,h_2)\mathfrak{R}(e_1,e_2,\cdot,\cdot) \in \mathrm{Sym}^2\mathbf{E}^* \cong \mathrm{Sym}^2\mathbf{E}\,, \end{aligned} \tag{3.9}$$

acting as endomorphisms on $\mathbf{H}\otimes\mathbf{E}$. Analyzing these terms leads to the following formula for the pull–back of the curvature 2–form Ω of the connection ω to P:

Lemma 3.10 *The curvature 2–form of the connection ω on S pulled back to P is given by*

$$\pi_S^*\Omega \;=\; \frac{\kappa}{16n(n+2)}\Big(\langle\,\theta_M\wedge i\theta_M\,\rangle i + \langle\,\theta_M\wedge j\theta_M\,\rangle j + \langle\,\theta_M\wedge k\theta_M\,\rangle k\Big)\,,$$

where by definition $\langle\,\theta_M\wedge i\theta_M\,\rangle(X,Y) := 2\langle\,\theta_M(X),\, i\theta_M(Y)\,\rangle$.

Proof. Instead of calculating the curvature on S directly we will calculate the curvature of the bundle $\mathrm{Sym}^2\mathbf{H}$, which can be associated to S or P inheriting the same connection due to $\pi_S^*\omega = \omega_M^{\mathfrak{sp}(1)}$. Its curvature considered as an $\mathfrak{sp}(1)$–valued 2–form on P is thus $\pi_S^*\Omega$. Obviously only R^H acts non–trivially on $\mathrm{Sym}^2\mathbf{H}$ and neglecting for a moment that it is defined for sections of vector bundles we may consider it as a real $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$–equivariant morphism

$$\mathbb{C}\otimes_{\mathbb{R}}\Lambda^2T \xrightarrow{\cong} \Lambda^2(H\otimes E) \xrightarrow{R^H} \mathrm{Sym}^2H \xrightarrow{\cong} \mathbb{C}\otimes_{\mathbb{R}}\mathfrak{sp}(1)\,,$$

where the first isomorphism is the extension of $\mathbb{C}\otimes_{\mathbb{R}}T\cong H\otimes E$ and the second is the canonical isomorphism $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{sp}(1)\cong\mathrm{Sym}^2H$, which makes H the defining representation of $\mathbf{Sp}(1)$. To make this isomorphism explicit in the standard picture we choose the canonical base j, -1 of $\mathbb{H}$ with $\sigma_{\mathbb{H}}(j,-1)=1$ and find for the infinitesimal action of i, j and $k\in\mathrm{Im}\;\mathbb{H}$:

$$\begin{array}{lll} i:\; 1\mapsto -i \,=\, i(-1) & \qquad j:\; 1\mapsto -j & \qquad k:\; 1\mapsto -k \,=\, i(-j) \\ \quad\;\; j\mapsto \;\;\,k \,=\, i(j) & \qquad \quad\;\; j\mapsto \;\;1 & \qquad \quad\;\; j\mapsto -i \,=\, i(-1)\,. \end{array}$$

Hence, the isomorphism $\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{sp}(1)\cong\mathrm{Sym}^2\mathbb{H}$ maps i to $i(1\,j)$, j to $\frac{1}{2}(j^2+1^2)$ and k to $\frac{i}{2}(j^2-1^2)$. Accordingly, the morphism R^H reads in the standard picture

$$\begin{aligned} R^H_{1\otimes_{\mathbb{R}}v_1,1\otimes_{\mathbb{R}}v_2} &= \tfrac{1}{2}\big(R^H_{j\otimes v_1,j\otimes v_2} - R^H_{1\otimes jv_1,j\otimes v_2} - R^H_{j\otimes v_1,1\otimes jv_2} + R^H_{1\otimes jv_1,1\otimes jv_2}\big) \\ &= \tfrac{1}{2}\big(\sigma_{\mathbb{H}^n}(v_1,v_2)j^2 - \sigma_{\mathbb{H}^n}(jv_1,v_2)1\,j - \sigma_{\mathbb{H}^n}(v_1,jv_2)1\,j + \sigma_{\mathbb{H}^n}(jv_1,jv_2)1^2\big) \\ &= \mathrm{Re}\;\sigma_{\mathbb{H}^n}(v_1,v_2)\cdot\tfrac{1}{2}(j^2+1^2) + \mathrm{Im}\;\sigma_{\mathbb{H}^n}(v_1,v_2)\cdot\tfrac{i}{2}(j^2-1^2) - \mathrm{Im}\;\sigma_{\mathbb{H}^n}(v_1,jv_2)\cdot i(1\,j) \\ &= -\langle\,v_1,\,jv_2\,\rangle j - \langle\,v_1,\,kv_2\,\rangle k - \langle\,v_1,\,iv_2\,\rangle i \end{aligned}$$

for v_1, $v_2\in\mathbb{H}^n$. Being $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$–equivariant R^H can be made a $\mathbb{C}\otimes_{\mathbb{R}}(P\times_{\mathrm{Ad}}\mathfrak{sp}(1))\cong\mathrm{Sym}^2\mathbf{H}$–valued 2–form in a straightforward way and becomes the R^H defined above. Alternatively, $-\frac{\kappa}{8n(n+2)}R^H$ can be thought of as the horizontal $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$–equivariant $\mathfrak{sp}(1)$–valued curvature form $\pi_S^*\Omega$ on P and is given by the stated formula. The additional factor $\frac{1}{2}$ comes from the definition of the wedge product. $\square$

4 The Levi–Civitá Connection on S and $\widehat{M}$

Let $\pi : S \to M$ be the canonical $\mathbf{SO}(3)$–bundle over M defined in (3.6), with connection form ω and Riemannian metric

$$g_S \;=\; \tfrac{16n(n+2)}{\kappa}\, B(\omega,\,\omega) \;+\; \pi^* g_M\,,$$

where B is the standard metric on $\mathfrak{sp}(1) \cong \mathrm{Im}\,\mathbb{H}$, and κ denotes the scalar curvature of M. This metric is Einstein, and if we rescale the metrics of M and S so that $\kappa = 16n(n+2)$, then $(S,\, g_S)$ has a natural Sasakian 3–structure. The structure group of S reduces to $\mathbf{Sp}(n)$, and we can embed the principal $\mathbf{Sp}(n)$–bundle P into the frame bundle of S in such a way that the soldering form $\theta_S \in \Gamma(T^*P \otimes (\mathfrak{sp}(1) \oplus T))$ of S on P is given by

$$\theta_S \;=\; \sqrt{\tfrac{16n(n+2)}{\kappa}}\,\omega_M^{\mathfrak{sp}(1)} \;\oplus\; \theta_M\,. \tag{4.11}$$

In this way the Riemannian metric of S is associated to the standard metric $B \oplus \langle\,,\,\rangle$ on $\mathfrak{sp}(1) \oplus T$. In terms of covariant derivatives the Levi–Civitá connection of g_S is easily computed, and we obtain:

Lemma 4.12 *Let U, V be vertical vector fields given as fundamental vector fields induced by elements of the Lie algebra $\mathfrak{sp}(1)$. Further, let X^h, Y^h be the horizontal lifts of vector fields X, Y on M and $\{e_\nu\}_{\nu=1,\ldots,4n}$ a locally defined horizontal orthonormal frame on S. Then the only non–zero covariant derivatives are given by*

$$\begin{aligned}
\nabla_U V &= \tfrac{1}{2}\,[U,\,V]\\
\nabla_{X^h} Y^h &= (\nabla_X Y)^h \;-\; \tfrac{1}{2}\,\Omega(X^h,\,Y^h)\\
\nabla_{X^h} U &= \nabla_U X^h \;=\; \tfrac{1}{2}\textstyle\sum_{\nu=1}^{4n} g_S\big(\Omega(X^h,\,e_\nu),\,U\big)\,e_\nu\,.
\end{aligned}$$

Here and in the sequel we identify elements of $\mathfrak{sp}(1)$ with their associated fundamental vector fields. The Levi–Civitá connection of S is determined by a $\mathfrak{so}(\mathfrak{sp}(1) \oplus T)$–valued 1–form ω_S on the orthonormal frame bundle of S, but as the structure group reduces to $\mathbf{Sp}(n)$, it is sufficient to know its restriction to the $\mathbf{Sp}(n)$–reduction P again denoted by ω_S.

Lemma 4.13 *The connection form ω_S on P can be written as*

$$\omega_S \;=\; \omega_M^{\mathfrak{sp}(n)} \;+\; \tfrac{1}{2}\,\mathrm{ad}\big(\omega_M^{\mathfrak{sp}(1)}\big) \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\big(i\,\theta_M \wedge i \;+\; j\,\theta_M \wedge j \;+\; k\,\theta_M \wedge k\big)\,.$$

Proof. For the proof we identify vector fields on S with equivariant functions on P, i. e.

$$\begin{aligned}
\Gamma(TS) &\;\cong\; \mathcal{C}^\infty(P,\,\mathfrak{sp}(1) \oplus T)^{\mathbf{Sp}(n)}\\
A &\;\mapsto\; \widehat{A}\,.
\end{aligned}$$

With respect to this identification the covariant derivative translates as

$$\widehat{\nabla_A B} \;=\; d\widehat{B}(\widetilde{A}) \;+\; \omega_S(\widetilde{A})\,\widehat{B}\,,$$

where $\widetilde{A}$ denotes an arbitrary lift of A to a vector field on P. We will use this formula and Lemma 4.12 to compute all non–zero terms $\omega_S(\widetilde{A})\,\widehat{B}$, which then combine to give the stated expression for ω_S. The definition of the soldering form θ_S immediately implies that the function $\widehat{A}$ is given by $\theta_S(\widetilde{A})$. In particular, we have for a fundamental vector field U

$$\widehat{U} \;=\; \theta_S(\widetilde{U}) \;=\; \sqrt{\tfrac{16n(n+2)}{\kappa}}\,\pi_S^*\omega(\widetilde{U}) \;=\; \sqrt{\tfrac{16n(n+2)}{\kappa}}\,\omega(U) \;=\; \sqrt{\tfrac{16n(n+2)}{\kappa}}\,U \;\in \mathfrak{sp}(1)\,. \tag{4.14}$$

Let X^h, Y^h denote the horizontal lifts of the vector fields X, Y on M. Because of $\pi \circ \pi_S = \pi_M$ we can assume $\widetilde{X^h} = \widetilde{X}$. Then,

$$\widehat{\nabla_{X^h} Y^h} \;=\; d\widehat{Y}(\widetilde{X}) \;+\; \omega_S(\widetilde{X})\,\widehat{Y}\,.$$

Using Lemma 4.12, Lemma 3.10 and equation (4.14) we find

$$\begin{aligned}
\omega_S(\widetilde{X})\,\widehat{Y} \;&=\; \widehat{(\nabla_X Y)^h} \;-\; \tfrac{1}{2}\,\Omega\widehat{(X^h,\,Y^h)} \;-\; d\widehat{Y}(\widetilde{X}) \\
&=\; \omega_M(\widetilde{X})\,\widehat{Y} \;-\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\Big(\big\langle\theta_M(\widetilde{X}),\,i\,\theta_M(\widetilde{Y})\big\rangle\, i \;+\; \big\langle\theta_M(\widetilde{X}),\,j\,\theta_M(\widetilde{Y})\big\rangle\, j \\
&\qquad\qquad + \big\langle\theta_M(\widetilde{X}),\,k\,\theta_M(\widetilde{Y})\big\rangle\, k\Big) \\
&=\; \omega_M(\widetilde{X})\,\widehat{Y} \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\Big(\big\langle i\,\theta_M(\widetilde{X}),\,\widehat{Y}\big\rangle\, i \;+\; \big\langle j\,\theta_M(\widetilde{X}),\,\widehat{Y}\big\rangle\, j \;+\; \big\langle k\,\theta_M(\widetilde{X}),\,\widehat{Y}\big\rangle\, k\Big) \\
&=\; \omega_M(\widetilde{X})\,\widehat{Y} \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\Big(i\,\theta_M(\widetilde{X})\,\wedge\, i \;+\; j\,\theta_M(\widetilde{X})\,\wedge\, j \;+\; k\,\theta_M(\widetilde{X})\,\wedge\, k\Big)\,\widehat{Y}\,.
\end{aligned}$$

Let U, V be fundamental vector fields. Then $\widehat{V}$ is a constant function and $d\widehat{V}(\widetilde{U}) = \widetilde{U}(\widehat{V}) = 0$. Hence,

$$\widehat{\nabla_U V} \;=\; \omega_S(\widetilde{U})\widehat{V} \;=\; \tfrac{1}{2}\,\widehat{[U,\,V]} \;=\; \tfrac{1}{2}\,[\widehat{U},\,\widehat{V}] \;=\; \tfrac{1}{2}\,\mathrm{ad}\big(\omega_M^{\mathfrak{sp}(1)}(\widetilde{U})\big)\,\widehat{V}\,.$$

Finally, let U be fundamental, X^h a horizontal lift of X and $\{e_\nu\}_{\nu=1,\ldots,4n}$ a horizontal orthonormal frame on S, with $E_\nu = \theta_S(\widetilde{e_\nu}) = \widehat{e_\nu}$. Then $d\widehat{U}(\widetilde{X}) = \widetilde{X}(\widehat{U}) = 0$ and we obtain

$$\begin{aligned}
\omega_S(\widetilde{X})\,\widehat{U} \;&=\; \widehat{\nabla_{X^h} U} \;=\; \tfrac{1}{2}\,\textstyle\sum_{\nu=1}^{4n}\, g_S\big(\Omega(X^h,\,e_\nu),\,U\big)\,E_\nu \\
&=\; \tfrac{\kappa}{16n(n+2)}\,\textstyle\sum_{\nu=1}^{4n}\,\Big(\big\langle\theta_M(\widetilde{X}),\,i\,E_\nu\big\rangle\,g_S(i,\,U)\,E_\nu \;+\; \big\langle\theta_M(\widetilde{X}),\,j\,E_\nu\big\rangle\,g_S(j,\,U)\,E_\nu \\
&\qquad\qquad + \big\langle\theta_M(\widetilde{X}),\,k\,E_\nu\big\rangle\,g_S(k,\,U)\,E_\nu\Big) \\
&=\; -\,\textstyle\sum_{\nu=1}^{4n}\,\Big(\big\langle i\,\widehat{X},\,E_\nu\big\rangle\,B(i,\,U)\,E_\nu \;+\; \big\langle j\,\widehat{X},\,E_\nu\big\rangle\,B(j,\,U)\,E_\nu \;+\; \big\langle k\,\widehat{X},\,E_\nu\big\rangle\,B(k,\,U)\,E_\nu\Big) \\
&=\; -\,\textstyle\sum_{\nu=1}^{4n}\,\Big\langle\big(i\,B(i,\,U) \;+\; j\,B(j,\,U) \;+\; k\,B(k,\,U)\big)\,\widehat{X},\,E_\nu\Big\rangle\,E_\nu \\
&=\; -\,\omega_M^{\mathfrak{sp}(1)}(\widetilde{U})\,\widehat{X}\,.
\end{aligned}$$

Combining these three calculations leads to

$$\omega_S \;=\; \omega_M \;+\; \tfrac{1}{2}\,\mathrm{ad}\big(\omega_M^{\mathfrak{sp}(1)}\big) \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\big(i\,\theta_M\,\wedge\, i \;+\; j\,\theta_M\,\wedge\, j \;+\; k\,\theta_M\,\wedge\, k\big) \;-\; \omega_M^{\mathfrak{sp}(1)}\,.$$

In this formula the last summand $-\omega_M^{\mathfrak{sp}(1)}$ acts by the infinitesimal $\mathfrak{sp}(1)$–action on T and thus cancels the action of $\omega_M^{\mathfrak{sp}(1)}$ as part of the first summand ω_M. So we end up with the stated formula for ω_S. □

Besides the manifold S we also need to consider the cone $\widehat{M}$ over S, i. e. the warped product $\widehat{M} := \mathbb{R}^+ \times_{t^2} S$ with metric

$$\widehat{g} \;=\; \tfrac{16n(n+2)}{\kappa}\,dt^2 \;+\; t^2\,g_S.$$

The structure group of $\widehat{M}$ reduces to $\mathbf{Sp}(n)$, and we can embed the principal $\mathbf{Sp}(n)$–bundle $P_{\widehat{M}} := \mathbb{R}^+ \times P$ into the bundle of orthonormal frames of $\widehat{M}$ in such a way that the $\mathbb{R} \oplus \mathfrak{sp}(1) \oplus T$–valued soldering form on $P_{\widehat{M}}$ is given by

$$\theta_{\widehat{M}} \;=\; \sqrt{\tfrac{16n(n+2)}{\kappa}}\,(-dt) \;\oplus\; t\,\theta_S \;=\; \sqrt{\tfrac{16n(n+2)}{\kappa}}\Big(-\,dt \;\oplus\; t\,\omega_M^{\mathfrak{sp}(1)}\Big) \;\oplus\; t\,\theta_M\,.$$

This convention makes the inward pointing vector field $\Xi := -\sqrt{\frac{\kappa}{16n(n+2)}}\,\frac{\partial}{\partial t}$ correspond to $\mathit{1} := \theta_{\widehat{M}}(\Xi) \in \mathbb{R}$. This may be surprising at first but it turns out that only this orientation is compatible with a hyperkähler structure on $\widehat{M}$ introduced later. With this choice of soldering form the Riemannian metric $\widehat{g}$ is associated to the standard metric $\langle\,,\,\rangle \oplus B \oplus \langle\,,\,\rangle$ on $\mathbb{R} \oplus \mathfrak{sp}(1) \oplus T$.

Lemma 4.15 *The restriction $\omega_{\widehat{M}}$ of the Levi–Civitá connection of $\widehat{M}$ to the reduction $P_{\widehat{M}}$ of the bundle of orthonormal frames reads:*

$$\begin{aligned}
\omega_{\widehat{M}} &= \omega_S + \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\theta_S \wedge \mathit{1} \\
&= \omega_M^{\mathfrak{sp}(n)} + \tfrac{1}{2}\,\mathrm{ad}\big(\omega_M^{\mathfrak{sp}(1)}\big) + \omega_M^{\mathfrak{sp}(1)} \wedge \mathit{1} \\
&\quad + \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\big(\theta_M \wedge \mathit{1} + i\,\theta_M \wedge i + j\,\theta_M \wedge j + k\,\theta_M \wedge k\big)\,.
\end{aligned}$$

In particular, the connection form $\omega_{\widehat{M}}$ is the pull–back of a well defined 1–form on P.

Proof. The proof is similar to the proof of the corresponding formula for ω_S. Let $\widehat{\nabla}$ denote the covariant derivative for the Levi–Civitá connection of $\widehat{g}$. The only non–vanishing terms are

$$\widehat{\nabla}_X Y \;=\; \nabla^S_X Y \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\tfrac{1}{t}\,\widehat{g}(X,\,Y)\,\Xi \qquad \text{and} \qquad \widehat{\nabla}_X \Xi \;=\; \widehat{\nabla}_\Xi X \;=\; -\sqrt{\tfrac{\kappa}{16n(n+2)}}\,\tfrac{1}{t}\,X\,.$$

Using the same notation as in the proof of Lemma 4.13 we obtain

$$\omega_{\widehat{M}}(\widetilde{X})\widehat{Y} \;=\; \omega_S(\widetilde{X})\widehat{Y} \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\tfrac{1}{t}\,\big\langle\theta_{\widehat{M}}(\widetilde{X}),\,\theta_{\widehat{M}}(\widetilde{Y})\big\rangle\,\widehat{\Xi} \;=\; \omega_S(\widetilde{X})\widehat{Y} \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\big\langle\theta_S(\widetilde{X}),\,\widehat{Y}\big\rangle\,\mathit{1}\,.$$

Having this expression for horizontal vector fields X, Y we immediately derive

$$\omega_{\widehat{M}} \;=\; \omega_S \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\theta_S \wedge \mathit{1} \;=\; \omega_S \;+\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\theta_M \wedge \mathit{1} \;+\; \omega_M^{\mathfrak{sp}(1)} \wedge \mathit{1}\,. \quad \square$$

An interesting application of the formulas given above relates the curvature tensor $\widehat{R}$ of the manifold $(\widehat{M},\,\widehat{g})$ to the hyperkähler part R^{hyper} of the curvature tensor of $(M,\,g)$. The proof will be given in appendix B.

Proposition 4.16 *The curvature tensor $\widehat{R}$ is horizontal with respect to $\widehat{\pi} : \widehat{M} \to M$. Its only non–vanishing terms are*

$$\widehat{R}(X,\,Y) \;=\; R^{hyper}(\widehat{\pi}_* X,\,\widehat{\pi}_* Y)\,.$$

The right–hand side acts on the orthogonal complement $(T^V\widehat{M})^\perp$ of the vertical tangent bundle, which is canonical isomorphic to $\widehat{\pi}^* TM$.

The manifold $\widehat{M}$ has been previously studied by A. Swann using the notation $\mathcal{U}(M)$ [Swa91]. He constructs $\widehat{M}$ as the $\mathbb{Z}_2$–quotient of the total space of the locally defined bundle $\mathbf{H}$ with zero section removed. The metric $\widehat{g}$ is a member of the family of hyperkähler metrics on $\widehat{M}$ introduced in [Swa91]. In particular, the proposition above is implicit in his work (see also [Swa97]).

5 Quaternionic Killing Spinors

In this chapter we recall the quaternionic Killing equation introduced in [KSW97b]. It will provide us with an equivalent formulation of the limit case. First, we have to collect some facts for the spinor bundle and Clifford multiplication on a quaternionic Kähler manifold (cf. [KSW97a]).

The spinor bundle of a $4n$–dimensional quaternionic Kähler spin manifold decomposes into a sum of $n+1$ subbundles, which can be expressed using the locally defined bundles $\mathbf{E}$ and $\mathbf{H}$ (cf. [BaS83], [HiM95] or [Wan89]). For this we have to introduce the bundles $\Lambda^s_\circ\mathbf{E}$. They are associated to the irreducible $\mathbf{Sp}(n)$–representations on the spaces $\Lambda^s_\circ E$ of primitive s–vectors, which are the kernels of the contraction $\sigma_E \lrcorner \,:\, \Lambda^s E \longrightarrow \Lambda^{s-2} E$ with the symplectic form σ_E. With this notation the spinor bundle can be written

$$\mathbf{S}(M) \;=\; \bigoplus_{r=0}^{n} \mathbf{S}_r(M) \;:=\; \bigoplus_{r=0}^{n} \mathrm{Sym}^r\mathbf{H} \otimes \Lambda^{n-r}_\circ\mathbf{E}\,. \tag{5.17}$$

In order to define the Clifford multiplication we have to fix notations for modified contraction and multiplication on $\mathrm{Sym}^r\mathbf{H}$ and $\Lambda^s_\circ\mathbf{E}$. Contraction preserves the primitive spaces, i. e. if η is in $\Lambda^s_\circ E$ then $e^\sharp \lrcorner\, \eta \,\in\, \Lambda^{s-1}_\circ E$, where $e^\sharp := \sigma_E(e,\,\cdot) \in E^*$ denotes the dual of $e \in E$. However, this is not true for the wedge product and the projection $e \wedge_\circ \eta$ of $e \wedge \eta$ onto $\Lambda^{s+1}_\circ E$ is given by

$$e \wedge_\circ \eta = e \wedge \eta \;-\; \tfrac{1}{n-s+1}\, L_E \wedge (e^\sharp \lrcorner\, \eta)\,,$$

where L_E is the canonical bivector associated to σ_E under the isomorphism $\Lambda^2 E \cong \Lambda^2 E^*$. Let $h\cdot$ denote the symmetric product with $h \in H$, and for $h^\sharp := \sigma_H(h,\cdot) \in H^*$ we define $h^\sharp \lrcorner_\circ : \mathrm{Sym}^r H \to \mathrm{Sym}^{r-1} H$ by $h^\sharp \lrcorner_\circ := \frac{1}{r} h^\sharp \lrcorner$. Let $h \otimes e \in \mathbf{H} \otimes \mathbf{E} = TM^{\mathbb{C}}$ be a tangent vector. Then, the Clifford multiplication $\mu(h \otimes e) : \mathbf{S}(M) \to \mathbf{S}(M)$ is given by

$$\mu(h \otimes e) \quad = \quad \sqrt{2}\,(h \cdot \otimes e^\sharp \lrcorner \;\; + \;\; h^\sharp \lrcorner_\circ \otimes e \wedge_\circ\,)\,. \tag{5.18}$$

In particular, it maps the subbundle $\mathbf{S}_r(M)$ to the sum $\mathbf{S}_{r-1}(M) \oplus \mathbf{S}_{r+1}(M)$ and thus splits into two components

$$\mu^+_- : \quad TM \otimes \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r+1}(M) \quad \text{and} \quad \mu^-_+ : \quad TM \otimes \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r-1}(M)\,,$$

with $\mu^+_-(e \otimes h) = \sqrt{2}\,(h \cdot \otimes e^\sharp \lrcorner\;)$ and $\mu^-_+(e \otimes h) = \sqrt{2}\,(h^\sharp \lrcorner_\circ \otimes e \wedge_\circ\,)$. There are two operations defined similar to Clifford multiplication

$$\begin{array}{rccc} \mu^+_+ : & TM \otimes \mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E} & \longrightarrow & \mathrm{Sym}^{r+1}\mathbf{H} \otimes \Lambda^{s+1}_\circ\mathbf{E} \\ & h \otimes e \otimes \psi & \longmapsto & \sqrt{2}\,(h \cdot \otimes e \wedge_\circ)\psi \end{array}$$

and

$$\begin{array}{rccc} \mu^-_- : & TM \otimes \mathrm{Sym}^r\mathbf{H} \otimes \Lambda^s_\circ\mathbf{E} & \longrightarrow & \mathrm{Sym}^{r-1}\mathbf{H} \otimes \Lambda^{s-1}_\circ\mathbf{E} \\ & h \otimes e \otimes \psi & \longmapsto & \sqrt{2}\,(h^\sharp \lrcorner_\circ \otimes e^\sharp \lrcorner\;)\psi\,. \end{array}$$

Using these notations the Dirac operator D can be written $D = D^+_- + D^-_+$ with

$$D^+_- := \mu^+_- \circ \nabla : \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r+1}(M) \qquad D^-_+ := \mu^-_+ \circ \nabla : \mathbf{S}_r(M) \longrightarrow \mathbf{S}_{r-1}(M)\,.$$

The square of the Dirac operator respects the splitting of the spinor bundle, i. e. $D^2 : \mathbf{S}_r(M) \longrightarrow \mathbf{S}_r(M)$ and we have $D^+_- D^+_- = 0 = D^-_+ D^-_+$.

A quaternionic Killing spinor is by definition (cf. [KSW97b]) a section $\psi = (\psi_0,\, \psi_1,\, \psi_-)$ of the Killing bundle

$$\mathbf{S}^{Killing}(M) \;:=\; \mathbf{S}_0(M) \,\oplus\, \mathbf{S}_1(M) \,\oplus\, \Lambda^{n-2}_\circ\mathbf{E} \;\cong\; \Lambda^n_\circ\mathbf{E} \,\oplus\, (\mathbf{H} \otimes \Lambda^{n-1}_\circ\mathbf{E}) \,\oplus\, \Lambda^{n-2}_\circ\mathbf{E}\,,$$

satisfying the following *quaternionic Killing equation* for some parameter $\lambda \neq 0$ and all tangent vectors X:

$$\begin{array}{rcl} \nabla_X\psi_0 & = & -\,\frac{\lambda}{n+3}\,\mu^-_+(X)\,\psi_1 \\ \nabla_X\psi_1 & = & -\,\frac{\lambda}{4n}\,\mu^+_-(X)\,\psi_0 \;+\; \frac{3\lambda}{2(n+3)}\,\mu^+_+(X)\,\psi_- \\ \nabla_X\psi_- & = & -\,\frac{\lambda}{4n}\,\mu^-_-(X)\,\psi_1\,. \end{array}$$

We remark that if $(\psi_0,\, \psi_1,\, \psi_-)$ is a solution for parameter λ, then $(\psi_0,\, -\psi_1,\, \psi_-)$ is a solution for parameter $-\lambda$. In [KSW97b] we showed that for any solution $\psi \neq 0$ the spinor $\psi_0 + \psi_1$ is an eigenspinor for the minimal eigenvalue

$$\lambda \;=\; \pm\sqrt{\tfrac{\kappa}{4}\,\tfrac{n+3}{n+2}}\,.$$

In particular, only for two values of the parameter $\lambda \neq 0$ there can possibly exist non–trivial solutions. Conversely, any eigenspinor for the minimal eigenvalue is of the form $\psi_0 + \psi_1 \in \Gamma(\mathbf{S}_0(M) \oplus \mathbf{S}_1(M))$, and the augmented eigenspinor (ψ_0, ψ_1, ψ_-) with $\psi_- := \frac{1}{4\lambda}\,\frac{n+3}{n+4}\,(\mu^-_- \circ \nabla)\psi_1 \in \Gamma(\Lambda^{n-2}_\circ\mathbf{E})$ is a solution of the quaternionic Killing equation. Obviously, solutions are sections parallel with respect to a modified connection. Its curvature is precisely the hyperkähler part R^{hyper} of the curvature of M. Since this part vanishes on the quaternionic projective space the Killing bundle of $\mathbb{H}P^n$ is trivialized by augmented eigenspinors with minimal eigenvalue. Proposition 4.16 then motivates to lift a quaternionic Killing spinor to a parallel spinor on $\widehat{M}$.

For our purpose of characterizing the limit case it is more convenient to consider an equivalent version of the quaternionic Killing equation. Let $\psi = (\psi_0, \psi_1, \psi_-)$ be a solution of the original equation, then the scaled section

$$\psi^{scal} \;:=\; (\psi_0^{scal}, \psi_1^{scal}, \psi_-^{scal}) \;=\; \Big(\sqrt{\tfrac{n+3}{4n}}\,\psi_0,\; \psi_1,\; -\sqrt{\tfrac{4n}{n+3}}\,\psi_-\Big)$$

is a solution of the equation

$$\nabla_X \psi^{scal} \;=\; -\sqrt{\tfrac{\kappa}{16n(n+2)}}\, A_X\, \psi^{scal}\,,$$

where ψ^{scal} is considered as column vector with three entries and A_X denotes the matrix

$$A_X \;=\; \begin{pmatrix} 0 & \mu_+^-(X) & 0 \\ \mu_-^+(X) & 0 & \frac{3}{2}\,\mu_+^+(X) \\ 0 & -\,\mu_-^-(X) & 0 \end{pmatrix}.$$

For the remainder of this article, the index denoting the scaling will be omitted. The following lemma shows that A_X can be interpreted as part of an $\mathfrak{sp}(n+1)$–action.

Lemma 5.19 *Let $F = H \oplus E$ be the defining representation of $\mathbf{Sp}(n+1)$ with symplectic form $\sigma_F = \sigma_H + \sigma_E$. Restricted to the subgroup $\mathbf{Sp}(1)\times\mathbf{Sp}(n)$ the $\mathbf{Sp}(n+1)$–representation $\Lambda^s_\circ F$ decomposes into*

$$\Lambda^s_\circ F \cong \Lambda^s_\circ E \,\oplus\, (H\otimes\Lambda^{s-1}_\circ E) \,\oplus\, \Lambda^{s-2}_\circ E\,,$$

which descends to a well defined representation of $\mathbf{Sp}(1)\cdot\mathbf{Sp}(n)$ if s is even. Explicitly, the isomorphism $\iota : \Lambda^s_\circ E \,\oplus\, (H\otimes\Lambda^{s-1}_\circ E) \,\oplus\, \Lambda^{s-2}_\circ E \longrightarrow \Lambda^s_\circ F$ is given by

$$\iota\big(\phi_0 \oplus (h\otimes\phi_1)\oplus\phi_-\big) \;=\; \phi_0 \,+\, (h\wedge\phi_1) \,+\, (L_H - \tfrac{1}{n-s+2}L_E)\wedge\phi_-\,.$$

Similarly, $\mathrm{Sym}^2 F \cong \mathbb{C}\otimes_{\mathbb{R}}\mathfrak{sp}(n+1)$ *decomposes into* $\mathrm{Sym}^2 H \,\oplus\, (H\otimes E)\,\oplus\,\mathrm{Sym}^2 E$*. For $s = n$, the subspace $H\otimes E \subset \mathrm{Sym}^2 F$ acts on $\Lambda^n_\circ F$ via $(h\otimes e)\,\phi \;=\; \frac{1}{\sqrt{2}}\,A_{h\otimes e}\,\phi$.*

Proof. It is clear that ι is an injective map to $\Lambda^s F$. It remains to show that its image is already contained in $\Lambda^s_\circ F$. Since $\sigma_F = \sigma_H + \sigma_E$ the statement follows from

$$\sigma_F\,\lrcorner\,(L_H - \tfrac{1}{n-s+2}\,L_E)\wedge\phi_- \;=\; [\sigma_H\,\lrcorner\,,\, L_H\wedge]\,\phi_- \;-\; \tfrac{1}{n-s+2}\,[\sigma_E\,\lrcorner\,,\, L_E\wedge]\,\phi_- \;=\; 0\,,$$

where we used the relations $[\sigma_H\,\lrcorner\,,\,L_H\wedge]\,\phi_- \;=\; \phi_-$ and $[\sigma_E\,\lrcorner\,,\,L_E\wedge]\,\phi_- \;=\; (n-s+2)\,\phi_-$. Comparing dimensions shows that ι is in addition surjective, hence it defines an isomorphism.

The action of an element $f_1\,f_2 \in \mathrm{Sym}^2 F$ on F is given by $(f_1\,f_2)(f) = \sigma_F(f_1,\,f)f_2 \,+\, \sigma_F(f_2,\,f)\,f_1$. It extends as derivation to $\Lambda^*_\circ F$ and can be explicitly written as $(f_1\,f_2)(\omega) \;=\; (f_2\wedge f_1^\sharp\lrcorner \;+\; f_1\wedge f_2^\sharp\lrcorner\,)(\omega)$. Hence, for $h\in H$ and $e\in E$ considered as elements of F, the element $h\otimes e\in H\otimes E$ is identified with $h\,e\in\mathrm{Sym}^2 F$, and the action on $\iota(\phi) = \phi_0\oplus(a\wedge\phi_1)\oplus(L_H - \frac{1}{2}L_E)\wedge\phi_- \in \Lambda^n_\circ F$ is given by

$$\begin{aligned}
(h\otimes e)\,\iota(\phi_0) \;&=\; h\wedge e^\sharp\lrcorner\,\phi_0 \\
&=\; \tfrac{1}{\sqrt{2}}\,\iota\big(\,\mu_-^+(h\otimes e)\phi_0\,\big) \\[1ex]
(h\otimes e)\,\iota(a\otimes\phi_1) \;&=\; (h\wedge e^\sharp\lrcorner \;+\; e\wedge h^\sharp\lrcorner\,)(a\wedge\phi_1) \\
&=\; -\,h\wedge a\wedge e^\sharp\lrcorner\;\phi_1 \;+\; \sigma_H(h,\,a)\,e\wedge\,\phi_1 \\
&=\; -\sigma_H(h,\,a)\,\big(L_H \;-\; \tfrac{1}{2}L_E\big)\;\wedge\,e^\sharp\lrcorner\;\phi_1 \;+\; \sigma_H(h,\,a)\,e\wedge_\circ\,\phi_1 \\
&=\; \tfrac{1}{\sqrt{2}}\,\iota\Big(\mu_+^-(h\otimes e)\,(a\otimes\phi_1) \;-\; \mu_-^-(h\otimes e)\,(a\otimes\phi_1)\Big)
\end{aligned}$$

and

$$\begin{aligned}(h\otimes e)\,\iota(\phi_-) &= -\tfrac{1}{2}\,h\wedge e^\sharp\lrcorner\,(L_E\wedge\phi_-)\;+\;e\wedge h^\sharp\lrcorner\,(L_H\wedge\phi_-)\\ &= \tfrac{1}{2}\,h\wedge e\wedge\phi_-\;-\;\tfrac{1}{2}\,h\wedge L_E\wedge(e^\sharp\lrcorner\;\phi_-)\;-\;e\wedge h\wedge\phi_-\\ &= \tfrac{3}{2}\,h\wedge e\wedge_\circ\phi_-\\ &= \tfrac{3}{2\sqrt{2}}\,\iota\big(\,\mu_+^+(h\otimes e)\phi_-\,\big)\,.\end{aligned}$$

Hence, we see that operation of $(h\otimes e)$ on $\iota(\phi)$ is just application of the matrix $\frac{1}{\sqrt{2}}A_{h\otimes e}$ to the column vector ϕ. $\square$

To stress the origin of the operation $\frac{1}{\sqrt{2}}A_X$ from a group action, we introduce $\star: H\otimes E\otimes\Lambda^n_\circ F\to\Lambda^n_\circ F$ for the infinitesimal action of $H\otimes E$ on $\Lambda^n_\circ F$. With this notation the quaternionic Killing equation reads

$$\nabla_X\psi\;=\;-\sqrt{\tfrac{\kappa}{16n(n+2)}}\,A_X\psi\;=\;-\sqrt{\tfrac{\kappa}{8n(n+2)}}\,\Phi(X)\star\psi\,, \tag{5.20}$$

where Φ is the isomorphism defined in Lemma 2.3.

6 The Geometry of $\widehat{M}$ and Application to Spinors

The aim of this section is to show that the quaternionic Killing equation, considered as a differential equation on equivariant functions on P can be interpreted in three different ways: first, of course, when we think of its solutions as sections of the Killing–bundle $\mathbf{S}^{Killing}(M)$ on M they are quaternionic Killing spinors. Interpreted as a section of the spinor bundle on S, the solutions are Killing spinors, and finally solutions pulled back to $P_{\widehat{M}}=\mathbb{R}^+\times P$ are parallel sections of the spinor bundle of $\widehat{M}$.

6.1 The Hyperkähler Structure of $\widehat{M}$

First we recall that the structure group of both S and $\widehat{M}$ reduce to $\mathbf{Sp}(n)$, i. e. the tangent bundles may be associated to the principal $\mathbf{Sp}(n)$–bundles P and $P_{\widehat{M}}$ through the $\mathbf{Sp}(n)$–representations $\mathfrak{sp}(1)\oplus T$ and $\mathbb{R}\oplus\mathfrak{sp}(1)\oplus T$ respectively. However, as $\mathbf{Sp}(n)$–representation T can be identified with E according to Lemma 2.1 reflecting the isomorphism

$$\pi^*TM\cong P\times_{\mathbf{Sp}(n)}T\cong P\times_{\mathbf{Sp}(n)}E$$

of vector bundles on S. More important, this identification is respected by the Levi–Civitá connection, because the infinitesimal $\mathfrak{sp}(1)$–action on T present in $\omega_M=\omega_M^{\mathfrak{sp}(1)}\oplus\omega_M^{\mathfrak{sp}(n)}$ is canceled in the connection form ω_S of S. For this reason we will consequently identify T with E and consider E as a euclidean vector space with scalar product $\langle\,\cdot,\cdot\,\rangle=\mathrm{Re}\;\sigma_E(\cdot,J\cdot)$.

In the same vein we combine the obvious identification $\mathbb{R}\oplus\mathfrak{sp}(1)\cong\mathbb{R}\oplus\mathrm{Im}\;\;\mathbb{H}=\mathbb{H}$ with the isomorphism $\mathbb{H}\to H$ of defining representations of $\mathbf{Sp}(1)$ unique up to sign to get an isometry from the standard metric on $\mathbb{R}\oplus\mathfrak{sp}(1)$ to H with scalar product $\mathrm{Re}\;\sigma_H(\cdot,J\cdot)$ sending *1*, *i*, *j*, *k* to $\mathbf{1},\mathbf{I},\mathbf{J},\mathbf{K}\in H$. In this way we get an isometry

$$\mathbb{R}\oplus\mathfrak{sp}(1)\oplus T\longrightarrow F$$

with the defining representation $F=H\oplus E$ of $\mathbf{Sp}(n+1)$ considered as a euclidean vector space with scalar product $\mathrm{Re}\;\sigma_F(\cdot,J\cdot)$. As this isometry is $\mathbf{Sp}(n)$–equivariant by construction the tangent bundles of $\widehat{M}$ and S are associated to the $\mathbf{Sp}(n)$–representations F and $\{\mathbf{1}\}^\perp=:(\mathrm{Im}\;\;H)\oplus E\subset F$.

Though the structure group of $\widehat{M}$ reduces to $\mathbf{Sp}(n)$, the holonomy of $\widehat{M}$ does not. Nevertheless, we will show that it is a subgroup of $\mathbf{Sp}(n+1)$, i. e. $\widehat{M}$ is hyperkähler. For this purpose we group the summands of the connection form $\omega_{\widehat{M}}$ of $\widehat{M}$ given in Lemma 4.15 as follows

$$\begin{aligned}\omega_{\widehat{M}}^{\mathfrak{sp}(1)} &:= \tfrac{1}{2}\,\mathrm{ad}(\omega_M^{\mathfrak{sp}(1)})\;+\;\omega_M^{\mathfrak{sp}(1)}\wedge\mathbf{1}\\ \omega_{\widehat{M}}^{\mathfrak{sp}(n)} &:= \omega_M^{\mathfrak{sp}(n)}\\ \omega_{\widehat{M}}^{\mathbb{H}^n} &:= \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\big(\theta_M\wedge\mathbf{1}+i\theta_M\wedge\mathbf{I}+j\theta_M\wedge\mathbf{J}+k\theta_M\wedge\mathbf{K}\big)\,.\end{aligned}$$

Thus, the Levi–Civitá connection can be written $\omega_{\widehat{M}} = \omega_{\widehat{M}}^{\mathfrak{sp}(1)} + \omega_{\widehat{M}}^{\mathfrak{sp}(n)} + \omega_{\widehat{M}}^{\mathbb{H}^n}$.

Lemma 6.21 *The actions of $\omega_{\widehat{M}}^{\mathfrak{sp}(1)}$ and $\omega_{M}^{\mathfrak{sp}(1)}$ are the same, i. e.*

$$\omega_{\widehat{M}} = \omega_{M}^{\mathfrak{sp}(1)} + \omega_{M}^{\mathfrak{sp}(n)} + \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\big(\theta_M \wedge \mathbf{1} + i\theta_M \wedge \mathbf{I} + j\theta_M \wedge \mathbf{J} + k\theta_M \wedge \mathbf{K}\big)\,. \tag{6.22}$$

In particular, the connection form $\omega_{\widehat{M}}$ takes values in $\mathfrak{sp}(n+1)$.

Proof. Note that for $q \in \mathbb{H}$ and imaginary $z \in \mathrm{Im}\ \mathbb{H}$ we have $\frac{1}{2}(zq + qz) = (\mathrm{Re}\ q)z - \langle\, q,\, z\,\rangle = -(z \wedge 1)q$. Using this algebraic identity the infinitesimal $\mathfrak{sp}(1)$–action on $\mathbb{H}$ in the standard picture can be written as

$$-qz = \tfrac{1}{2}\mathrm{ad}(z)q + (z \wedge 1)q\,.$$

Hence, a particular merit of the identifications above is that the two summands $\frac{1}{2}\mathrm{ad}(\omega_M^{\mathfrak{sp}(1)})$ and $\omega_M^{\mathfrak{sp}(1)} \wedge \mathbf{1}$ of the Levi–Civitá connection of $\widehat{M}$ acting on $\mathbb{R} \oplus \mathfrak{sp}(1)$ combine into the infinitesimal action of $\omega_M^{sp(1)}$ on H. Consequently, the summands $\omega_{\widehat{M}}^{\mathfrak{sp}(1)}$ and $\omega_{\widehat{M}}^{\mathfrak{sp}(n)}$ take values in $\mathfrak{sp}(n+1)$, i. e. in the infinitesimal quaternionic linear isometries of F. The same is true for $\omega_{\widehat{M}}^{\mathbb{H}^n}$ because of its $\mathbb{H}$–linearity. □

With the Levi–Civitá connection being $\mathfrak{sp}(n+1)$–valued the manifold $\widehat{M}$ is hyperkähler, and we may use the description of the spinor bundle for the more general quaternionic Kähler manifolds given in (5.17). Consider a complex vector space $\mathbb{C}^2$ endowed with a symplectic form $\sigma_{\mathbb{C}^2}$ and a positive quaternionic structure J. Choosing an isomorphism to the group of all symplectic transformations of $\mathbb{C}^2$ commuting with J would make $\mathbb{C}^2$ the defining representation of $\mathbf{Sp}(1)$. However, on a hyperkähler manifold this $\mathbf{Sp}(1)$–symmetry is not a local "gauged" symmetry, but a purely global one. For this reason the trivial $\mathbb{C}^2$–bundle on $\widehat{M}$ plays the role of H on M. Accordingly, the spinor bundle of $\widehat{M}$ is associated to the $\mathbf{Sp}(n)$–representation

$$\Sigma \;=\; \bigoplus_{r=0}^{n+1} \Sigma_r \;=\; \bigoplus_{r=0}^{n+1} \mathrm{Sym}^r\mathbb{C}^2 \otimes \Lambda_\circ^{n+1-r} F\,,$$

where $\mathbf{Sp}(n)$ operates trivially on $H \subset F$. The Clifford multiplication with complex tangent vectors in $\mathbb{C}^2 \otimes F$ is then given by the formula (5.18). To describe the Clifford multiplication with real tangent vectors however, we have to choose an isomorphism among the family of isometries $\Phi : \mathbb{C} \otimes_{\mathbb{R}} F \to \mathbb{C}^2 \otimes F$ defined in equation (2.2). For quaternionic Kähler manifolds this isometry is uniquely fixed by the additional local $\mathbf{Sp}(1)$–symmetry up to sign, but this is no longer true in the hyperkähler case. In fact, we get a family of Clifford multiplications depending on the choice of Φ, i. e. of a canonical base $p,\, q$ of $\mathbb{C}^2$ satisfying $Jp = q$ and $\sigma_{\mathbb{C}^2}(\,p,\, q\,) = 1$. All these are intertwined by the global $\mathbf{Sp}(1)$–symmetry acting on $\mathbb{C}^2$. In this sense, to define the Clifford multiplication with real tangent vectors $f \in F$, we first have to apply the isomorphism

$$\Phi : 1 \otimes_{\mathbb{R}} f \longmapsto \tfrac{1}{\sqrt{2}}\big(p \otimes f - q \otimes Jf\big)\,. \tag{6.23}$$

Note that this isomorphism points out the ambivalence of the vector bundle associated to the defining representation F of $\mathbf{Sp}(n+1)$. Normally, this vector bundle is the real tangent bundle, but in the description of the spinor bundle it plays a role strictly analogous to the isotropic subspace $T^{0,1}M$ of the complexified tangent bundle on Kähler manifolds.

To describe the spinor bundle on S we recall that the spinor module of $\mathbf{Spin}(4n+4)$ decomposes into the two half–spin modules $\Sigma^{\pm}$. When restricted to $\mathbf{Sp}(n+1)$ the representations $\Sigma^{\pm}$ decompose further into a sum of certain Σ_r. Since Clifford multiplication maps Σ_r to $\Sigma_{r-1} \oplus \Sigma_{r+1}$ and interchanges Σ^+ with Σ^-, we conclude

$$\Sigma^+ \;=\; \bigoplus_{\substack{r=0\\ r\equiv 1\,(2)}}^{n+1} \mathrm{Sym}^r\mathbb{C}^2 \otimes \Lambda_\circ^{n+1-r} F \qquad\qquad \Sigma^- \;=\; \bigoplus_{\substack{r=0\\ r\equiv 0\,(2)}}^{n+1} \mathrm{Sym}^r\mathbb{C}^2 \otimes \Lambda_\circ^{n+1-r} F\,,$$

at least if n is even. Due to Lemma 5.19 Σ^+ and Σ^- are equivalent as $\mathbf{Sp}(n)$–representations, and the spinor module of S is associated to the $\mathbf{Sp}(n)$–principal bundle P through either one, e. g. Σ^+. The Clifford multiplication with $f \in \{\mathbf{1}\}^{\perp} \subset F$ is then given by $f \underset{S}{\cdot} := (f \wedge \mathbf{1})\cdot = f \cdot \mathbf{1}\cdot$.

6.2 Reinterpretation of the Quaternionic Killing Equation

In this section we will translate the quaternionic Killing equation (5.20) on M into the equation for a parallel spinor on $\widehat{M}$. Let ψ be a quaternionic Killing spinor on M, which we will consider as an equivariant function on P, i. e. $\psi \in \mathcal{C}^\infty(P, \Lambda^n_\circ F)^{\mathbf{Sp}(n)}$. Then the quaternionic Killing equation reads

$$\big(d + \omega_M + \sqrt{\tfrac{\kappa}{8n(n+2)}}\,\theta_M^{H\otimes E}\star\big)\psi \;=\; 0\,, \tag{6.24}$$

where by definition $\theta_M^{H\otimes E} := \Phi \circ \theta_M$. The subbundle $\mathbf{S}_1(\widehat{M})$ of the spinor bundle of $\widehat{M}$ is associated to $P_{\widehat{M}}$ by the representation $\mathbb{C}^2 \otimes \Lambda^n_\circ F$. To construct a section of this bundle, we proceed as follows: we lift ψ to $P_{\widehat{M}}$ by extending it constantly along the $\mathbb{R}^+$–direction and choose an arbitrary constant vector $\xi \in \mathbb{C}^2$. Then, the function $\xi \otimes \psi$ defines a section in $\mathbf{S}_1(\widehat{M})$. As function on $P_{\widehat{M}}$ it satisfies

$$\big(d + \omega_M + \sqrt{\tfrac{\kappa}{8n(n+2)}}\,\mathrm{id} \otimes \theta_M^{H\otimes E}\star\big)(\xi \otimes \psi) \;=\; 0\,,$$

where the connection and soldering form are considered to be pulled back to $P_{\widehat{M}}$. Obviously, they do not act on ξ but only on ψ. Using the crucial Lemma 6.25 below we can replace $\sqrt{\tfrac{\kappa}{8n(n+2)}}\,\mathrm{id} \otimes \theta_M^{H\otimes E}\star$ by $\omega_{\widehat{M}}^{\mathbb{H}^n}$, so that the resulting equation on $P_{\widehat{M}}$ reads

$$\big(d + \omega_M^{\mathfrak{sp}(1)} + \omega_M^{\mathfrak{sp}(n)} + \omega_{\widehat{M}}^{\mathbb{H}^n}\big)(\xi \otimes \psi) \;=\; \big(d + \omega_{\widehat{M}}\big)(\xi \otimes \psi) \;=\; 0\,.$$

Hence, $\xi \otimes \psi$ defines a parallel spinor on $\widehat{M}$.

We remark that the parallel spinor $\xi \otimes \psi$ also gives rise to a Killing spinor on S. This follows of course from the general equivalence between Killing spinors on a Riemannian manifold and parallel spinors on its cone, as described by C. Bär (cf. [Bär93]) and briefly summarized in appendix A. Nevertheless, we will include this construction since it is an immediate consequence of our approach. We have seen above that $\xi \otimes \psi$, considered as function on $P_{\widehat{M}}$, satisfies $(d + \omega_{\widehat{M}})(\xi \otimes \psi) \;=\; 0$. If we substitute the expression $\omega_{\widehat{M}}$ given in Lemma 4.15 we obtain

$$\big(d + \omega_S + \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\theta_S \wedge \mathbf{1}\big)(\xi \otimes \psi) \;=\; 0\,.$$

Interpreted as equation on P this is just the Killing equation, i.e.

$$(d + \omega_S)\,(\xi \otimes \psi) \;=\; -\tfrac{1}{2}\sqrt{\tfrac{\kappa}{16n(n+2)}}\,\theta_S \underset{S}{\cdot} (\xi \otimes \psi)\,.$$

The additional factor $\frac{1}{2}$ is due to the action of the orthogonal Lie algebra on the spinor bundle. Before closing this section we formulate the lemma needed above.

Lemma 6.25

$$\sqrt{\tfrac{\kappa}{8n(n+2)}}\,\mathrm{id} \otimes \theta_M^{H\otimes E}\star \;=\; \omega_{\widehat{M}}^{\mathbb{H}^n}\,.$$

The proof needs an additional proposition which is analogous to Proposition 2.3 in [KSW97a].

Proposition 6.26 *Let $p,\, q$ be a base of $\mathbb{C}^2 \cong \mathbb{H}$ with $\sigma(p, q) = 1$ and $f_1, f_2 \in F$. Then we have the following identity of operators on the spinor bundle:*

$$(p \otimes f_1) \wedge (q \otimes f_2) - (q \otimes f_1) \wedge (p \otimes f_2) \;=\; \mathrm{id} \otimes f_1 \cdot f_2\,.$$

As element of $\mathfrak{so}(4n+4)$, the left hand side acts on the spinor module via the isomorphism $\mathfrak{so}(4n+4) \cong \mathfrak{spin}(4n+4)$ sending $e_1 \wedge e_2$ to $\frac{1}{2}(e_1\, e_2 + \langle e_1,\, e_2 \rangle)$.

We remark that the proof of these two technical Propositions amounts to prove the decomposition (5.17) of the spinor bundle. With the help of this proposition it is easy to prove Lemma 6.25.

Proof. Let p, q be the canonical base of $\mathbb{C}^2$ used to define Clifford multiplication. We can then extend the isomorphism $\mathbb{C} \otimes_{\mathbb{R}} F \cong \mathbb{C}^2 \otimes F$ in equation (6.23) to $\mathbb{C} \otimes_{\mathbb{R}} \Lambda^2 F \cong \Lambda^2(\mathbb{C}^2 \otimes F)$ to get

$$\begin{aligned}
(\theta_M \wedge \mathbf{1}) &= \tfrac{1}{2}\,(p \otimes \theta_M + q \otimes J\theta_M) \wedge (p \otimes \mathbf{1} + q \otimes \mathbf{J}) \\
(i\theta_M \wedge \mathbf{I}) &= -\tfrac{1}{2}\,(p \otimes \theta_M - q \otimes J\theta_M) \wedge (p \otimes \mathbf{1} - q \otimes \mathbf{J}) \\
(j\theta_M \wedge \mathbf{J}) &= \tfrac{1}{2}\,(p \otimes J\theta_M - q \otimes \theta_M) \wedge (p \otimes \mathbf{J} - q \otimes \mathbf{1}) \\
(k\theta_M \wedge \mathbf{K}) &= -\tfrac{1}{2}\,(p \otimes J\theta_M + q \otimes \theta_M) \wedge (p \otimes \mathbf{J} + q \otimes \mathbf{1})\,.
\end{aligned}$$

In the second and fourth line we use that J is conjugate linear. By summing up the four equations, some terms cancel, and we obtain

$$\begin{aligned}
\omega^{\mathbb{H}^n}_{\widehat{M}} = \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\Big(&(p \otimes \theta_M) \wedge (q \otimes \mathbf{J}) - (q \otimes \theta_M) \wedge (p \otimes \mathbf{J}) \\
&- (p \otimes J\theta_M) \wedge (q \otimes \mathbf{1}) + (q \otimes J\theta_M) \wedge (p \otimes \mathbf{1})\Big)\,.
\end{aligned}$$

Applying the proposition above yields the following equivalence

$$\begin{aligned}
\omega^{\mathbb{H}^n}_{\widehat{M}} &= \sqrt{\tfrac{\kappa}{16n(n+2)}}\,(\mathrm{id} \otimes \theta_M \cdot \mathbf{J} - \mathrm{id} \otimes J\theta_M \cdot \mathbf{1}) \\
&= \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\mathrm{id} \otimes (\mathbf{J} \otimes \theta_M - \mathbf{1} \otimes J\theta_M)\star \quad = \quad \sqrt{\tfrac{\kappa}{8n(n+2)}}\,\mathrm{id} \otimes \theta_M^{H \otimes E} \star\,. \qquad \square
\end{aligned}$$

7 Proof of the Theorem

The fact that a quaternionic Killing spinor on M translates into a parallel spinor on the hyperkähler manifold $\widehat{M}$ is crucial to the proof of the main theorem.

Theorem 1 *Let M be a compact quaternionic Kähler manifold of quaternionic dimension n and positive scalar curvature $\kappa > 0$. If there is an eigenspinor for the Dirac operator with eigenvalue λ satisfying*

$$\lambda^2 = \frac{\kappa}{4}\frac{n+3}{n+2}\,,$$

then M is isometric to the quaternionic projective space.

Proof. After the work done in the preceding sections the proof of this theorem reduces to a simple holonomy argument. The spinor bundle of the hyperkähler manifold $\widehat{M}$ is associated to the $\mathbf{Sp}(n)$–representation

$$\Sigma = \bigoplus_{r=0}^{n+1} \Sigma_r = \bigoplus_{r=0}^{n+1} \mathrm{Sym}^r\mathbb{C}^2 \otimes \Lambda_\circ^{n+1-r} F\,,$$

where $F = H \oplus E$, and $\mathbf{Sp}(n)$ operates trivially on H. The holonomy of $\widehat{M}$ is contained in $\mathbf{Sp}(n+1)$ and operates trivially on $\mathrm{Sym}^r\mathbb{C}^2$. The subbundle $\mathbf{S}_{n+1}(\widehat{M})$ of $\mathbf{S}(\widehat{M})$ associated to $\Sigma_{n+1} = \mathrm{Sym}^{n+1}\mathbb{C}^2$ is consequently trivialized by $n+2 = \dim\,(\mathrm{Sym}^{n+1}\mathbb{C}^2)$ linearly independent parallel spinors.

If M admits a quaternionic Killing spinor, there are additional parallel spinors on $\widehat{M}$, which are sections of the subbundle $\mathbf{S}_1(\widehat{M})$ associated to the representation $\mathbb{C}^2 \otimes \Lambda^n_\circ F$.

Due to a result of Wang [Wan89], on a manifold with holonomy equal to $\mathbf{Sp}(n+1)$ there are exactly $n+2$ linearly independent parallel spinors, just those trivializing $\mathbf{S}_{n+1}(\widehat{M})$. The additional parallel spinor constructed out of a quaternionic Killing spinor reduces the holonomy further. According to Berger's list this can only happen if $\widehat{M}$ is reducible or locally symmetric. In the first case, as consequence of a theorem of Gallot [Gal79], $\widehat{M}$ has to be flat. But so it is in the second case, because it is hyperkähler, hence Ricci–flat, and in addition locally symmetric. Therefore $\widehat{M}$ is flat which forces M to be isometric to the quaternionic projective space. $\square$

A Spinors on Cones

In this appendix we will describe how to lift spinors on a Riemannian spin manifold N to spinors on its cone $\widehat{N} := \mathbb{R}^+ \times N$. In particular, we will show that Killing spinors on N translate into parallel spinors on $\widehat{N}$. This construction is originally due to C. Bär [Bär93].

Let (N, g_N) be a spin manifold of dimension n and let $\pi : \widehat{N} = \mathbb{R}^+ \times N \to N$ be the cone over N endowed with the warped product metric $g_{\widehat{N}} = dt^2 + e^{2\lambda}\pi^* g_N$. The soldering and connection form on the principal bundle $P := P_{\mathbf{Spin}(n)}N$ associated to the chosen spin structure are denoted by θ_N and ω_N. Obviously, the cone is again spin and the spin structure reduces to the principal $\mathbf{Spin}(n)$–bundle $\widehat{P} := \mathbb{R}^+ \times P$. Forms on P give rise to forms on $\widehat{P}$ by extending them constantly along the $\mathbb{R}^+$–direction. It easy to see that the soldering resp. the connection form of $\widehat{N}$ on $\widehat{P}$ are given by

$$\theta_{\widehat{N}} = dt + e^\lambda \theta_N \quad \text{resp.} \quad \omega_{\widehat{N}} = \omega_N - e^\lambda \frac{\partial \lambda}{\partial t}\, \theta_N \wedge \Xi,$$

where $\Xi = \frac{\partial}{\partial t}$ denotes the vertical unit vector.

A spinor ψ on N can be interpreted as a $\mathbf{Spin}(n)$–equivariant function on P with values in the spinor module Σ_n with a fixed Clifford module structure. With the canonical isomorphism $\mathbb{C}\mathbf{l}_n \cong \mathbb{C}\mathbf{l}^0_{n+1}$, $e_i \mapsto e_i \cdot \Xi$ in mind, we can consider Σ_n as an $\mathbb{C}\mathbf{l}^0_{n+1}$– and therefore as an $\mathbf{Spin}(n+1)$–representation. The values of ψ have now to be interpreted as lying in the $\mathbf{Spin}(n+1)$–representation. Let ψ be a Killing spinor on N. Interpreted as function on P, it satisfies:

$$d\psi + (\omega_N - \mu\theta_N)\,\psi \;=\; 0\,,$$

where μ is the Killing constant. Extending ψ constantly in $\mathbb{R}^+$–direction and using the isomorphism of the Clifford algebras above defines a spinor on $\widehat{N}$. As a function on $\widehat{P}$, it satisfies

$$d\psi + (\omega_N - \mu\pi^*\theta_N \Xi)\,\psi \;=\; 0\,.$$

If the warping function is chosen such that $\mu = \frac{1}{2} e^\lambda \frac{\partial\lambda}{\partial t}$, the expression in brackets is equal to $\omega_{\widehat{N}}$, and therefore ψ has to be parallel on $\widehat{N}$. The other way round, it is also clear that, if ψ is a parallel spinor on $\widehat{N}$, then its associated function on $\widehat{P}$ is constant in $\mathbb{R}^+$–direction and it projects onto a Killing spinor on N.

B The Curvature Tensor of $\widehat{M}$

In this appendix we will prove of Proposition 4.16 relating the curvature tensor of $\widehat{M}$ to the hyperkähler part R^{hyper} of the curvature of M. We recall that the connection form of $\widehat{M}$ restricted to $P_{\widehat{M}}$ is pulled back from P and so is its curvature $\Omega_{\widehat{M}}$. We have seen in section 6.1 that the most convenient way to read the connection form $\omega_{\widehat{M}}$ given in Lemma 4.15 is

$$\omega_{\widehat{M}} \;=\; \omega^{\mathfrak{sp}(1)}_{\widehat{M}} \;+\; \omega^{\mathfrak{sp}(n)}_{\widehat{M}} \;+\; \omega^{\mathbb{H}^n}_{\widehat{M}}\;,$$

where $\omega^{\mathfrak{sp}(1)}_{\widehat{M}}$ and $\omega^{\mathfrak{sp}(n)}_{\widehat{M}}$ are the pull–backs of $\omega^{\mathfrak{sp}(1)}_{M}$ and $\omega^{\mathfrak{sp}(n)}_{M}$ respectively, and

$$\omega^{\mathbb{H}^n}_{\widehat{M}} \;:=\; \sqrt{\tfrac{\kappa}{16n(n+2)}}\,\big(\theta_M \wedge \mathbf{1} + i\theta_M \wedge \mathbf{I} + j\theta_M \wedge \mathbf{J} + k\theta_M \wedge \mathbf{K}\big)\,.$$

Defining $[\alpha \wedge \beta](X, Y) := [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)] = [\beta \wedge \alpha](X, Y)$ for Lie algebra valued 1–forms α, β, the curvature 2–form Ω_M of M on P can be written

$$\begin{aligned}\Omega_M \;&=\; d\omega_M + \tfrac{1}{2}\big[\omega_M \wedge \omega_M\big]\\ &=\; \Big(d\omega^{\mathfrak{sp}(1)}_M + \tfrac{1}{2}\big[\omega^{\mathfrak{sp}(1)}_M \wedge \omega^{\mathfrak{sp}(1)}_M\big]\Big) + \Big(d\omega^{\mathfrak{sp}(n)}_M + \tfrac{1}{2}\big[\omega^{\mathfrak{sp}(n)}_M \wedge \omega^{\mathfrak{sp}(n)}_M\big]\Big)\,.\end{aligned}$$

Of course, $[\omega^{\mathfrak{sp}(1)}_M \wedge \omega^{\mathfrak{sp}(n)}_M] = 0$ since $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$ centralize each other in $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$. Using the naturality of the exterior differential and $\omega^{\mathfrak{sp}(1)}_M = \pi^*_S\omega$ we conclude that the first summand is equal to $\pi^*_S\Omega$.

Thus, it corresponds to $-\frac{\kappa}{8n(n+2)}\,R^H$ in the sense of decomposition (3.8) as shown in the proof of Lemma 3.10. We conclude that the second summand corresponds to $-\frac{\kappa}{8n(n+2)}\,R^E\ +\ R^{hyper}$. The curvature $\Omega_{\widehat{M}}$ of $\widehat{M}$ can be calculated similarly. With $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$ centralizing each other in $\mathfrak{sp}(n+1)$ we still have $[\omega_{\widehat{M}}^{\mathfrak{sp}(1)}\wedge\omega_{\widehat{M}}^{\mathfrak{sp}(n)}]=0$, and $\Omega_{\widehat{M}}$ is the sum of the three terms:

$$\begin{array}{ccccc} d\omega_{\widehat{M}}^{\mathfrak{sp}(1)} & + & \frac{1}{2}\,[\omega_{\widehat{M}}^{\mathfrak{sp}(1)}\wedge\omega_{\widehat{M}}^{\mathfrak{sp}(1)}] & + & \frac{1}{2}\,[\omega_{\widehat{M}}^{\mathbb{H}^n}\wedge\omega_{\widehat{M}}^{\mathbb{H}^n}]^{\mathfrak{sp}(1)} \\ d\omega_{\widehat{M}}^{\mathbb{H}^n} & + & [\omega_{\widehat{M}}^{\mathfrak{sp}(1)}\wedge\omega_{\widehat{M}}^{\mathbb{H}^n}] & + & [\omega_{\widehat{M}}^{\mathfrak{sp}(n)}\wedge\omega_{\widehat{M}}^{\mathbb{H}^n}] \\ d\omega_{\widehat{M}}^{\mathfrak{sp}(n)} & + & \frac{1}{2}\,[\omega_{\widehat{M}}^{\mathfrak{sp}(n)}\wedge\omega_{\widehat{M}}^{\mathfrak{sp}(n)}] & + & \frac{1}{2}\,[\omega_{\widehat{M}}^{\mathbb{H}^n}\wedge\omega_{\widehat{M}}^{\mathbb{H}^n}]^{\mathfrak{sp}(n)} \end{array} \tag{B.27}$$

where $[\omega_{\widehat{M}}^{\mathbb{H}^n}\wedge\omega_{\widehat{M}}^{\mathbb{H}^n}]$ is projected onto its two components in $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$ according to the Cartan decomposition $\mathfrak{sp}(n+1)=(\mathfrak{sp}(1)\oplus\mathfrak{sp}(n))\oplus\mathbb{H}^n$. Using the formula

$$[\,a_1\wedge b_1,\,a_2\wedge b_2\,]=\langle a_1,a_2\rangle b_1\wedge b_2-\langle b_1,a_2\rangle a_1\wedge b_2-\langle a_1,b_2\rangle b_1\wedge a_2+\langle b_1,b_2\rangle a_1\wedge a_2$$

we find

$$\begin{aligned}
&\tfrac{1}{2}[\omega_{\widehat{M}}^{\mathbb{H}^n}\wedge\omega_{\widehat{M}}^{\mathbb{H}^n}]\\
&=\ \tfrac{\kappa}{32n(n+2)}\Big(\theta_M\wedge\theta_M\ +\ i\theta_M\wedge i\theta_M\ +\ j\theta_M\wedge j\theta_M\ +\ k\theta_M\wedge k\theta_M\\
&\qquad\qquad+2\langle\theta_M\wedge i\theta_M\rangle\mathbf{1}\wedge\mathbf{I}\ +\ 2\langle\theta_M\wedge j\theta_M\rangle\mathbf{1}\wedge\mathbf{J}\ +\ 2\langle\theta_M\wedge k\theta_M\rangle\mathbf{1}\wedge\mathbf{K}\\
&\qquad\qquad+2\langle i\theta_M\wedge j\theta_M\rangle\mathbf{I}\wedge\mathbf{J}\ +\ 2\langle i\theta_M\wedge k\theta_M\rangle\mathbf{I}\wedge\mathbf{K}\ +\ 2\langle j\theta_M\wedge k\theta_M\rangle\mathbf{J}\wedge\mathbf{K}\Big)\\
&=\ \tfrac{\kappa}{32n(n+2)}\Big(\theta_M\wedge\theta_M\ +\ i\theta_M\wedge i\theta_M\ +\ j\theta_M\wedge j\theta_M\ +\ k\theta_M\wedge k\theta_M\Big) && \text{(B.28)}\\
&\quad+\tfrac{\kappa}{16n(n+2)}\Big(\langle\theta_M\wedge i\theta_M\rangle(\mathbf{1}\wedge\mathbf{I}-\mathbf{J}\wedge\mathbf{K})+\langle\theta_M\wedge j\theta_M\rangle(\mathbf{1}\wedge\mathbf{J}-\mathbf{K}\wedge\mathbf{I})+\langle\theta_M\wedge k\theta_M\rangle(\mathbf{1}\wedge\mathbf{K}-\mathbf{I}\wedge\mathbf{J})\Big) && \text{(B.29)}
\end{aligned}$$

We remark that by construction of the base $\mathbf{1}$, $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ the infinitesimal action of i, j and $k\in\mathfrak{sp}(1)$ on H can be written

$$\begin{array}{llllllllll} i: & \mathbf{1}\mapsto-\mathbf{I} & \mathbf{J}\mapsto\ \ \mathbf{K} & \quad j: & \mathbf{1}\mapsto-\mathbf{J} & \mathbf{K}\mapsto\ \ \mathbf{I} & \quad k: & \mathbf{1}\mapsto-\mathbf{K} & \mathbf{I}\mapsto\ \ \mathbf{J} \\ & \mathbf{I}\mapsto\ \ \mathbf{1} & \mathbf{K}\mapsto-\mathbf{J} & & \mathbf{J}\mapsto\ \ \mathbf{1} & \mathbf{I}\mapsto-\mathbf{K} & & \mathbf{K}\mapsto\ \ \mathbf{1} & \mathbf{J}\mapsto-\mathbf{I} \end{array},$$

i. e. i corresponds to $-(\mathbf{1}\wedge\mathbf{I}-\mathbf{J}\wedge\mathbf{K})$. Thus, the summand (B.29) is equal to $-\pi_S^*\Omega$ according to Lemma 3.10, i. e. equal to $\frac{\kappa}{8n(n+2)}R^H$. Without proof we state that the summand (B.28) is equal to $\frac{\kappa}{8n(n+2)}R^E$ as this is certainly true on the quaternionic projective space. Consequently, in the decomposition (B.27) of $\Omega_{\widehat{M}}$ the first term vanishes as does the second, because straightforward calculations show that it depends linearly on the torsion of the Levi–Civitá connection ω_M of M. Hence, the curvature $\Omega_{\widehat{M}}$ reduces to

$$\Omega_{\widehat{M}}=d\omega_{\widehat{M}}^{\mathfrak{sp}(n)}\ +\ \tfrac{1}{2}\,[\omega_{\widehat{M}}^{\mathfrak{sp}(n)}\wedge\omega_{\widehat{M}}^{\mathfrak{sp}(n)}]\ +\ \tfrac{\kappa}{8n(n+2)}R^E=R^{hyper}\,.$$

In this way $\Omega_{\widehat{M}}$ operates only on the subbundle $P_{\widehat{M}}\times_{\mathbf{Sp}(n)}E$ of the tangent bundle of $\widehat{M}$. We remark that this subbundle is canonically isomorphic to $P_{\widehat{M}}\times_{\mathbf{Sp}(n)}T$, i. e. to $\widehat{\pi}^*TM$. □

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